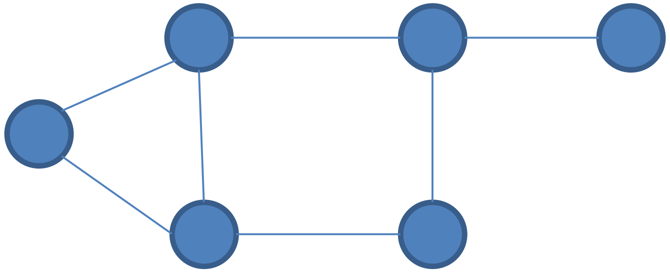
DATA MINING
LECTURE 12

Coverage

Approximation Algorithms

Example

- Promotion campaign on a social network
  - We have a social network as a graph.
  - People are more likely to buy a product if they have a friend who has the product.
  - We want to offer the product for free to some people such that every person in the graph is covered: they have a friend who has the product.
  - We want the number of free products to be as small as possible



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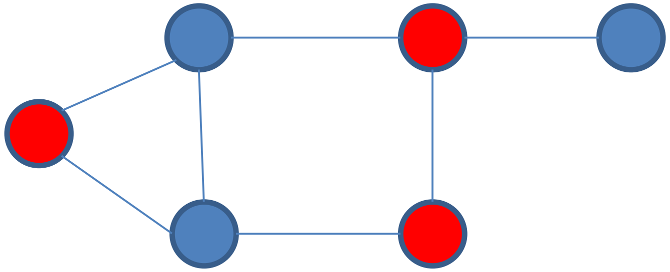

One possible selection

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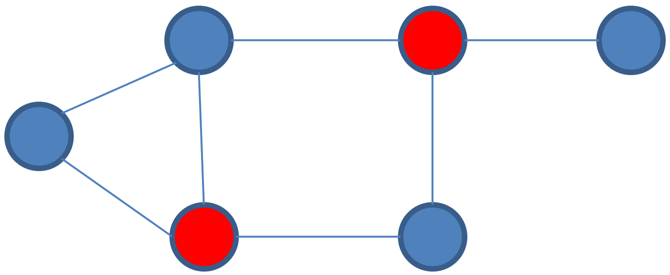


A better selection

Dominating set

- Our problem is an instance of the dominating set problem

- Dominating Set: Given a graph $G = (V, E)$, a set of vertices $D \subseteq V$ is a dominating set if for each node u in V, either u is in D, or u has a neighbor in D.

- The Dominating Set Problem: Given a graph $G = (V, E)$ find a dominating set of minimum size.

Set Cover

- The dominating set problem is a special case of the Set Cover problem

- The Set Cover problem:
  - We have a universe of elements $U = \{x_1, \dots, x_N\}$
  - We have a collection of subsets of U, $\boldsymbol{S} = \{S_1, \dots, S_n\}$, such that $\bigcup_i S_i = U$
  - We want to find the smallest sub-collection $\boldsymbol{C} \subseteq \boldsymbol{S}$ of $\boldsymbol{S}$, such that $\bigcup_{S_i \in \boldsymbol{C}} S_i = U$
    - The sets in $\boldsymbol{C}$ cover the elements of U

An application of Set Cover

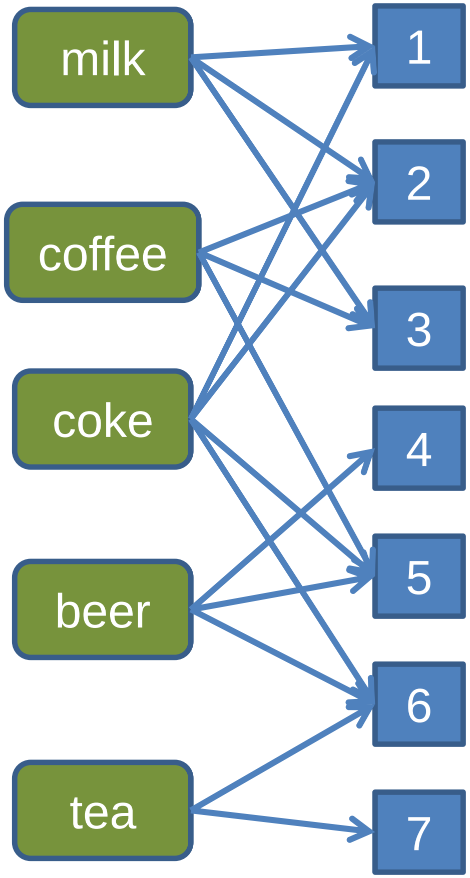
- Suppose that we want to create a catalog (with coupons) to give to customers of a store:
  - We want for every customer, the catalog to contain a product bought by the customer (this is a small store)
- How can we model this as a set cover problem?

