DATA MINING
LINK ANALYSIS RANKING

PageRank – Random walks

HITS

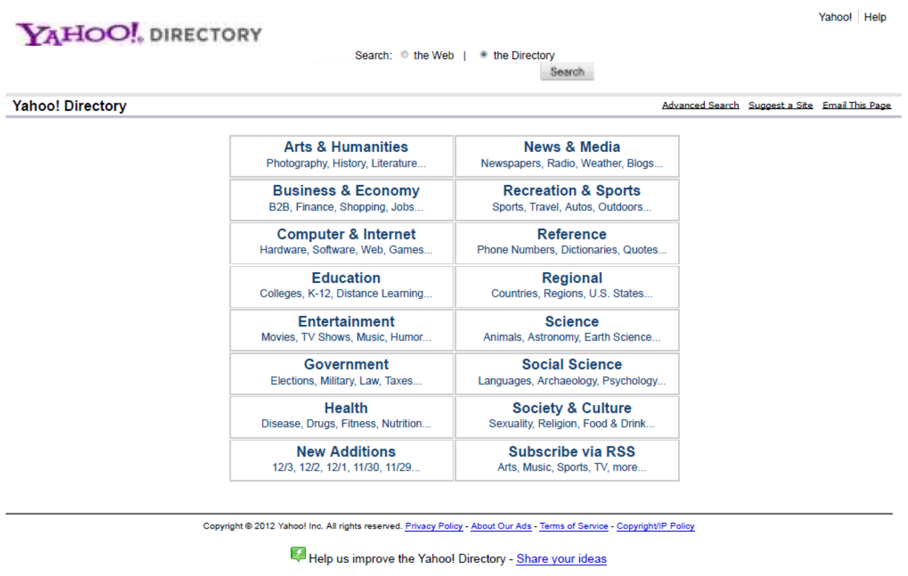
Absorbing Random Walks and Label Propagation

Network Science

- A number of complex systems can be modeled as networks (graphs).
  - The Web
  - (Online) Social Networks
  - Biological systems
  - Communication networks (internet, email)
  - The Economy
- We cannot truly understand such complex systems unless we understand the underlying network.
  - Everything is connected, studying individual entities gives only a partial view of a system
- Data mining for networks is a very popular area
  - Applications to the Web is one of the success stories for network data mining.

A case study: Searching the web

- First try: Manually curated Web Directories



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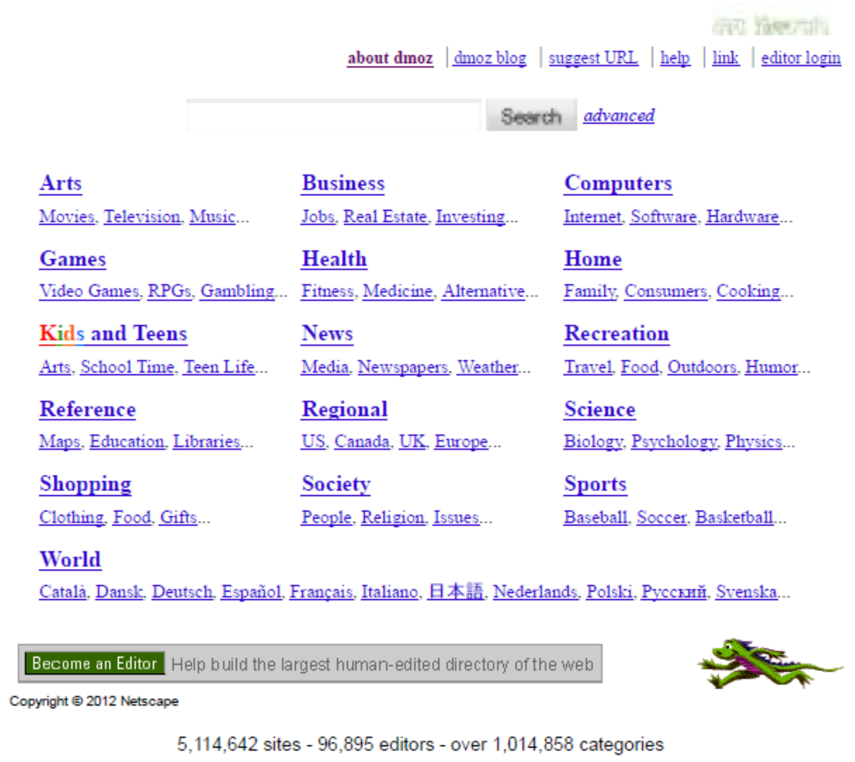

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A case study: Searching the web

- Second try: Web Search
  - Information Retrieval investigates:
    - Find relevant docs in a small and trusted set e.g., Newspaper articles, Patents, etc. (“needle-in-a-haystack”)
    - Limitation of keywords (synonyms, polysemy, etc)
  - But: Web is huge, full of untrusted documents, random things, web spam, etc.
    - Everyone can create a web page of high production value
    - Rich diversity of people issuing queries
    - Dynamic and constantly-changing nature of web content

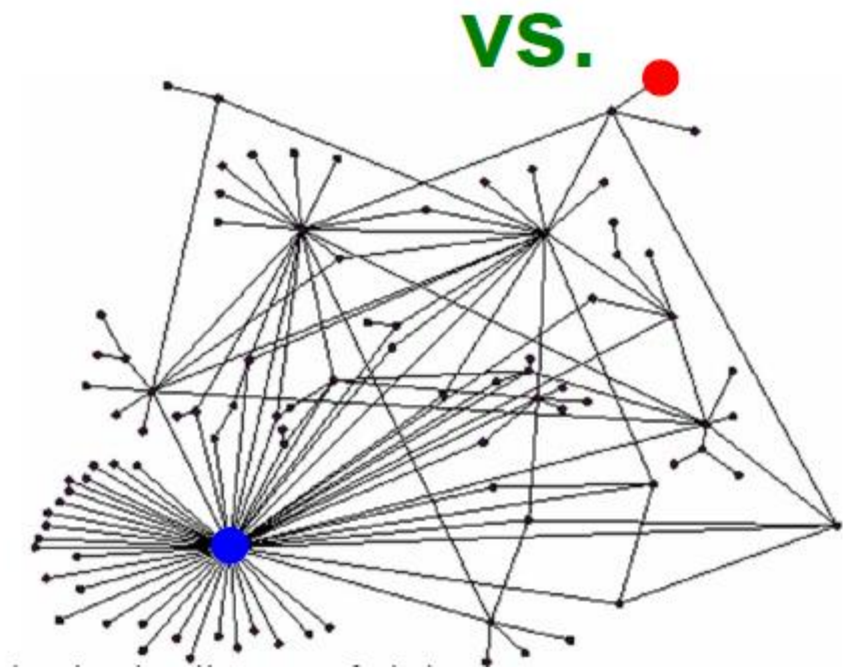
A case study: Searching the web

- Third try (the Google era): using the web graph
  - Sift from relevance to authoritativeness
  - It is not only important that a page is relevant, but that it is also important on the web

- For example, what kind of results would we like to get for the query "game of thrones"?

Link Analysis Ranking

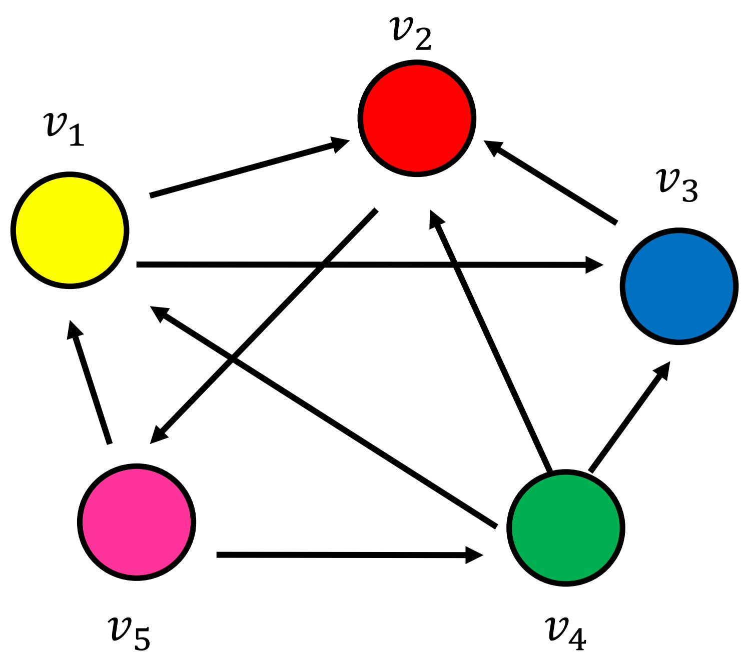
- Use the graph structure in order to determine the relative importance of the nodes
  - Applications: Ranking on graphs (Web, Twitter, FB, etc)
- Intuition: An edge from node p to node q denotes endorsement
  - Node p endorses/recommends/confirms the authority/centrality/importance of node q
  - Use the graph of recommendations to assign an authority value to every node



What is the simplest way to measure importance of a page on the web?

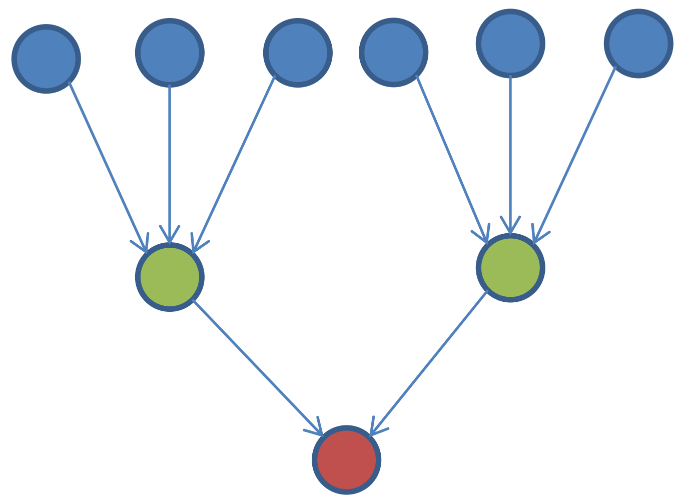
Rank by Popularity

- Rank pages according to the number of incoming edges (in-degree, degree centrality)



1. Red Page
2. Yellow Page
3. Blue Page
4. Purple Page
5. Green Page

Popularity



- It is not important only how many link to you, but how important are the people that link to you.
- Good authorities are pointed by good authorities
  - Recursive definition of importance

PAGERANK

PageRank

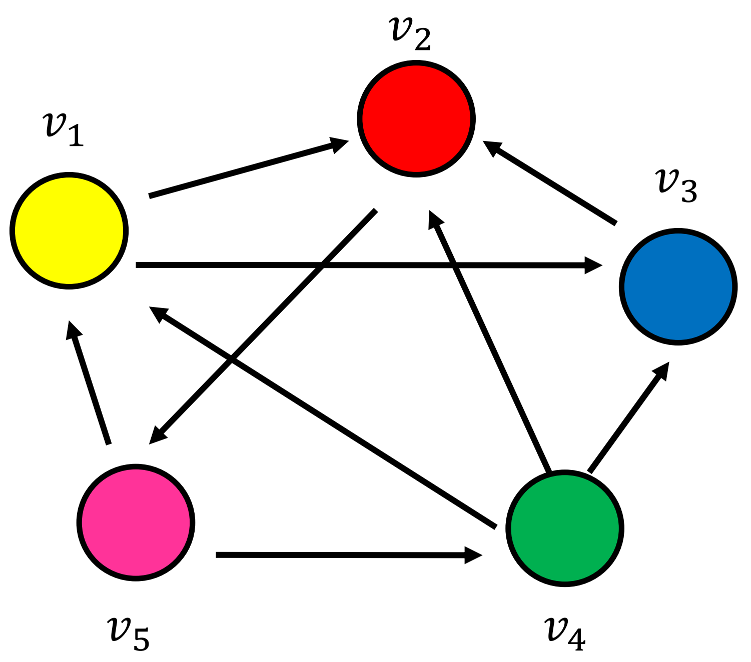
- Good authorities should be pointed by good authorities
  - The value of a node is the value of the nodes that point to it.
- How do we implement that?
  - Assume that we have a unit of authority to distribute to all nodes.
  - Node i gets a fraction w_i of that authority weight
  - Each node distributes the authority value they have to their neighbors
  - The authority value of each node is the sum of the authority fractions it collects from its neighbors.

$$w_i = \sum_{j \to i} \frac{1}{|N_{out}(j)|} w_j$$

Recursive definition

Example

$$w_i = \sum_{j \to i} \frac{1}{|N_{out}(j)|} w_j$$



$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$

$w_1 + w_2 + w_3 + w_4 + w_5 = 1$

We can obtain the weights by solving this system of equations

Computing PageRank weights

- A simpler way to compute the weights is by iteratively updating the weights using the equations
- PageRank Algorithm

```
Initialize all PageRank weights to $w_i^0 = \frac{1}{n}$
Repeat:
    $w_i^t = \sum_{j \to i} \frac{1}{|N_{out}(j)|} w_j^{t-1}$
Until the weights do not change
```

- This process converges

Example

$w_1 = 1/3\ w_4 + 1/2\ w_5$

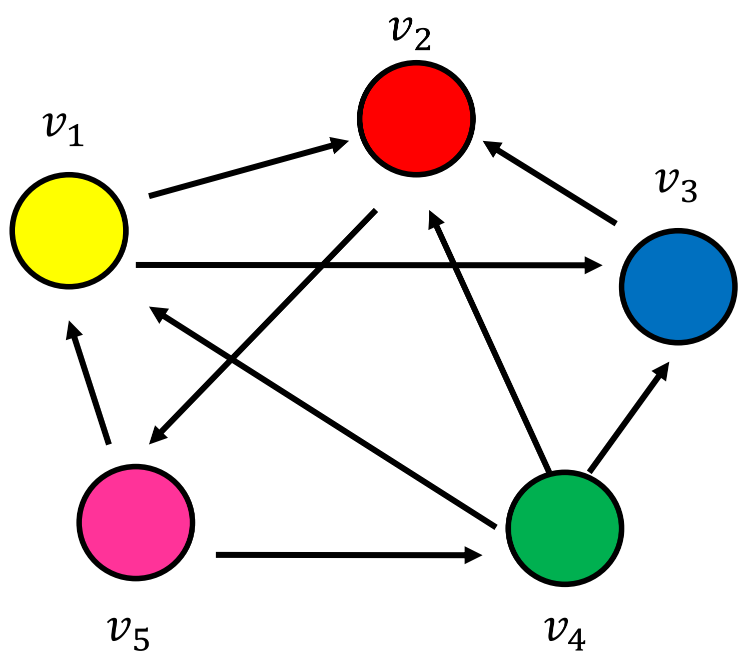
$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$

	w_1	w_2	w_3	w_4	w_5
t=0	0.2	0.2	0.2	0.2	0.2
t=1	0.16	0.36	0.16	0.1	0.2
t=2	0.13	0.28	0.11	0.1	0.36
t=3	0.22	0.22	0.1	0.18	0.28
t=4	0.2	0.27	0.17	0.14	0.22



Think of the weight as a fluid: there is constant amount of it in the graph, but it moves around until it stabilizes

Example

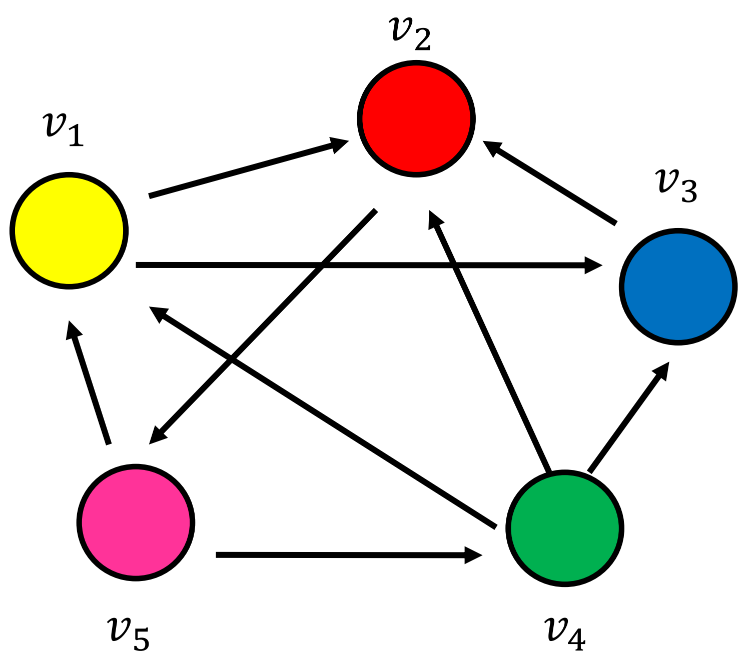
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



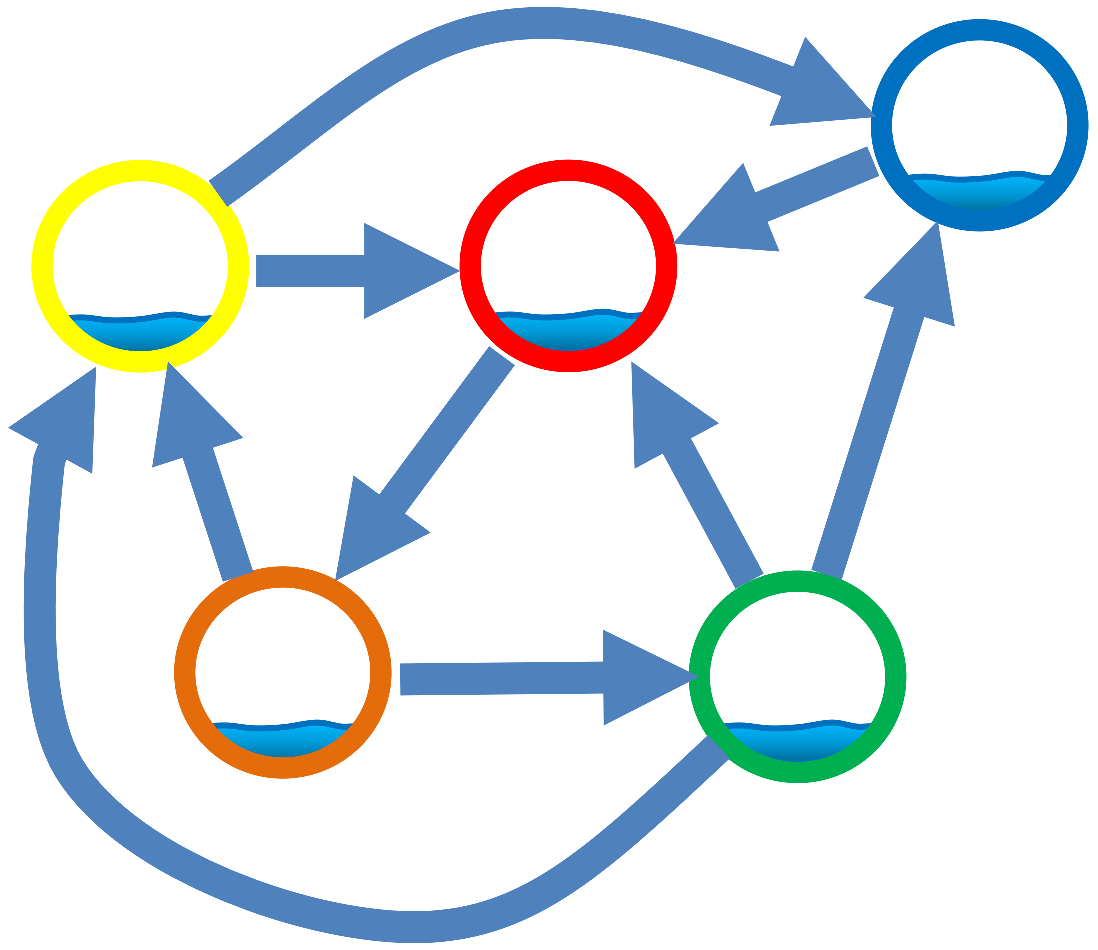
	w_1	w_2	w_3	w_4	w_5
t=25	0.18	0.27	0.13	0.13	0.27

