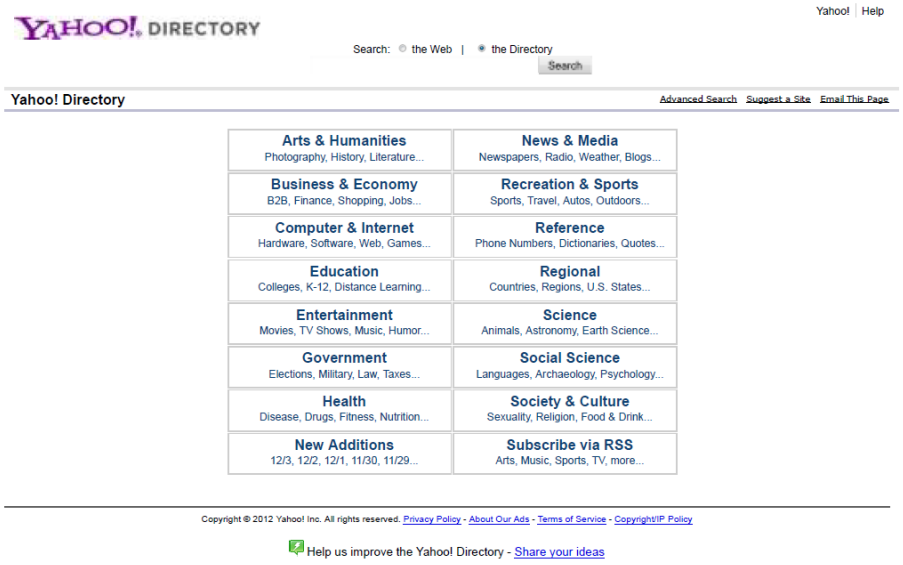
Online Social Networks and Media

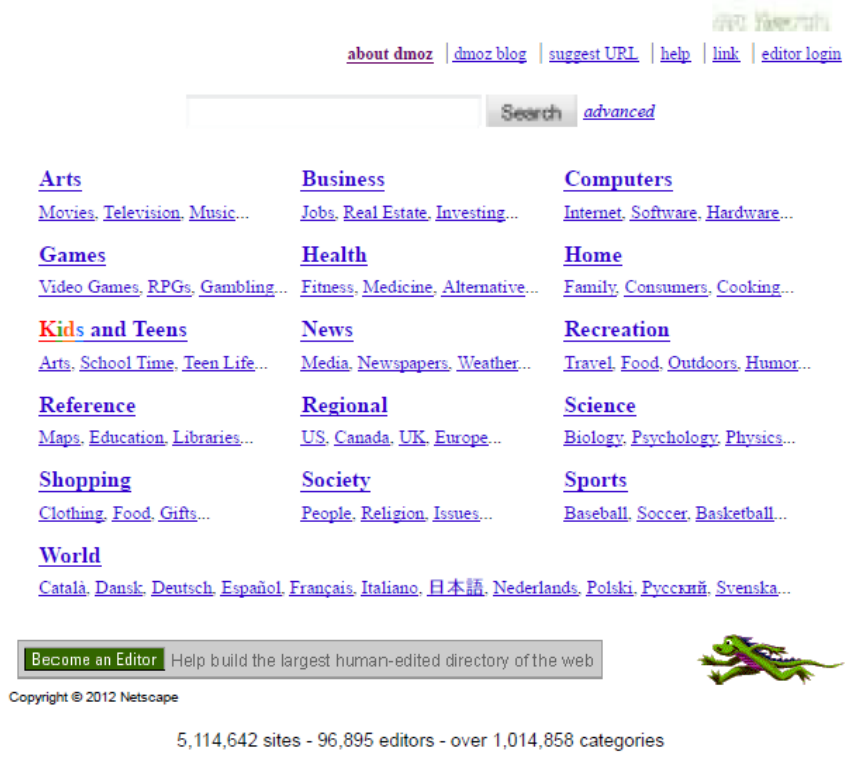
Link Analysis and Web Search

How to Organize the Web

First try: Human curated Web directories

Yahoo, DMOZ, LookSmart





How to organize the web

- Second try: Web Search
  - Information Retrieval investigates:
    - Find relevant docs in a small and trusted set e.g., Newspaper articles, Patents, etc. (“needle-in-a-haystack”)
    - Limitation of keywords (synonyms, polysemy, etc)

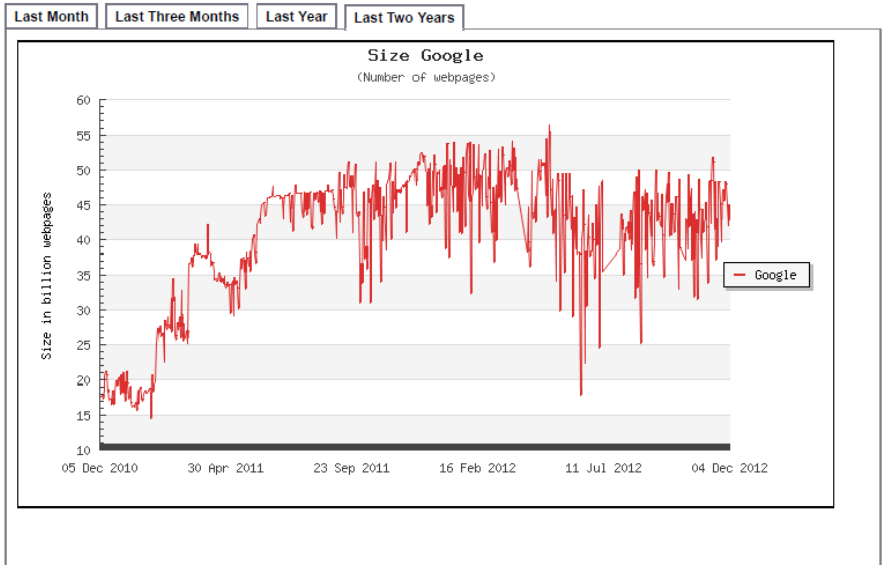
  But: Web is huge, full of untrusted documents, random things, web spam, etc.

  - Everyone can create a web page of high production value
  - Rich diversity of people issuing queries
  - Dynamic and constantly-changing nature of web content