Applications

- The universe U of elements is the set of customers of a store.
- Each set corresponds to a product p sold in the store:

  $S_p = \{customers\ that\ bought\ p\}$

- Set cover: Find the minimum number of products (sets) that cover all the customers (elements of the universe)



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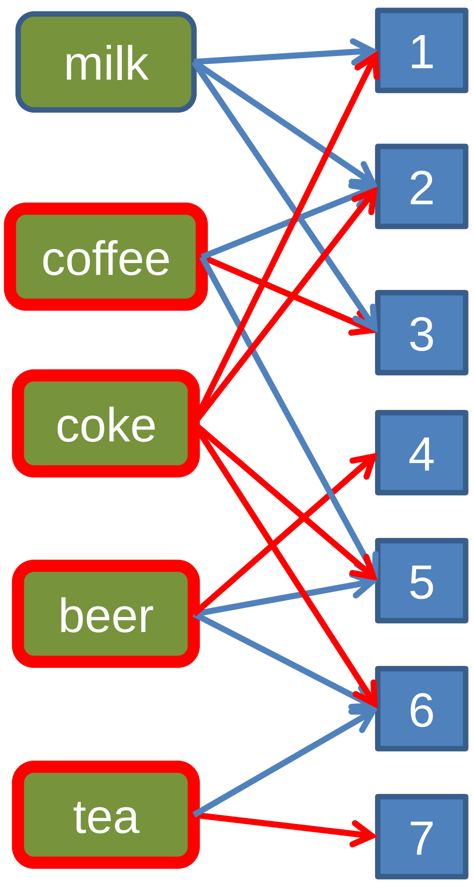

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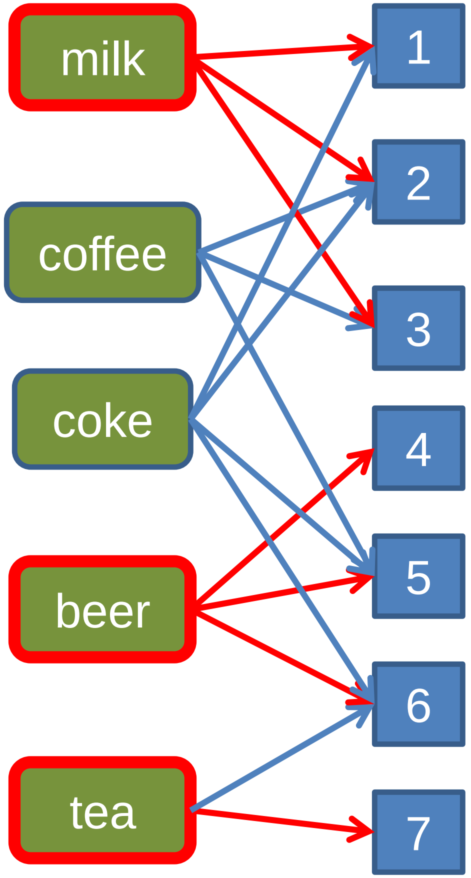


Applications

- Dominating Set (or Promotion Campaign) as Set Cover:
  - The universe U is the set of nodes V
  - Each node u defines a set S_u consisting of the node u and all of its neighbors
  - We want the minimum number of sets S_u (nodes) that cover all the nodes in the graph.
- Many more…

Best selection variant

- Suppose that we have a budget K of how big our set cover can be
  - We only have K products to give out for free.
  - We want to cover as many customers as possible.

- Maximum-Coverage Problem: Given a universe of elements $\boldsymbol{U}$, a collection $\boldsymbol{S}$ of subsets of $\boldsymbol{U}$, and a budget K, find a sub-collection $\boldsymbol{C} \subseteq \boldsymbol{S}$ of size at most K, such that the number of covered elements $\bigcup_{S_i \in \boldsymbol{C}} S_i$ is maximized.

Complexity

- Both the Set Cover and the Maximum Coverage problems are NP-complete
  - What does this mean?
  - Why do we care?

- There is no algorithm that can guarantee finding the best solution in polynomial time
  - Can we find an algorithm that can guarantee to find a solution that is close to the optimal?
  - Approximation Algorithms.

Approximation Algorithms

- For an (combinatorial) optimization problem, where:
  - X is an instance of the problem,
  - OPT(X) is the value of the optimal solution for X,
  - ALG(X) is the value of the solution of an algorithm ALG for X

ALG is a good approximation algorithm if the ratio of OPT(X) and ALG(X) is bounded for all input instances X

- Minimum set cover: input X = (U,**S**) is the universe of elements and the set collection, OPT(X) is the size of minimum set cover, ALG(X) is the size of the set cover found by an algorithm ALG.

- Maximum coverage: input X = (U,**S**,K) is the input instance, OPT(X) is the coverage of the optimal algorithm, ALG(X) is the coverage of the set found by an algorithm ALG.

Approximation Algorithms

- For a minimization problem, the algorithm ALG is an α-approximation algorithm, for $\alpha > 1$, if for all input instances X,

$$ALG(X) \leq \alpha OPT(X)$$

- In simple words: the algorithm ALG is at most α times worse than the optimal.

- α is the approximation ratio of the algorithm – we want α to be as close to 1 as possible
  - Best case: $\alpha = 1 + \epsilon$ and $\epsilon \to 0$, as $n \to \infty$ (e.g., $\epsilon = \frac{1}{n}$)
  - Good case: $\alpha = O(1)$ is a constant (e.g., $\alpha = 2$)
  - OK case: $\alpha = O(\log n)$
  - Bad case $\alpha = O(n^{\epsilon})$

Approximation Algorithms

- For a maximization problem, the algorithm ALG is an α-approximation algorithm, for $\alpha < 1$, if for all input instances X,

$$ALG(X) \geq \alpha OPT(X)$$

- In simple words: the algorithm ALG achieves at least α percent of what the optimal achieves.