Think of the weight as a fluid: there is constant amount of it in the graph, but it moves around until it stabilizes

The PageRank algorithm

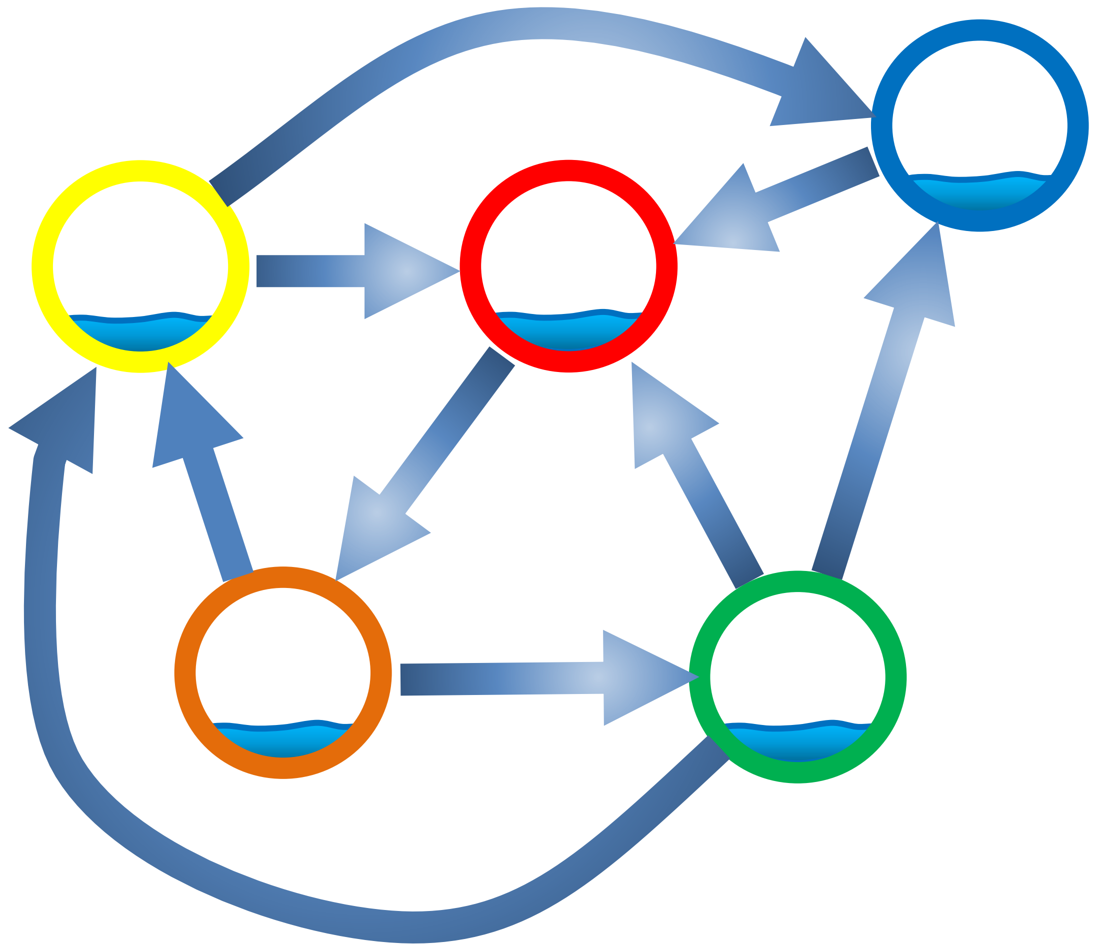
Think of the nodes in the graph as containers of capacity of 1 liter.

We distribute a liter of liquid equally to all containers



The PageRank algorithm

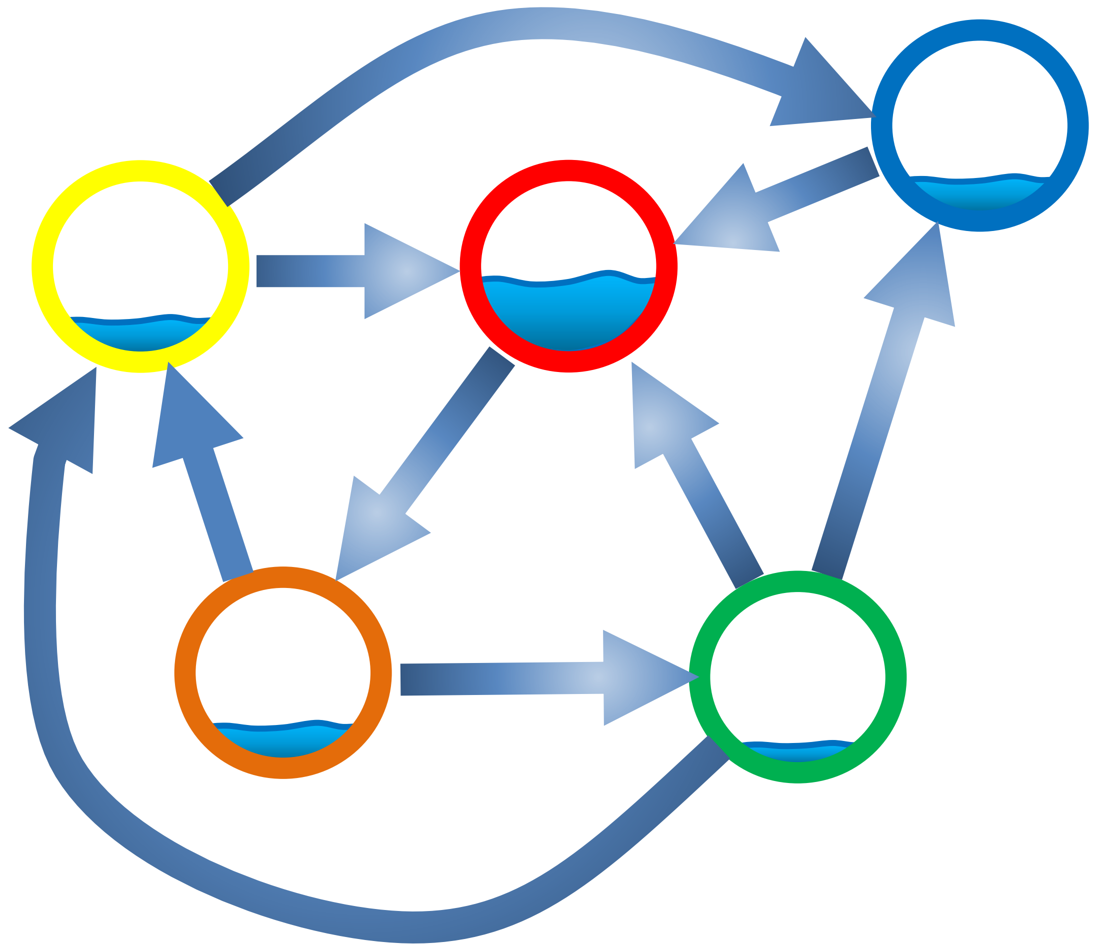
The edges act like pipes that transfer liquid between nodes.



The PageRank algorithm

The edges act like pipes that transfer liquid between nodes.

The contents of each node are distributed to its neighbors.



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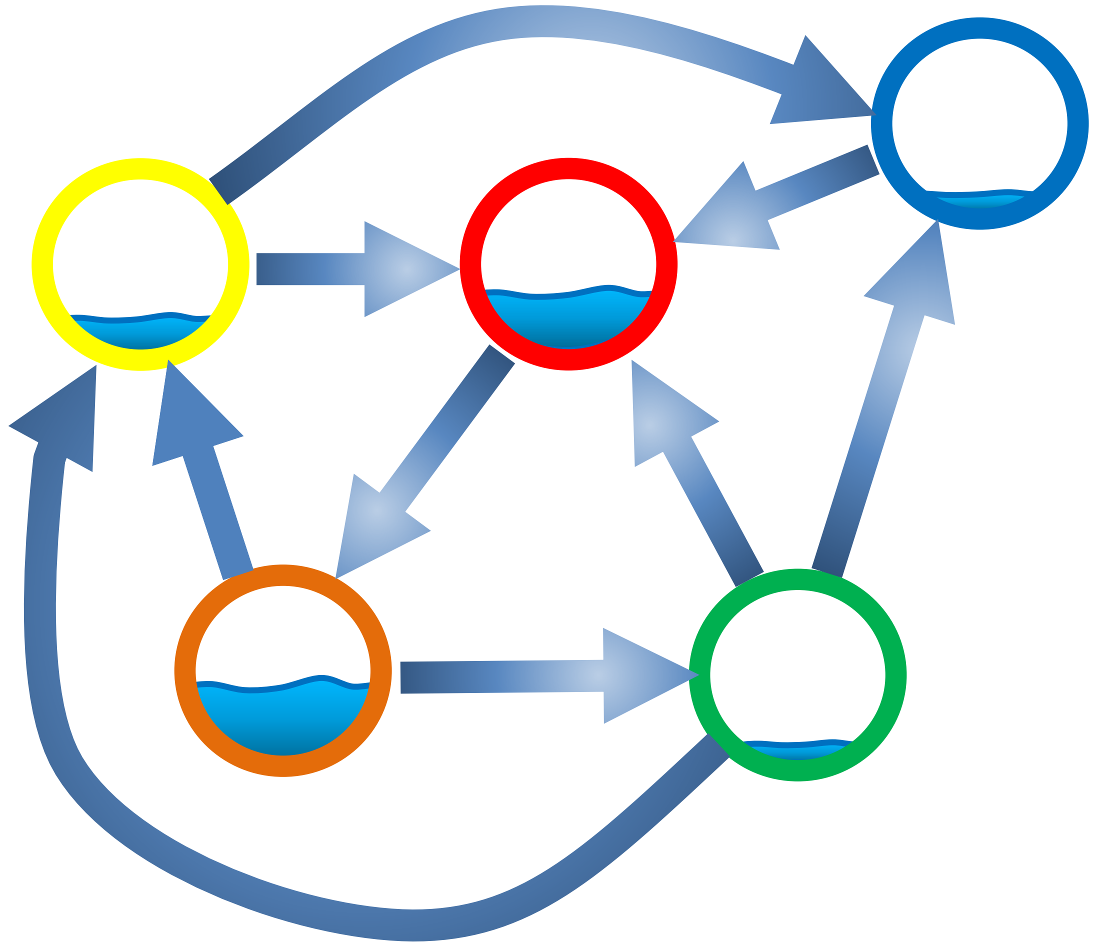

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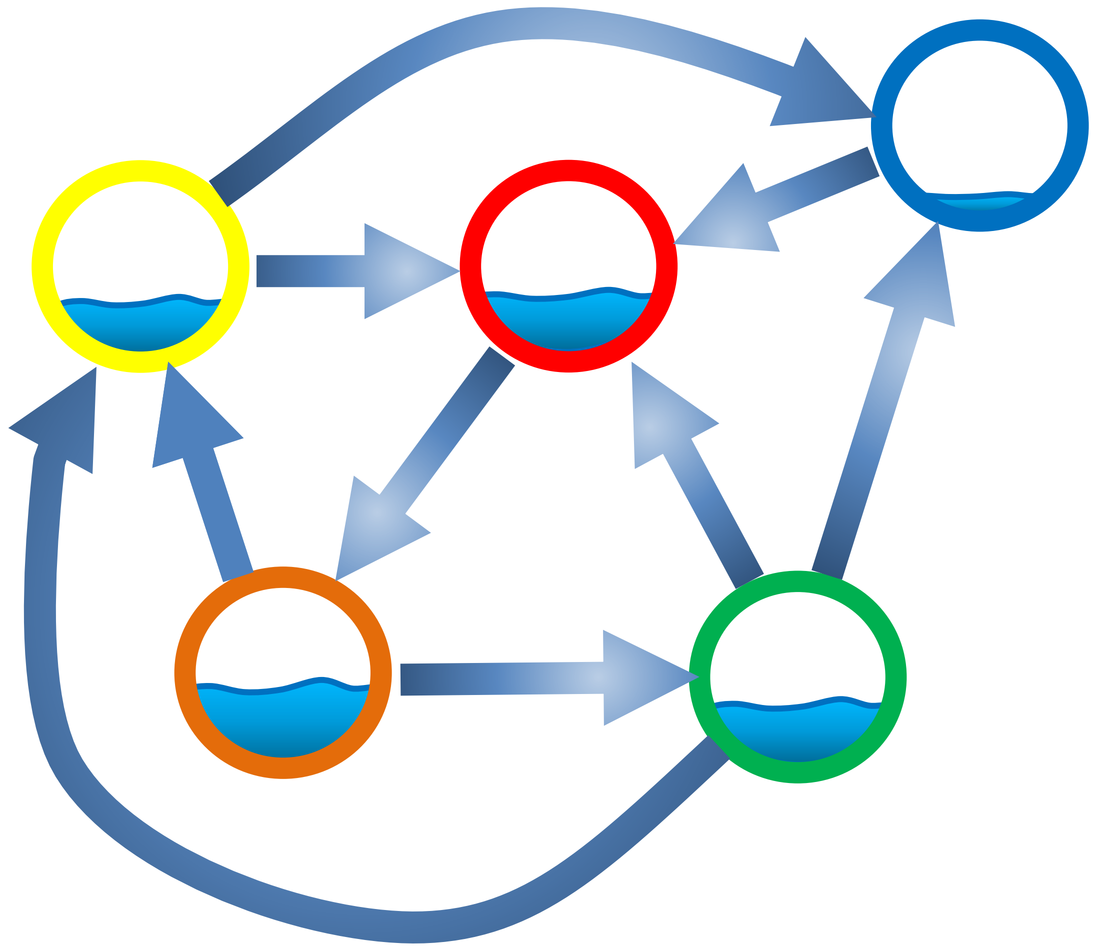
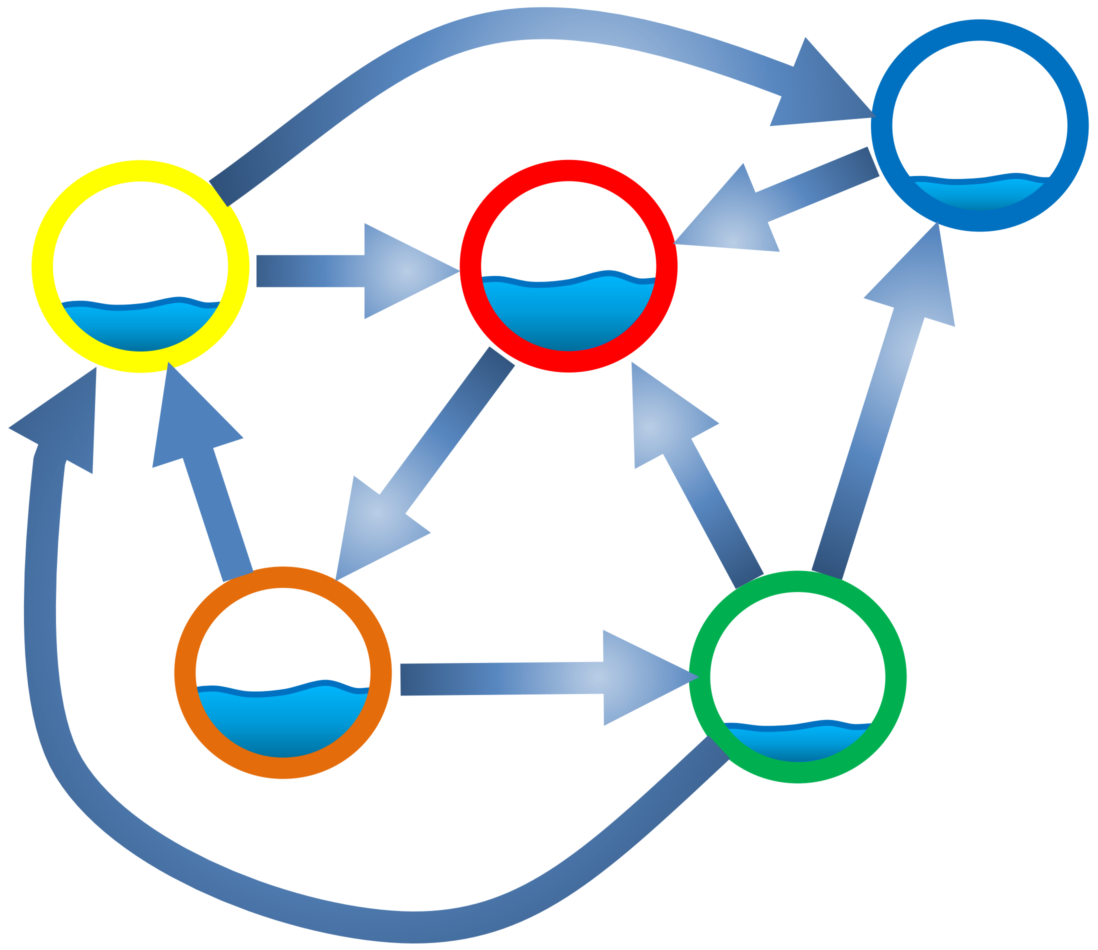


The PageRank algorithm

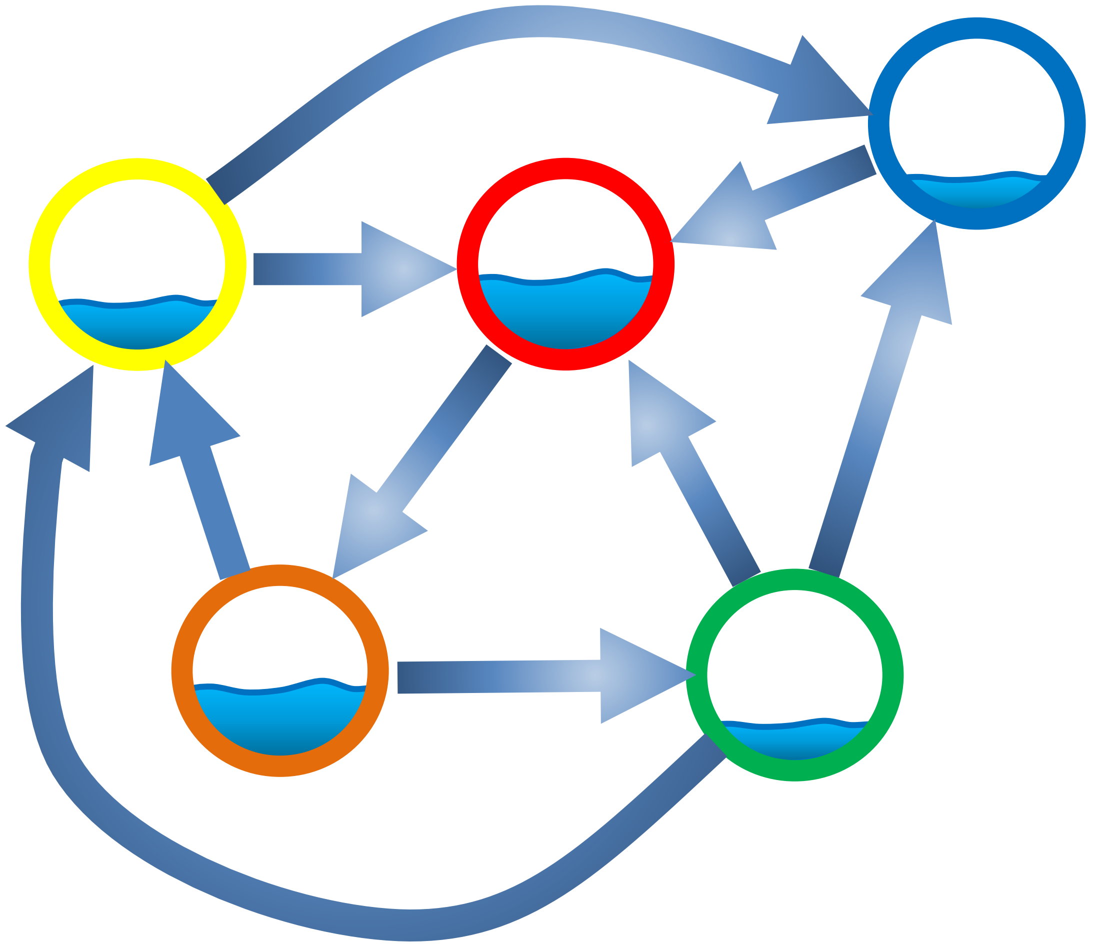
The system will reach an equilibrium state where the amount of liquid in each node remains constant.