Size of the Search Index

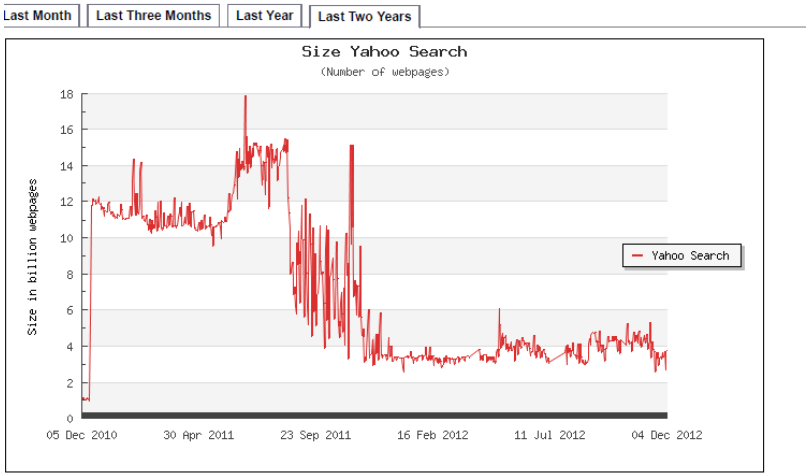
Google

The size of the World Wide Web:
Estimated size of Google's index



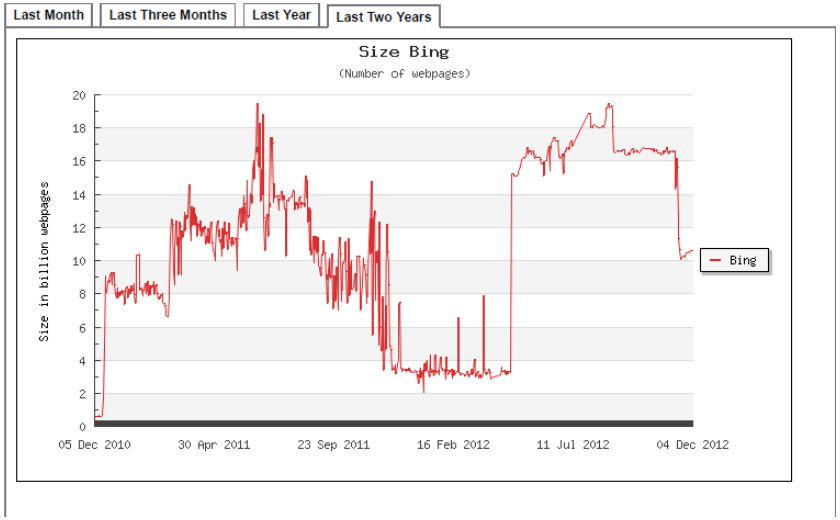
YAHOO! SEARCH

The size of the World Wide Web:
Estimated size of Yahoo Search index



bing

The size of the World Wide Web:
Estimated size of Bing index



http://www.worldwidewebsize.com/

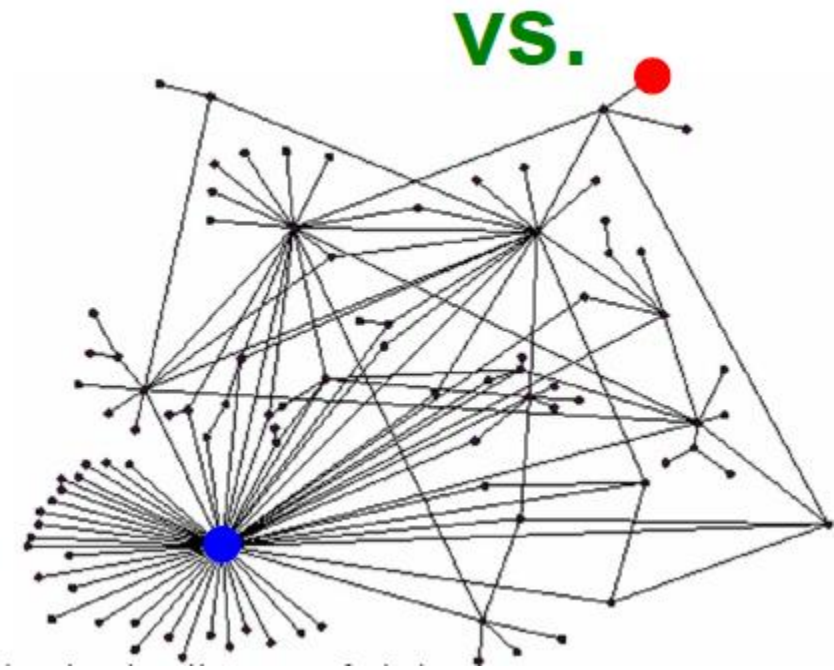
How to organize the web

- Third try (the Google era): using the web graph
  - Swift from relevance to *authoritativeness*
  - It is not only important that a page is relevant, but that it is also important on the web
- For example, what kind of results would we like to get for the query “greek newspapers”?

Link Analysis

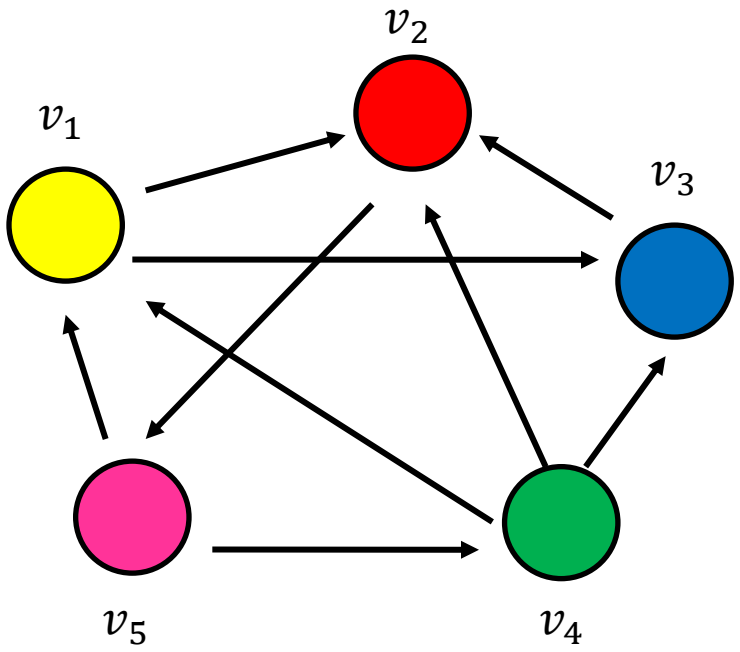
- Not all web pages are equal on the web
- The links act as endorsements:
  - When page *p* links to *q* it endorses the content of the content of *q*

What is the simplest way to measure importance of a page on the web?



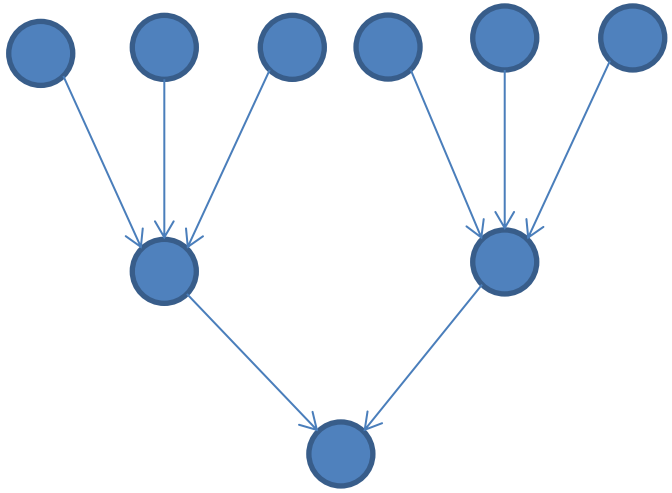
Rank by Popularity

- Rank pages according to the number of incoming edges (in-degree, degree centrality)



1. **Red Page**
2. **Yellow Page**
3. **Blue Page**
4. **Purple Page**
5. **Green Page**

Popularity



- It is not important only how many link to you, but also how important are the people that link to you.
- Good authorities are pointed by good authorities
  - Recursive definition of importance

THE PAGERANK ALGORITHM

PageRank

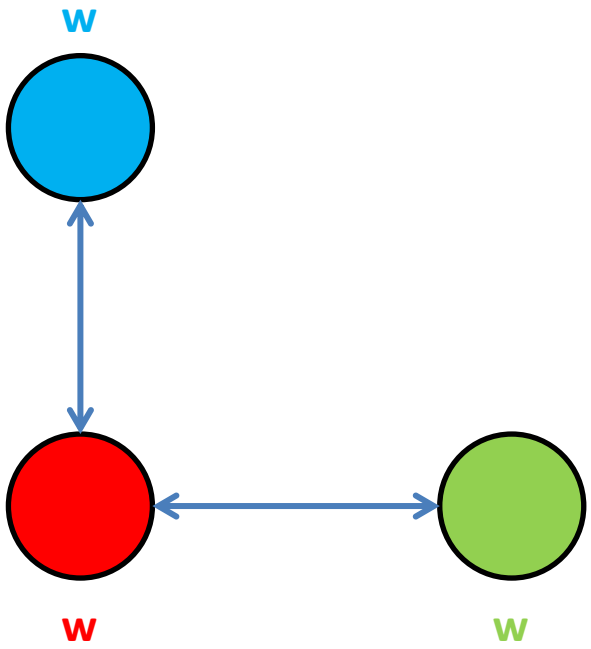
- Good authorities should be pointed by good authorities
  - The value of a node is the value of the nodes that point to it.
- How do we implement that?
  - Assume that we have a unit of authority to distribute to all nodes.
    - Initially each node gets $\frac{1}{n}$ amount of authority
  - Each node distributes the authority value they have to their neighbors
  - The authority value of each node is the sum of the authority fractions it collects from its neighbors.

