DATA MINING
DATA EXPLORATION AND STATISTICS

Exploratory data analysis

Basic Statistics

Exploratory data analysis

- In many cases after collecting the data we want to know "what do the data look like?"
- This simple question is hard to answer when dealing with millions of records with millions of attributes
- To answer it we perform measurements that capture properties of the data and give an aggregate picture
- We also produce plots with distributions of these metrics

Exploratory analysis of data – Summary Statistics

- Summary statistics: numbers that summarize properties of the data

- Summarized properties include frequency, location and spread
  - Examples: location - mean
    spread - standard deviation

- Most summary statistics can be calculated in a single pass through the data

- Computing data statistics is one of the first steps in understanding our data

Frequency and Mode

- The frequency of an attribute value is the percentage of time the value occurs in the data set
  - For example, given the attribute ‘gender’ and a representative population of people, the gender ‘female’ occurs about 50% of the time.
- The mode of an attribute is the most frequent attribute value
- The notions of frequency and mode are typically used with categorical data or discrete numerical data

- We can visualize the data frequencies using a value histogram
- Frequency, and frequency histogram are the empirical analogues of probability and probability distribution

Example

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Cheat**
1	Yes	Single	125K	**No**
2	No	Married	100K	**No**
3	No	Single	70K	**No**
4	Yes	Married	120K	**No**
5	No	Divorced	10000K	**Yes**
6	No	NULL	60K	**No**
7	Yes	Divorced	220K	**NULL**
8	No	Single	85K	**Yes**
9	No	Married	90K	**No**
10	No	Single	90K	**No**

Marital Status

Single	Married	Divorced	NULL
4	3	2	1

Attribute value frequencies

Mode: Single

Example

Tid	Refund	Marital Status	Taxable Income	Cheat
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	10000K	Yes
6	No	NULL	60K	No
7	Yes	Divorced	220K	NULL
8	No	Single	85K	Yes
9	No	Married	90K	No
10	No	Single	90K	No

Marital Status

Single	Married	Divorced	NULL
40%	30%	20%	10%

Attribute value distribution

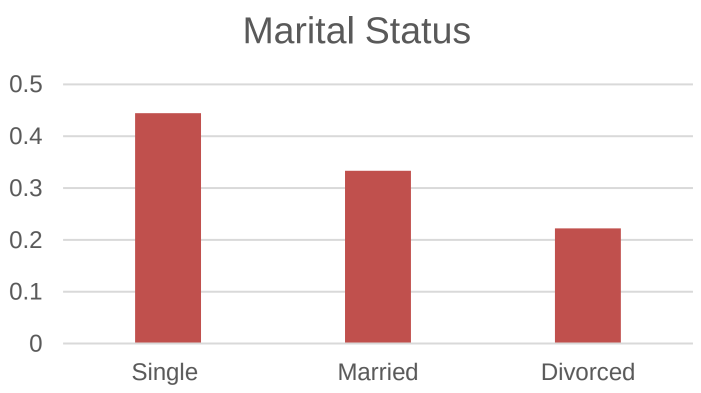
Example

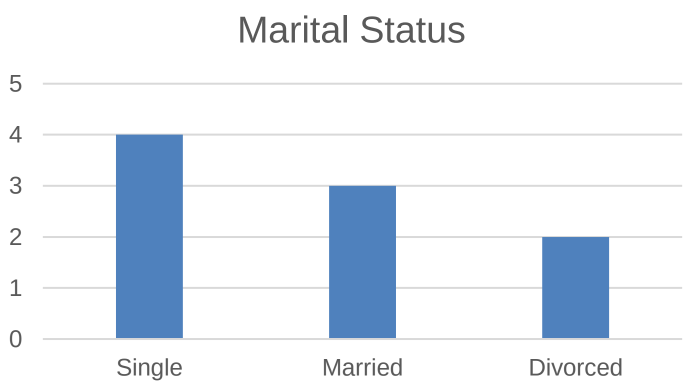
Tid	Refund	Marital Status	Taxable Income	Cheat
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	10000K	Yes
6	No	NULL	60K	No
7	Yes	Divorced	220K	NULL
8	No	Single	85K	Yes
9	No	Married	90K	No
10	No	Single	90K	No

We can choose to ignore NULL values

Marital Status

Single	Married	Divorced
45%	33%	22%





Attribute value histogram
(we could also plot the frequency values)

Data histograms

Tid	Refund	Marital Status	Taxable Income	Cheat
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	10000K	Yes
6	No	NULL	60K	No
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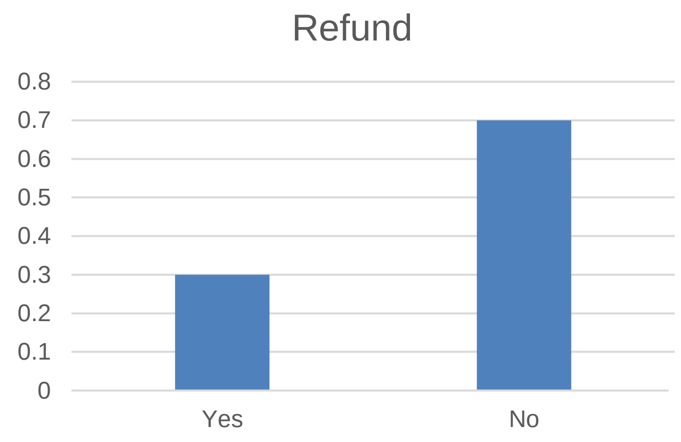

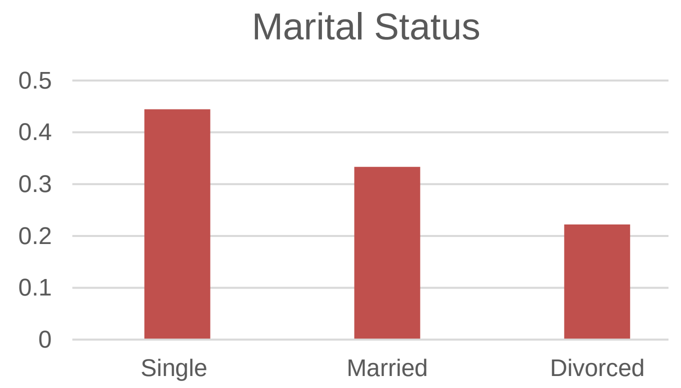

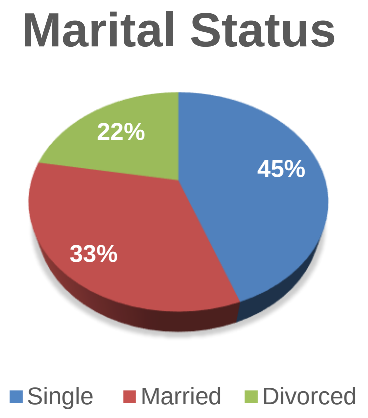

Data histograms

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Cheat**
1	Yes	Single	125K	**No**
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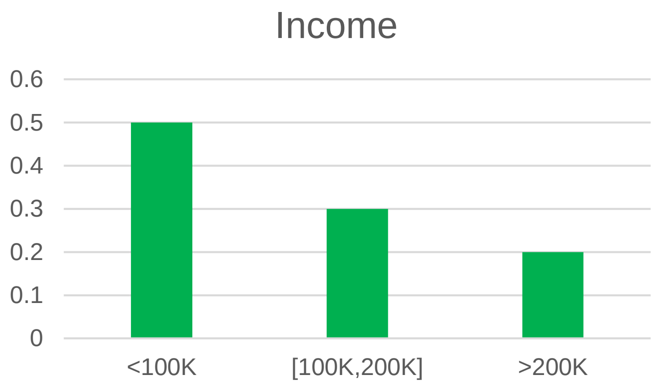
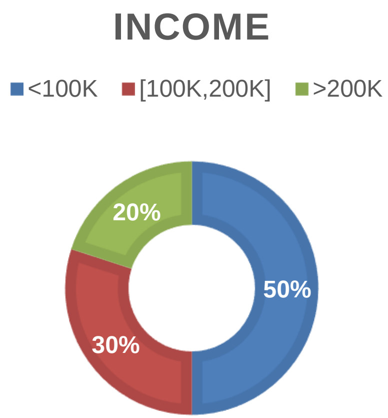
For real numerical values we use binning to create the histogram





In most plotting libraries, we specify the number of bins and the method creates an equiwidth histogram

Percentiles

- For continuous data, the notion of a percentile is more useful.

Given an ordinal or continuous attribute x and a number p between 0 and 100, the p^{th} percentile is a value x_p of x such that p% of the observed values of x are less or equal than x_p.

- For instance, the 80th percentile is the value $x_{80\%}$ that is greater or equal than 80% of all the values of x we have in our data.

- The percentiles are the empirical analogue of the cumulative probability distribution function

Example

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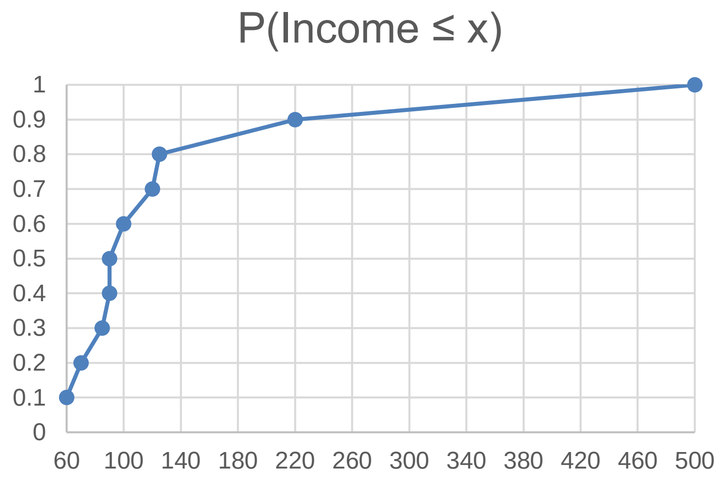
7	Yes	Divorced	220K	**NULL**
8	No	Single	85K	**Yes**
9	No	Married	90K	**No**
10	No	Single	90K	**No**

	Taxable Income
1	10000K
2	220K
3	125K
4	120K
5	100K
6	90K
7	90K
8	85K
9	70K
10	60K

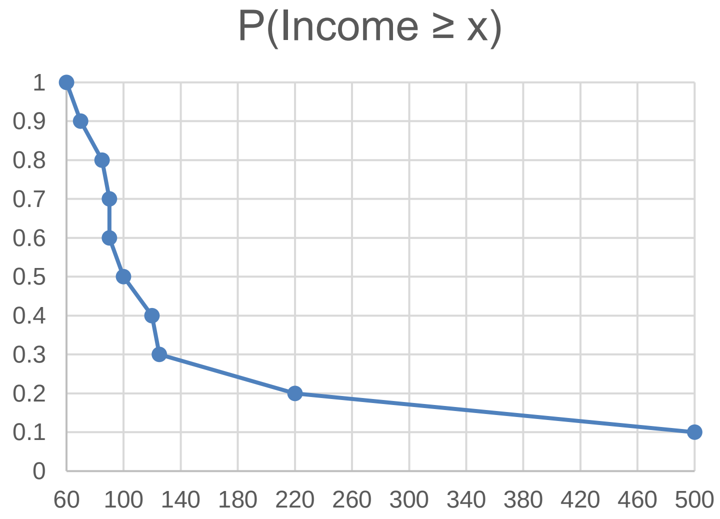
$x_{80\%}$ = 125K

Plotting the cumulative distribution

	Taxable Income
1	500K
2	220K
3	125K
4	120K
5	100K
6	90K
7	90K
8	85K
9	70K
10	60K



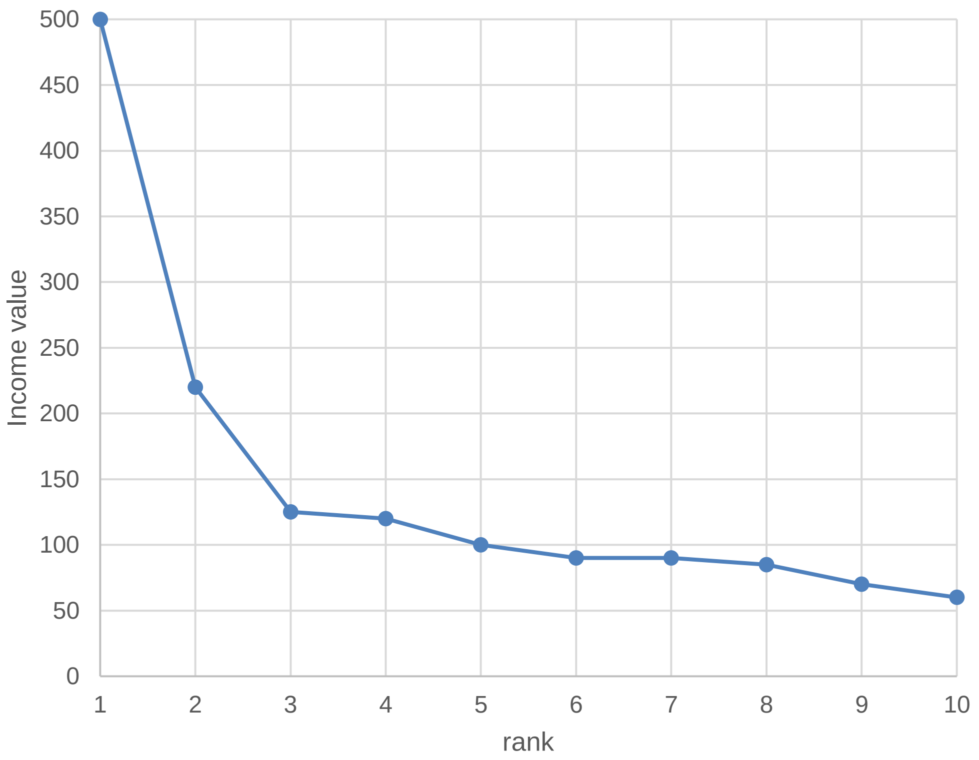
Plotting the fraction of entries that have value less or equal to x, for all possible values x of income in the data