The PageRank algorithm

The amount of liquid in each node determines the importance of the node.

Large quantity means large incoming flow from nodes with large quantity of liquid.



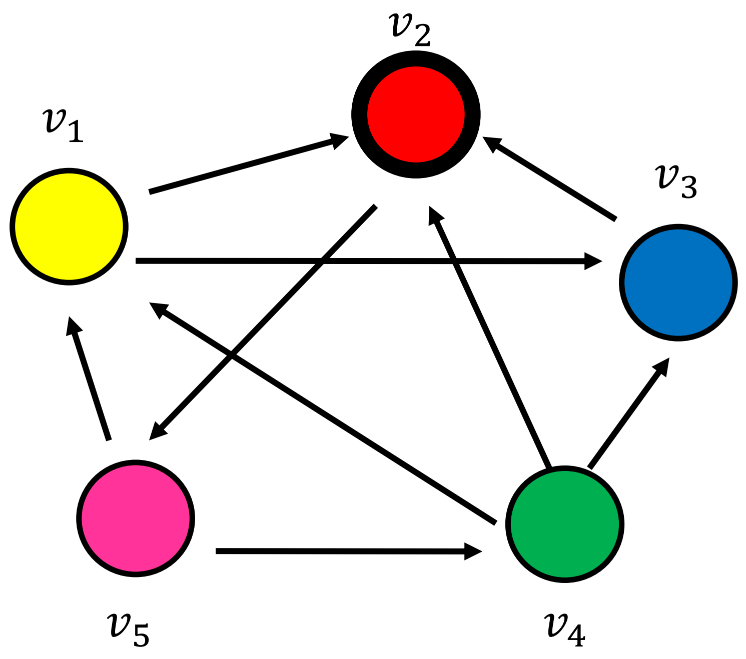
Random Walks on Graphs

- The algorithm defines a random walk on the graph

- Random walk:
  - Start from a node chosen uniformly at random with probability $\frac{1}{n}$.
    - Pick one of the outgoing edges uniformly at random
    - Move to the destination of the edge
    - Repeat.

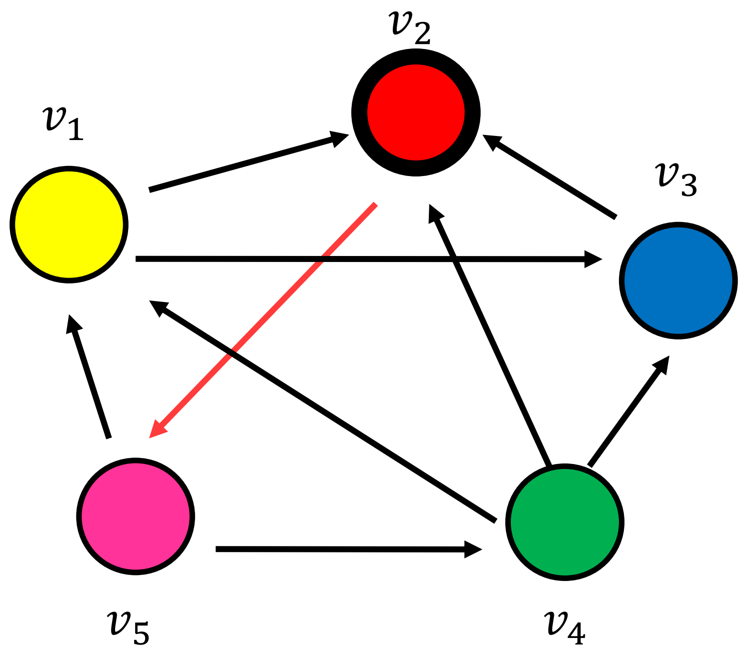
Example

- Step 0



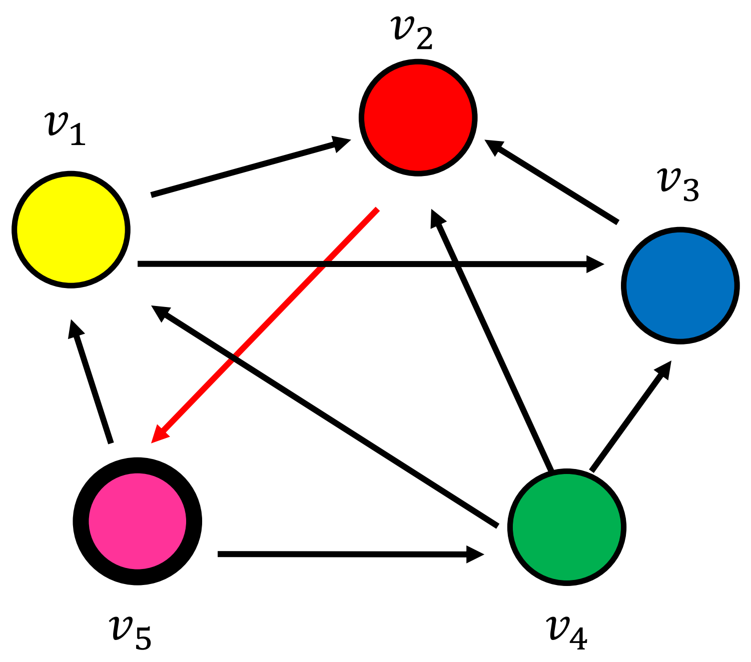
Example

- Step 0



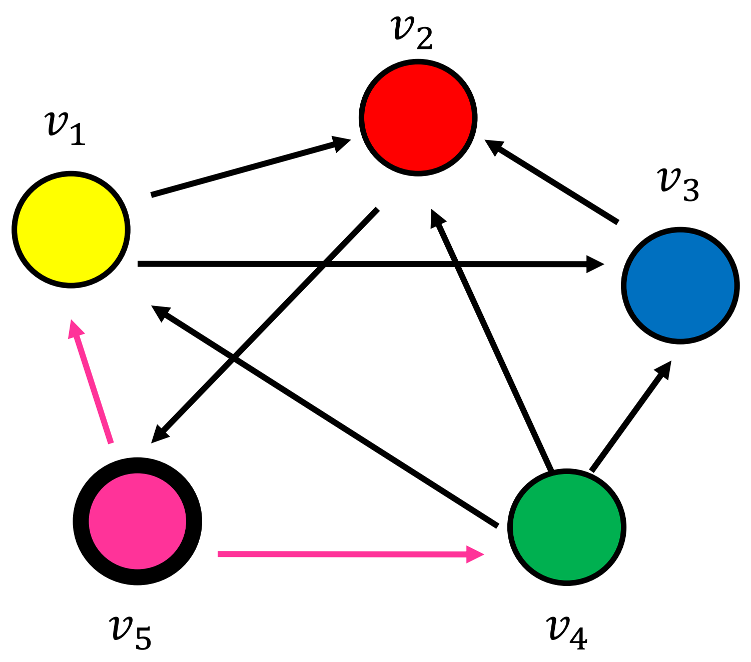
Example

- Step 1



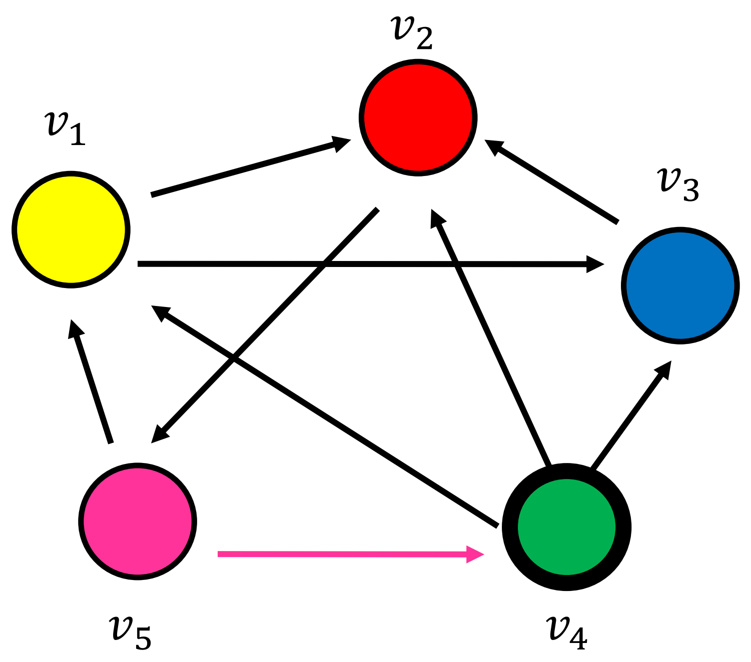
Example

- Step 1



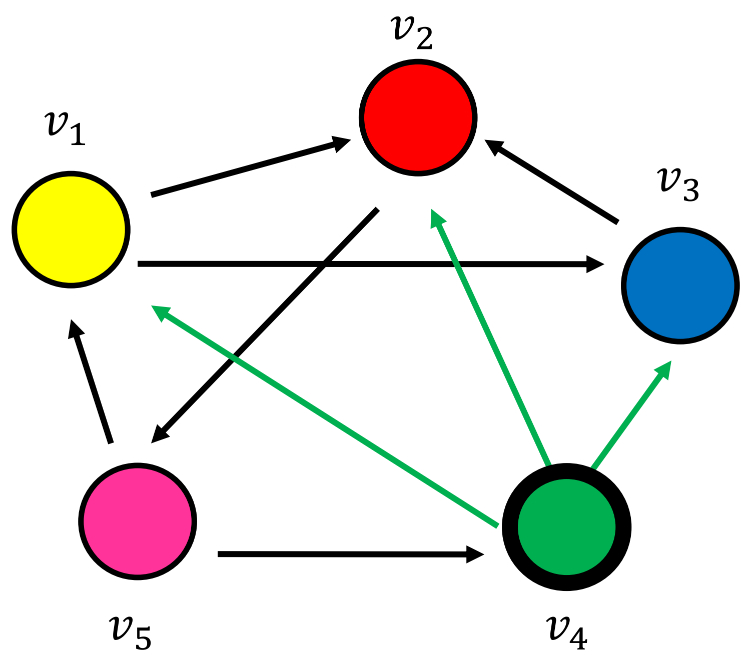
Example

- Step 2



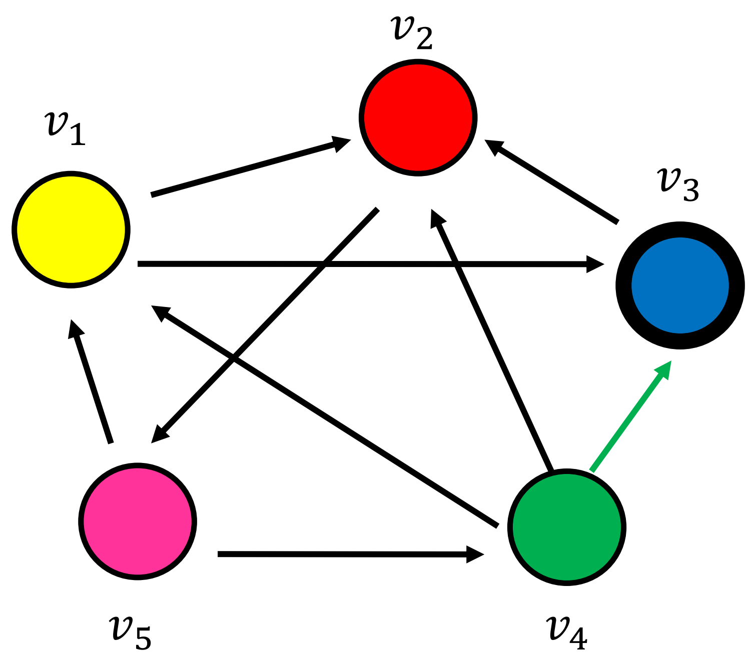
Example

- Step 2



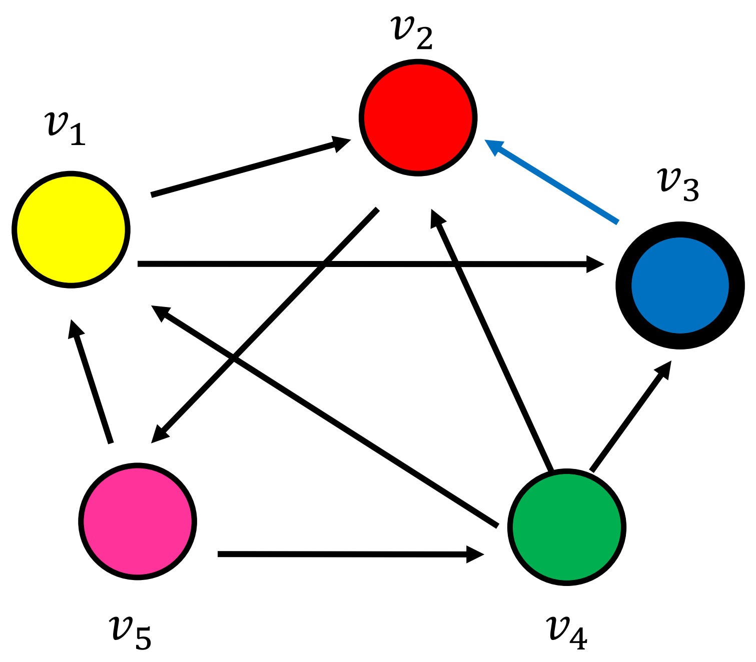
Example

- Step 3



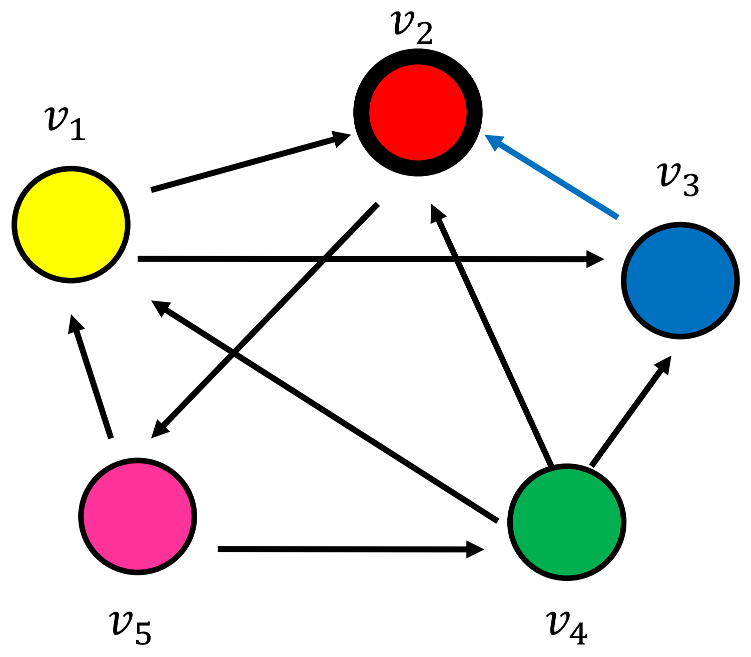
Example

- Step 3



Example

- Step 4…



Random walk

- Question: what is the probability p_i^t of being at node i after t steps?

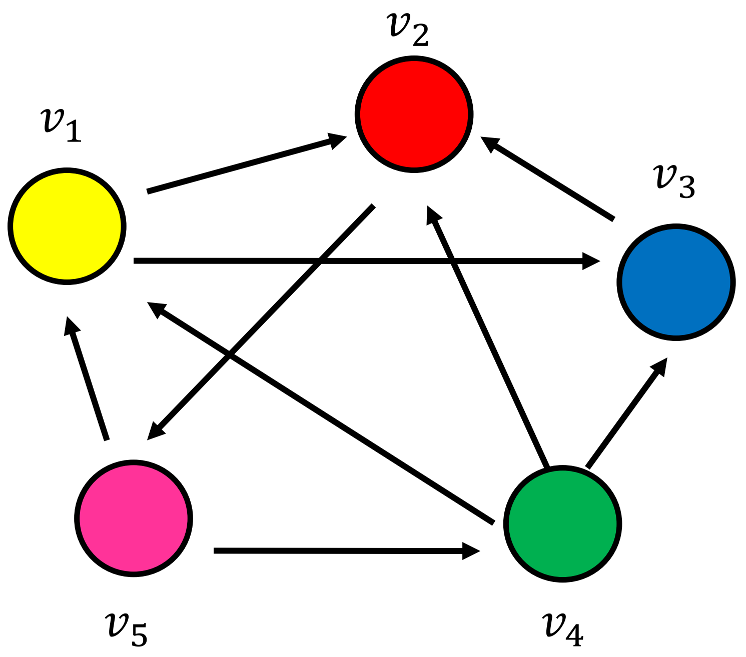
$$p_1^0 = \frac{1}{5} \qquad p_1^t = \frac{1}{3}p_4^{t-1} + \frac{1}{2}p_5^{t-1}$$

$$p_2^0 = \frac{1}{5} \qquad p_2^t = \frac{1}{2}p_1^{t-1} + p_3^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_3^0 = \frac{1}{5} \qquad p_3^t = \frac{1}{2}p_1^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_4^0 = \frac{1}{5} \qquad p_4^t = \frac{1}{2}p_5^{t-1}$$

$$p_5^0 = \frac{1}{5} \qquad p_5^t = p_2^{t-1}$$



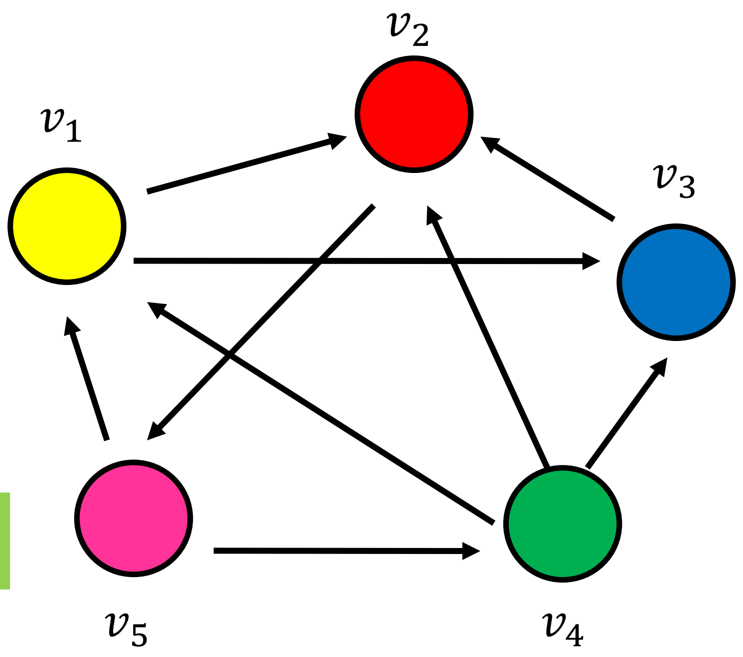
$$p_i^t = \sum_{j \to i} \frac{1}{|N_{out}(j)|} p_j^{t-1}$$

The equations are the same as those for the PageRank iterative computation

Random walk

- At convergence:

$$p_i = \sum_{j \to i} \frac{1}{|N_{out}(j)|} p_j$$



We get the same equation as for PageRank

The PageRank of node i is the probability that the random walk is at node i after a very large (infinite) number of steps

Markov chains

- A Markov chain describes a discrete time stochastic process over a set of states
$$S = \{s_1, s_2, \dots, s_n\}$$
according to a transition probability matrix $P = \{P_{ij}\}$
  - P_{ij} = probability of moving from state i to state j