$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

w_v: the PageRank value of node v

Recursive definition

A simple example



$w + w + w = 1$

$w = w + w$

$w = \frac{1}{2} w$

$w = \frac{1}{2} w$

- Solving the system of equations we get the authority values for the nodes
  - $w = \frac{1}{2}$ $w = \frac{1}{4}$ $w = \frac{1}{4}$

A more complex example

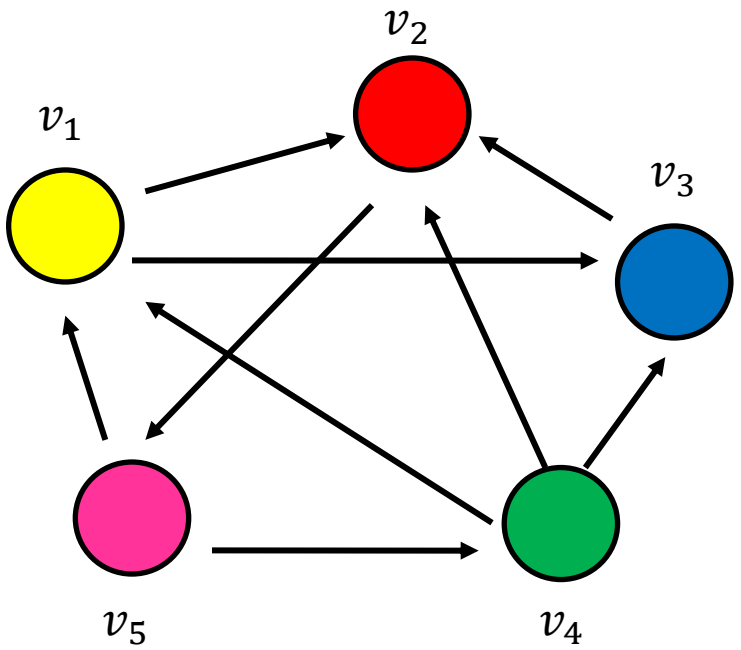
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

Computing PageRank weights

- A simple way to compute the weights is by iteratively updating the weights
- PageRank Algorithm

Initialize all PageRank weights to $\frac{1}{n}$

Repeat:

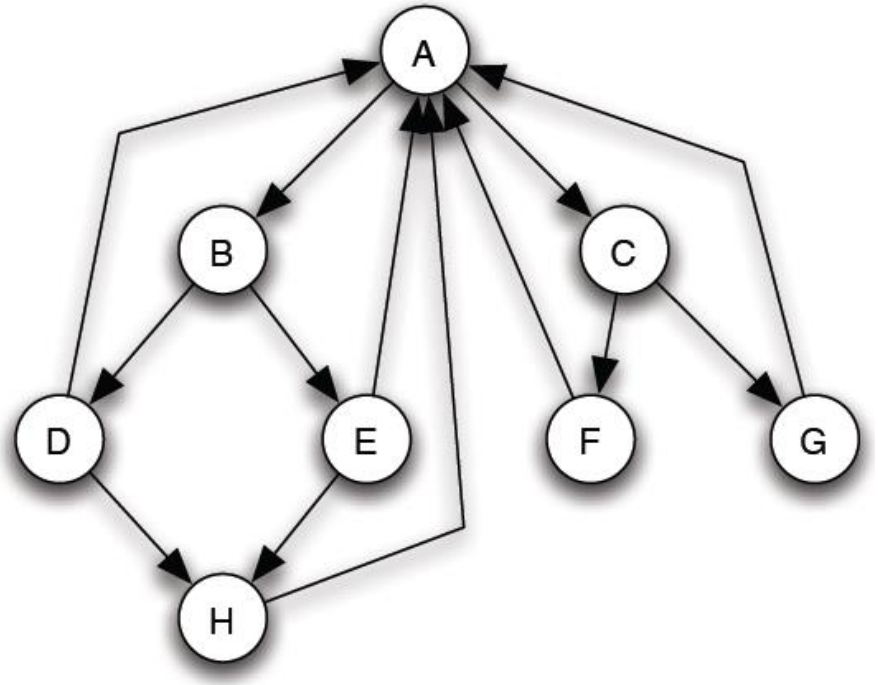
$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

Until the weights do not change

- This process converges

PageRank

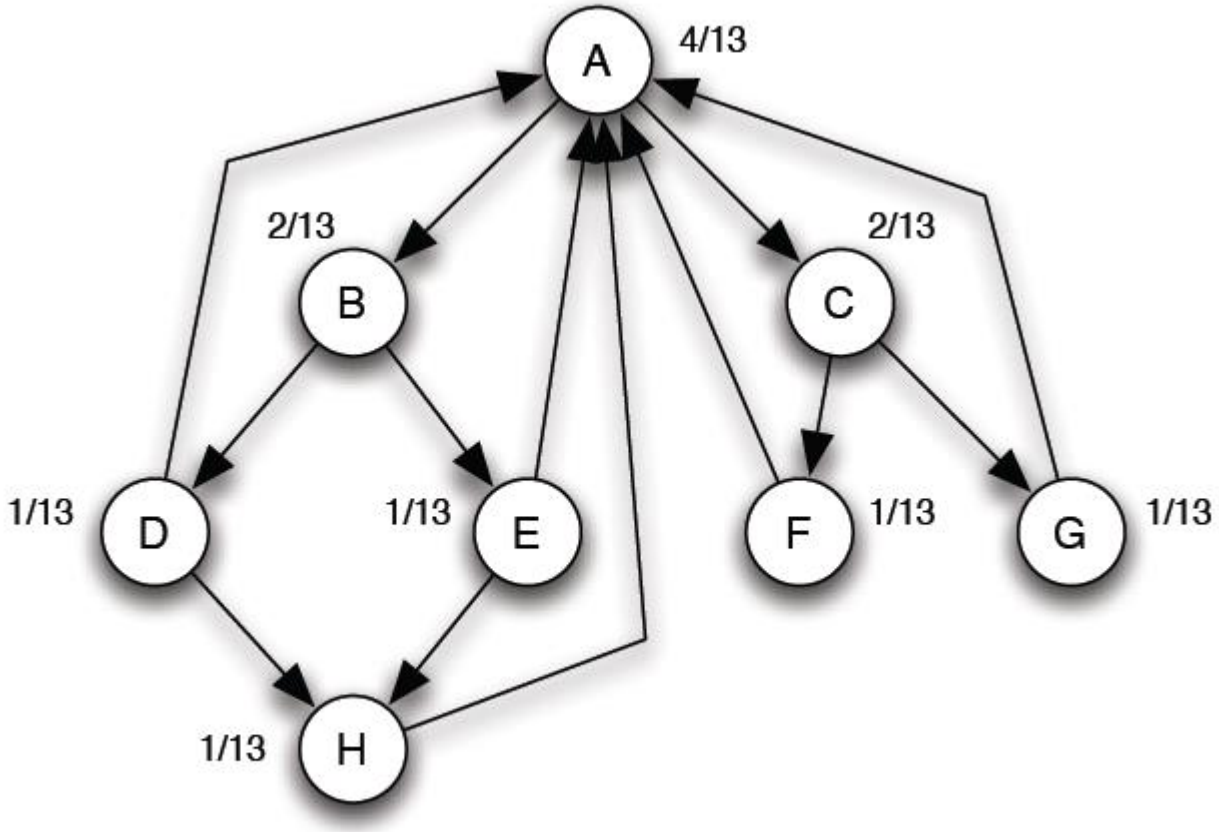
Initially, all nodes PageRank 1/8



Step	A	B	C	D	E	F	G	H
1	1/2	1/16	1/16	1/16	1/16	1/16	1/16	1/8
2	3/16	1/4	1/4	1/32	1/32	1/32	1/32	1/16