Plotting the fraction of entries that have value greater or equal to x, for all possible values x of income in the data

Rank-Value plot

	Taxable Income
1	500K
2	220K
3	125K
4	120K
5	100K
6	90K
7	90K
8	85K
9	70K
10	60K



Plotting the values of the income (y-axis) against their rank (x-axis)

The rank of a value is its order when all values are sorted in decreasing order

Also known as Zipf plot

Frequency-count plots

- In some cases, we have to put some more work
- Example: market-basked data

Id	Basket contents
1	milk, coffee
2	milk, coffee, sugar
3	milk, coffee, sugar, cookies
4	milk, tea, bread, butter, jam
5	milk, bread, butter, honey
6	milk, cream, honey, flour, eggs
7	milk, coffee, eggs, bacon
8	milk
9	milk, coffee, sugar, eggs, bacon, bread
10	eggs, bacon, bread

How do we describe this data?

Frequency-count plots

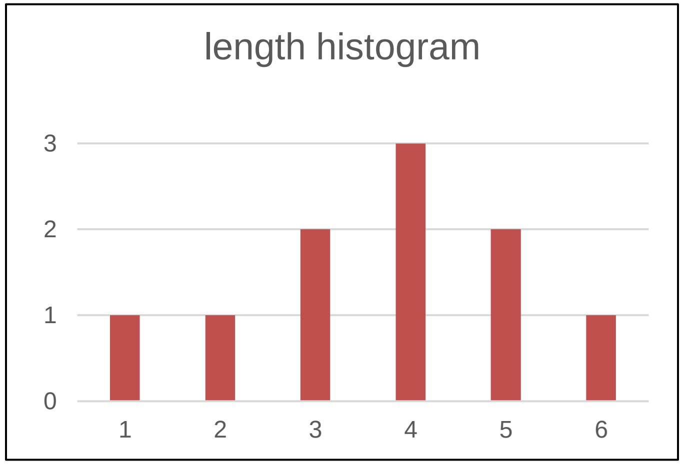
- Example: market-basked data

Id	Basket contents
1	milk, coffee
2	milk, coffee, sugar
3	milk, coffee, sugar, cookies
4	milk, tea, bread, butter, jam
5	milk, bread, butter, honey
6	milk, cream, honey, flour, eggs
7	milk, coffee, eggs, bacon
8	milk
9	milk, coffee, sugar, eggs, bacon, bread
10	eggs, bacon, bread

Basket length

Id	length
1	2
2	3
3	4
4	5
5	4
6	5
7	4
8	1
9	6
10	3

length	count
1	1
2	1
3	2
4	3
5	2
6	1



Frequency-count plots

- Example: market-basked data

Id	Basket contents
1	milk, coffee
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3	milk, coffee, sugar, cookies
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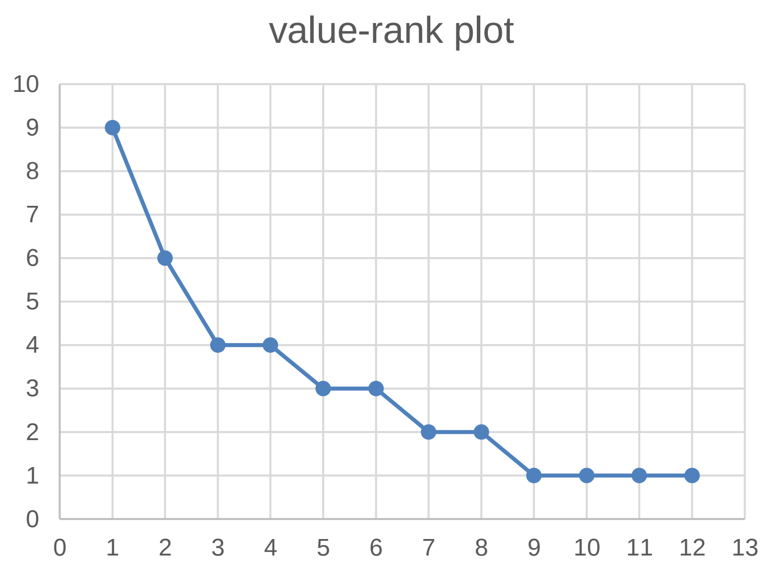
6	milk, coffee, cream, honey, eggs
7	milk, coffee, eggs, bacon
8	milk
9	milk, coffee, sugar, eggs, bacon, bread
10	eggs, bacon, bread

Item counts

Item	count
milk	9
coffee	6
eggs	4
bread	4
sugar	3
bacon	3
butter	2
honey	2
cookies	1
tea	1
jam	1
cream	1



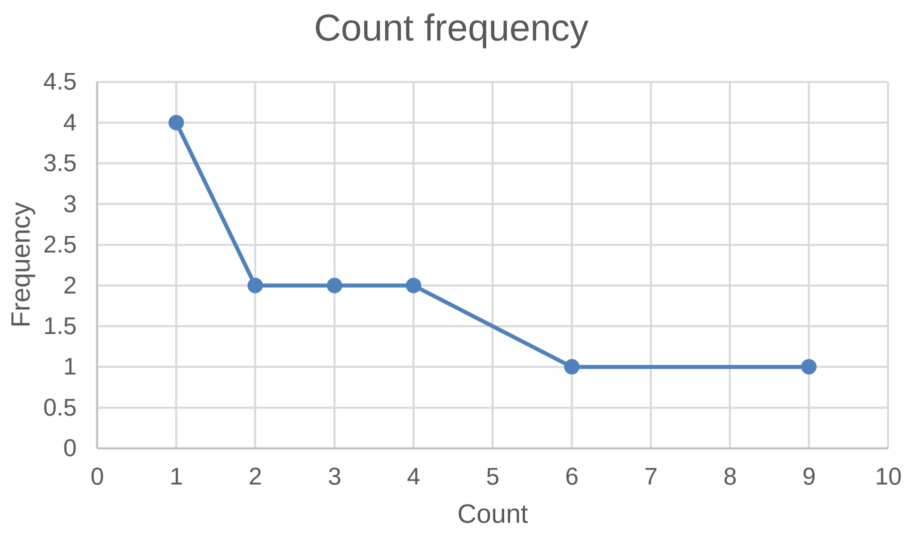
Frequency-count plots

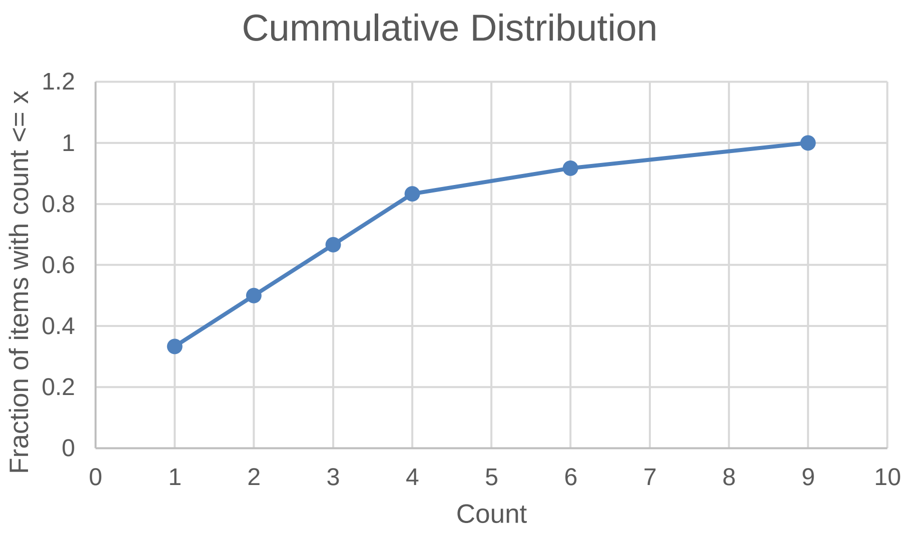
- Example: market-basked data

Item	count
milk	9
coffee	6
eggs	4
bread	4
sugar	3
bacon	3
butter	2
honey	2
cookies	1
tea	1
jam	1
cream	1

Count histogram

Count	Frequency
1	4
2	2
3	2
4	2
6	1
9	1





Measures of Location: Mean and Median

- The mean is the most common measure of the location of a set of points.

$$\text{mean}(x) = \overline{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$$

- However, the mean is very sensitive to outliers.
- Thus, the median is also commonly used.

$$\text{median}(x) = \begin{cases} x_{(r+1)} & \text{if } m \text{ is odd, i.e., } m = 2r+1 \\ \frac{1}{2}(x_{(r)} + x_{(r+1)}) & \text{if } m \text{ is even, i.e., } m = 2r \end{cases}$$

- Or the trimmed mean: the mean after removing min and max values

Example

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10	No	Single	90K	**No**

Mean: 1090K

Trimmed mean (remove min, max): 105K

Median: (90+100)/2 = 95K

Measures of Spread: Range and Variance

- Range is the difference between the max and min

- The variance or standard deviation is the most common measure of the spread of a set of points.

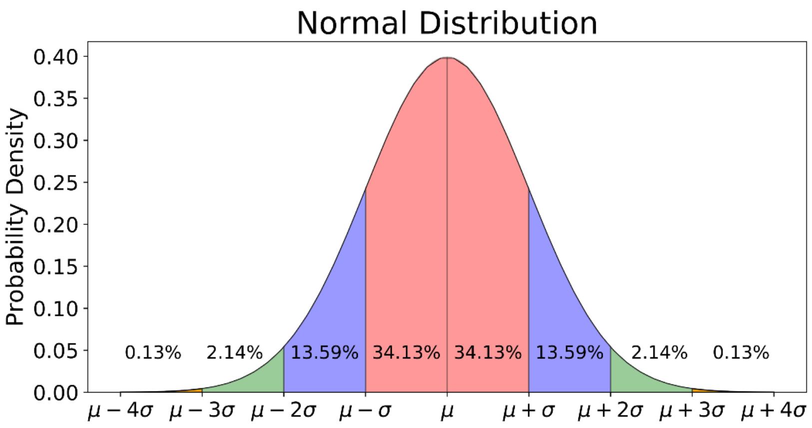
$$var(x) = \frac{1}{m-1}\sum_{i=1}^{m}(x - \bar{x})^2$$

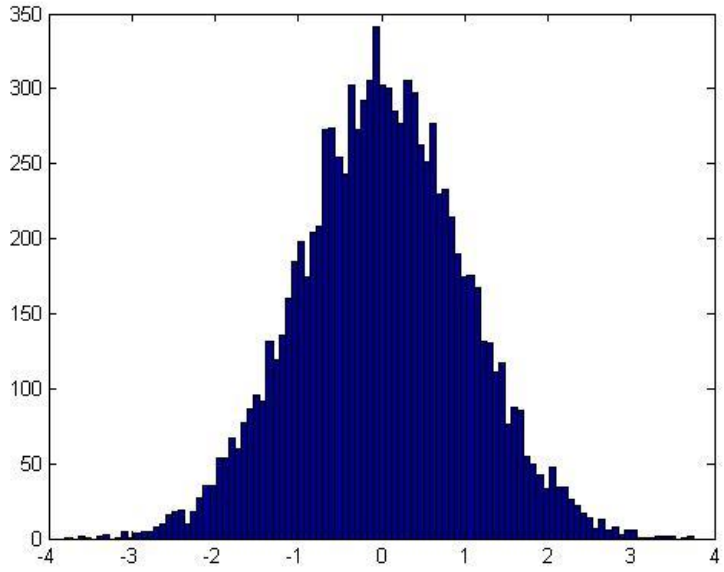
$$\sigma(x) = \sqrt{var(x)}$$

m or $m-1$?
When computing the sample variance m-1 is used
For large data it does not make much difference