- Matrix P has the property that the entries of all rows sum to 1
$$\sum_j P[i,j] = 1$$
A matrix with this property is called stochastic

Markov chains

- The stochastic process proceeds in steps and moves between the states:
  - State probability distribution: The vector $p^t = (p_1^t, p_2^t, \dots, p_n^t)$ that stores the probability distribution of being at state s_i after t steps
- Memorylessness property: The next state of the chain depends only at the current state and not on the past of the process (first order MC)
  - Higher order MCs are also possible
- We can compute the vector p^t at step t using a vector-matrix multiplication

$$p^t = p^{t-1}P$$

Stationary distribution

- The stationary distribution of a random walk with transition matrix P, is a probability distribution π, such that $\pi = \pi P$

- The stationary distribution is an eigenvector of matrix P
  - the principal left eigenvector of P – stochastic matrices have maximum eigenvalue 1

- Markov Chain Theory: The random walk converges to a unique stationary distribution independent of the initial vector under some conditions

Random walks

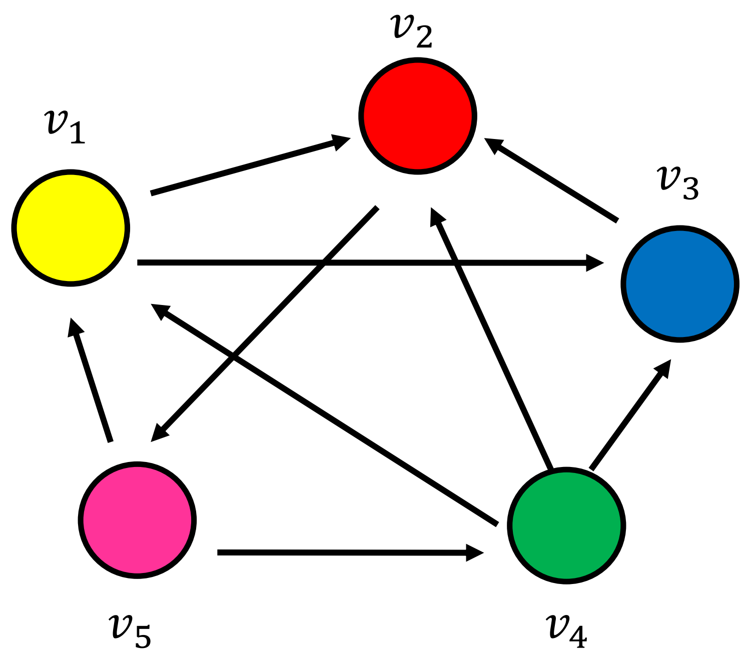
- Markov Chains are equivalent to random walks
  - The set of states S is the set of nodes of the graph G
  - The transition probability matrix is the probability that we follow an edge from one node to another

$$P[i,j] = \frac{1}{|\mathrm{N}_{out}(i)|}$$

The Pagerank random walk and Markov Chain

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The Pagerank random walk and Markov Chain

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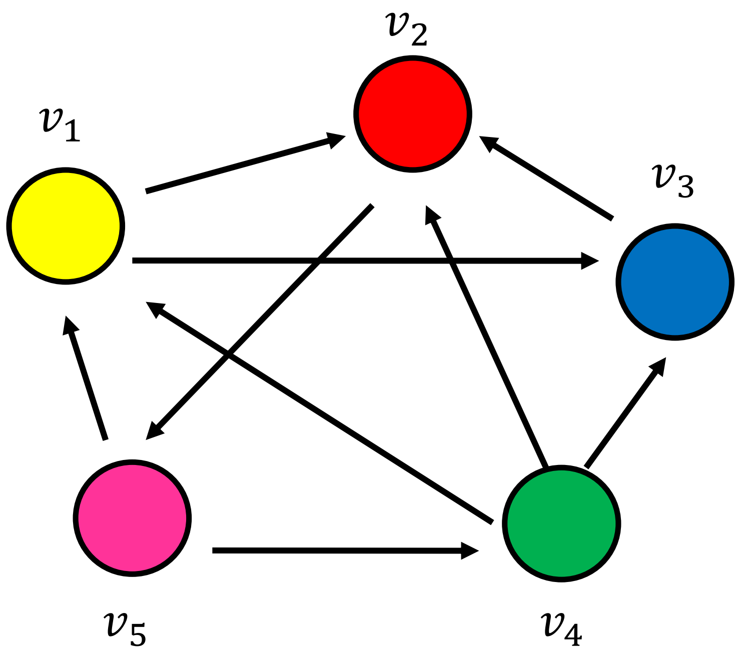


$$p_1^t = \frac{1}{3} p_4^{t-1} + \frac{1}{2} p_5^{t-1}$$

$$p_2^t = \frac{1}{2} p_1^{t-1} + p_3^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_3^t = \frac{1}{2} p_1^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_4^t = \frac{1}{2} p_5^{t-1}$$

$$p_5^t = p_2^{t-1}$$

$$\boxed{p^t = p^{t-1} P}$$

Computing the stationary distribution

- The Power Method, same as the PageRank computation

```
Initialize $p^0$ to some distribution
Repeat
    $p^t = p^{t-1}P$
Until convergence
```

- After many iterations $p^t \rightarrow \pi$ regardless of the initial vector p^0 if the graph is strongly connected, and not bipartite.

- Power method because it computes $p^t = p^0 P^t$

- The rate of convergence is determined by the second eigenvalue λ_2

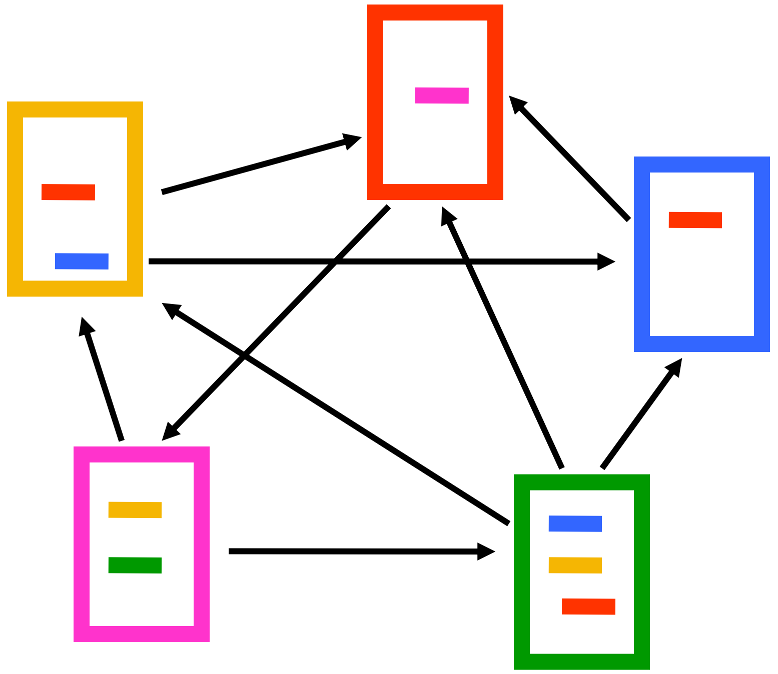
The stationary distribution

- π is the left eigenvector of transition matrix P
- $\pi(i)$: the probability of being at node i after very large (infinite) number of steps
- $\pi(i)$: the fraction of times that the random walk visited state i as $t \to \infty$
- $\pi = p^0 P^\infty$, where P is the transition matrix, p^0 the original vector
  - $P(i,j)$: probability of going from i to j in one step
  - $P^2(i,j)$: probability of going from i to j in two steps (sum of probabilities of all paths of length 2)
  - $P^\infty(i,j) = \pi(j)$: probability of going from i to j in infinite steps – starting point does not matter.

The PageRank random walk

- Vanilla random walk
  - make the adjacency matrix stochastic and run a random walk

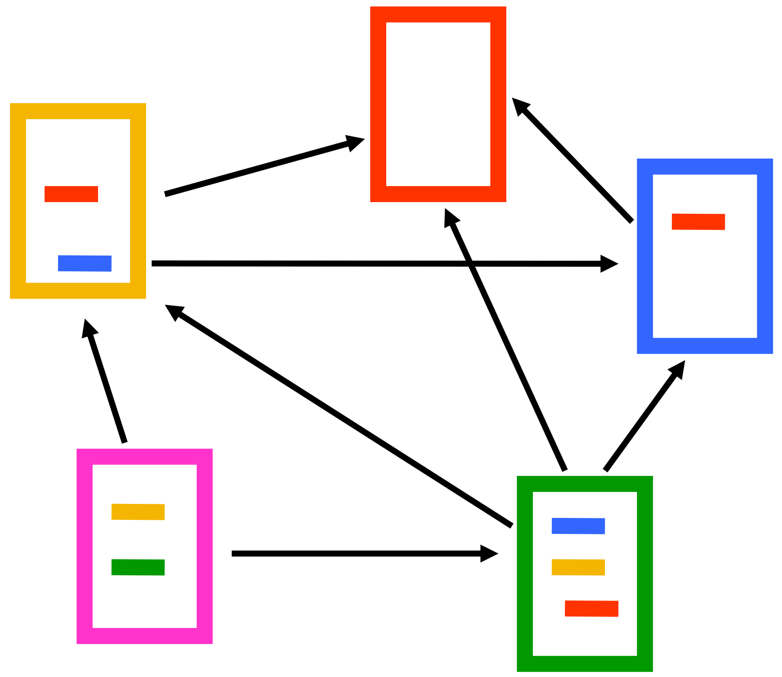
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- What about sink nodes?
  - what happens when the random walk moves to a node without any outgoing inks?

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

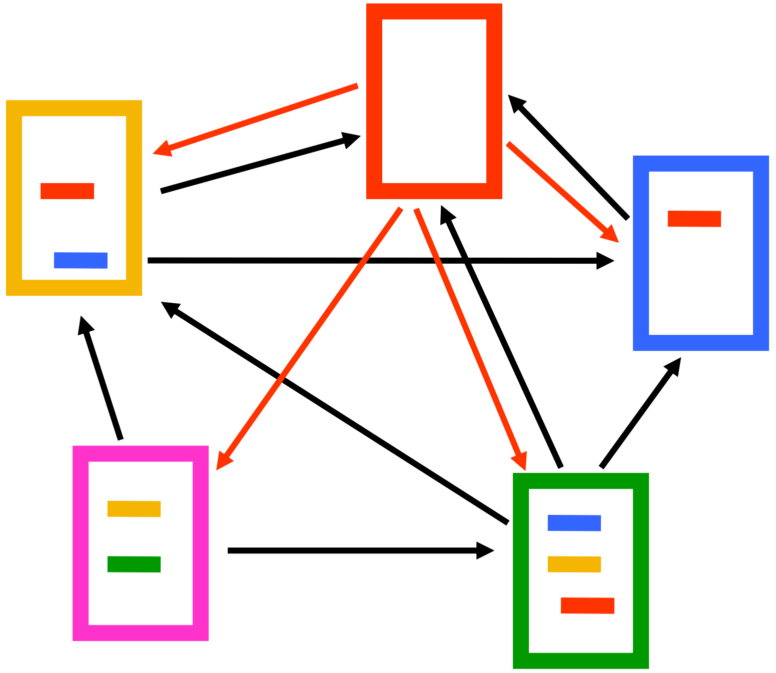
- Replace these row vectors with a vector u
  - typically, the uniform vector

$$P' = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

Outer product

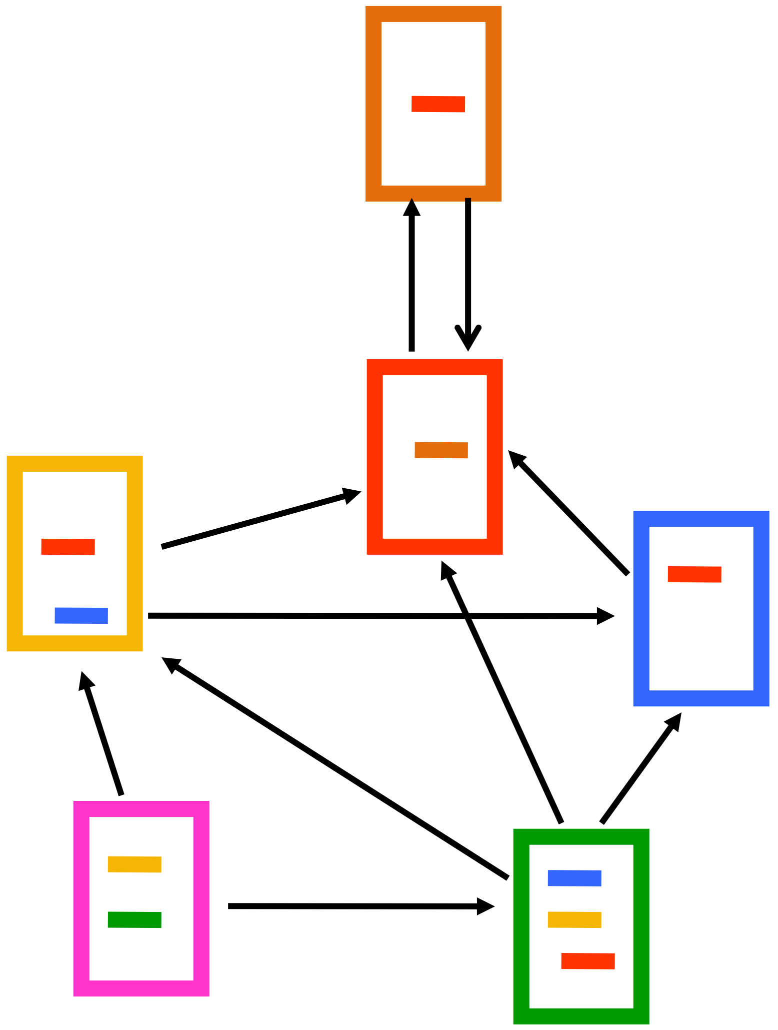
$$P' = P + du^T \qquad d = \begin{cases} 1 & \text{if i is sink} \\ 0 & \text{otherwise} \end{cases}$$

u: The jump vector



The PageRank random walk

- What about loops?
  - Spider traps



The PageRank random walk

- At every step with (fixed) probability α perform a random jump to a node selected according the distribution vector u
  - Typically, to a uniform vector
  - Guarantees irreducibility, convergence
- You can think of the random jump as a restart of the random walk
  - Restart the walk from the chosen node (every $1/\alpha$ steps in expectation)

$$P'' = (1-\alpha)\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + \alpha \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

$P'' = (1-\alpha)P' + \alpha \mathbf{1} u^T$, where **1** is the vector of all 1s

α: jump probability

Random walk with restarts

The PageRank weights

- For the PageRank weights we have

$$p_i = (1-\alpha)\sum_{j\to i}\frac{1}{|N_{out}(j)|}p_j + \alpha u_i$$

  - $\alpha = 0.15$ in most cases

- In matrix-vector terms, if p is the stationary distribution:

$$p^T = p^T(1-\alpha)P + \alpha u^T$$

- Solving for p:

$$p^T = \alpha u^T(I - (1-\alpha)P)^{-1}$$

Stationary distribution with random jump

- If u is the jump vector

$$
\begin{aligned}
p^0 &= u \\
p^1 &= (1-\alpha)p^0P + \alpha u = (1-\alpha)uP + \alpha u \\
p^2 &= (1-\alpha)p^1P + \alpha u = (1-\alpha)^2uP^2 + (1-\alpha)\alpha uP + \alpha u \\
p^3 &= (1-\alpha)p^2P + \alpha u = (1-\alpha)^3uP^3 + (1-\alpha)^2\alpha uP^2 + (1-\alpha)\alpha uP + \ \alpha u \\
p^k &= (1-\alpha)^k uP^k + (1-\alpha)^{k-1}\alpha uP^{k-1} + \cdots + (1-\alpha)\alpha uP + \ \alpha u \\
p^\infty &= \alpha u + (1-\alpha)\alpha uP + (1-\alpha)^2\alpha uP^2 + \ \cdots = \alpha(I-(1-\alpha)P)^{-1}u
\end{aligned}
$$

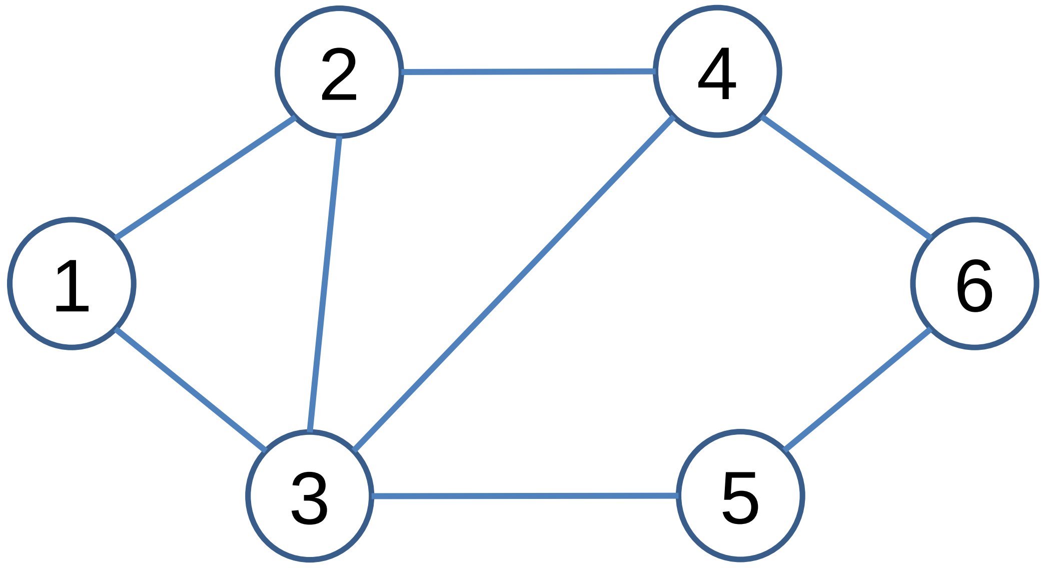
- Explanation: From the last step trace the last restart :
  - With probability α we just restarted in the last step
  - With probability $(1-\alpha)\alpha$ we restarted one step before and then did a random walk step
  - With probability $(1-\alpha)^2\alpha$ we restarted two steps before and then did two random walk steps
  - Etc…

- Conclusion: you are not likely to walk very far
  - The probability that you did k steps after the last restart $(1-\alpha)^k$ drops exponentially with k
  - On average the random walk restarts every $1/\alpha$ steps

Random walks with restarts

- In Random walks with restarts the shorter paths are more important, since the weight decreases exponentially
  - This changes the stationary distribution. When (re)starting from some node x, nodes close to x have higher probability
- Restart vector:
  - If u is not uniform, we can bias the random walk towards the nodes that are close to the restart nodes
- Personalized Pagerank:
  - Always restart to some node x, e.g., the home page of a user
- Topic-Specific Pagerank
  - Restart to nodes about a specific topic, e.g., Greek pages, University home pages
    - Anti-spam
- Random Walks with restarts is a general technique for measuring closeness on graphs.