- ✓ As a kind of “fluid” that circulates through the network
- ✓ The total PageRank in the network remains constant (no need to normalize)

PageRank: equilibrium



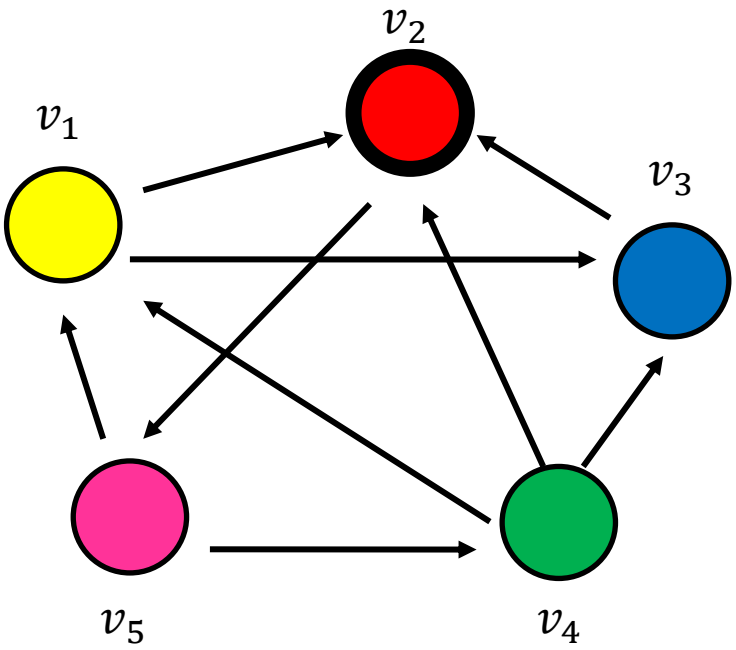
- A simple way to *check* whether an assignment of numbers forms an equilibrium set of PageRank values: check that they sum to 1, and that when apply the Basic PageRank Update Rule, we get the same values back.
- If the network is strongly connected, then there is a unique set of equilibrium values.

Random Walks on Graphs

- The algorithm defines a random walk on the graph
- Random walk:
  - Start from a node chosen uniformly at random with probability $\frac{1}{n}$.
  - Pick one of the outgoing edges uniformly at random
  - Move to the destination of the edge
  - Repeat.
- The PageRank of node v is the probability that the random walk is at node v after a very large number of steps.
- The Random Surfer model
  - Users wander on the web, following links.