Normal Distribution

- $\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$





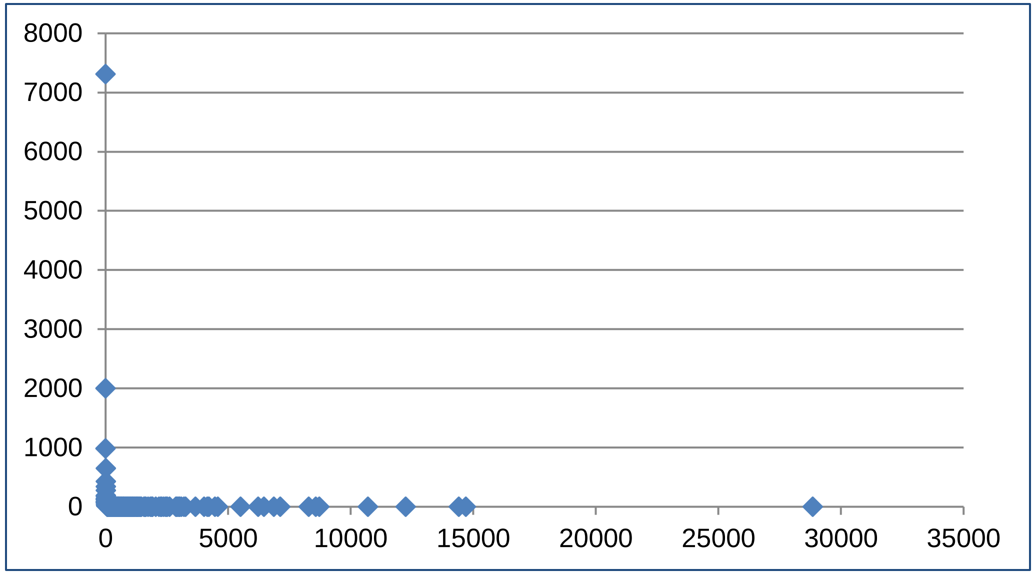
This is a value histogram

- An important distribution that characterizes many quantities and has a central role in probabilities and statistics.
- Appears also in the central limit theorem: the distribution of the sum of IID random variables.
- Fully characterized by the mean μ and standard deviation σ

Not everything is normally distributed

- Plot of number of words with x number of occurrences

y: number of words with x number of occurrences



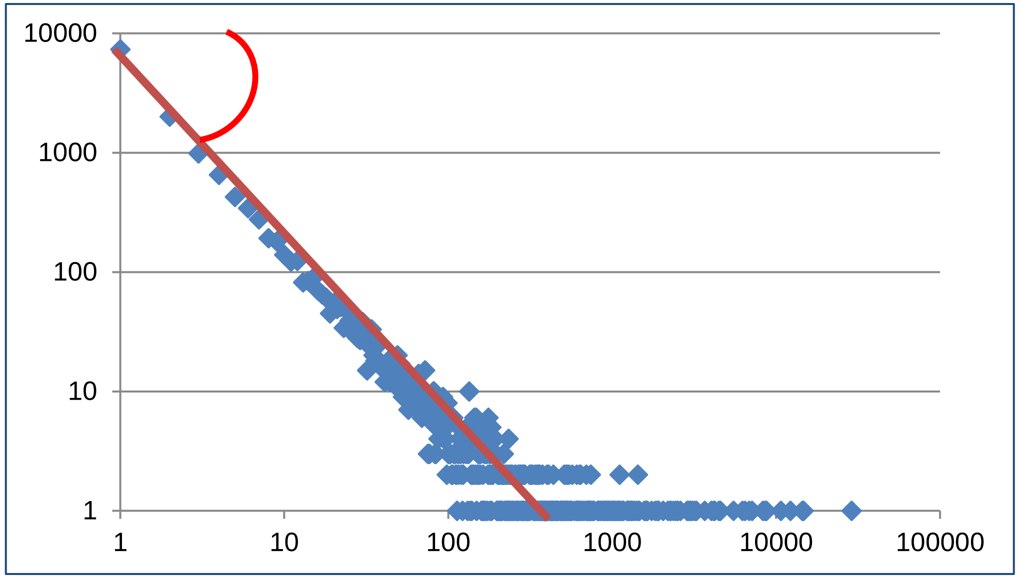
x: number of occurrences

- If this was a normal distribution we would not have number of occurrences as large as 28K

Power-law distribution

- We can understand the distribution of words if we take the log-log plot

y: logarithm of number of words with x number of occurrences



x: logarithm of number of occurrences

Power-law distribution:

$$p(k) = k^{-a}$$

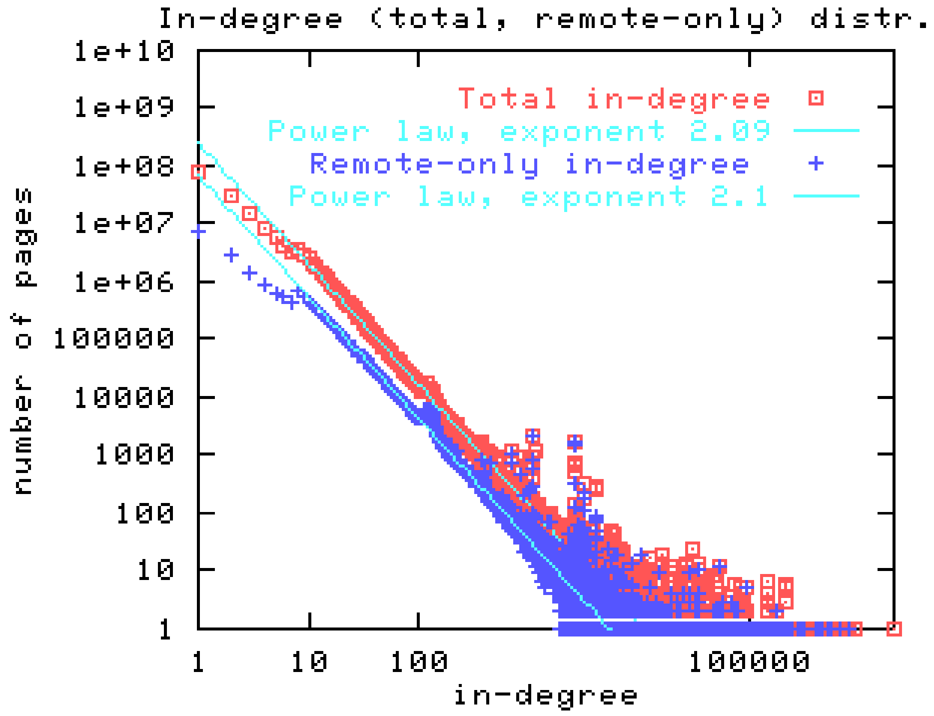
The slope of the line gives us the exponent α

Linear relationship in the log-log space

$$\log p(x = k) = -a \log k$$

Power-laws are everywhere

- Incoming and outgoing links of web pages, number of friends in social networks, number of occurrences of words, file sizes, city sizes, income distribution, popularity of products and movies
  - Signature of human activity?
  - A mechanism that explains everything?
  - Rich get richer process

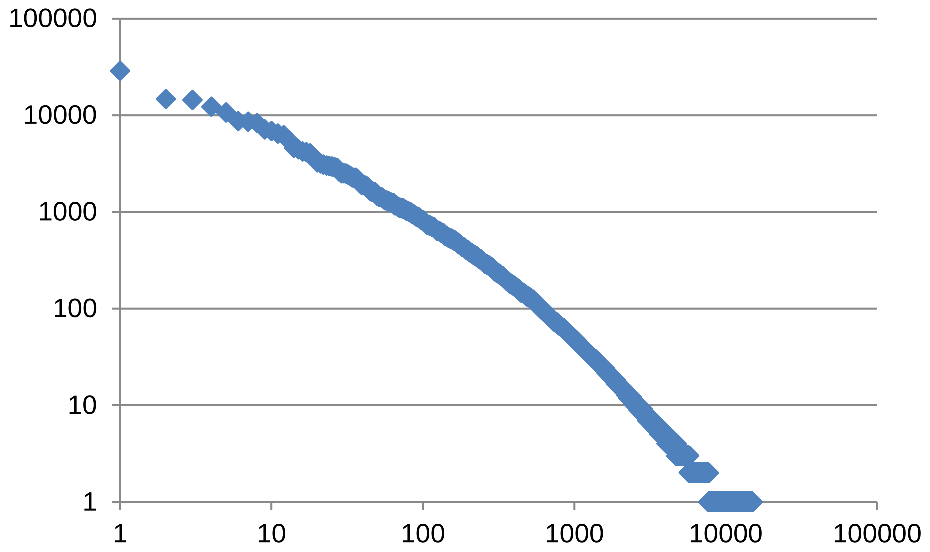


- Related distribution: log-normal
  - Taking the log of the values gives a normal distribution

Zipf's law

- Power laws can be detected also by a linear relationship in the log-log space for the rank-value plot

y: number of occurrences of the r-th most frequent word



r: rank of word according to frequency (1st, 2nd …)

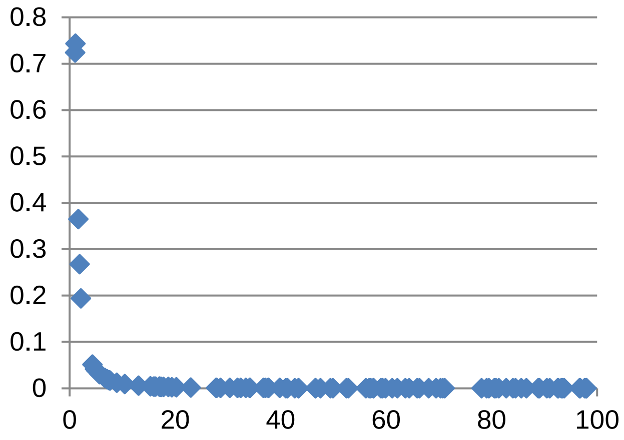
Zipf distribution:
$f(r) = r^{-\beta}$

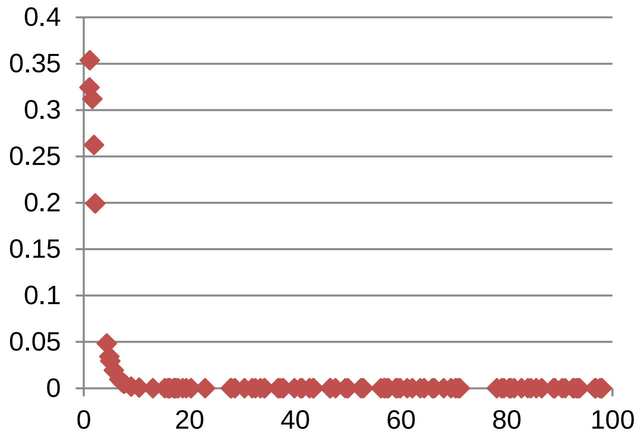
- $f(r)$: Frequency of the r-th most frequent word

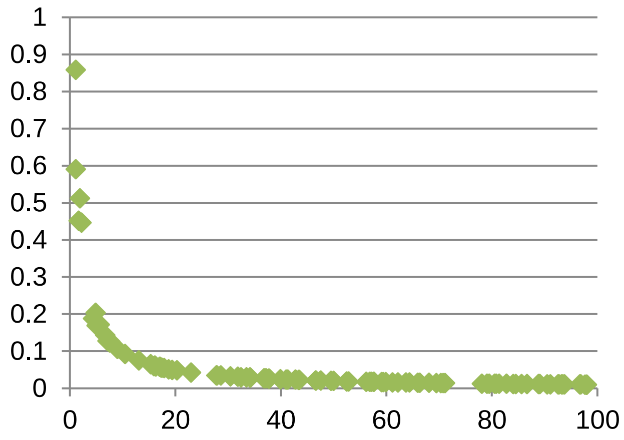
$$\log f(r) = -\beta \log r$$

The importance of correct representation

- Consider the following three plots which are histograms of values. What do you observe? What can you tell of the underlying function?