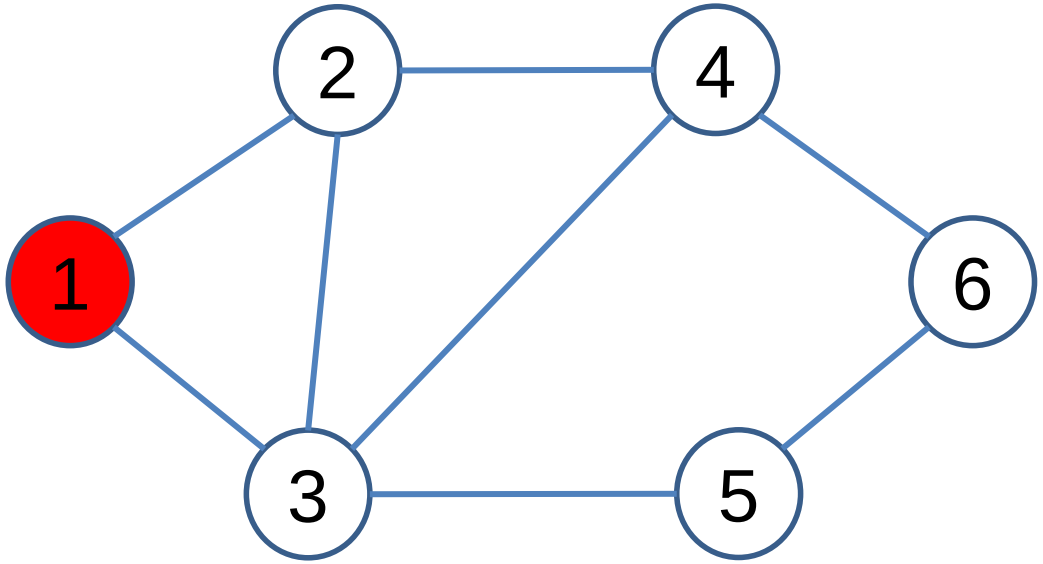
Personalized Pagerank Example



- Global Pagerank vector (uniform jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$)

$$[0.13, 0.18, 0.24, 0.18, 0.13, 0.13]$$

Personalized Pagerank Example



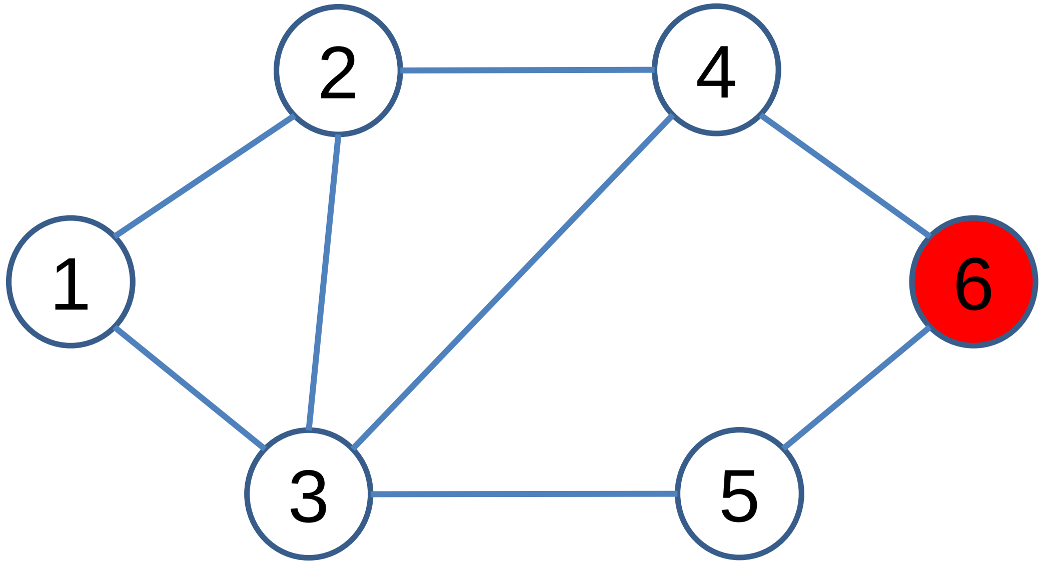
- Global Pagerank vector (jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$) :

$$[0.13, 0.18, 0.24, 0.18, 0.13, 0.13]$$

- Personalized Pagerank for node 1 (jump vector [1,0,0,0,0,0]):

$$[0.26, 0.20, 0.24, 0.14, 0.08, 0.07]$$

Personalized Pagerank Example



- Global Pagerank vector (jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$) :

  $$[0.13, 0.18, 0.24, 0.18, 0.13, 0.13]$$

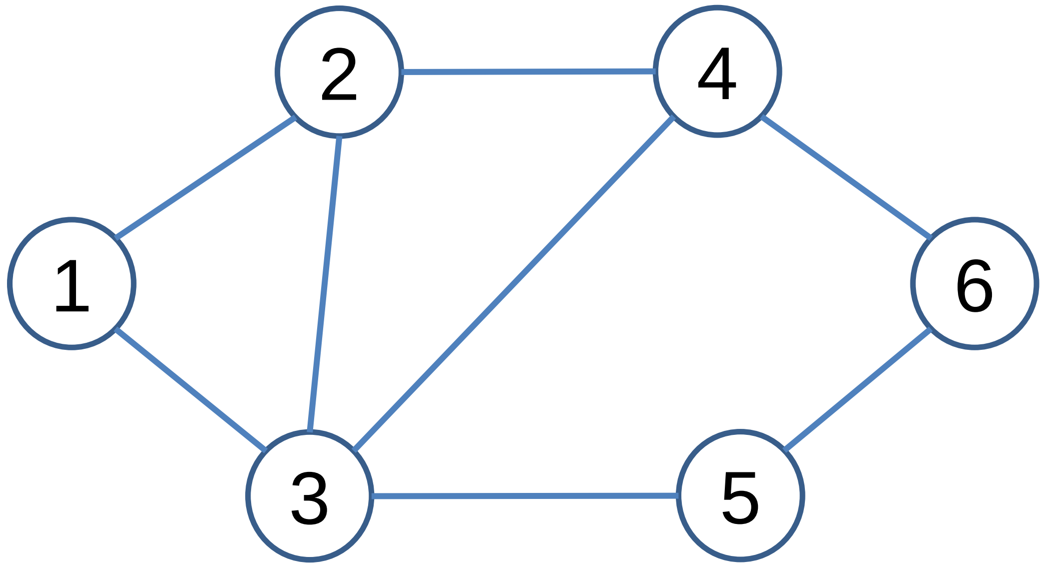
- Personalized Pagerank from node 1 (jump vector $[1,0,0,0,0,0]$):

  $$[0.26, 0.20, 0.24, 0.14, 0.08, 0.07]$$

- Personalized Pagerank for node 6 (jump vector $[0,0,0,0,0,1]$):

  $$[0.07, 0.13, 0.19, 0.19, 0.15, 0.27]$$

Personalized Pagerank Example



With $a = 0.5$

- Global Pagerank vector (jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$) :
  $[0.14, 0.17, 0.21, 0.18, 0.15, 0.15]$
- Personalized Pagerank from node 1 (jump vector $[1,0,0,0,0,0]$):
  $[0.55, 0.17, 0.18, 0.05, 0.03, 0.02]$
- Personalized Pagerank for node 6 (jump vector $[0,0,0,0,0,1]$):
  $[0.02, 0.04, 0.07, 0.16, 0.15, 0.56]$

Random walks on undirected graphs

- For undirected graphs, the stationary distribution is proportional to the degrees of the nodes
  - In this case a random walk is the same as degree popularity

- This is no longer true if we do random jumps
  - Now the short paths play a greater role, and the previous distribution does not hold.

Pagerank implementation

- Store the graph in adjacency list, or list of edges
- Keep current pagerank values and new pagerank values
- Go through edges and update the values of the destination nodes.
- Repeat until the difference (L_1 or L_∞ difference) is below some small value ε.

A (Matlab/Numpy-friendly) PageRank algorithm

- Performing vanilla power method is now too expensive – the matrix is not sparse

```
$q^0 = u$
$t = 1$
repeat
    $q^t = (P'')^T q^{t-1}$
    $\delta = \|q^t - q^{t-1}\|$
    $t = t + 1$
until $\delta < \varepsilon$
```

Efficient computation of $y = (P'')^T x$

$$y = (1-\alpha)P^T x$$
$$\beta = \|x\|_1 - \|y\|_1$$
$$y = y + \beta u$$

P = normalized adjacency matrix

$P' = P + du^T$, where d_i is 1 if i is sink and 0 o.w.

$P'' = (1-\alpha)P' + \alpha \mathbf{1} u^T$, where $\mathbf{1}$ is the vector of all 1s

Pagerank history

- Huge advantage for Google in the early days
  - It gave a way to get an idea for the value of a page, which was useful in many different ways
    - Put an order to the web.
  - After a while it became clear that the anchor text was probably more important for ranking
  - Also, link spam became a new (dark) art
- Flood of research
  - Numerical analysis got rejuvenated
  - Huge number of variations
  - Efficiency became a great issue.
  - Huge number of applications in different fields
    - Random walk is often referred to as PageRank.

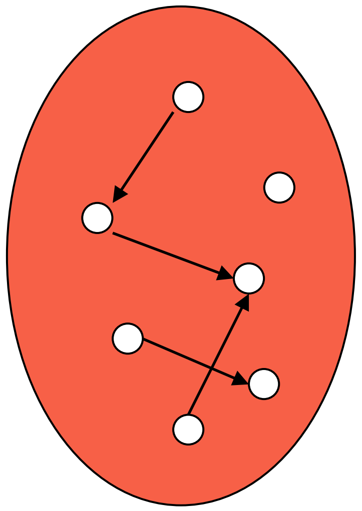
THE HITS ALGORITHM

The HITS algorithm

- Another algorithm proposed around the same time as Pagerank for using the hyperlinks to rank pages
  - Kleinberg: then an intern at IBM Almaden
  - IBM never made anything out of it

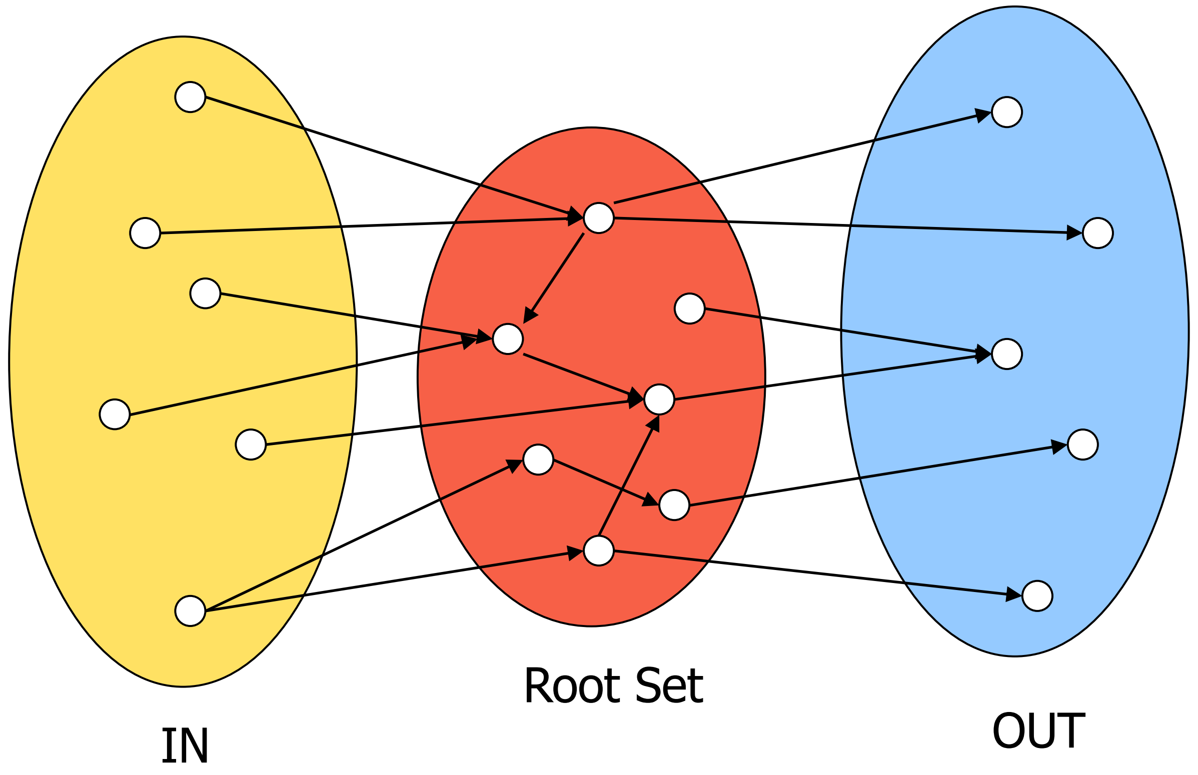
Query dependent input

Root set obtained from a text-only search engine

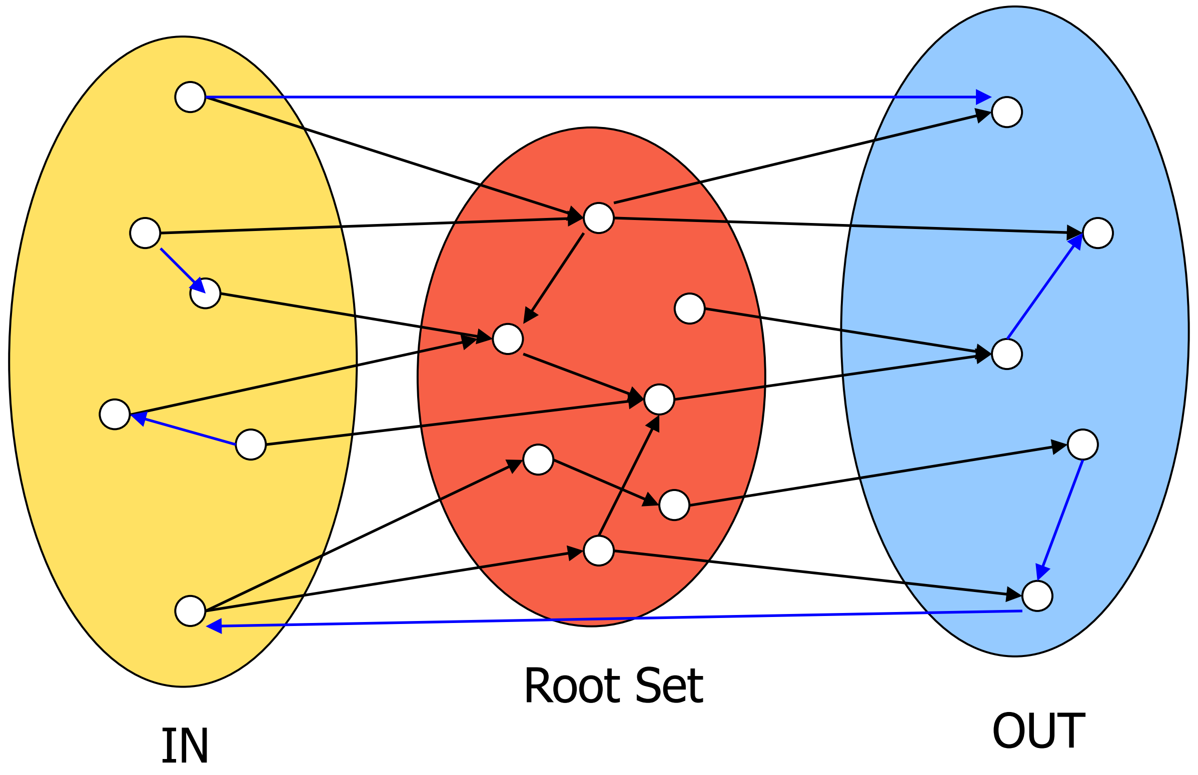


Root Set

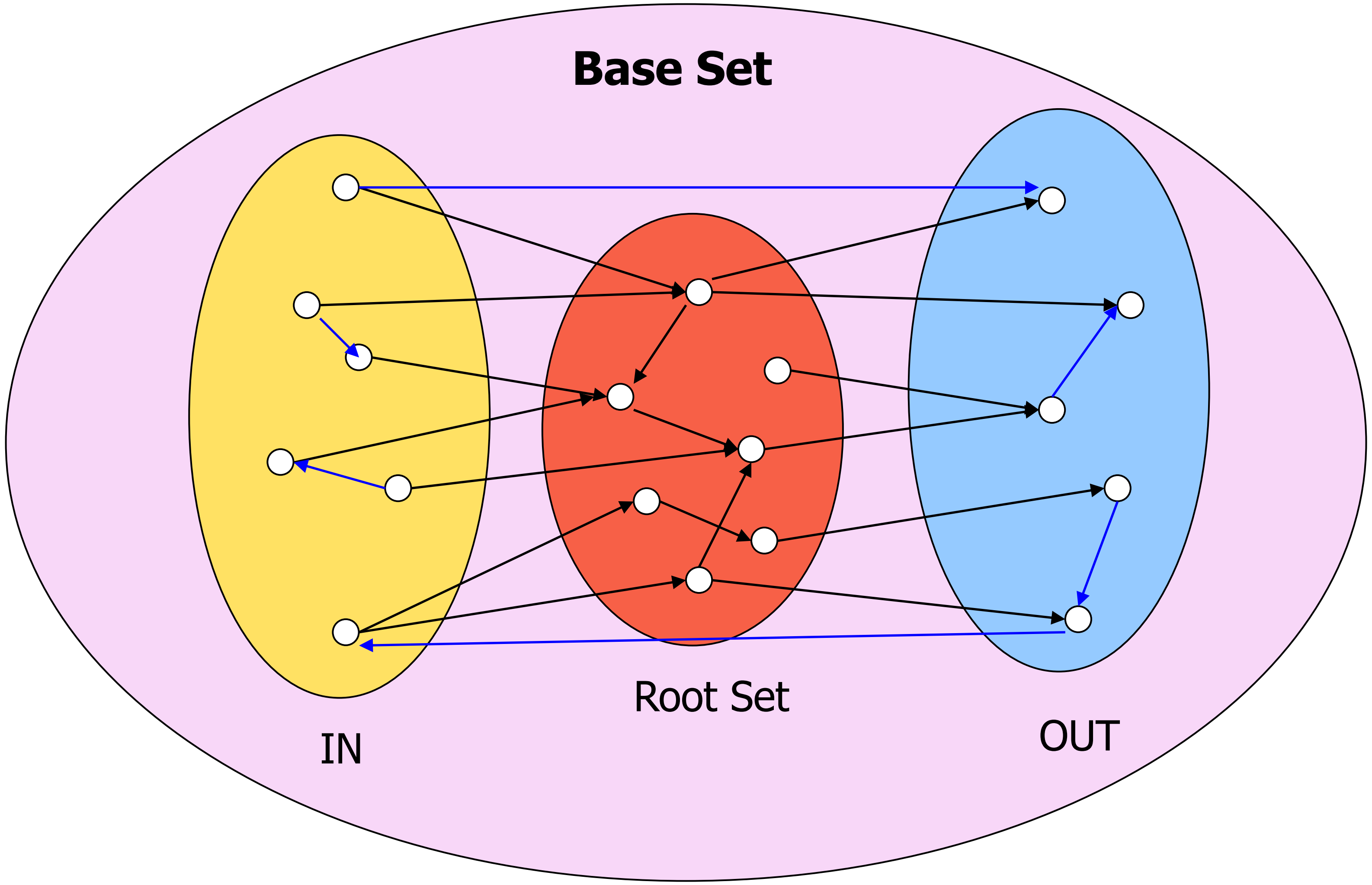
Query dependent input



Query dependent input

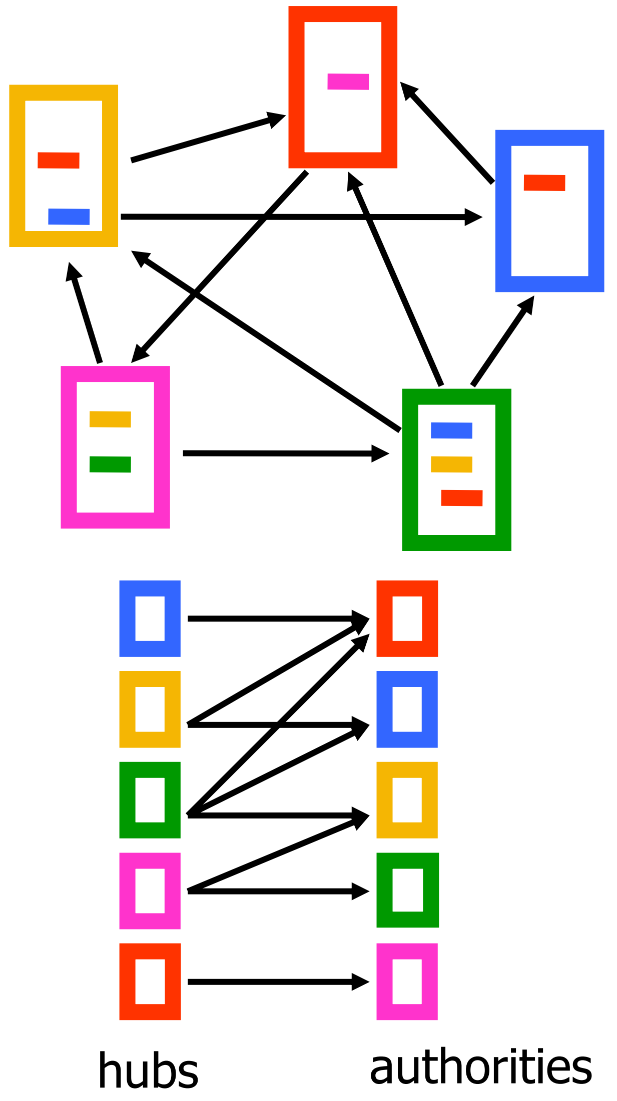


Query dependent input



Hubs and Authorities [K98]

- Authority is not necessarily transferred directly between authorities
- Pages have double identity
  - hub identity
  - authority identity
- Good hubs point to good authorities
- Good authorities are pointed by good hubs



Hubs and Authorities

- Two kind of weights:
  - Hub weight
  - Authority weight

- The hub weight is the sum of the authority weights of the authorities pointed to by the hub

- The authority weight is the sum of the hub weights that point to this authority.