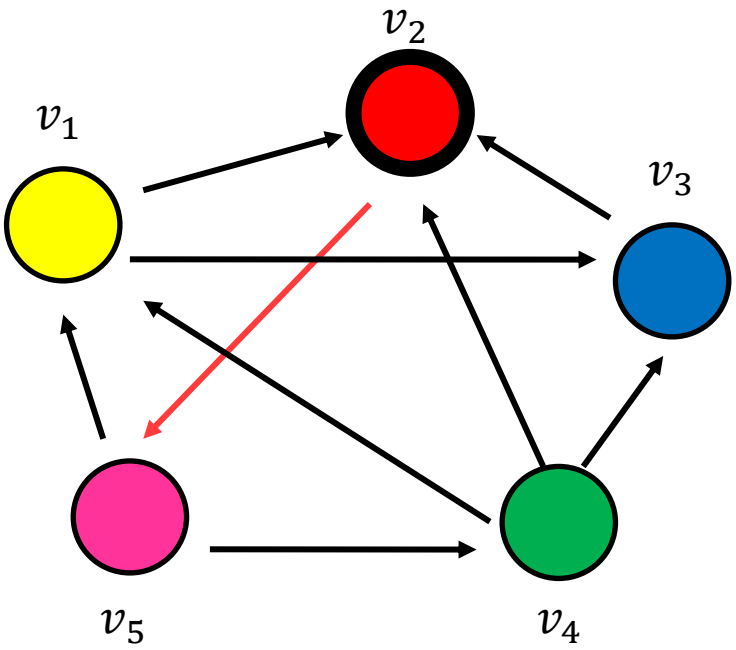
Example

- Step 0



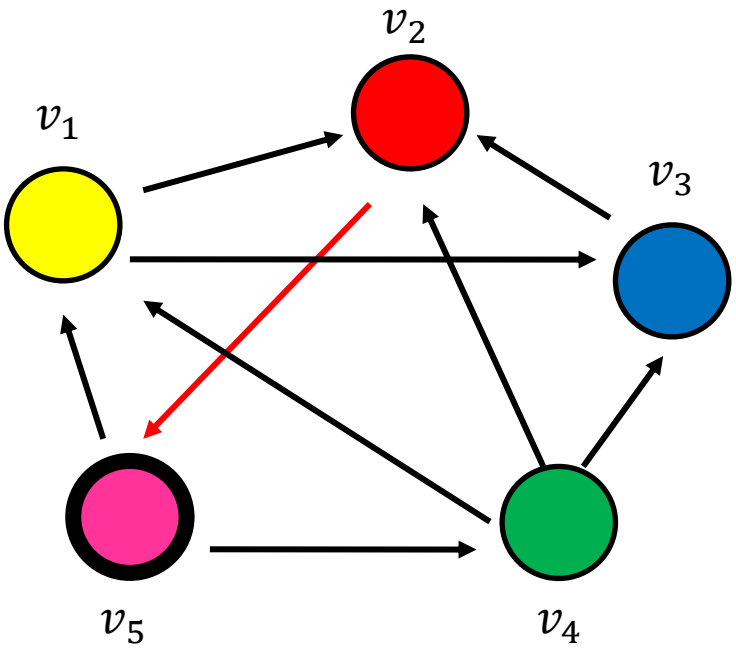
Example

- Step 0



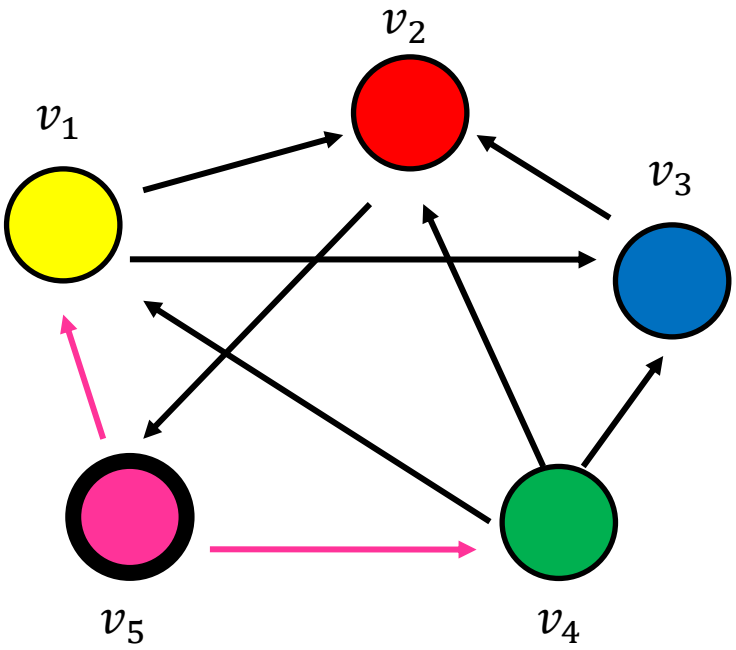
Example

- Step 1



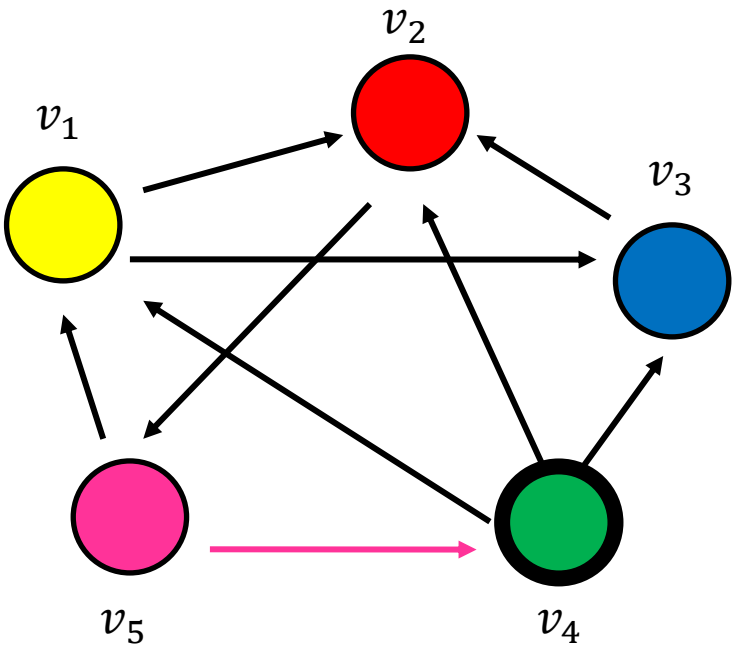
Example

- Step 1



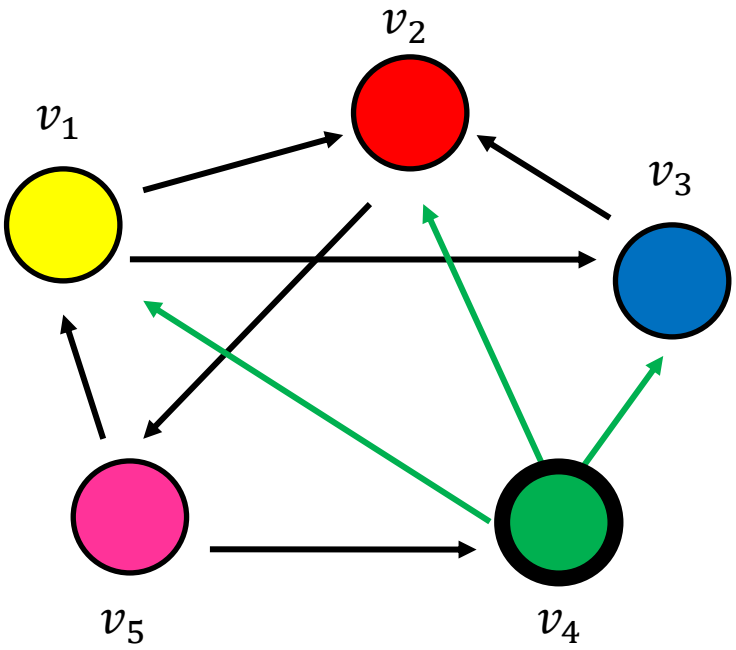
Example

- Step 2



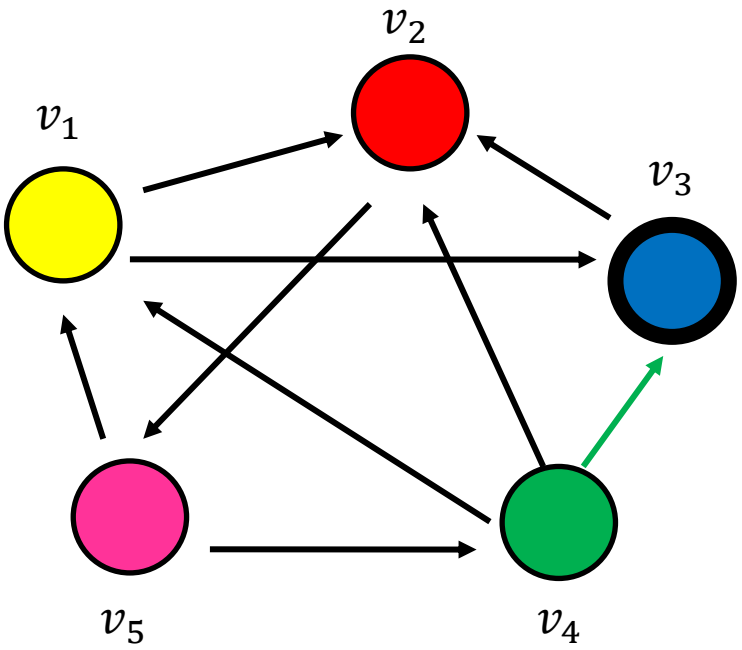
Example

- Step 2



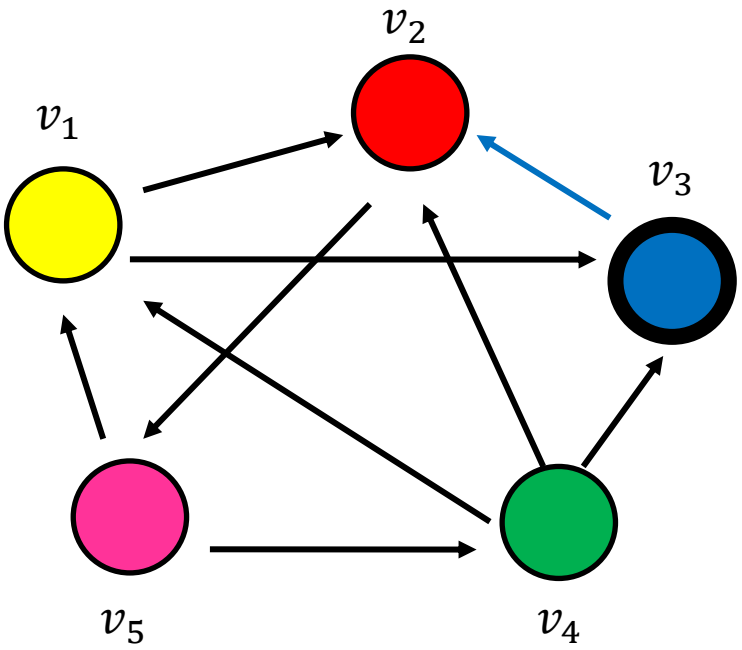
Example

- Step 3



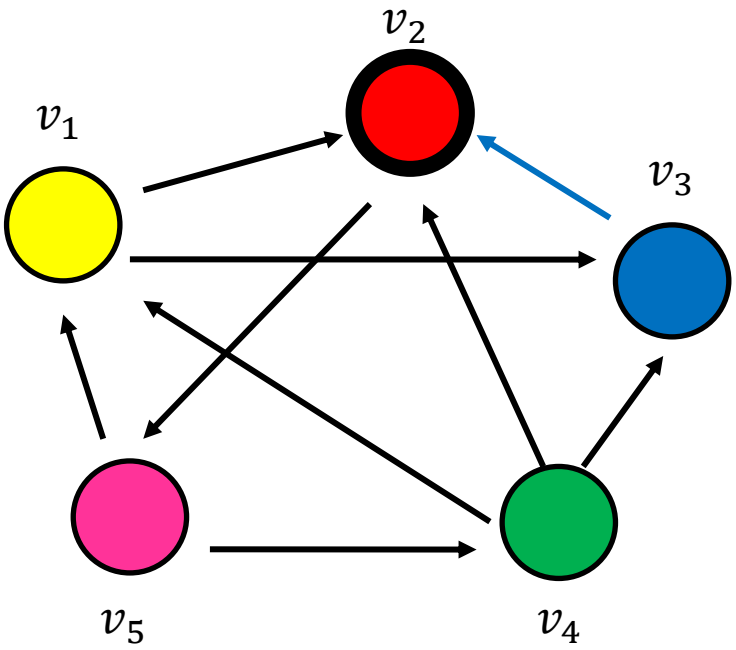
Example

- Step 3



Example

- Step 4…



Random walk

- Question: what is the probability p_i^t of being at node i after t steps?