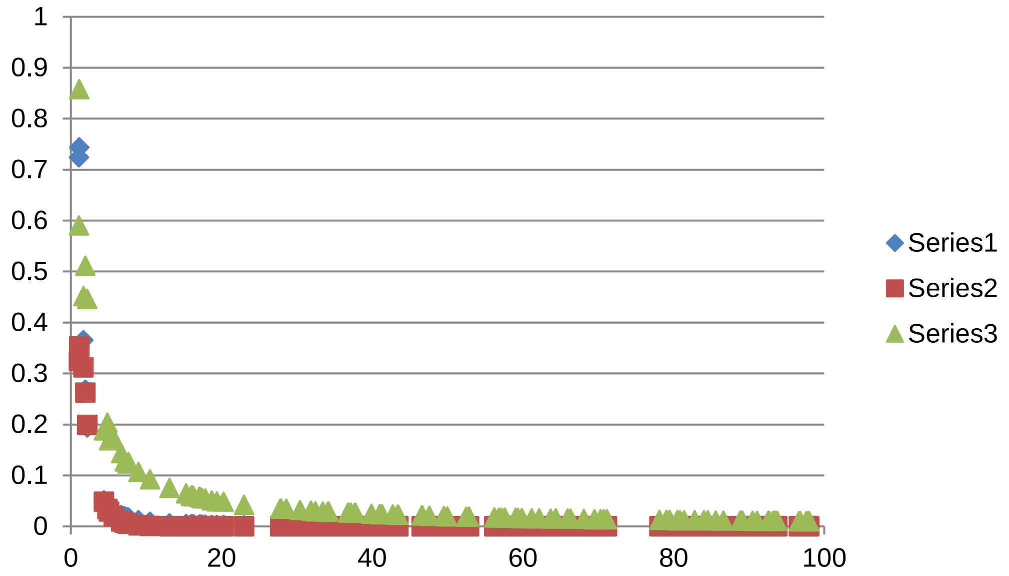






The importance of correct representation

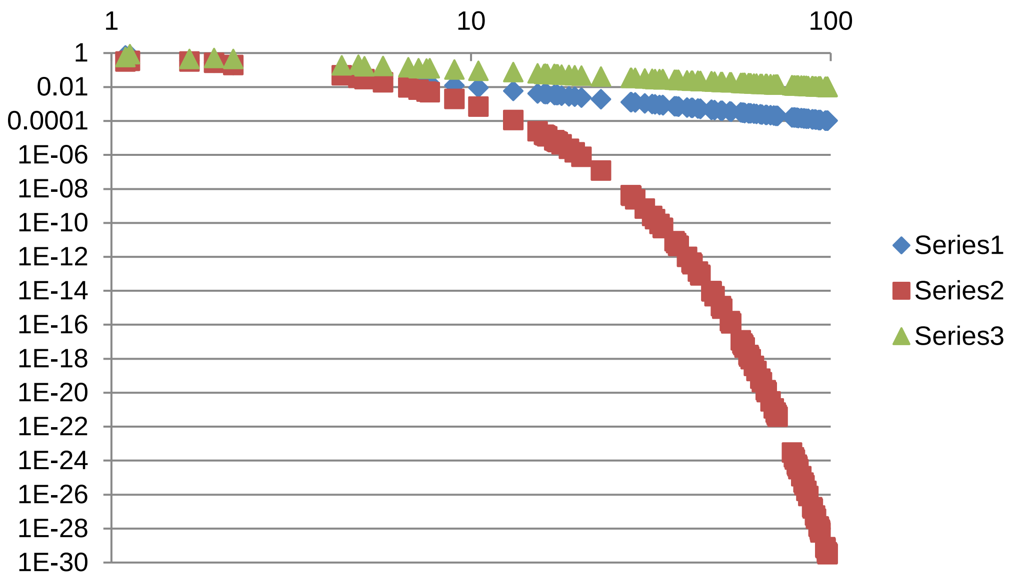
- Putting all three plots together makes it clearer to see the differences



- Green falls more slowly. Blue and Red seem more or less the same

The importance of correct representation

- Making the plot in log-log space makes the differences more clear



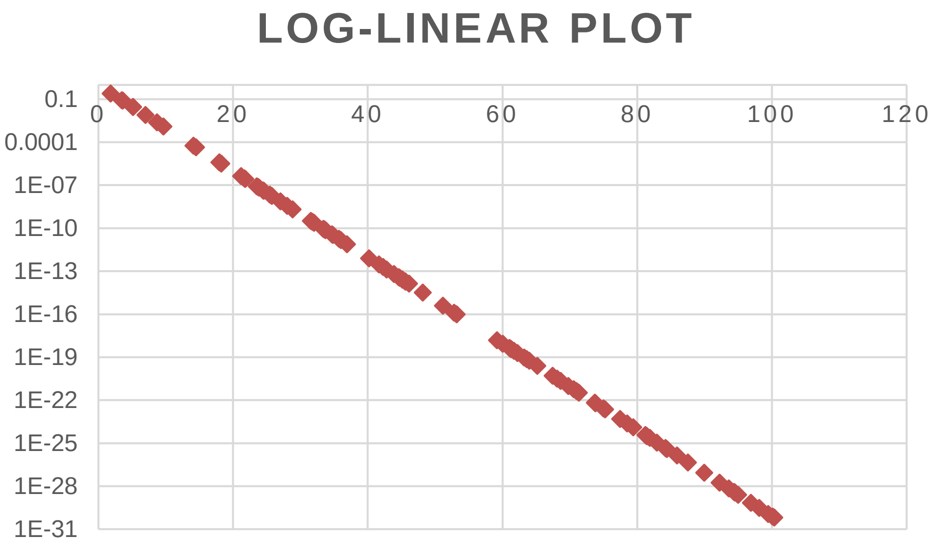
Linear relationship in log-log means polynomial in linear-linear
The slope in the log-log is the exponent of the polynomial

Exponential relationship remains exponential in log-log

- Green and Blue form straight lines. Red drops exponentially.
  - $y = \frac{1}{2x+\epsilon}$ $\log y \approx -\log x + c$
  - $y = \frac{1}{x^2+\epsilon}$ $\log y \approx -2\log x + c$
  - $y = 2^{-x} + \epsilon$ $\log y \approx -x + c = -10^{\log x} + c$

The importance of correct representation

- To confirm the exponential relationship of the Red function we can do a log-linear plot



In a log-linear plot, we plot the logarithm of the Y values against the X values

- $y = 2^{-x} + \epsilon$ $\quad\quad\quad$ $\log y \approx -x + c = -10^{\log x} + c$

- When we plot $z = \log y$ against x, we should se a linear relationship

Attribute relationships

- In many cases it is interesting to look at two attributes together to understand if they are correlated. For example:
  - How does marital status relate with tax cheating?
  - Does refund correlate with average income?
  - Is there a relationship between years of study and income?
- How do we measure and visualize these relationships?

Correlating categorical attributes

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8	No	Single	85K	**Yes**
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Confusion or Contingency Matrix

	No	**Yes**
Single	2	1
Married	4	0
Divorced	2	1

Correlating categorical attributes

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8	No	Single	85K	Yes
9	No	Married	90K	No
10	No	Single	90K	No

Confusion Matrix

	No	**Yes**
Single	2	1
Married	4	0
Divorced	2	1

Joint Distribution Matrix

	No	**Yes**
Single	0.2	0.1
Married	0.4	0.0
Divorced	0.2	0.1

	No	**Yes**
Single	0.2	0.1
Married	0.4	0.0
Divorced	0.2	0.1

It can also be represented as a heatmap

Correlating categorical attributes

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8	No	Single	85K	**Yes**
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10	No	Single	90K	**No**

Joint Distribution Matrix

	No	**Yes**	
Single	0.2	0.1	0.3
Married	0.4	0.0	0.4
Divorced	0.2	0.1	0.3
	0.8	0.2	1

Marginal distribution for Marital Status

Marginal distribution for Cheat

Correlating categorical attributes

Tid	Refund	Marital Status	Taxable Income	Cheat
1	Yes	Single	125K	No
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8	No	Single	85K	Yes
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10	No	Single	90K	No

How do we know if there are interesting correlations?

Joint Distribution Matrix P

	No	Yes	
Single	0.2	0.1	0.3
Married	0.4	0.0	0.4
Divorced	0.2	0.1	0.3
	0.8	0.2	1

Independence Matrix I

	No	Yes	
Single	0.24	0.06	0.3
Married	0.32	0.08	0.4
Divorced	0.24	0.06	0.3
	0.8	0.2	1

Compare the values P_{xy} with I_{xy}

The product of the two marginal values 0.3*0.8

Correlating categorical attributes – Values correlation

We now want to find out if the co-occurrence of two values is interesting. We will check if the events of their occurrence are independent

Tid	Refund	Marital Status	Taxable Income	Cheat
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3	No	Single	70K	No
4	Yes	Married	120K	No
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8	No	Single	85K	Yes
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10	No	Single	90K	No

Joint Distribution Matrix P

	No	Yes	
Single	0.2	0.1	0.3
Married	0.4	0.0	0.4
Divorced	0.2	0.1	0.3
	0.8	0.2	1

Independence Matrix I

	No	Yes	
Single	0.24	0.06	0.3
Married	0.32	0.08	0.4
Divorced	0.24	0.06	0.3
	0.8	0.2	1

Finding interesting pairs of values:

- If $P(x,y) \gg I(x,y)$ there is positive correlation (e.g, Married, No)
- If $P(x,y) \ll I(x,y)$ there is negative correlation (e.g., Single, No)
- Otherwise, there is no correlation

The quantity $\frac{P(x,y)}{I(x,y)} = \frac{P(x,y)}{P(x)P(y)}$ is called Lift, or Pointwise Mutual Information

Correlating categorical attributes – Chi-square test

We now want to test if there is a correlation between the attributes Marital Status and Cheat

Tid	Refund	Marital Status	Taxable Income	Cheat
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	10000K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	90K	No
10	No	Single	90K	No

We will view the two attributes as random variables, and we will test if the random variables are independent

We will compare the Observed Contingency Matrix against the Expected Contingency Matrix if the attributes were independent

Observed Contingency Matrix C

	No	Yes
Single	20	10
Married	40	0
Divorced	20	10

Expected Contingency Matrix E

	No	Yes
Single	24	6
Married	32	8
Divorced	24	6

Correlating categorical attributes – Chi-square test

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Cheat**
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	10000K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	90K	No
10	No	Single	90K	No

Observed
Contingency Matrix C

	No	**Yes**
Single	20	10
Married	40	0
Divorced	20	10

Expected
Contingency Matrix E

	No	**Yes**
Single	24	6
Married	32	8
Divorced	24	6

Pearson x^2 Independence Test Statistic:
Compare the relative difference between the two contingency tables:

$$Chi2 = \sum_x \sum_y \frac{\left(C_{xy} - E_{xy}\right)^2}{E_{xy}}$$

For the two attributes to be correlated this should be large. But how large is large enough? When is the difference statistically significant?