- α is the approximation ratio of the algorithm – we want α to be as close to 1 as possible
  - Best case: $\alpha = 1 - \epsilon$ and $\epsilon \to 0$, as $n \to \infty$(e.g., $\epsilon = \frac{1}{n}$)
  - Good case: $\alpha = O(1)$ is a constant (e.g., $a = 0.5$)
  - OK case: $\alpha = O(\frac{1}{\log n})$
  - Bad case $\alpha = O(n^{-\epsilon})$

A simple approximation ratio for set cover

- Lemma: Any algorithm for set cover has approximation ratio $\alpha = |S_{max}|$, where S_{max} is the set in $\boldsymbol{S}$ with the largest cardinality
- Proof:
  - $OPT(X) \geq N/|S_{max}| \Rightarrow N \leq |S_{max}|OPT(X)$
  - $ALG(X) \leq N \leq |S_{max}|OPT(X)$

- This is true for any algorithm.
- Not a good bound since it may be that $|S_{max}| = O(N)$

An algorithm for Set Cover

- What is a natural algorithm for Set Cover?

- Greedy: each time add to the collection $\boldsymbol{C}$ the set S_i from $\boldsymbol{S}$ that covers the most of the remaining uncovered elements.

The GREEDY algorithm

GREEDY(U,S)

X= U

C = {}

while X is not empty do

For all $S_i \in \boldsymbol{S}$ let $\mathrm{gain}(S_i) = |S_i \cap X|$

Let S_* be such that $gain(S_*)$ is maximum

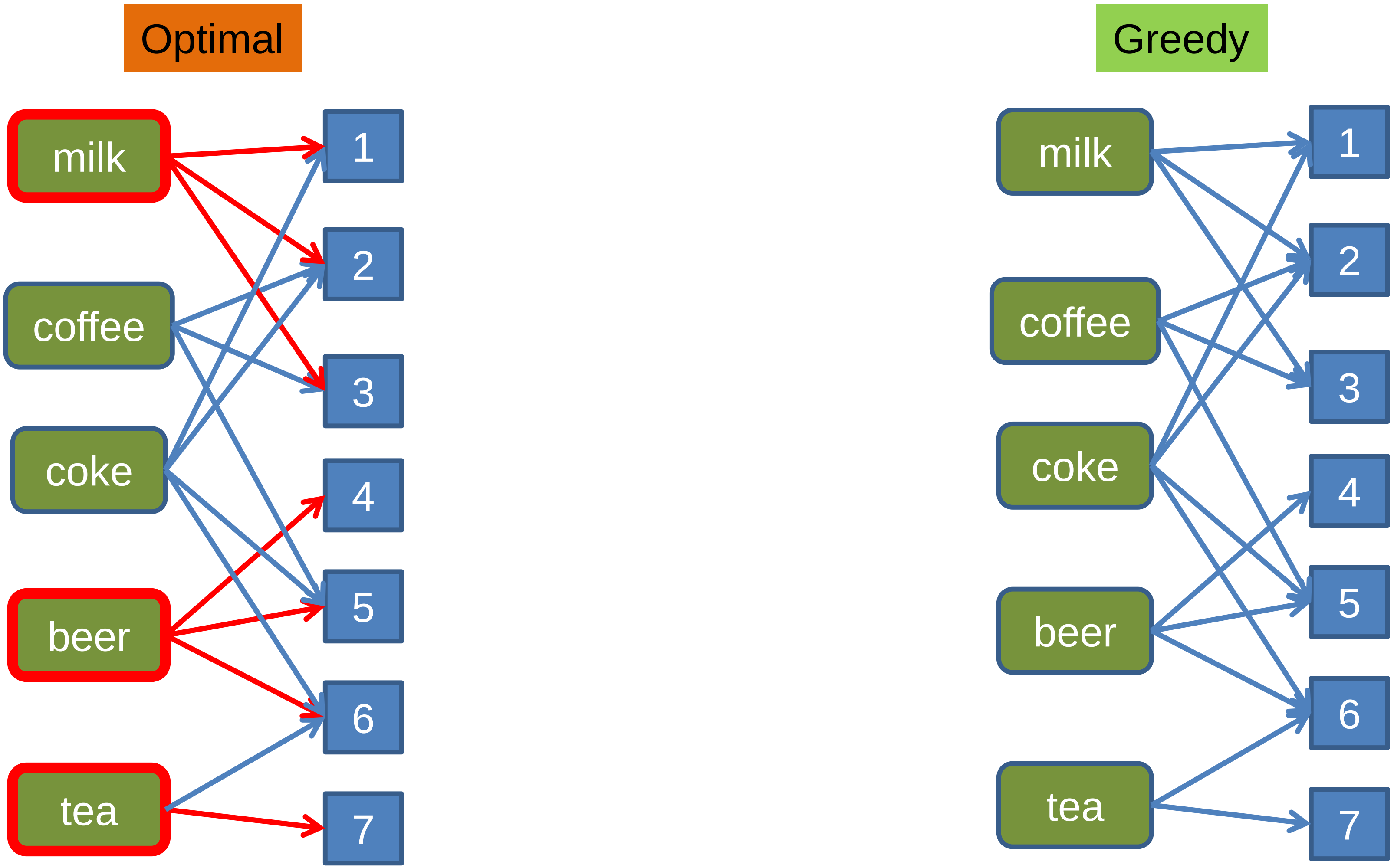
C = C U {S_*}

X = X\ S_*

S = ***S***\ S_*

The number of elements covered by S_i not already covered by C.