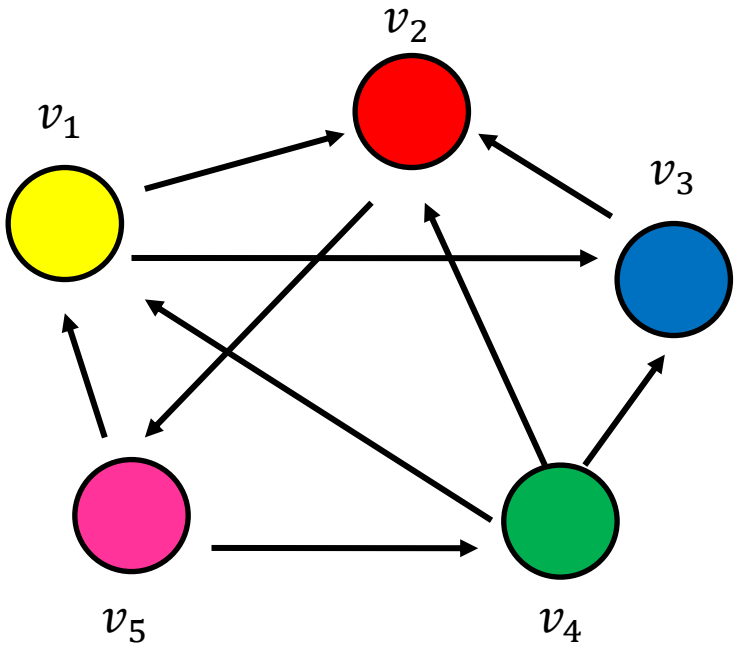
$$p_1^0 = \frac{1}{5} \qquad p_1^t = \frac{1}{3}p_4^{t-1} + \frac{1}{2}p_5^{t-1}$$

$$p_2^0 = \frac{1}{5} \qquad p_2^t = \frac{1}{2}p_1^{t-1} + p_3^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_3^0 = \frac{1}{5} \qquad p_3^t = \frac{1}{2}p_1^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_4^0 = \frac{1}{5} \qquad p_4^t = \frac{1}{2}p_5^{t-1}$$

$$p_5^0 = \frac{1}{5} \qquad p_5^t = p_2^{t-1}$$



Markov chains

- A Markov chain describes a discrete time stochastic process over a set of states

$$S = \{s_1, s_2, \dots, s_n\}$$

  according to a transition probability matrix $P = \{P_{ij}\}$
  - P_{ij} = probability of moving to state j when at state i

- Matrix P has the property that the entries of all rows sum to 1

$$\sum_j P[i,j] = 1$$

  A matrix with this property is called stochastic

- State probability distribution: The vector $p^t = (p_1^t, p_2^t, \dots, p_n^t)$ that stores the probability of being at state s_i after t steps
- Memorylessness property: The next state of the chain depends only at the current state and not on the past of the process (first order MC)
  - Higher order MCs are also possible
- Markov Chain Theory: After infinite steps the state probability vector converges to a unique distribution if the chain is irreducible (possible to get from any state to any other state) and aperiodic
  - These are the PageRank values

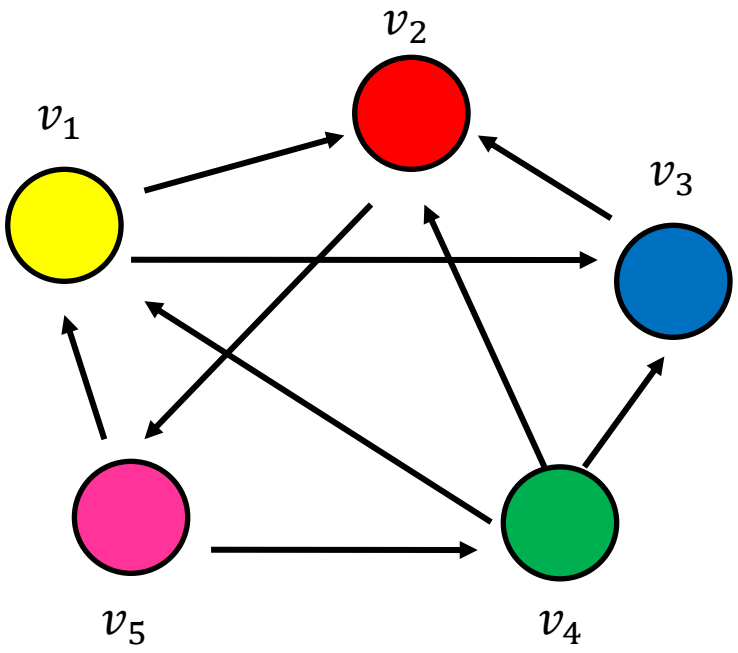
Random walks

- Random walks on graphs correspond to Markov Chains
  - The set of states S is the set of nodes of the graph G
  - The transition probability matrix is the probability that we follow an edge from one node to another

$$P[i,j] = 1/\deg_{out}(i)$$

An example

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

Node Probability vector

- The vector $p^t = (p_i^t, p_2^t, \dots, p_n^t)$ that stores the probability of being at node v_i at step t

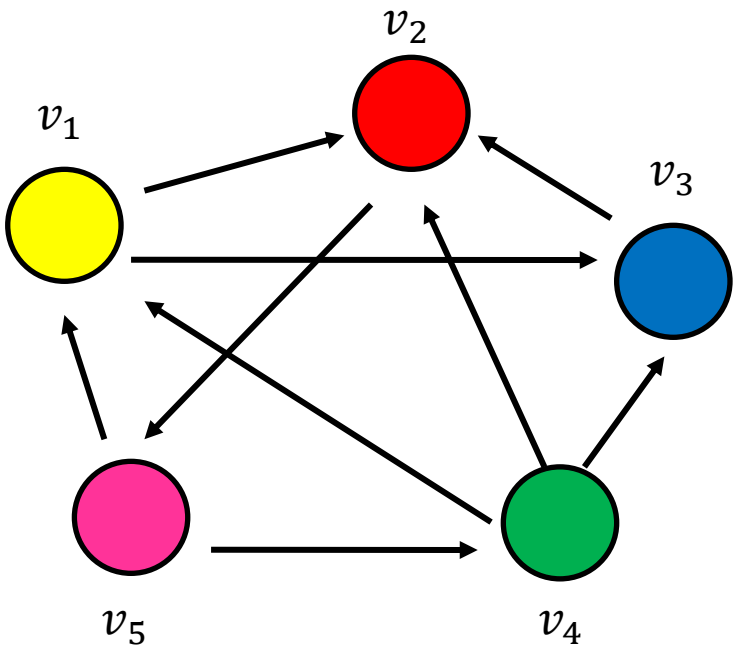
- p_i^0 = the probability of starting from state i (usually) set to uniform

- We can compute the vector p^t at step t using a vector-matrix multiplication

$$p^t = p^{t-1} P$$

An example

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



$$p_1^t = \frac{1}{3}p_4^{t-1} + \frac{1}{2}p_5^{t-1}$$

$$p_2^t = \frac{1}{2}p_1^{t-1} + p_3^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_3^t = \frac{1}{2}p_1^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_4^t = \frac{1}{2}p_5^{t-1}$$

$$p_5^t = p_2^{t-1}$$

Stationary distribution

- The stationary distribution of a random walk with transition matrix P, is a probability distribution π, such that $\pi = \pi P$
- The stationary distribution is an eigenvector of matrix P
  - the principal left eigenvector of P – stochastic matrices have maximum eigenvalue 1
- The probability π_i is the fraction of times that we visited state i as $t \rightarrow \infty$
- Markov Chain Theory: The random walk converges to a unique stationary distribution independent of the initial vector if the graph is strongly connected, and not bipartite.
  - In our case these are the PageRank values.