Statistical tests

- Statistical tests measure the value of a statistic S and determine its statistical significance
  - For example: the $Chi2$ value we computed, or the difference between two means (e.g., average grade) of two populations (e.g., students in cities vs students in rural areas).
- The magnitude of the value of S that is measured is also called the effect size
- The statistical significance of this value is measured with respect to a null hypothesis H_0
  - $H_0 =$ "The two attributes we consider in the $Chi2$ metric are independent"
  - $H_0 =$ "The difference of the means is zero"
- Under the null-hypothesis, we can estimate the distribution of our statistic measurement.
  - For example, the data the statistic $Chi2$ follows the χ^2 distribution
- For the statistic value $S = \theta$ $(\text{e.g.}, Chi2 = 0.9)$ we observe in our data, we compute the probability $P(S \geq \theta)$ under the null hypothesis
  - For most distributions there are tables that give these numbers for our data
- The value $P(S \geq \theta)$ is the p-value of our experiment:

> The p-value is the probability under H_0 (independence) of observing a value of the test statistic S the same as, or more extreme than the one that was actually observed

- We want the p-value to be smaller than the significance level α: ideally $\alpha = 0.01$, 0.05 is good, 0.1 is ok
  - This means that the observed value is interesting and we can reject the null hypothesis

Hypothesis Testing and p-values – A simple example

- A coin is tossed 20 times, and we get 16 heads.
- Hypothesis H_1=“The coin is not fair”
- Null Hypothesis H_0 = “The coin is fair” (probability 50% for head)
- p-value: What is the probability of getting a number of heads that is the same or more extreme than 16?
  - One-sided p-value: $\Pr(H \geq 16) = 0.0059$
  - Two-sided p-value: $\Pr(H \geq 16) + \Pr(H \leq 4) = 0.0118$
- With significance level $\alpha = 0.05$ we can conclude that we can reject the null hypothesis

P-values

- The p-value tells us the probability that the value we observe could appear in data generated under the null hypothesis.
  - The null hypothesis proposes a (random) model for the data generation
  - The p-value answers the question: “If the null hypothesis model was correct how likely would it be to observe the value we observe”?
- Be careful!
  - A p-value ϕ does not mean that the null hypothesis is correct with probability ϕ
    - A high p-value (e.g., 90%) does not mean that the null hypothesis is true, it only means that the data is consistent with the model of the null hypothesis
  - A p-value ϕ does not mean that our hypothesis is correct with probability $1 - \phi$
    - A p-value of 3% does not mean that our hypothesis is correct with probability 97%
    - It only means that the data is not consistent with the null hypothesis random model

Exact Fisher Test

- When our contingency table is 2×2 there is an exact formula for estimating the p-value of our observations

	Type 1	Type 2	
Blue	a	b	$a+b$
Red	c	d	$c+d$
	$a+c$	$b+d$	$a+b+c+d = n$

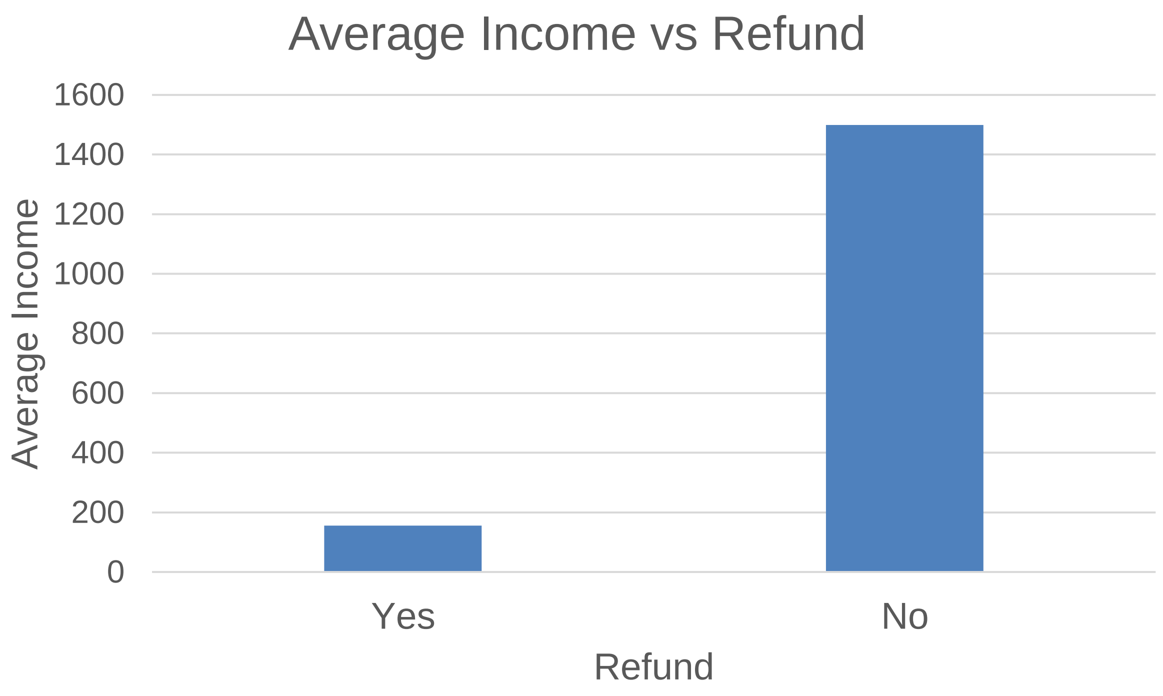
- The probability of value a given that we have fixed the marginal values

$$P(a) = \frac{\binom{a+c}{a}\binom{b+d}{b}}{\binom{n}{a+b}}$$

- Once we know a, we know all other values (given that the marginals are fixed).The probability $P(a)$ is the p-value for our observations

Correlating categorical and numerical attributes

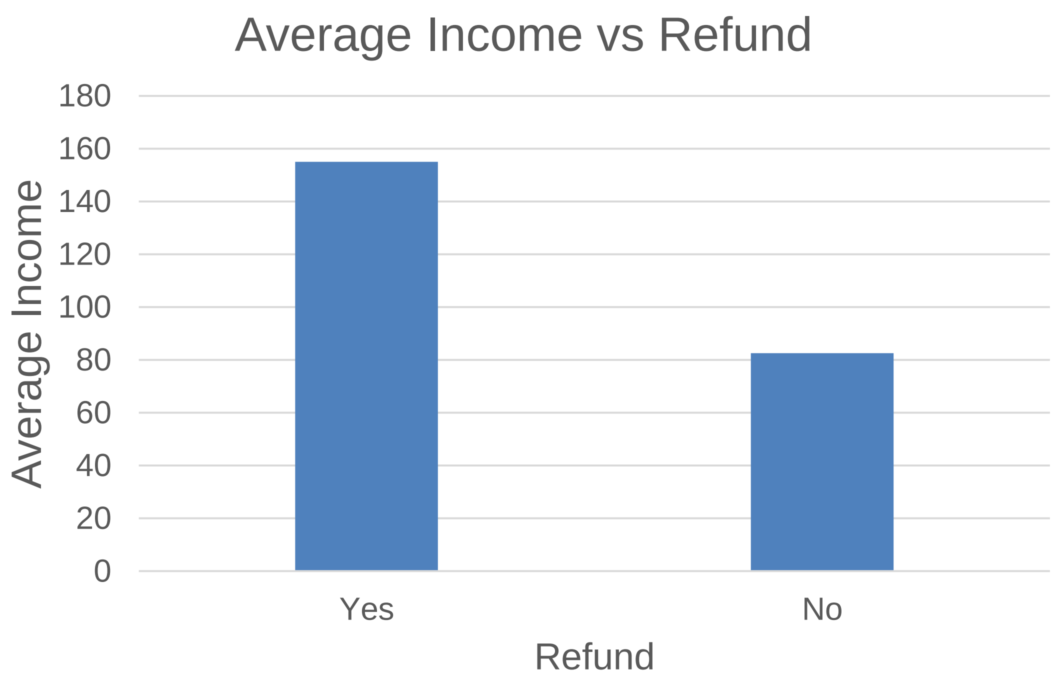
Tid	**Refund**	**Marital Status**	**Taxable Income**	**Cheat**
1	Yes	Single	125K	**No**
2	No	Married	100K	**No**
3	No	Single	70K	**No**
4	Yes	Married	120K	**No**
5	No	Divorced	10000K	**Yes**
6	No	NULL	60K	**No**
7	Yes	Divorced	220K	**NULL**
8	No	Single	85K	**Yes**
9	No	Married	90K	**No**
10	No	Single	90K	**No**



Categorical and numerical attributes

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Cheat**
1	Yes	Single	125K	**No**
2	No	Married	100K	**No**
3	No	Single	70K	**No**
4	Yes	Married	120K	**No**
5	No	Divorced	10000K	**Yes**
6	No	NULL	60K	**No**
7	Yes	Divorced	220K	**NULL**
8	No	Single	85K	**Yes**
9	No	Married	90K	**No**
10	No	Single	90K	**No**

After removing the outlier value

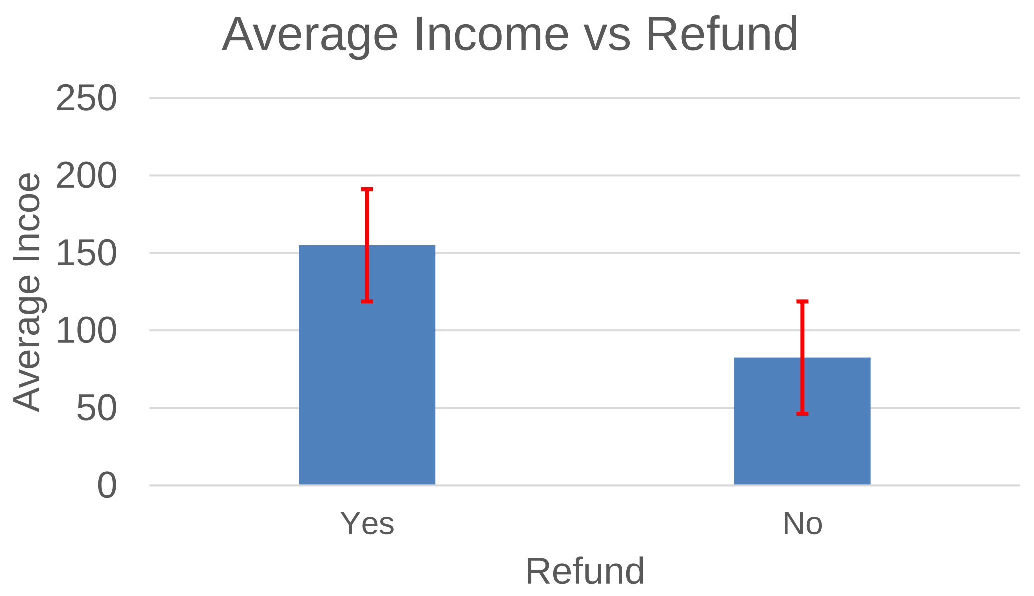


How informative are the means?

Categorical and numerical attributes

Tid	Refund	Marital Status	Taxable Income	Cheat
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	10000K	Yes
6	No	NULL	60K	No
7	Yes	Divorced	220K	NULL
8	No	Single	85K	Yes
9	No	Married	90K	No
10	No	Single	90K	No

Compute error bars



Error bars give a measure of the variability of the mean

Error bars

- Error bars may be:
  - The range
  - The standard deviation
  - The standard error
  - The 95% confidence interval

Descriptive error bars: They tell us something about the underlying distribution of the data

Inferential error bars: They tell us something about the quality of the estimation of the mean

- Inferential error bars get more informative the more data we collect.
- We should always specify what the error bars mean in a plot.

Standard Error (of the Mean)

- The Standard Error (SE) is usually defined for the mean of a sample of values X (it is also known as SEM – Standard Error of the Mean) and it is a measure of the deviation of the sample mean from the true mean.
- It is defined as:

$$se = \frac{\hat{\sigma}(X)}{\sqrt{n}}$$

where $\hat{\sigma}(X) =$ empirical standard deviation.

- As the sample size grows the SE is reduced (we have a better estimation of the mean)
- Computation follows from the fact that

$$se = \hat{\sigma}(\hat{\mu}), \hat{\mu} = \frac{1}{n}\sum_i X_i$$

- We assume that X_i are independent samples of the random variable X that come from the same distribution (they have the same variance). We use the fact that:

$$Var\left(\sum_i \alpha_i X_i\right) = \sum_i \alpha_i^2 \, Var(X_i) = \frac{1}{n^2}\sum_i Var(X) = \frac{1}{n} Var(X)$$

Confidence interval

- We want to estimate the average income μ which is a fixed value.
- We have a sample of the population and the measurements $\{X_i\}$ of incomes and we estimate the average income as:

$$\hat{\mu} = \frac{1}{n}\sum_i X_i$$

- The p-confidence interval of the value μ is an interval of values C_n such that