Greedy is not always optimal



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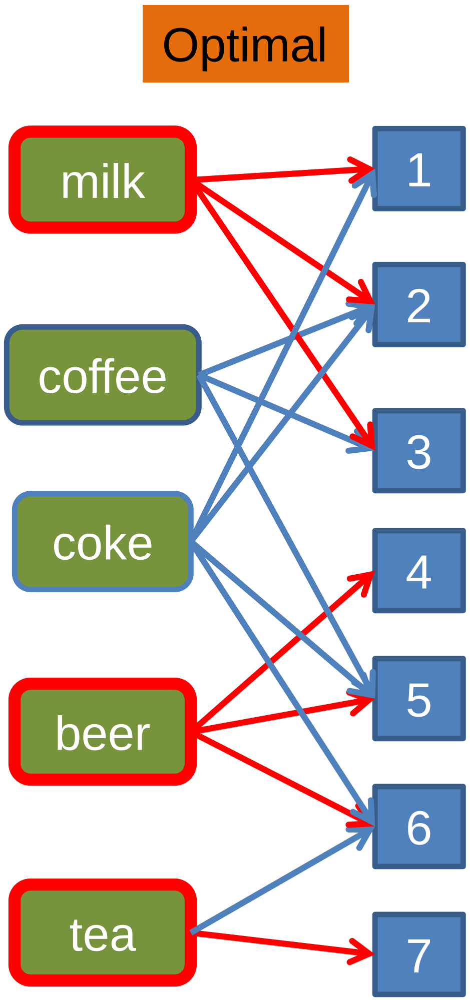

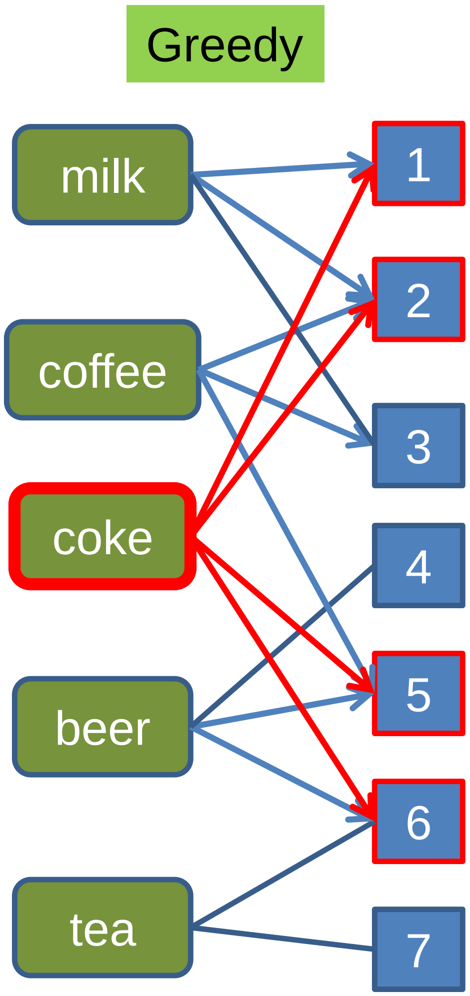

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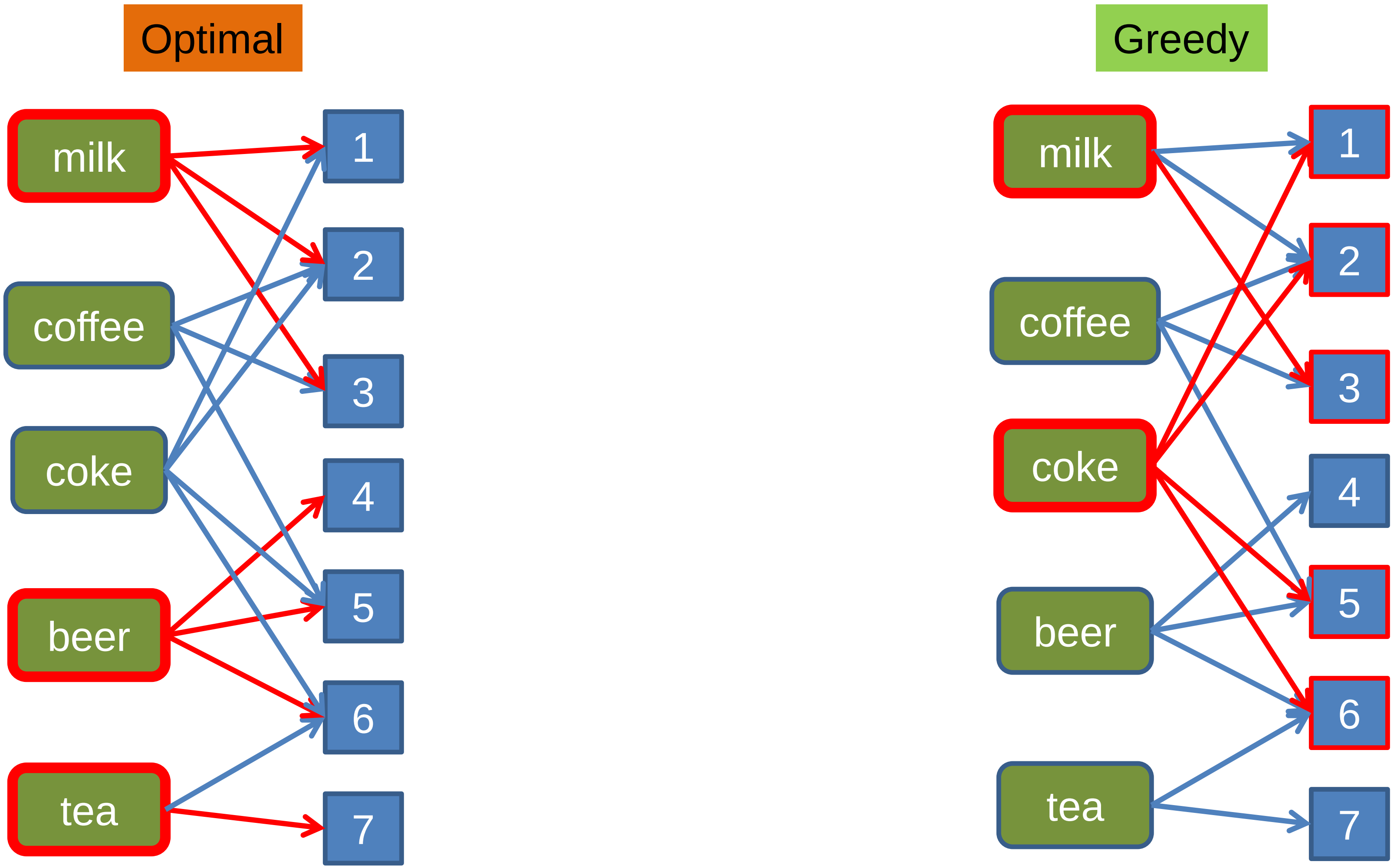