HITS Algorithm

- Initialize all weights to 1.
- Repeat until convergence
  - $\boldsymbol{O}$ operation : hubs collect the weight of the authorities

$$h_i^t = \sum_{j: i \to j} a_j^{t-1}$$

  - $\boldsymbol{I}$ operation: authorities collect the weight of the hubs

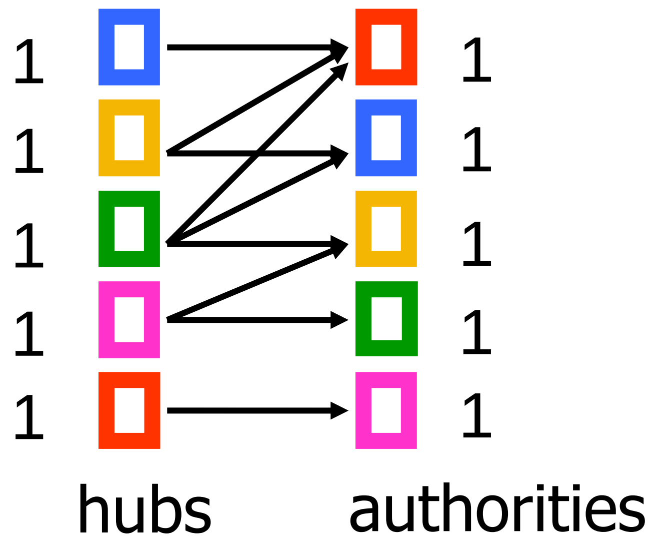
$$a_i^t = \sum_{j: j \to i} h_j^{t-1}$$

  - Normalize weights under some norm

The order of updates does not matter after many iterations.

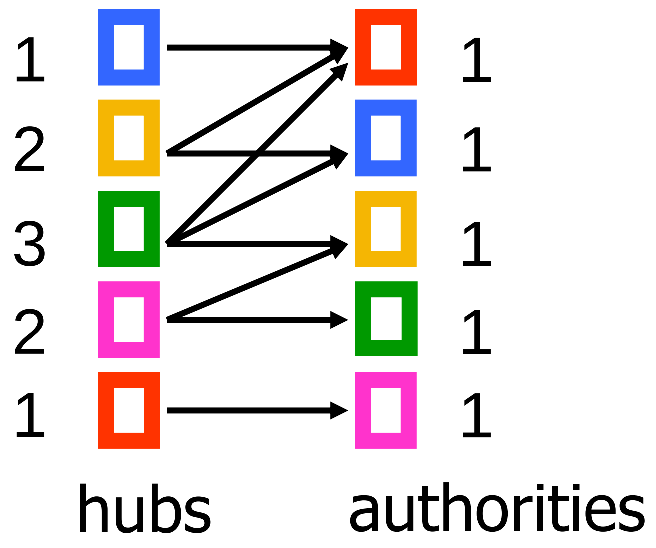
Example

Initialize



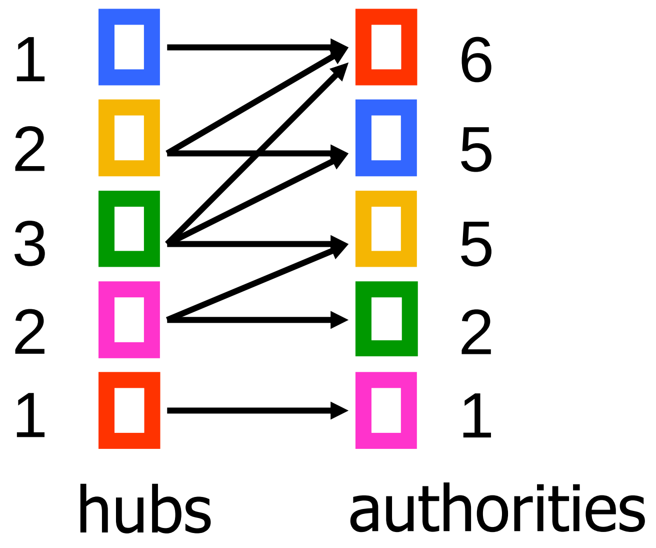
Example

Step 1: O operation



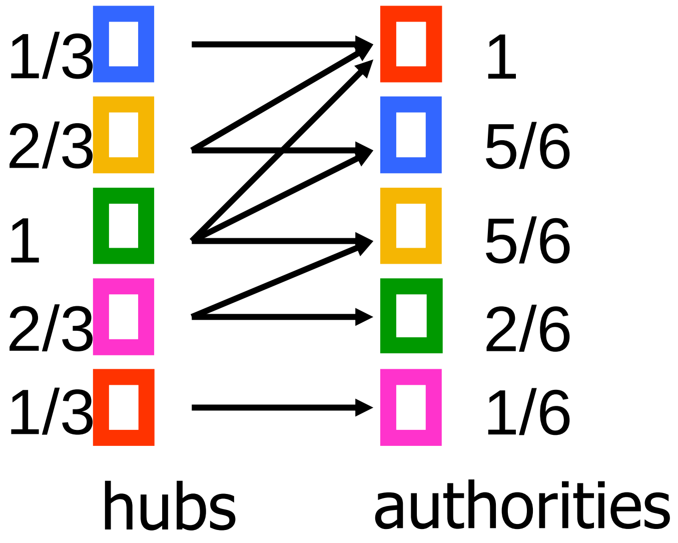
Example

Step 1: I operation



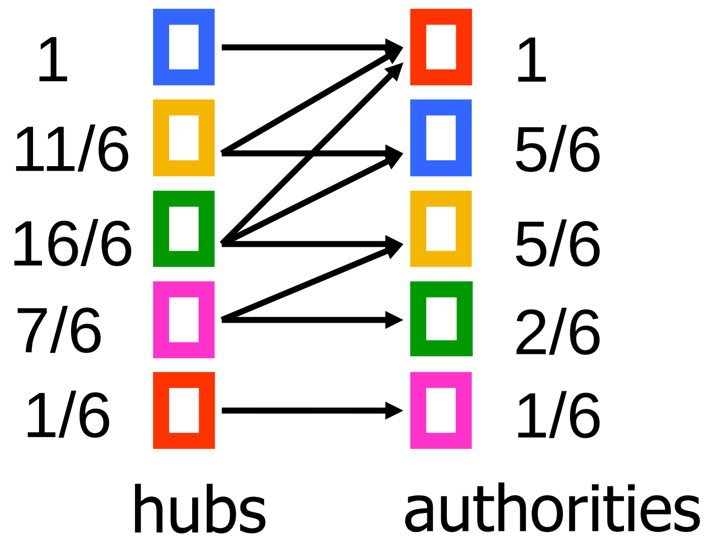
Example

Step 1: Normalization (Max norm)



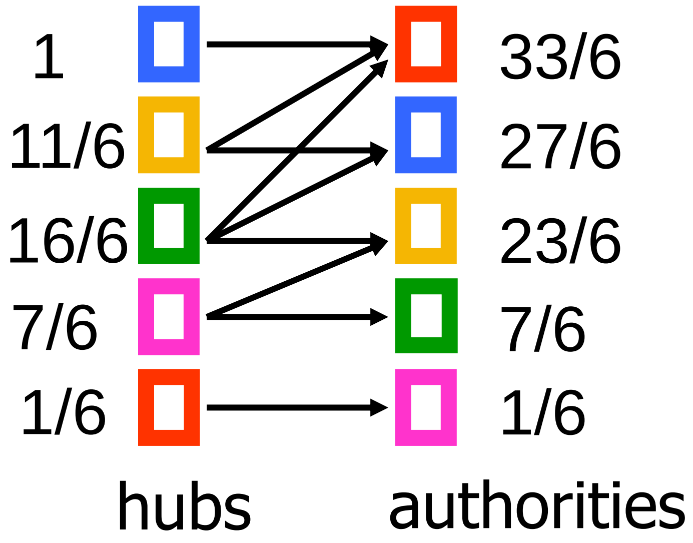
Example

Step 2: O step



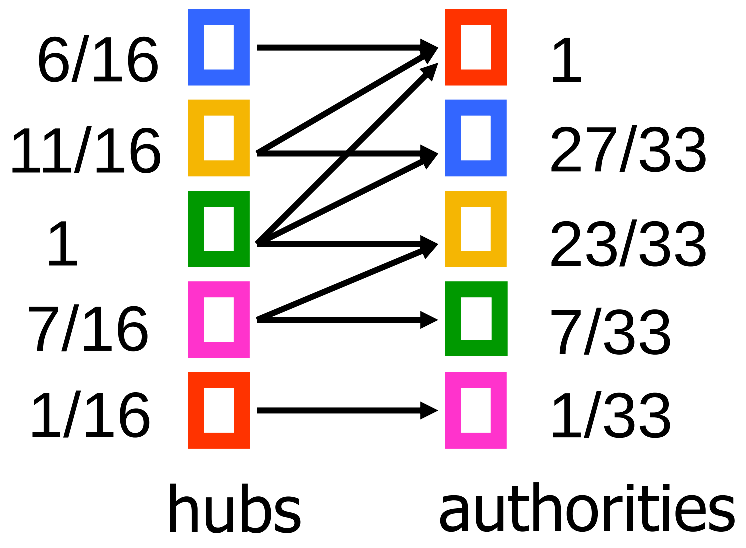
Example

Step 2: I step



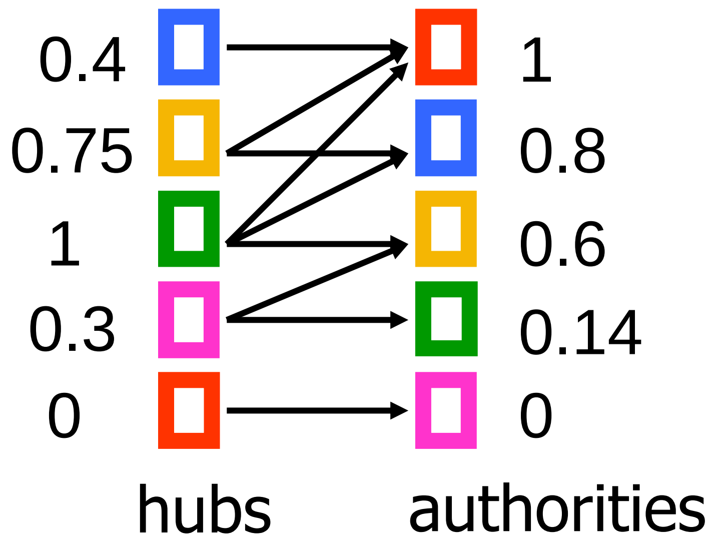
Example

Step 2: Normalization



Example

Convergence



HITS and eigenvectors

- The HITS algorithm is a power-method eigenvector computation
- In vector terms
  - $a^t = A^T h^{t-1}$ and $h^t = A a^{t-1}$
  - $a^t = A^T A a^{t-1}$ and $h^t = A A^T h^{t-1}$
  - Repeated iterations will converge to the eigenvectors

- The authority weight vector a is the eigenvector of $A^T A$
- The hub weight vector h is the eigenvector of $A A^T$

- The vectors a and h are the singular vectors of the matrix A

Singular Value Decomposition

$$A = U \quad \Sigma \quad V^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_r \end{bmatrix}$$

$$[n \times r]\,[r \times r]\,[r \times n]$$

- **r** : rank of matrix A
- $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$: singular values (square roots of eig-vals AA^T, A^TA)
- $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r$: left singular vectors (eig-vectors of AA^T)
- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$: right singular vectors (eig-vectors of A^TA)
- $$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

Why does the Power Method work?

- If a matrix R is real and symmetric, it has real eigenvalues and eigenvectors: $(\lambda_1, w_1), \ (\lambda_2, w_2), \ \ldots, (\lambda_r, w_r)$
  - r is the rank of the matrix
  - $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|$
- For any matrix R, the eigenvectors $w_1, w_2, \ldots, w_r$ of R define a basis of the vector space
  - For any vector x, $Rx = \alpha_1 w_1 + a_2 w_2 + \cdots + a_r w_r$
- After t multiplications we have:

$$R^t x = \lambda_1^{t-1} \alpha_1 w_1 + \lambda_2^{t-1} a_2 w_2 + \cdots + \lambda_2^{t-1} a_r w_r$$

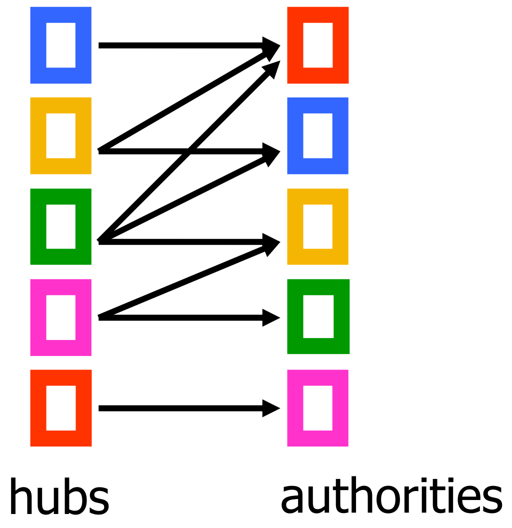
- Normalizing leaves only the term w_1.

OTHER ALGORITHMS

The SALSA algorithm

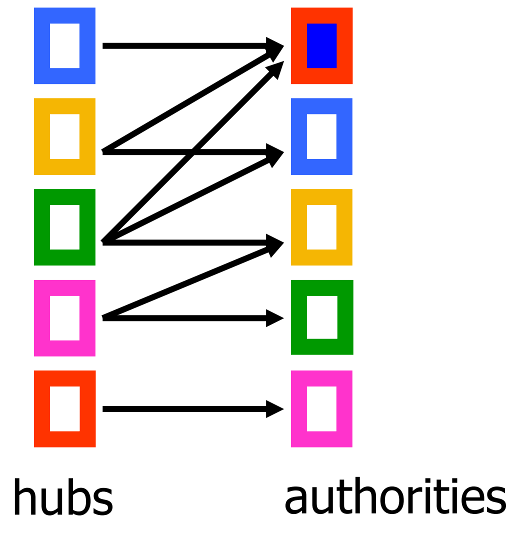
- Perform a random walk on the bipartite graph of hubs and authorities alternating between the two

- What does this random walk converges to?



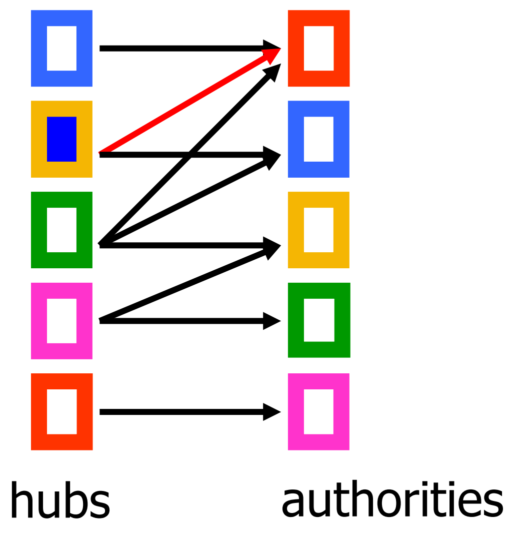
The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority



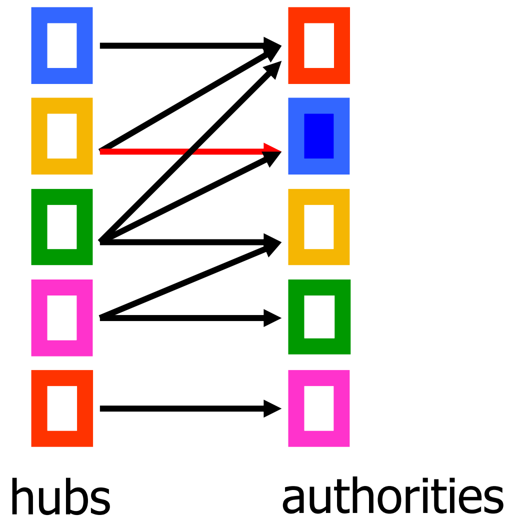
The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority
- Choose one of the in-coming links uniformly at random and move to a hub
  - e.g. move to the yellow authority with probability 1/3



The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority
- Choose one of the in-coming links uniformly at random and move to a hub
  - e.g. move to the yellow authority with probability 1/3
- Choose one of the out-going links uniformly at random and move to an authority
  - e.g. move to the blue authority with probability 1/2



The SALSA algorithm

- Formally we have probabilities:
  - a_i: probability of being at authority i
  - h_j: probability of being at hub j
- The probability of being at authority i is computed as:

$$a_i^t = \sum_{j \in N_{in}(i)} \frac{1}{d_{out}(j)} h_j^{t-1}$$

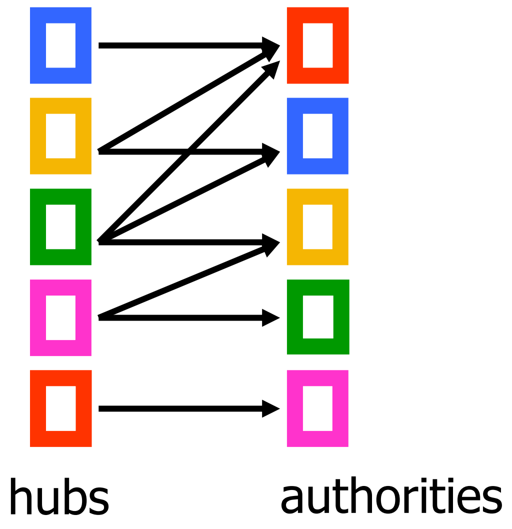
- The probability of being at hub j is computed as

$$h_j^t = \sum_{i \in N_{out}(j)} \frac{1}{d_{in}(i)} a_i^{t-1}$$

- Repeated computation converges

The SALSA algorithm

- In matrix terms
  - A_c = the matrix A where columns are normalized to sum to 1
  - A_r = the matrix A where rows are normalized to sum to 1
- The hub computation
  - $h = A_c a$
- The authority computation
  - $a = A_r^T h = A_r^T A_c a$
- In MC terms the transition matrix
  - $P = A_r A_c^T$



$$h_2 = 1/3\, a_1 + 1/2\, a_2$$

$$a_1 = h_1 + 1/2\, h_2 + 1/3\, h_3$$

Social network analysis

- Evaluate the centrality of individuals in social networks
  - degree centrality
    - the (weighted) degree of a node
  - distance centrality
    - the average (weighted) distance of a node to the rest in the graph

$$D_c(v) = \frac{1}{\sum_{u \neq v} d(v,u)}$$

  - betweenness centrality
    - the average number of (weighted) shortest paths that use node v

$$B_c(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

Counting paths – Katz 53

- The importance of a node is measured by the weighted sum of paths that lead to this node
- $A^m[i,j]$ = number of paths of length m from i to j
- Compute

$$P = bA + b^2A^2 + \cdots + b^mA^m + \cdots = (I - bA)^{-1} - I$$

- converges when $b < \lambda_1(A)$
- Rank nodes according to the column sums of the matrix P

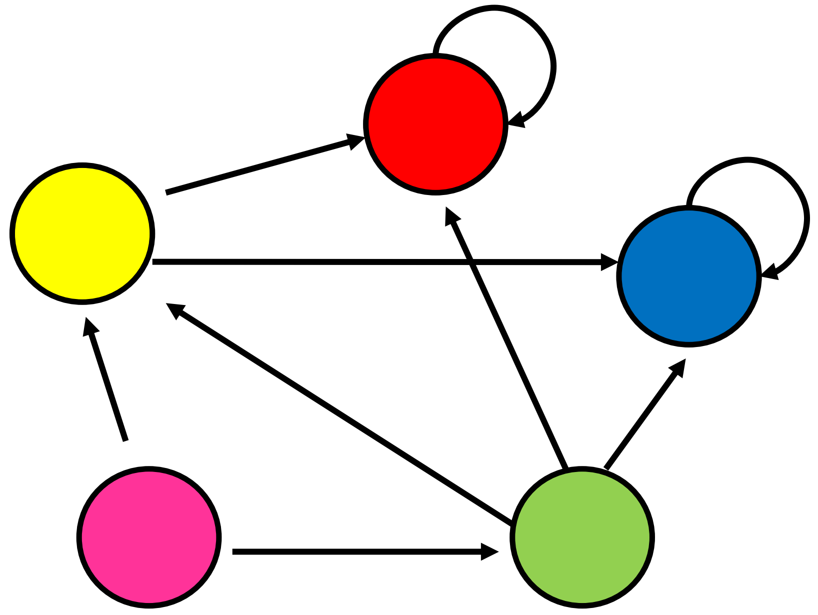
Bibliometrics

- Impact factor (E. Garfield 72)
  - counts the number of citations received for papers of the journal in the previous two years
- Pinsky-Narin 76
  - perform a random walk on the set of journals
  - P_{ij} = the fraction of citations from journal i that are directed to journal j

ABSORBING RANDOM WALKS

Random walk with absorbing nodes

- Absorbing nodes: nodes from which the random walk cannot escape.