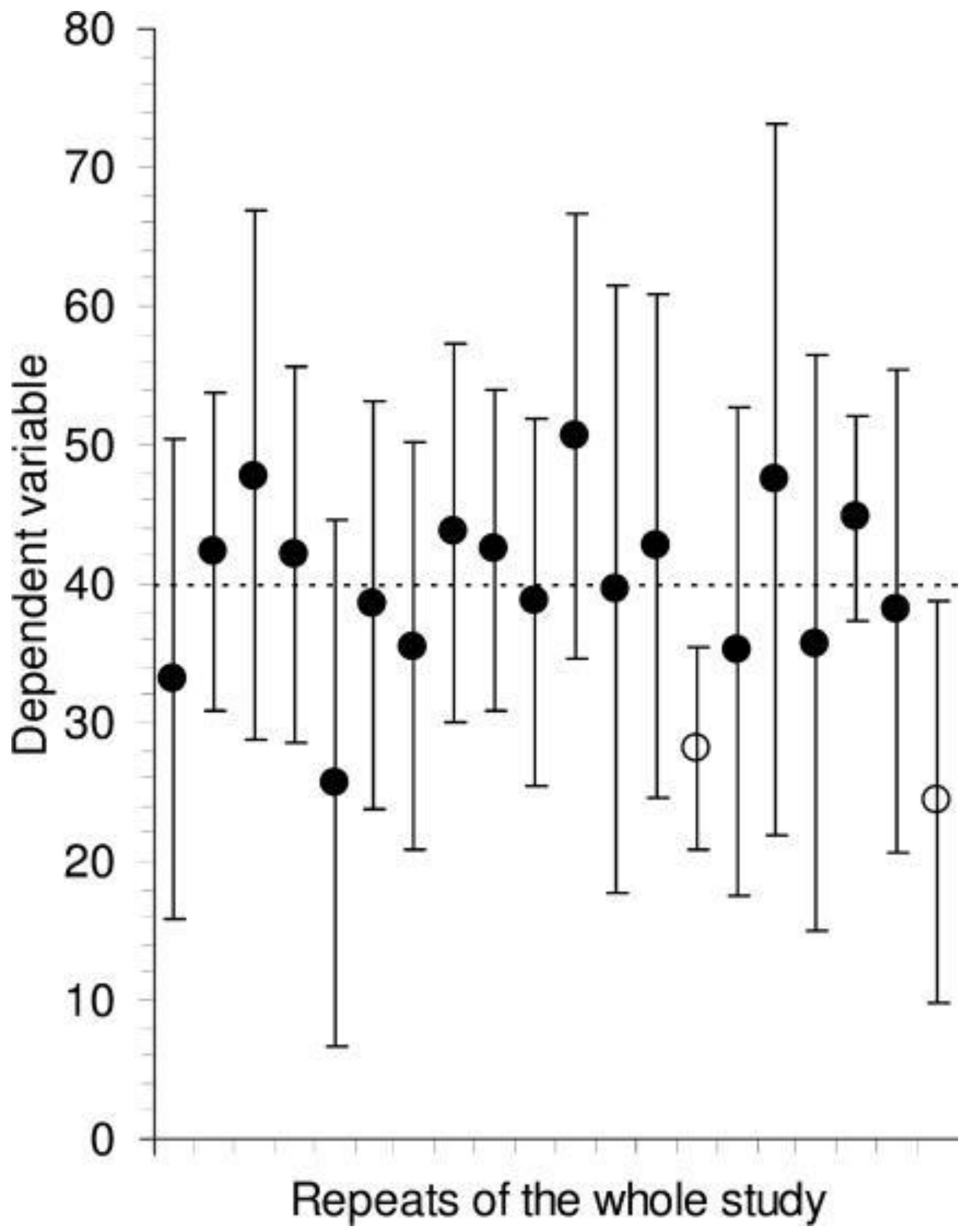
$$P(\mu \in C_n) \geq p$$

  - We usually ask for the 95% confidence interval
- Important: The probability is taken over the many different samples of the population
  - Different samples will generate different confidence intervals
  - There is a 95% chance that each of these intervals contains the true mean μ
  - It is incorrect to say that this is the probability that μ belongs to the interval
- The value $\hat{\mu}$ follows a normal distribution for large n. For normal distributions, the 95% confidence interval (for large enough n) is:

$$(\hat{\mu} - 2se, \hat{\mu} + 2se)$$

Example

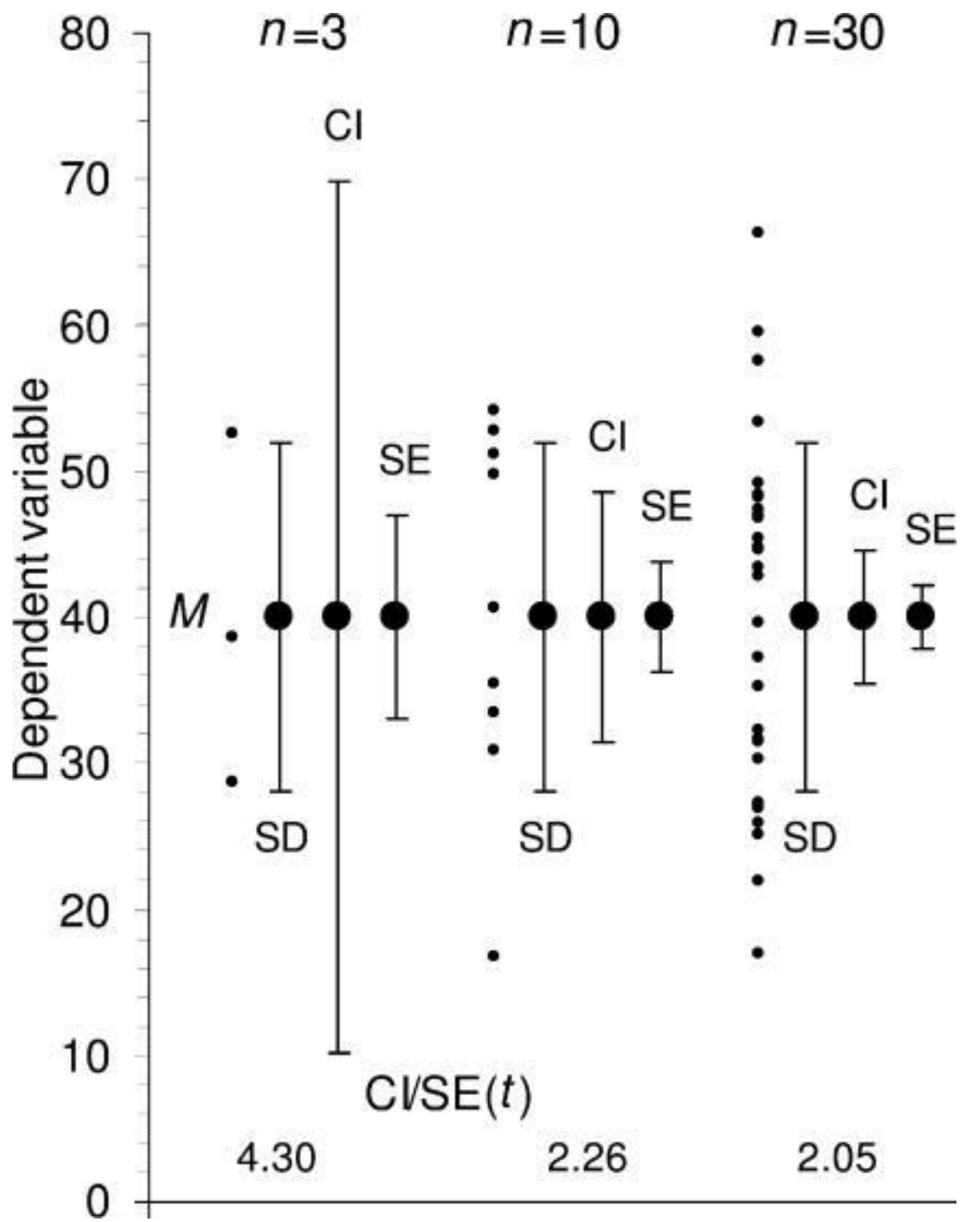
- If we obtain an estimate of the mean for 20 different population samples, we will obtain 20 different 95%-confidence intervals.
- We expect that 1/20 of these intervals will not contain the true mean (the dotted line)



Error bars example

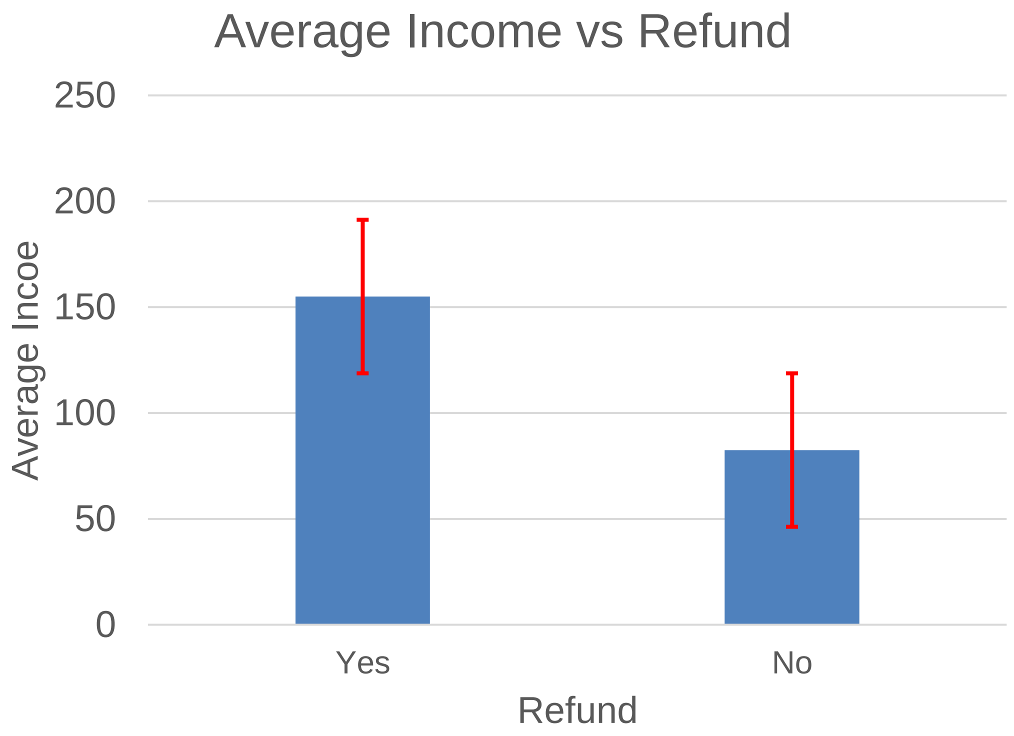
- The different error bars and how they change as the sample size increases

- Out of the four different error bars, the confidence interval is probably the most informative.



Statistical significance

- Given the means of two populations an important question is whether the difference we observe is statistically significant
- There are statistical tests for testing the difference of the means



Statistical significance via error bar overlap

- It is not always safe to declare that there is statistical significance when error bars do not overlap
  - We may have statistically significant differences when there is overlap, or no statistical significance when there is no overlap
- We can say that there is statistically significant difference of means when sample sizes are comparable, and the 95%-confidence intervals do not overlap
- There are a little more complex rules for the standard error.

Statistical tests – The Student t-test

- The Student t-test tests if the difference of the means of two samples is "big enough"

$$t = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$$

- Large t-value (effect size):
  - Large difference between the means
  - Small variance in the samples (more accurate measurements)
  - Large sample sizes (more reliable)

Statistical tests – The Student t-test

- The Student t-test produces a p-value:
  - Under the null hypothesis that the two distributions have zero difference in mean the t-value follows a Normal Distribution with zero mean
  - This is what we care about, the t-value is usually not looked at
- Student t-test additional implicit assumptions:
  - (near) Gaussian distribution of the data,
  - (near) same variance,
  - similar sample sizes.
- There is paired and unpaired Student t-test
  - Example of paired: behavior before and after a treatment.