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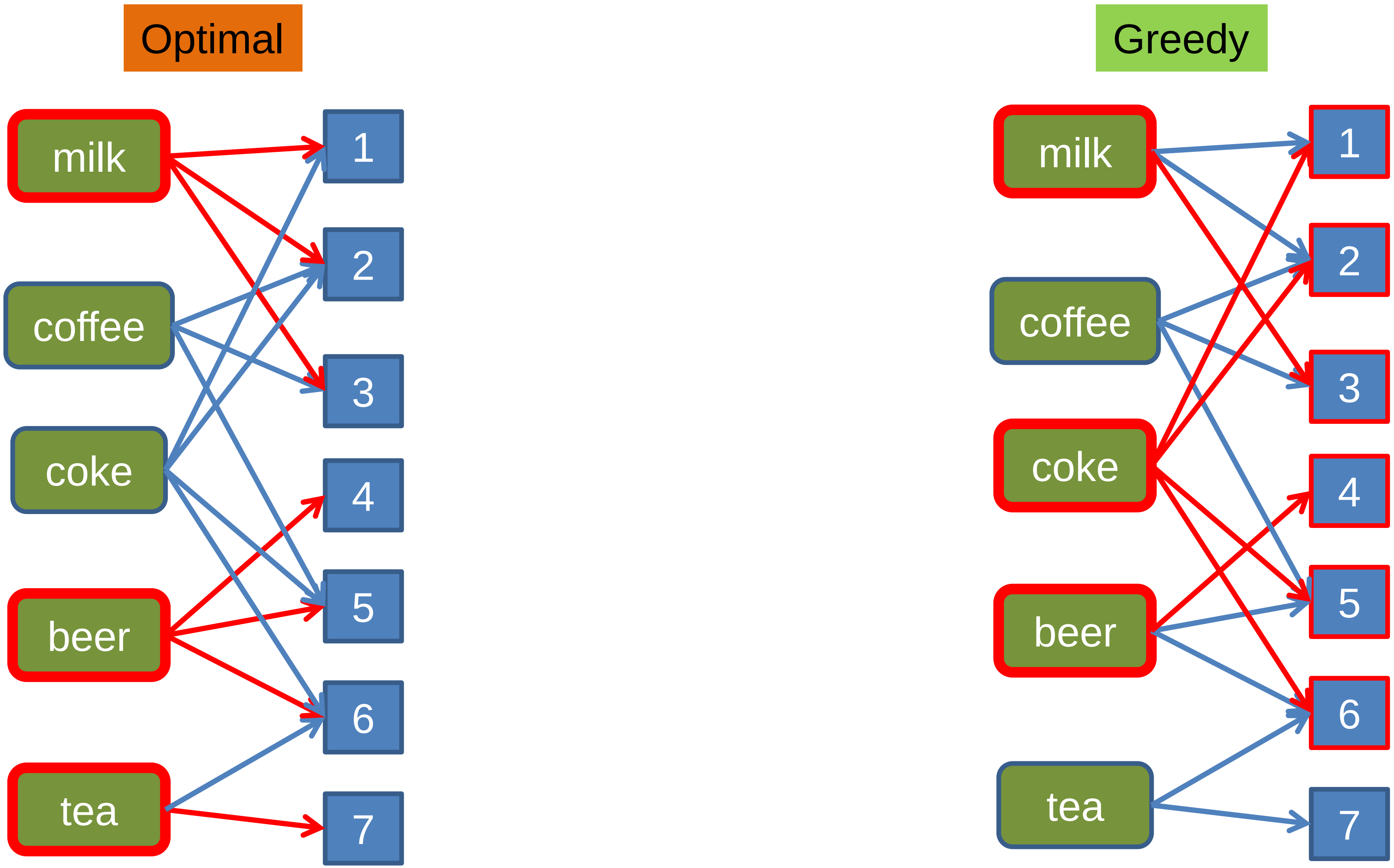