Computing the stationary distribution

- The Power Method

> Initialize p^0 to some distribution
> Repeat
> $p^t = p^{t-1}P$
> Until convergence

- After many iterations $p^t \rightarrow \pi$ regardless of the initial vector p^0
- Power method because it computes $p^t = p^0 P^t$

- Rate of convergence
  - determined by the second eigenvalue $\frac{|\lambda_2|}{|\lambda_1|}$

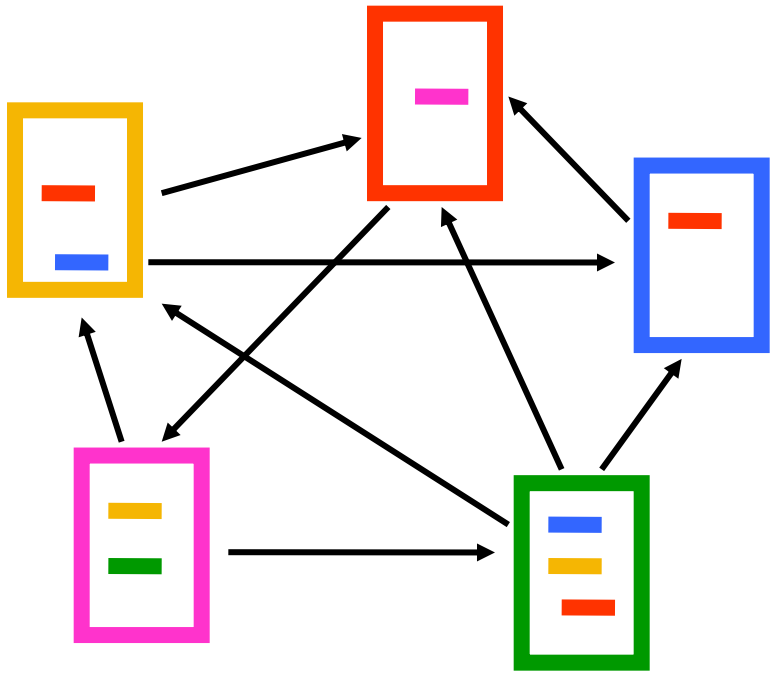
The stationary distribution

- What is the meaning of the stationary distribution π of a random walk?
- $\pi(i)$: the probability of being at node i after very large (infinite) number of steps
- $\pi = p_0 P^{\infty}$, where P is the transition matrix, p_0 the original vector
  - $P(i,j)$: probability of going from i to j in one step
  - $P^2(i,j)$: probability of going from i to j in two steps (probability of all paths of length 2)
  - $P^{\infty}(i,j) = \pi(j)$: probability of going from i to j in infinite steps – starting point does not matter.

The PageRank random walk

- Vanilla random walk
  - make the adjacency matrix stochastic and run a random walk

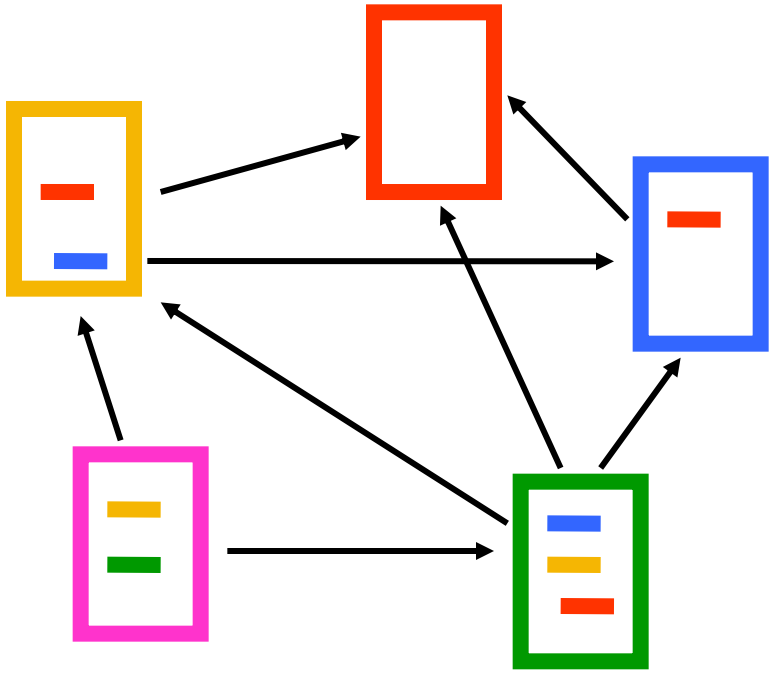
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- What about sink nodes?
  - what happens when the random walk moves to a node without any outgoing inks?

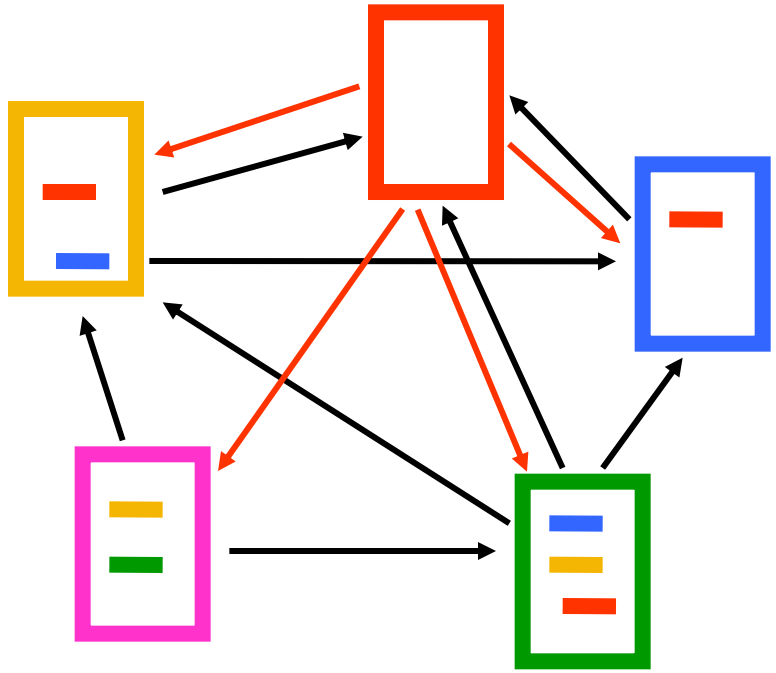
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- Replace these row vectors with a vector v
  - typically, the uniform vector