Statistical tests – The KS-test

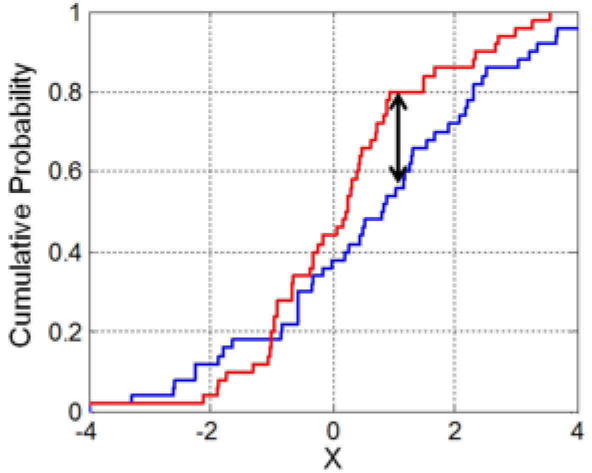
- The Kolomogorov-Smirnov (KS) test, tests if two samples come from the same distribution (or come from a specific distribution)
  - Take the cumulative distribution function (CDF) of the two distributions
  - Compute:

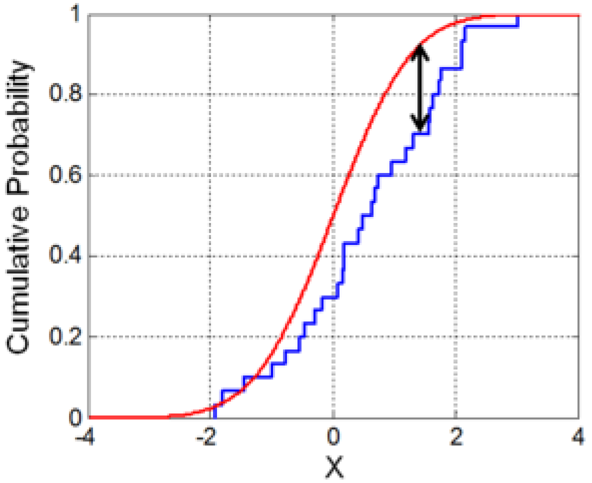
$$D(C_1, C_2) = \max_x |C_1(x) - C_2(x)|$$

- We can reject the null hypothesis if:

$$D(C_1, C_2) > c(\alpha)\sqrt{\frac{N_1+N_2}{N_1 N_2}}$$

  - α is the confidence level, $c(\alpha)$ is given by some tables

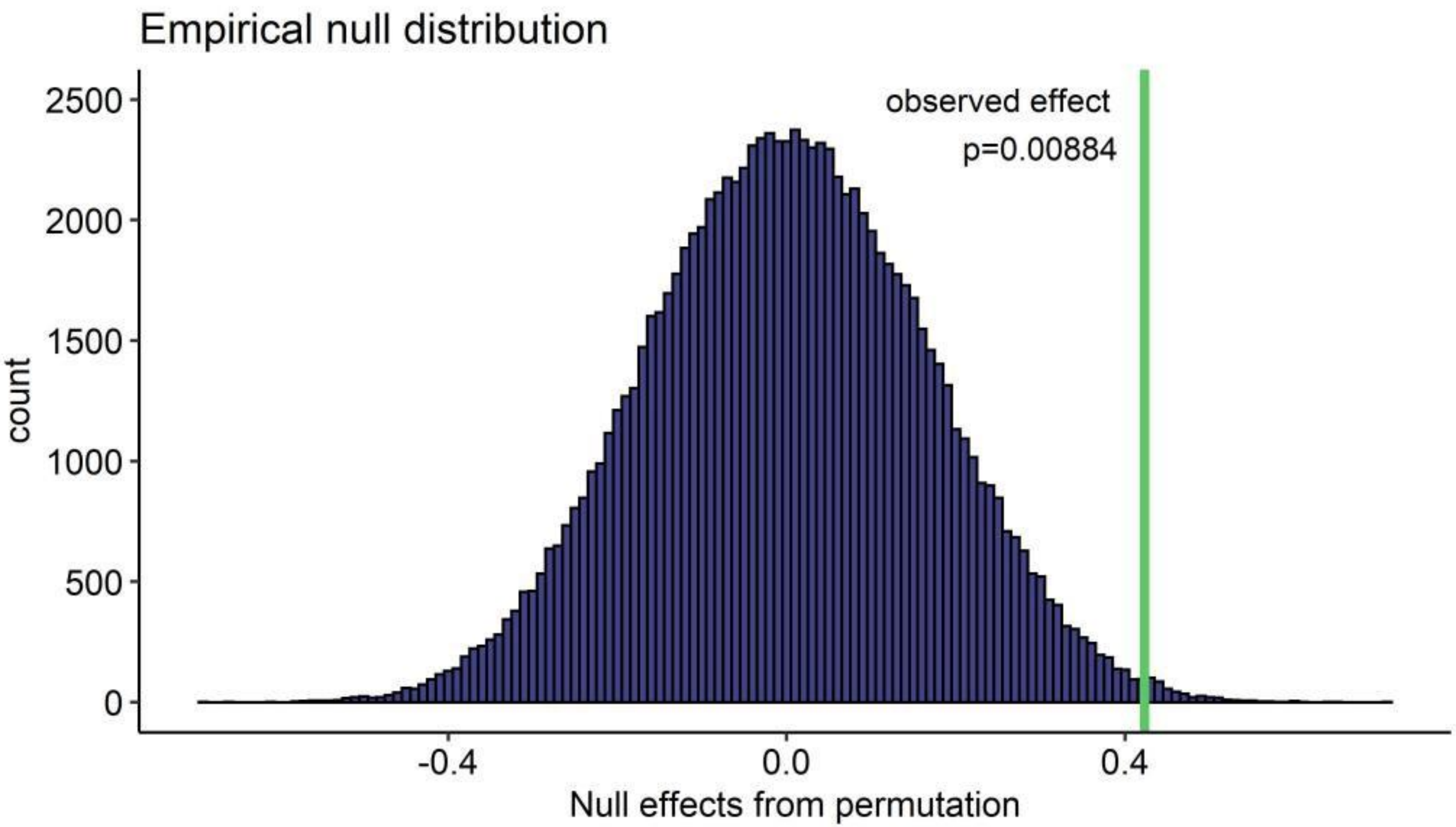




Hypothesis Testing – Permutation tests

- Most tests make some assumption about the underlying distribution of the data.
- A non-parametric statistical test is the permutation test
- Create random instances of the data by randomly permuting values
  - E.g., permute the Cheat labels randomly
- Compute the statistic of interest for the permuted data
  - E.g., the average income of the cheaters
- Repeat this several times (at least 1000)
- Compute the empirical p-value: the fraction of permutations where we have a value that is equal or more extreme than the one observed.

Example



Empirical null distribution
observed effect
p=0.00884
count
2500
2000
1500
1000
500
0
-0.4
0.0
0.4
Null effects from permutation

Correlating numerical attributes

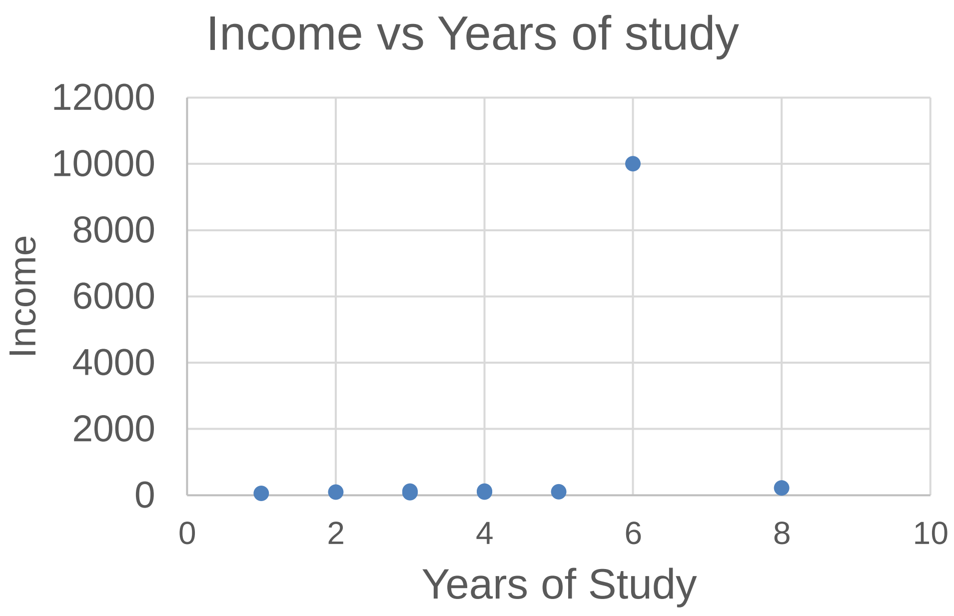
Tid	**Refund**	**Marital Status**	**Taxable Income**	**Years of Study**
1	Yes	Single	125K	4
2	No	Married	100K	5
3	No	Single	70K	3
4	Yes	Married	120K	3
5	No	Divorced	10000K	6
6	No	NULL	60K	1
7	Yes	Divorced	220K	8
8	No	Single	85K	3
9	No	Married	90K	2
10	No	Single	90K	4

Scatter plot:

X axis is one attribute, Y axis is the other

For each entry we have two values

Plot the entries as two-dimensional points



Correlating numerical attributes

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Years of Study**
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3	No	Single	70K	**3**
4	Yes	Married	120K	**3**
5	No	Divorced	10000K	**6**
6	No	NULL	60K	**1**
7	Yes	Divorced	220K	**8**
8	No	Single	85K	**3**
9	No	Married	90K	**2**
10	No	Single	90K	**4**

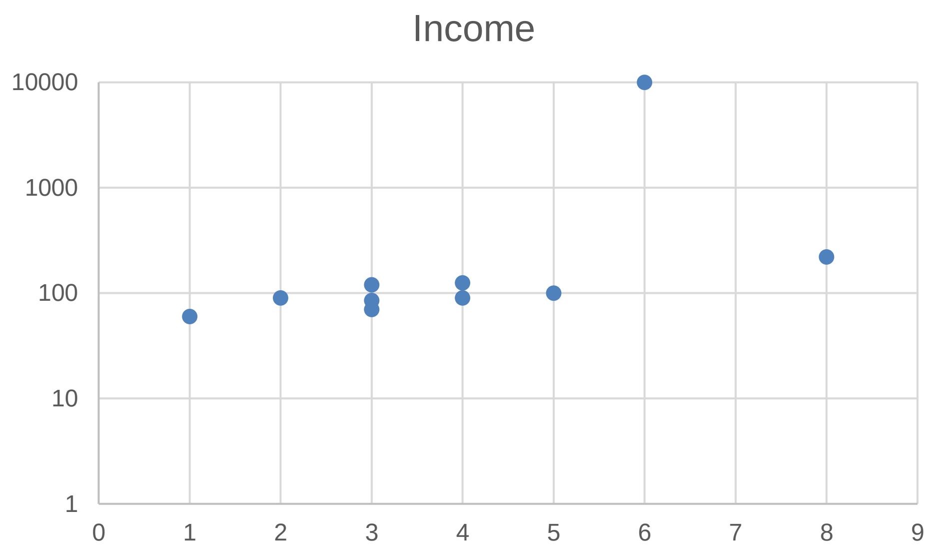
Scatter plot:

X axis is one attribute, Y axis is the other

For each entry we have two values

Plot the entries as two-dimensional points

Log-scale in y-axis makes the plot look a little better



Plotting attributes against each other

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Years of Study**
1	Yes	Single	125K	**4**
2	No	Married	100K	**5**
3	No	Single	70K	**3**
4	Yes	Married	120K	**3**
5	No	Divorced	10000K	**6**
6	No	NULL	60K	**1**
7	Yes	Divorced	220K	**8**
8	No	Single	85K	**3**
9	No	Married	90K	**2**
10	No	Single	90K	**4**

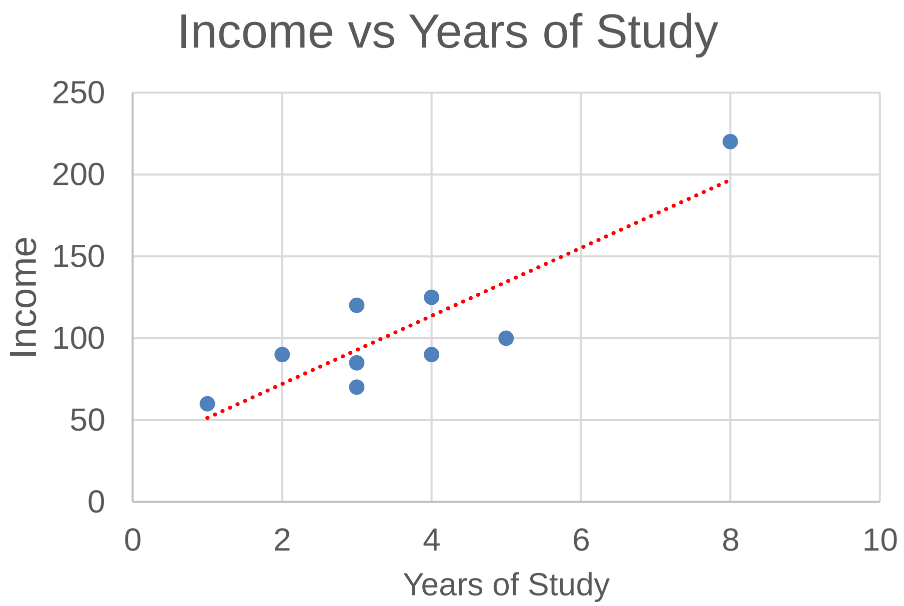
Scatter plot:

X axis is one attribute, Y axis is the other

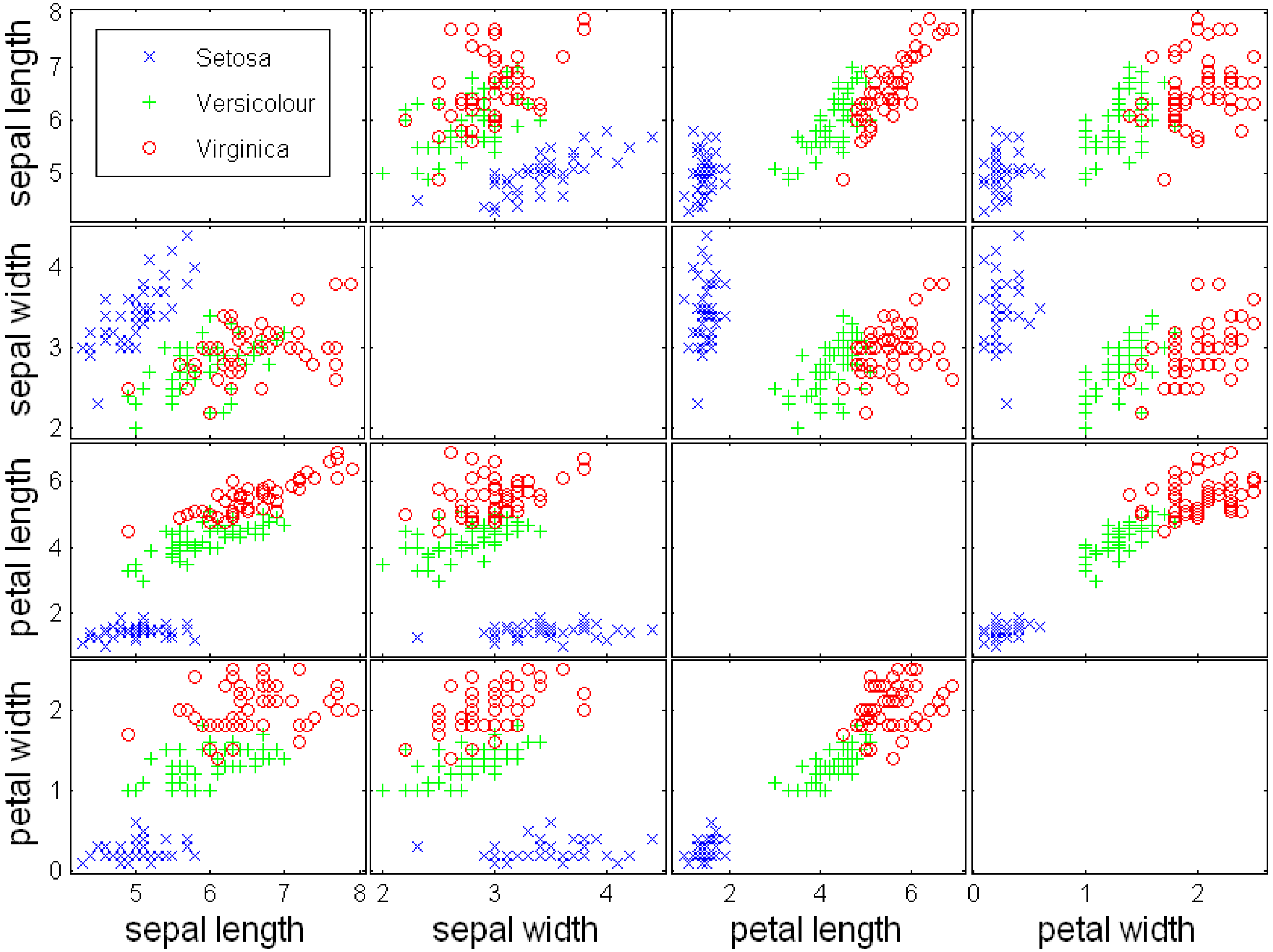
For each entry we have two values

Plot the entries as two-dimensional points

After removing the outlier value there is a clear correlation



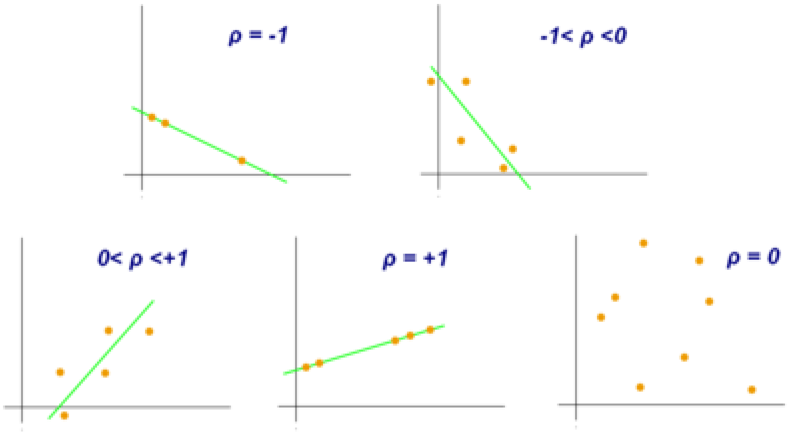
Scatter Plot Array of Iris Attributes



Measuring correlation

- Pearson correlation coefficient: measures the extent to which two variables are linearly correlated
  - $X = \{x_1, \ldots, x_n\}$
  - $Y = \{y_1, \ldots, y_n\}$
  - $corr(X, Y) = \frac{\sum_i (x_i - \mu_X)(y_i - \mu_Y)}{\sqrt{\sum_i (x_i - \mu_X)^2} \sqrt{\sum_i (y_i - \mu_Y)^2}}$

Must have pairs of observations



- It comes with a p-value
  - The p-value is the probability that the correlation was by chance.

Pearson correlation

- Assumptions:
  - Variables are normally distributed
  - No outliers
  - A linear relationship between the variables
- Caveats
  - For large samples p-values will always be small
  - Except for the p-value we need to also look at the effect size: the value of $r = corr(X,Y)$
- Interpretation
  - The value of r^2 measures the fraction of variance in one variable that is explained by the values of the other variable (shared variance)

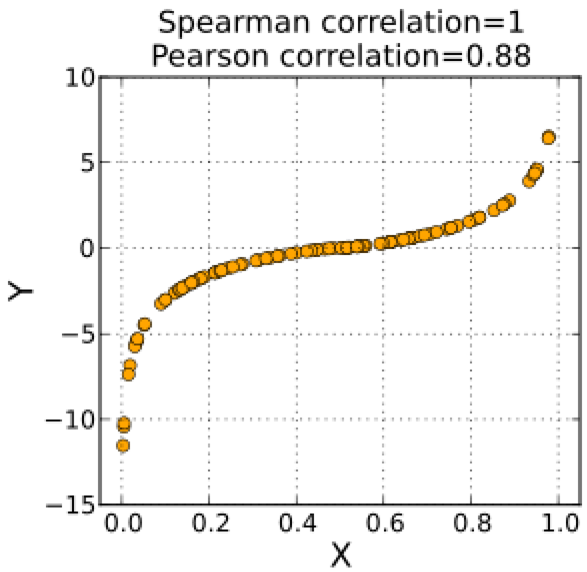
$$r = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

Rank correlation

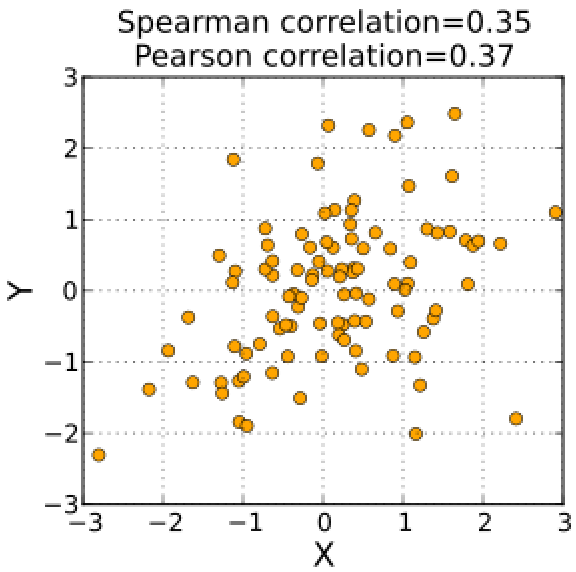
- Spearman rank correlation coefficient: tells us if two variable are rank-correlated
  - They place items in the same order – Pearson correlation of the rank vectors
    - From $X = \{x_1, \ldots, x_n\}$ we get $\{r_1^X, r_2^X, \ldots, r_n^X\}, r_i^X$ = rank of i^{th} observation in X
    - From $Y = \{y_1, \ldots, y_n\}$ we get $\{r_1^Y, r_2^Y, \ldots, r_n^Y\}, r_i^Y$ = rank of i^{th} observation in Y
    - $spearman(X,Y) = corr(r^X, r^Y) = \dfrac{\sum_i \left(r_i^X - \mu_{r^X}\right)\left(r_i^Y - \mu_{r^Y}\right)}{\sqrt{\sum_i \left(r_i^X - \mu_{r^X}\right)^2}\sqrt{\sum_i \left(r_i^X - \mu_{r^X}\right)^2}}$
  - For ranking without ties it looks at the differences between the ranks of the same items
    - $spearman(X,Y) = 1 - \dfrac{6\sum_i \left(r_i^X - r_i^Y\right)^2}{n(n^2-1)}$
- Spearman coefficient also comes with a p-value

Rank correlation

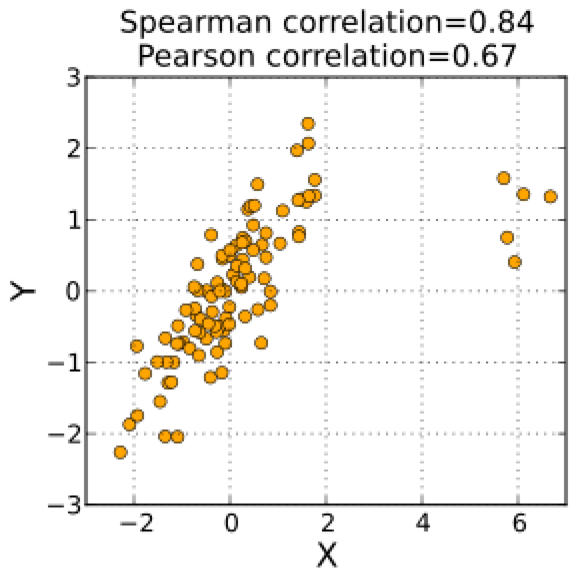
- Spearman coefficient does not assume a linear relationship, but a monotonic one



Monotonic but not linear relationship: Perfect Spearman correlation



Elliptical distribution Pearson and Spearman are more-or-less the same



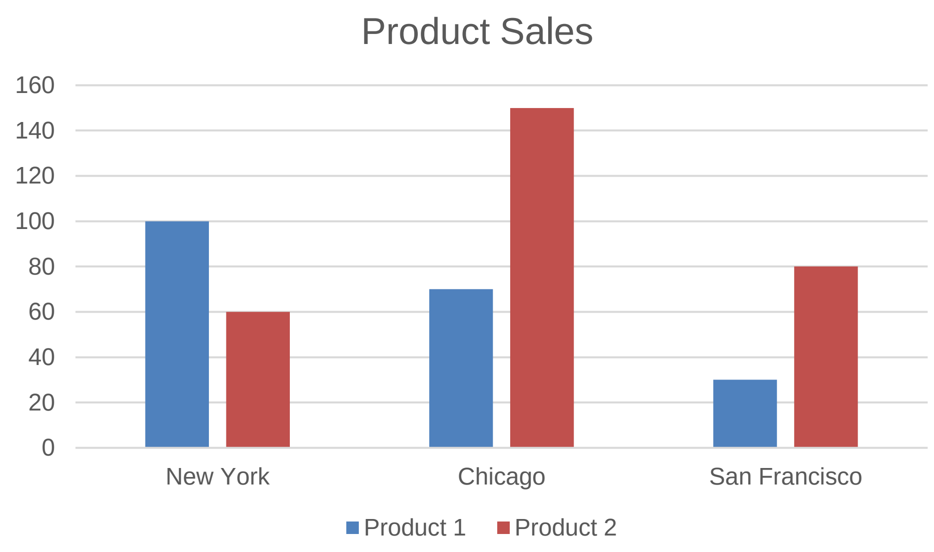
Pearson is more sensitive to outliers

Statistical significance vs Scientific significance

- Statistics place a lot of emphasis on the p-values and the statistical significance
- However, it may be that p-values are small but the finding is not of scientific interest
  - A difference or a correlation may be statistically significant, but too small to be of scientific interest
  - We need to evaluate the results beyond simply looking at the p-values.
  - We also need to look at the effect size, or the impact of the computed difference.

Plotting attributes together

City	Product 1	Product 2
New York	100	60
Chicago	70	150
San Francisco	30	80



How would you visualize the differences between the product sales per city?

Plotting attributes together

Year	Product 1	Product 2
2011	100	200
2012	200	250
2013	180	300
2014	300	350
2015	500	490
2016	600	500
2017	650	550
2018	640	540
2019	700	500
2020	200	100

How would you visualize the differences between the product sales over time?