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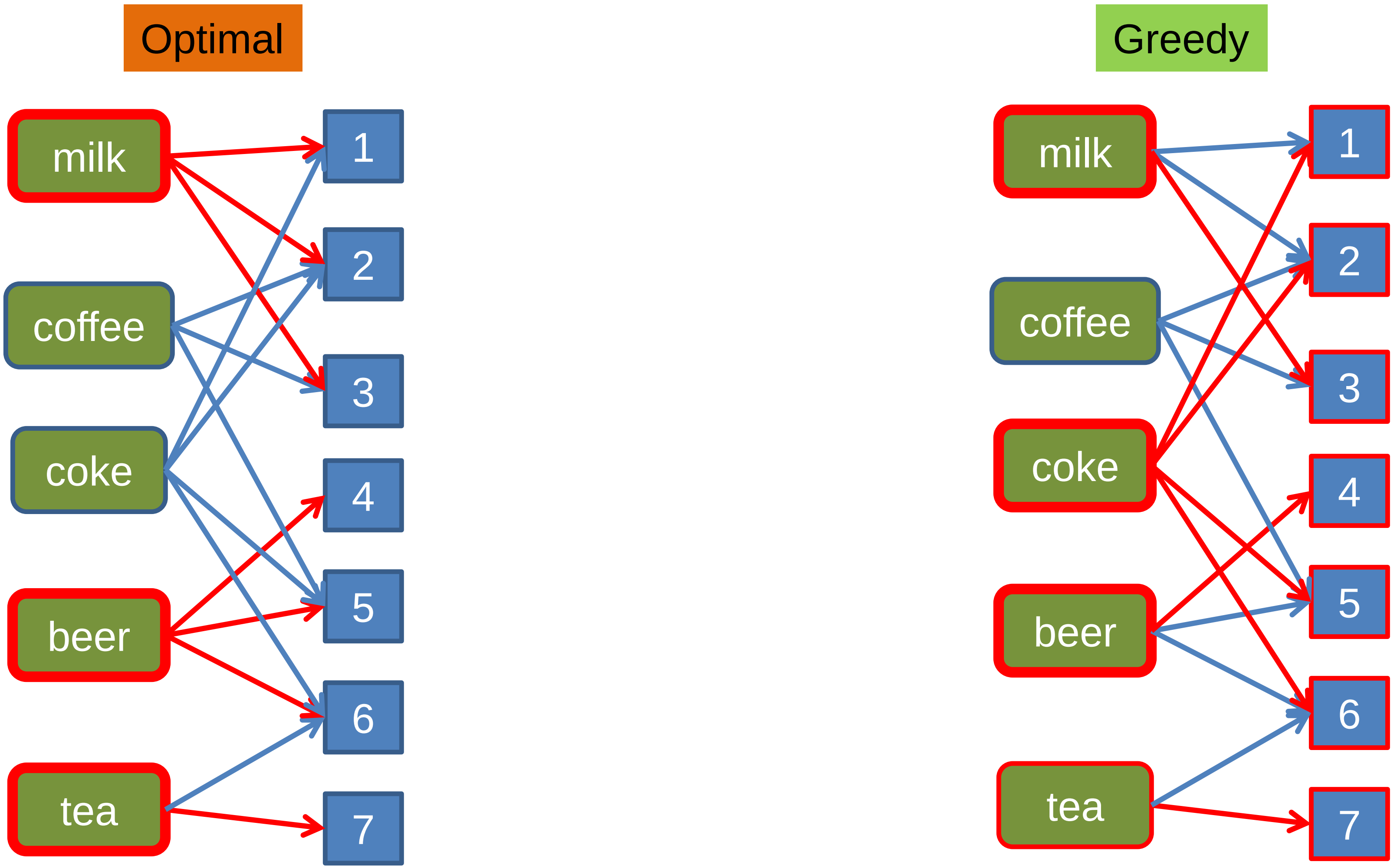
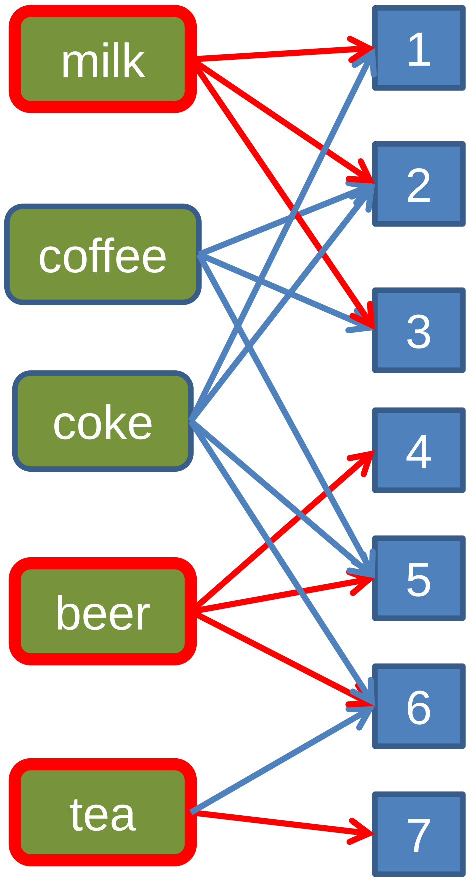
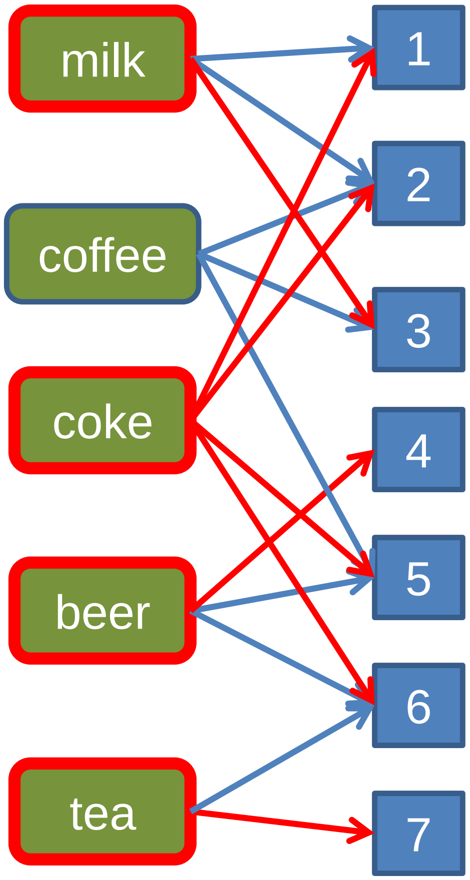