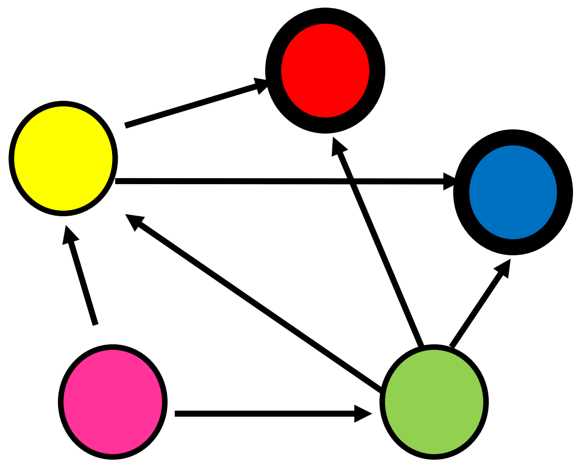


- Two absorbing nodes: the red and the blue.

P. G. Doyle, J. L. Snell. *Random Walks and Electrical Networks*. 1984

Absorption probability

- In a graph with more than one absorbing nodes a random walk that starts from a non-absorbing (transient) node t will be absorbed in one of them with some probability
  - For node t we can compute the probabilities of absorption



Absorption probability

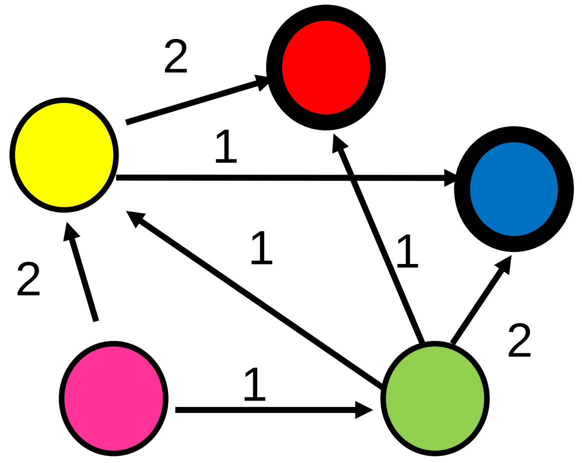
- Computing the probability of being absorbed:
  - The absorbing nodes have probability 1 of being absorbed in themselves and zero of being absorbed in another node.
  - For the non-absorbing nodes, take the (weighted) average of the absorption probabilities of your neighbors
    - if one of the neighbors is the absorbing node, it has probability 1
  - Repeat until convergence (= very small change in probs)

$$P(Red|Pink) = \frac{2}{3}P(Red|Yellow) + \frac{1}{3}P(Red|Green)$$

$$P(Red|Green) = \frac{1}{4}P(Red|Yellow) + \frac{1}{4}$$

$$P(Red|Yellow) = \frac{2}{3}$$

$$P(Red|Red) = 1 \,, P(Red|Blue) = 0$$



Absorption probability

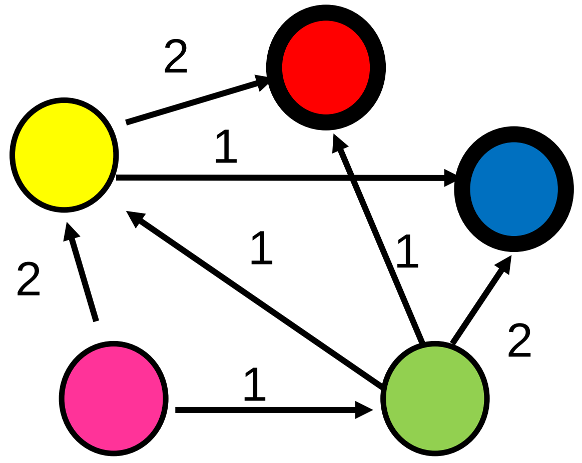
- Computing the probability of being absorbed:
  - The absorbing nodes have probability 1 of being absorbed in themselves and zero of being absorbed in another node.
  - For the non-absorbing nodes, take the (weighted) average of the absorption probabilities of your neighbors
    - if one of the neighbors is the absorbing node, it has probability 1
  - Repeat until convergence (= very small change in probs)

$$P(Blue|Pink) = \frac{2}{3}P(Blue|Yellow) + \frac{1}{3}P(Blue|Green)$$

$$P(Blue|Green) = \frac{1}{4}P(Blue|Yellow) + \frac{1}{2}$$

$$P(Blue|Yellow) = \frac{1}{3}$$

$$P(Blue|Blue) = 1\,, P(Blue|Red) = 0$$



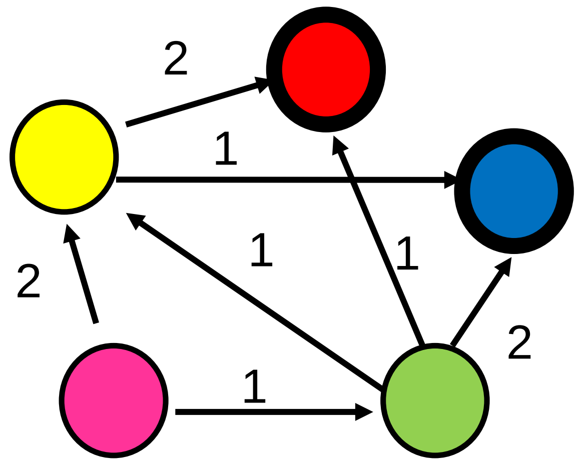
Absorption probability

- Computing the probability of being absorbed:
  - The absorbing nodes have probability 1 of being absorbed in themselves and zero of being absorbed in another node.
  - For the non-absorbing nodes, take the (weighted) average of the absorption probabilities of your neighbors
    - if one of the neighbors is the absorbing node, it has probability 1
  - Repeat until convergence (= very small change in probs)

General equation for the probability of transient node t being absorbed at absorbing node a

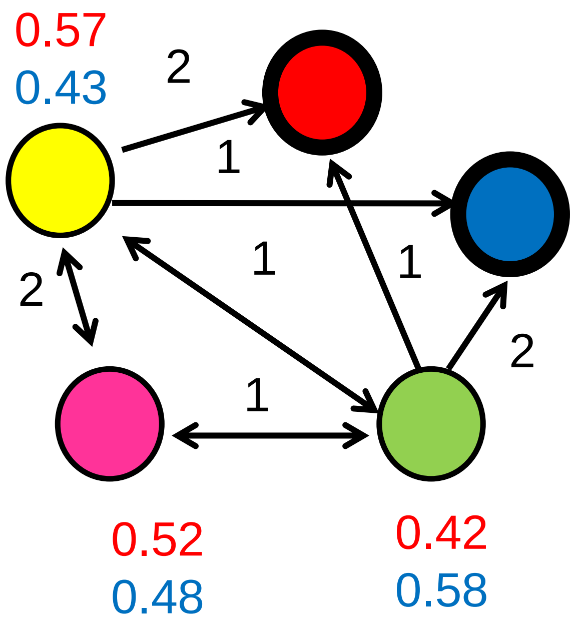
$$P(a|t) = \sum_{(t,x)\in E} P[t,x]P(a|x)$$

The weighted average of the neighbors



Absorption probabilities

- The absorption probabilities for red and blue



Absorption probabilities

- The absorption probability has several practical uses.
- Given a graph (directed or undirected) we can choose to make some nodes absorbing.
  - Simply direct all edges incident on the chosen nodes towards them and create a self-loop.
- The absorbing random walk provides a measure of proximity of transient nodes to the chosen nodes.
  - Useful for understanding proximity in graphs
  - Useful for propagation in the graph
    - E.g, on a social network some nodes are malicious, while some are certified, to which class is a transient node closer?

Linear Algebra

- The transition matrix of the random walk looks like this

$$P = \begin{array}{cc} \text{T} & \text{A} \\ \left[\begin{array}{c|c} P_{TT} & P_{TA} \\ \hline 0 & I \end{array}\right] \end{array}$$

T: transient

A: absorbing

- P_{TT}: transition probabilities between transient nodes
- P_{TA}: transition probabilities from transient to absorbing nodes
- Computing the absorption probabilities corresponds to iteratively multiplying matrix P with itself

Linear algebra

$$P^{\infty} = \begin{bmatrix} 0 & Q \\ 0 & I \end{bmatrix}$$

- The transient-to-absorbing matrix Q
  - $Q[i,k]$ = The probability of being absorbed in absorbing state a_k when starting from transient state t_i
- The fundamental matrix

$$F = I + P_{TT} + P_{TT}^2 + \cdots = \sum_{i=0}^{\infty} P_{TT}^i = (I - P_{TT})^{-1}$$

  - $F[i,j]$ = The sum (to infinity) of probabilities of visiting transient state t_j when starting from state t_i after any number of steps

    Also: The expected number of visits to transient state t_j when starting from state t_i after infinite number of steps

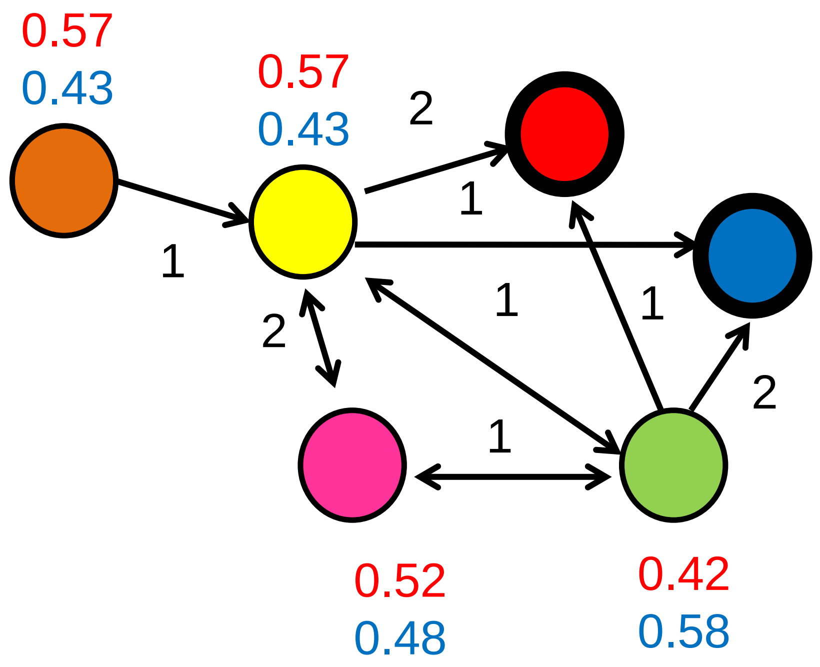
$$Q = FP_{TA} = P_{TA} + P_{TT}P_{TA} + P_{TT}^2 P_{TA} + \cdots$$

Penalizing long paths

- The orange node has the same probability of reaching red and blue as the yellow one

$$P(Red|Orange) = P(Red|Yellow)$$

$$P(Blue|Orange) = P(Blue|Yellow)$$



- Intuitively though it is further away

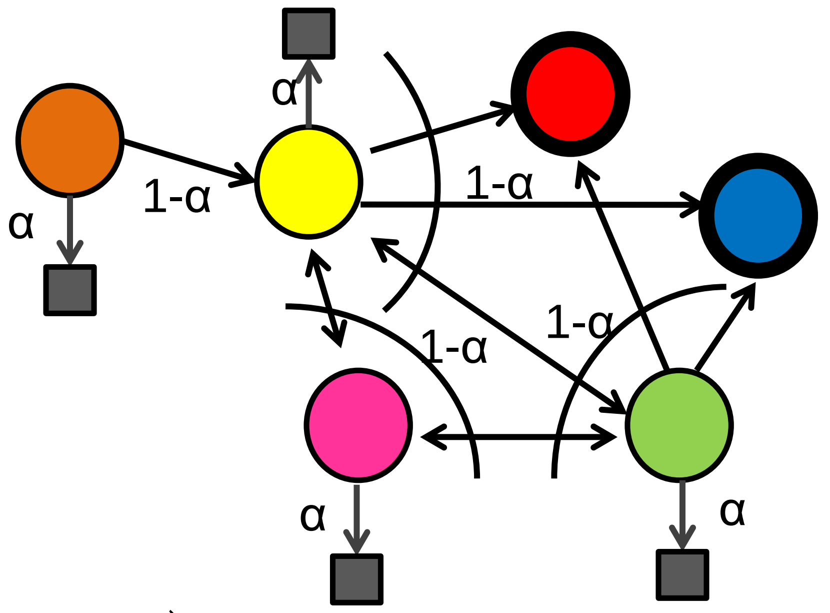
Penalizing long paths

- Add a universal absorbing node to which each node gets absorbed with probability α.

With probability α the random walk dies

With probability (1-α) the random walk continues as before

The longer the path from a node to an absorbing node the more likely the random walk dies along the way, the lower the absorbtion probability



e.g. $P(Red|Green) = (1-\alpha)\left(\frac{1}{5}P(Red|Yellow) + \frac{1}{5}P(Red|Pink) + \frac{1}{5}\right)$

Absorbing Random Walks and Random walks with restarts

- Adding a jump with probability α to a universal absorbing node seems similar to Pagerank

- The Random Walk With Restarts (RWS) and Absorbing Random Walk (ARW) are similar but not the same
  - RWS computes the probability of paths from the starting node u to a node v, while AWR the probability of paths from a node v, to the absorbing node u.
  - RWS defines a distribution over all nodes, while AWR defines a probability for each node
  - An absorbing node blocks the random walk, while restarts simply bias towards starting nodes
    - Makes a difference when having multiple (and possibly competing) absorbing nodes

- You can implement RWS as an absorbing walk, but not clear how to do the opposite

Propagating values

- Assume that Red has a positive value and Blue a negative value
  - Positive/Negative class, Positive/Negative opinion
- We can compute a value for all the other nodes by repeatedly averaging the values of the neighbors
  - The value of node u is the expected value at the point of absorption for a random walk that starts from u

$$V(Pink) = \frac{2}{3}V(Yellow) + \frac{1}{3}V(Green)$$

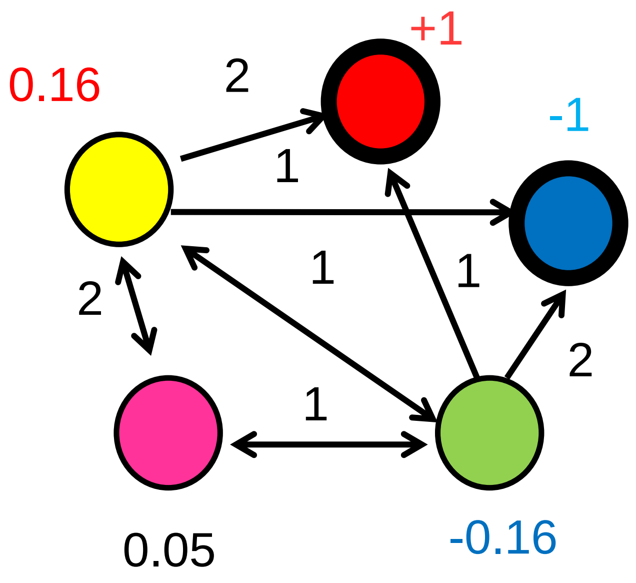
$$V(Green) = \frac{1}{5}V(Yellow) + \frac{1}{5}V(Pink) + \frac{1}{5} - \frac{2}{5}$$

$$V(Yellow) = \frac{1}{6}V(Green) + \frac{1}{3}V(Pink) + \frac{1}{3} - \frac{1}{6}$$

General equation for value propagation:

$$v(i) = \sum_{(i,j)\in E} P[i,j]v(j)$$

The value of i is the weighted average of the values of its neighbors



Linear algebra

- Computation of values is essentially multiplication of the matrix Q with the vector of values of the absorbing nodes
  - Recall: Matrix Q is the $T \times A$ matrix, and $Q[t, a]$ is the probability of transient node t being absorbed at absorbing node a

$$\boldsymbol{v} = Q\boldsymbol{s}$$

  - $\boldsymbol{s}$: vector of values of the absorbing nodes
  - $\boldsymbol{v}$: vector of values of the transient nodes