Greedy is not always optimal



- Adding Coke to the set is useless.
- We need Milk (or Coffee) and Beer to cover all customers



Approximation ratio of GREEDY

- Good news: GREEDY has approximation ratio:

$$\alpha = H(|S_{\max}|) = 1 + \ln|S_{\max}|, \qquad H(n) = \sum_{k=1}^{n} \frac{1}{k}$$

$$GREEDY(X) \leq (1 + \ln|S_{\max}|)OPT(X),$$ for all X

Maximum Coverage

- Greedy is also applicable here

GREEDY(U,S,K)

X = U

C = {}

while |***C***| < K

For all $S_i \in \boldsymbol{S}$ let $\mathrm{gain}(S_i) = |S_i \cap X|$

Let S_* be such that $gain(S_*)$ is maximum

C = C U {S_*}

X = X\ S_*

S= ***S***\ S_*

The number of elements covered by S_i not already covered by C.

Approximation Ratio for Max-K Coverage

- Better news! The GREEDY algorithm has approximation ratio $\alpha = 1 - \frac{1}{e}$

$$GREEDY(X) \geq \left(1 - \frac{1}{e}\right) OPT(X),$$ for all X

- (e is the basis of the natural logarithm)

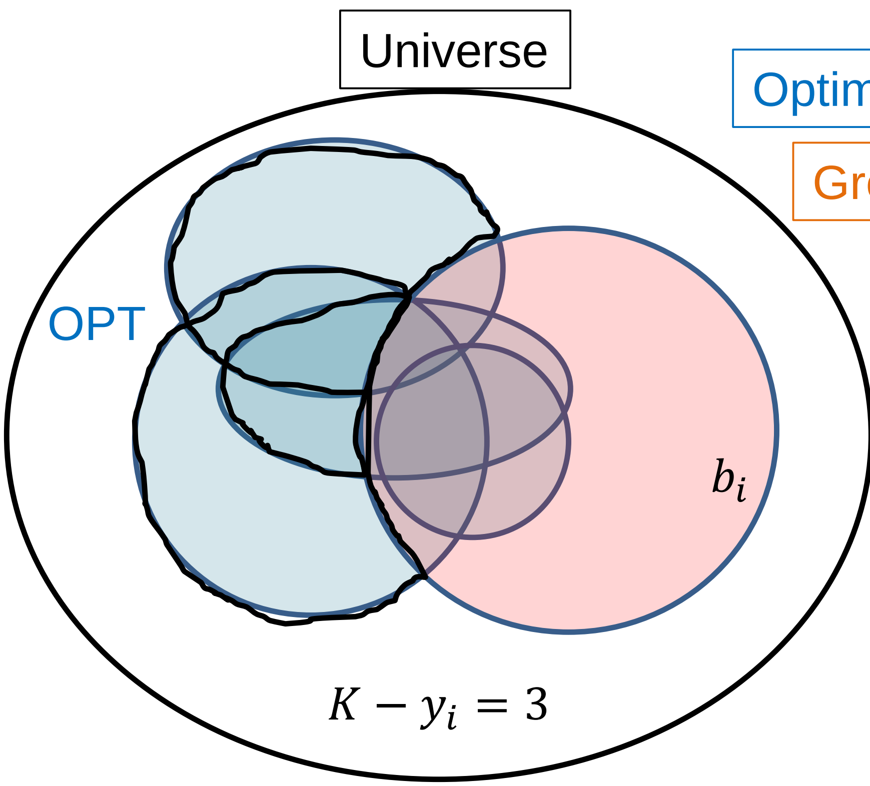
- The coverage of the Greedy solution is at least 63% that of the optimal

Proof of approximation ratios

- We will now give a proof of the approximation ratios for the SET-COVER and the MAX-COVERAGE
  - We start with MAX-COVERAGE

- Definitions:
  - OPT: size of the optimal solution
  - b_i: number of covered elements at iteration i of Greedy
  - a_i: number of newly covered elements at iteration i
  - $c_i = OPT - b_i$: difference between solution size of Optimal and Greedy solutions at iteration i.