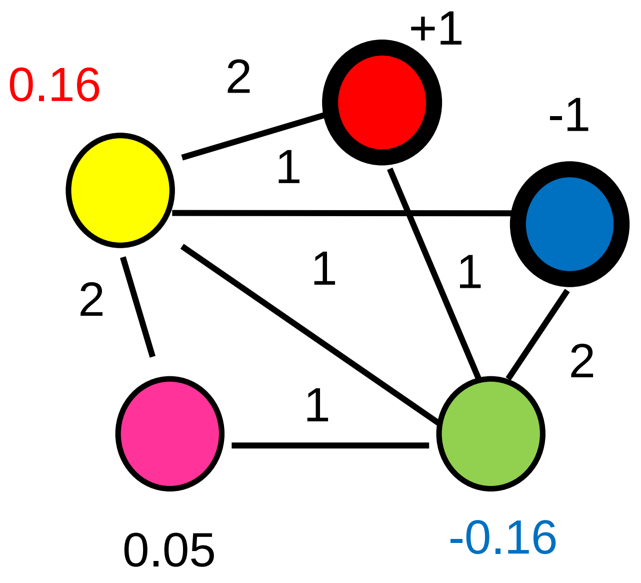
Electrical networks and random walks

- Our graph corresponds to an electrical network
- There is a positive voltage of +1 at the Red node, and a negative voltage -1 at the Blue node
- There are resistances on the edges inversely proportional to the weights (or conductance proportional to the weights)
- The computed values are the voltages at the nodes

$$V(Pink) = \frac{2}{3}V(Yellow) + \frac{1}{3}V(Green)$$

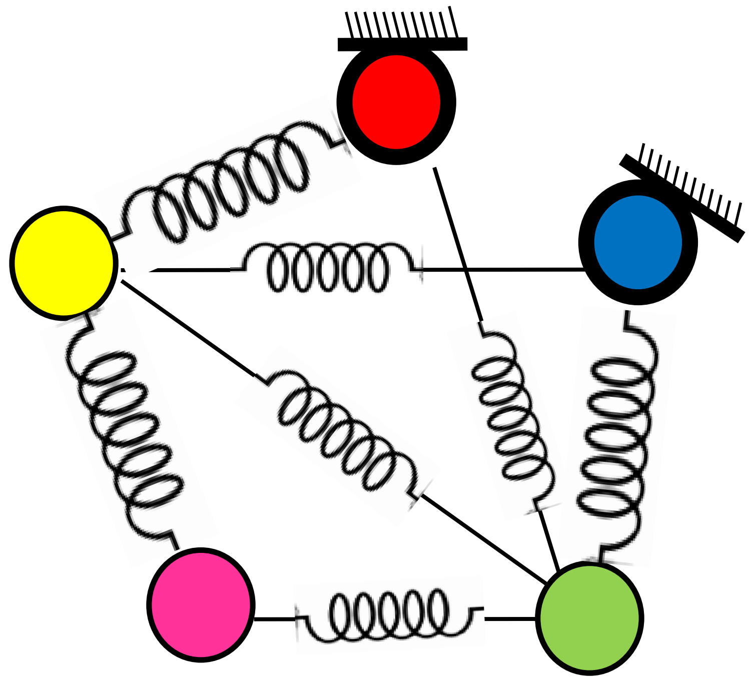
$$V(Green) = \frac{1}{5}V(Yellow) + \frac{1}{5}V(Pink) + \frac{1}{5} - \frac{2}{5}$$

$$V(Yellow) = \frac{1}{6}V(Green) + \frac{1}{3}V(Pink) + \frac{1}{3} - \frac{1}{6}$$



Springs and random walks

- Our graph corresponds to an spring system
- The Red node is pinned at position +1, while the Blue node is pinned at position -1 on a line.
- There are springs on the edges with hardness proportional to the weights
- The computed values are the positions of the nodes on the line



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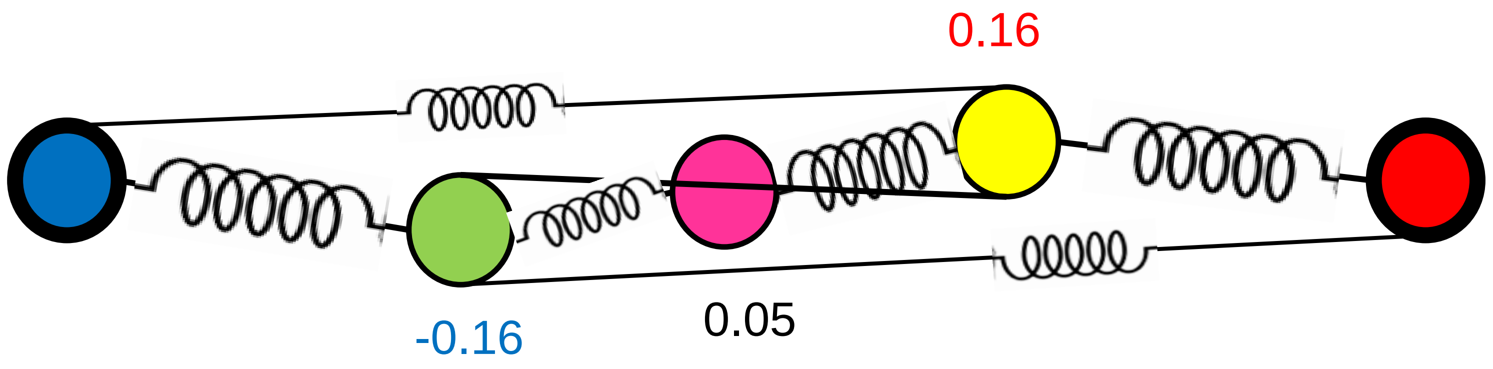


Opinion formation model (Friedkin and Johnsen)

- Opinions are values in $[-1, +1]$
- Every user i has an intrinsic opinion $s_i \in [-1, +1]$ and an expressed opinion $z_i \in [-1, +1]$
- The public opinion z_i of each user in the network is iteratively updated, each time taking the average of the expressed opinions of its neighbors and the intrinsic opinion of herself

$$z_i^t = \frac{w_{ii} s_i + \sum_{j \in N(i)} w_{ij} z_j^{t-1}}{w_{ij} + \sum_{j \in N(i)} w_{ij}}$$

Opinion formation as a game

- Assume that network users are rational (selfish) agents
- Each user has a personal cost for expressing an opinion

$$c(z_i) = w_{ii}(z_i - s_i)^2 + \sum_{j \in N(i)} w_{ij}\left(z_i - z_j\right)^2$$

Inconsistency cost: The cost for deviating from one’s intrinsic opinion

Conflict cost: The cost for disagreeing with the opinions in one’s social network

- Each user is selfishly trying to minimize her personal cost.

D. Bindel, J. Kleinberg, S. Oren. *How Bad is Forming Your Own Opinion?* Proc. 52nd IEEE Symposium on Foundations of Computer Science, 2011.

Opinion formation as a game

- The opinion z_i that minimizes the personal cost of user i

$$z_i = \frac{w_{ij} s_i + \sum_{j \in N(i)} w_{ij} z_j}{w_{ij} + \sum_{j \in N(i)} w_{ij}}$$

The Friedkin & Johnsen model

- In linear algebra terms (assume 0/1 weights):

$$(L + I)\boldsymbol{z} = \boldsymbol{s} \ \Rightarrow \boldsymbol{z} = (L + I)^{-1}\boldsymbol{s}$$

where L is the Laplacian of the graph.

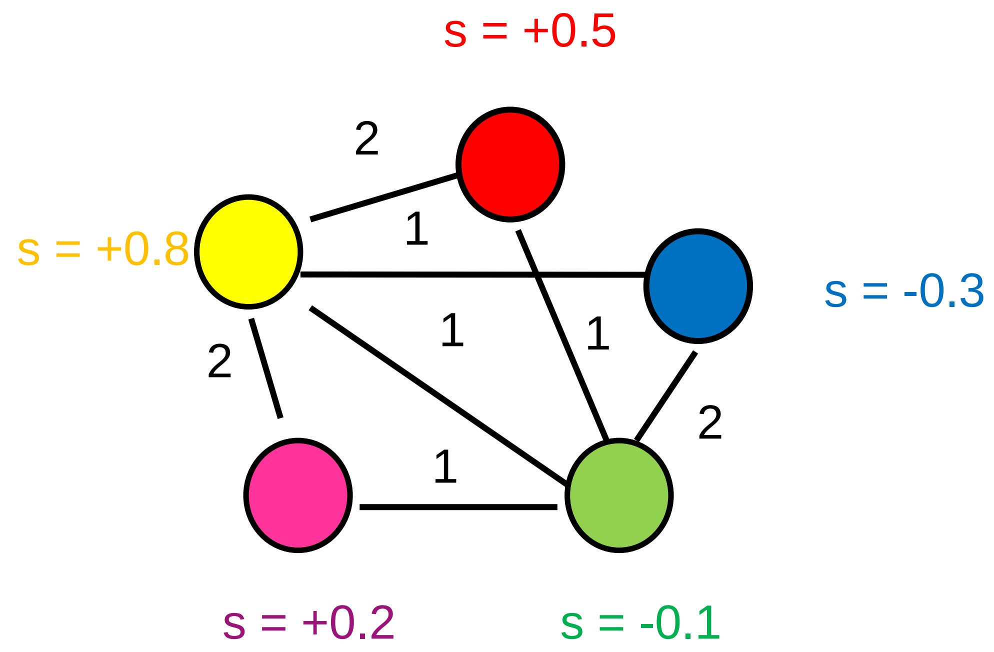
- The Laplacian is the negated adjacency matrix with the degree on the diagonal

$$L = D - A$$

  - D: diagonal matrix with the degrees on the diagonal

Example

- Social network with internal opinions



- There is a connection of this model with absorbing random walks and value propagation – can you see it?

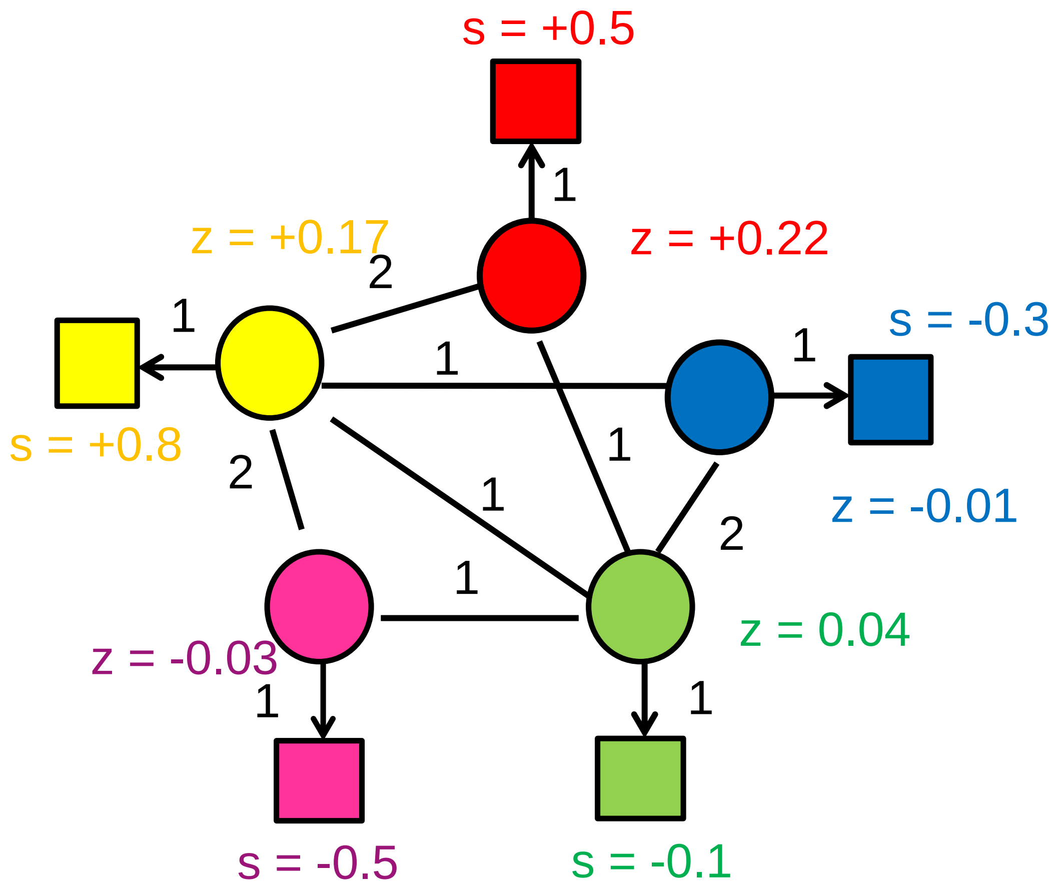
Opinion formation and absorbing random walks

Add to the network one absorbing node per user with value the intrinsic opinion of the user

Connect each transient node to her absorbing node with weight w_{ii}

The expressed opinion for each node is computed using the value propagation we described

- Repeated averaging



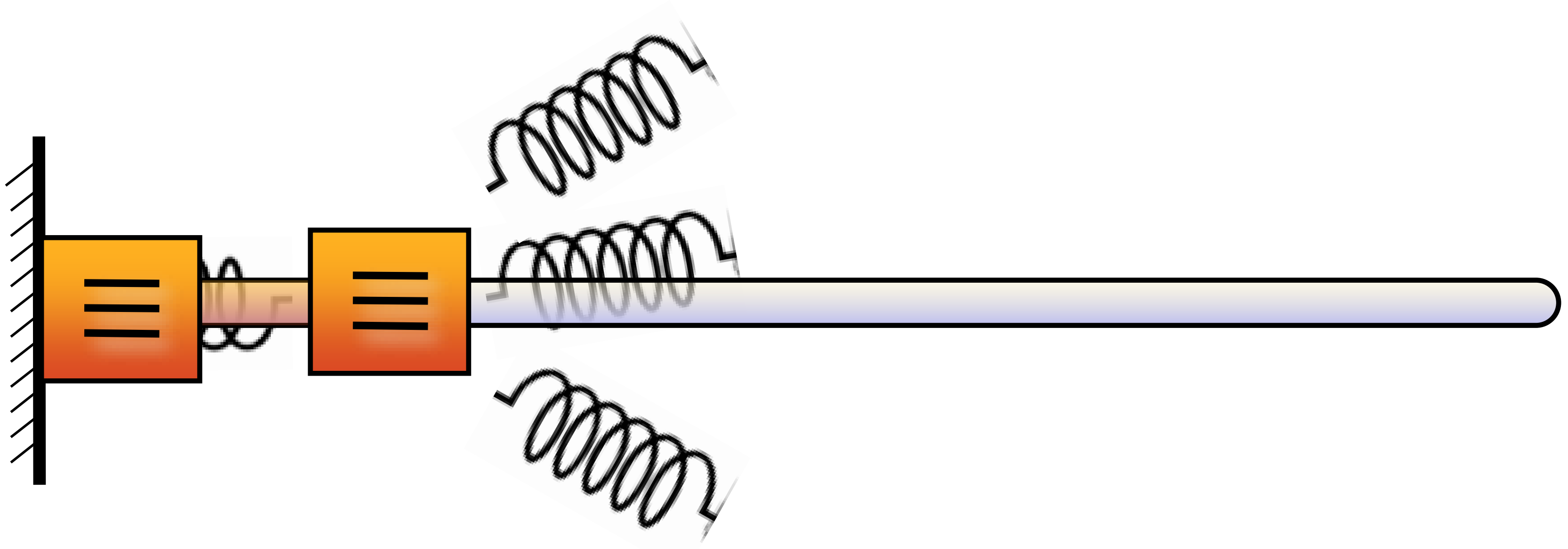
$$v(red) = \frac{0.5 + 2 \cdot v(yellow) + v(green)}{4}$$

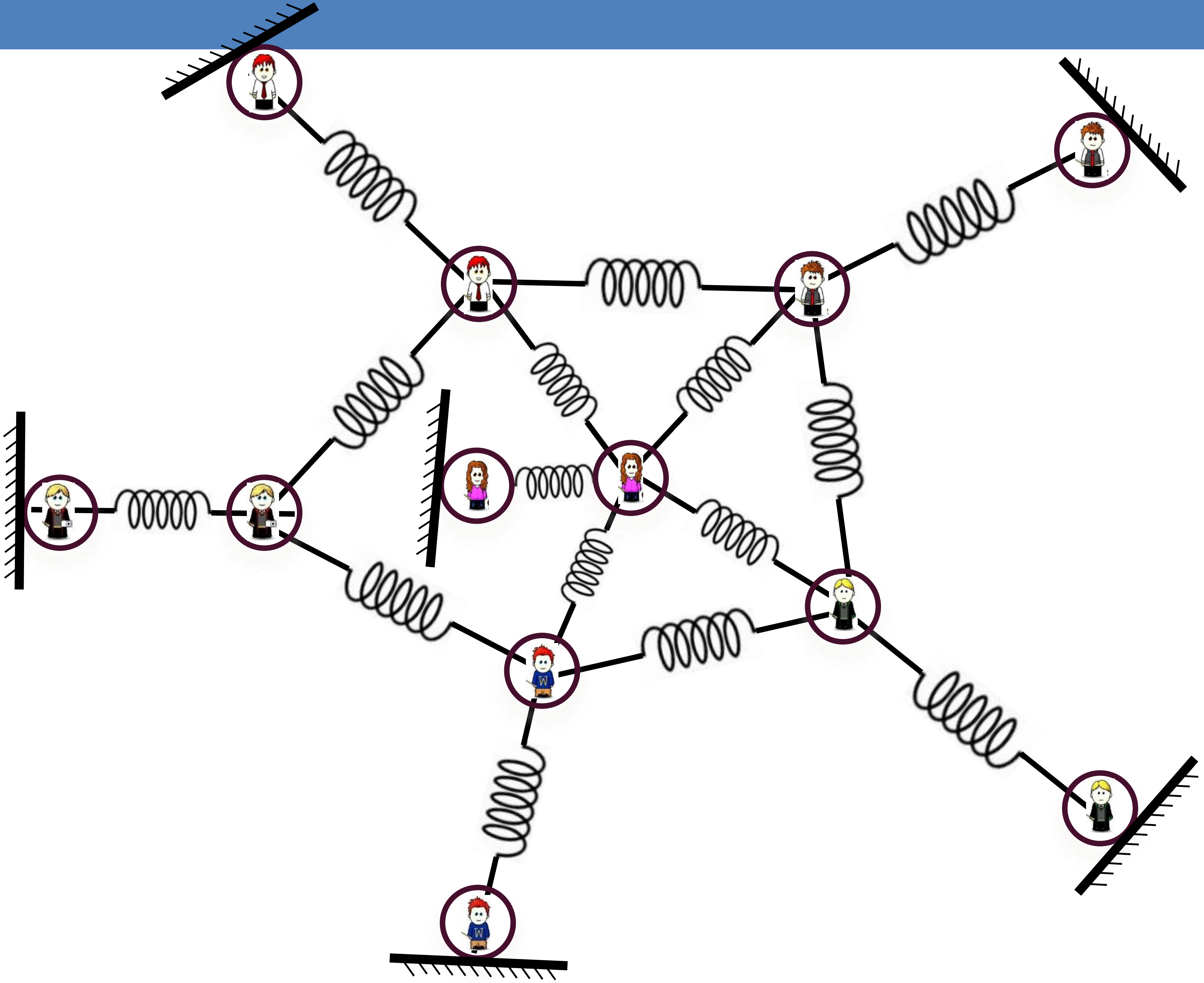
$$z_{red} = \frac{0.5 + 2 \cdot z_{yellow} + z_{green}}{4}$$

It is equal to the expected intrinsic opinion at the place of absorption

Opinion of a user

- For an individual user u
  - u's absorbing node is a stationary point
  - u's transient node is connected to the absorbing node with a spring.
  - The neighbors of u pull with their own springs.





Hitting time

- A related quantity: Hitting time H(u,v)
  - The expected number of steps for a random walk starting from node u to end up in v for the first time
    - Make node v absorbing and compute the expected number of steps to reach v
    - Assumes that the graph is strongly connected, and there are no other absorbing nodes.
- Commute time H(u,v) + H(v,u): often used as a distance metric
  - Proportional to the total resistance between nodes u, and v

Transductive learning

- If we have a graph of relationships and some labels on some nodes we can propagate them to the remaining nodes
  - Make the labeled nodes to be absorbing and compute the probability for the rest of the graph
  - E.g., a social network where some people are tagged as spammers
  - E.g., the movie-actor graph where some movies are tagged as action or comedy.
- This is a form of semi-supervised learning
  - We make use of the unlabeled data, and the relationships
- It is also called transductive learning because it does not produce a model, but just labels the unlabeled data that is at hand.
  - Contrast to inductive learning that learns a model and can label any new example

Implementation details

- Implementation is in many ways similar to the PageRank implementation
  - For an edge (u, v)instead of updating the value of v we update the value of u.
    - The value of a node is the average of its neighbors
  - We need to check for the case that a node u is absorbing, in which case the value of the node is not updated.
  - Repeat the updates until the change in values is very small.