- Lemma: $a_{i+1} \geq \frac{c_i}{K}$
- Proof:
  - For $i = 0$, it is simple to see since one of the K sets in the optimal solution has size at least $\frac{OPT}{K}$.
  - For larger i



Optimal solution for K = 5

Greedy solution at iteration i

x_i: intersection of optimal and Greedy solutions

$$x_i \leq b_i$$

y_i: number of optimal subsets fully included in the Greedy solution:

$$y_i \geq 0$$

There must be a set with

$$a_{i+1} \geq \frac{OPT - x_i}{K - y_i} \geq \frac{OPT - b_i}{K} = \frac{c_i}{K}$$

- Lemma: $c_{i+1} \leq \left(1 - \frac{1}{K}\right)^{i+1} OPT$

- Proof: By induction on i.
- Basis of induction: $c_0 \leq \left(1 - \frac{1}{K}\right) OPT$
  - Use the fact that $c_0 = OPT$, and $b_1 = a_1$
- Inductive Hypothesis: $c_i \leq \left(1 - \frac{1}{K}\right)^{i} OPT$
- Inductive step: $c_{i+1} \leq \left(1 - \frac{1}{K}\right)^{i+1} OPT$
  - Use the inductive hypothesis and that $b_{i+1} = \sum_{j=1}^{i+1} a_j$ and $c_{i+1} = c_i - a_i$

- Theorem: The Greedy algorithm has approximation ratio $\left(1-\frac{1}{e}\right)$
- Proof:

$$c_K \leq \left(1-\frac{1}{K}\right)^K OPT \leq \frac{1}{e} OPT$$

- The size of the Greedy solution is b_K

$$b_K = OPT - c_K \geq \left(1-\frac{1}{e}\right) OPT$$

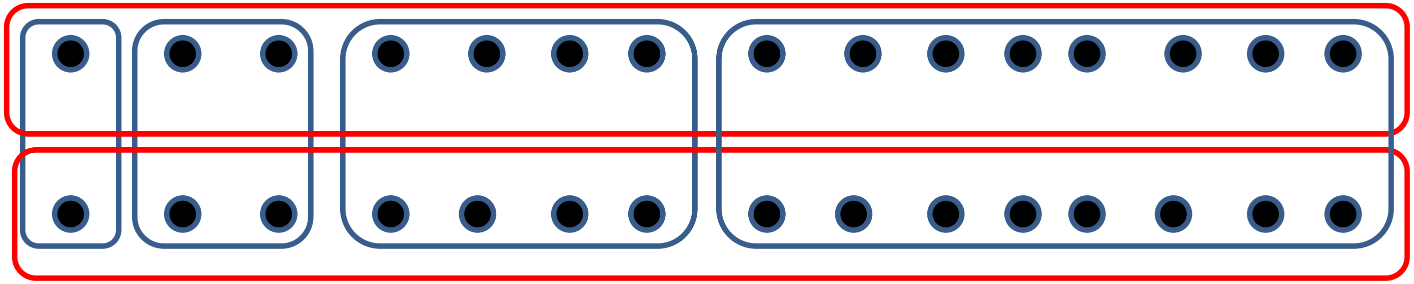
Proof for SET COVER

- In the case of SET COVER, we have that $OPT = n$
- Let k^* be the size of the optimal solution.

- We know that after i iterations: $c_i \leq \left(1 - \frac{1}{k^*}\right)^i n$.
- After $t = k^* \ln \frac{n}{k^*}$ iterations $c_t \leq k^*$ elements remain to be covered
  - We can cover those in at most k^* iterations
  - Total iterations are at most $k^*(\ln \frac{n}{k^*} + 1) \leq k^*(\ln n + 1)$

Lower bound

- The approximation ratio is tight up to a constant
  - Tight means that we can find a counter example with this ratio



- $OPT(X) = 2$
- $GREEDY(X) = \log N$
- $\alpha = \frac{1}{2}\log N$

Another proof of the approximation ratio for MAX-K COVERAGE

- For a collection of subsets C, let $F(C) = \left|\bigcup_{S_i \in \boldsymbol{C}} S_i\right|$ be the number of elements that are covered.
- The set function F has two properties:

- F is monotone:

$$F(A) \leq F(B) \; if \; A \subseteq B$$

- F is submodular:

$$F(A \cup \{S\}) - F(A) \geq F(B \cup \{S\}) - F(B) \;\; if \; A \subseteq B$$

- The addition of set S to a set of nodes has greater effect (more new covered items) for a smaller set.
  - The diminishing returns property

Optimizing submodular functions

- Theorem:

If we want to maximize a monotone and submodular function F under cardinality constraints (size of set at most K),

Then, the greedy algorithm that each time adds to the solution $\boldsymbol{C}$, the set S that maximizes the gain $F(\boldsymbol{C} \cup$

True for any monotone and submodular set function!

Other variants of Set Cover

- Hitting Set: select a set of elements so that you hit all the sets (the same as the set cover, reversing the roles)

- Vertex Cover: Select a set of vertices from a graph such that you cover all edges (for every edge an endpoint of the edge is in the set)
  - There is a 2-approximation algorithm

- Edge Cover: Select a set of edges that cover all vertices (for every vertex, there is one edge that has as endpoint this vertex)
  - There is a polynomial algorithm

OVERVIEW

Class Overview

- In this class you saw a set of tools for analyzing data
  - Frequent Itemsets, Association Rules
  - Sketching
  - Recommendation Systems
  - Clustering
  - Singular Value Decomposition
  - Classification
  - Link Analysis Ranking
  - Random Walks
  - Coverage
- All these are useful when trying to make sense of the data. A lot more tools exist.
- I hope that you found this interesting, useful and fun.