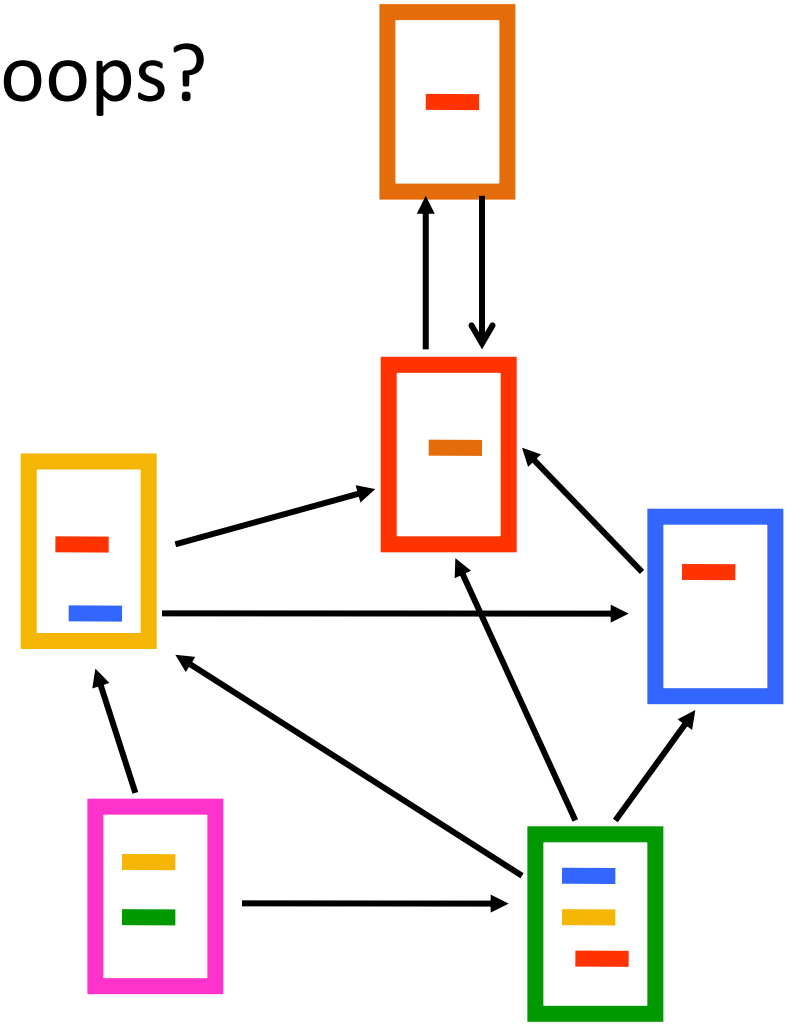
$$P' = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



$$P' = P + dv^T \qquad d = \begin{cases} 1 & \text{if } i \text{ is sink} \\ 0 & \text{otherwise} \end{cases}$$

The PageRank random walk

- What about loops?
  - Spider traps



The PageRank random walk

- Add a random jump to vector v with prob 1-α
  - typically, to a uniform vector
- Restarts after 1/(1-α) steps in expectation
  - Guarantees irreducibility, convergence

$$P'' = \alpha \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

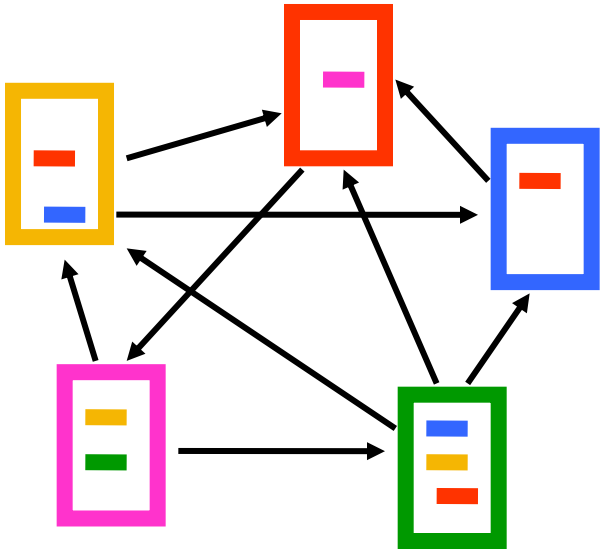
P'' = αP' + (1-α)uv^T, where u is the vector of all 1s

Random walk with restarts

PageRank algorithm [BP98]

- The Random Surfer model
  - pick a page at random
  - with probability $1 - \alpha$ jump to a random page
  - with probability α follow a random outgoing link
- Rank according to the stationary distribution
- $w_v = \alpha \sum_{u \to v} \frac{1}{d_{out}(u)} w_u + (1 - \alpha) \frac{1}{n}$

  $\alpha = 0.85$ in most cases
- We repeat this computation until convergece



1. **Red Page**
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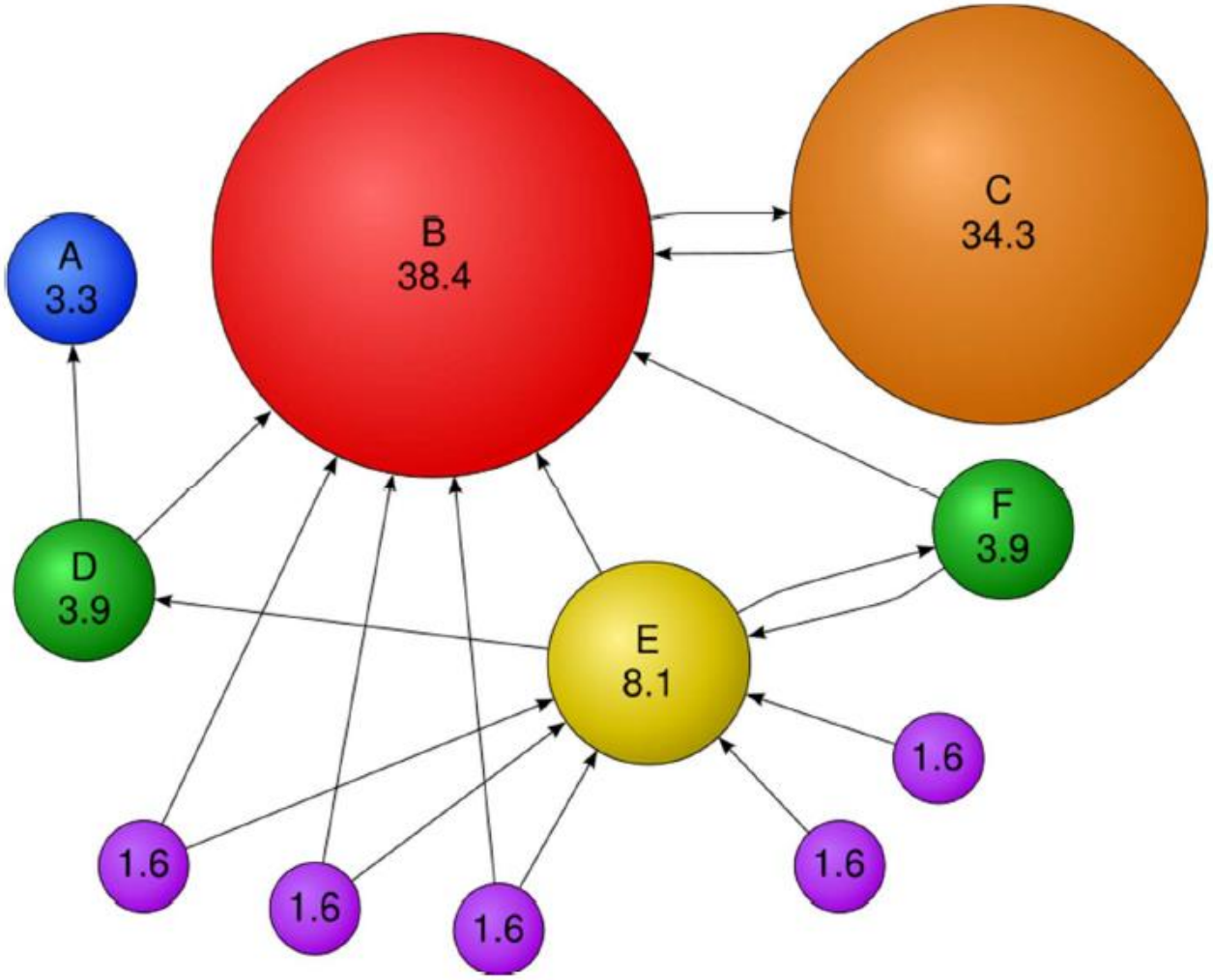
PageRank: Example



Stationary distribution with random jump

- If v is the jump vector

$$p^0 = v$$
$$p^1 = \alpha p^0 P' + (1-\alpha)v = \alpha v P' + (1-\alpha)v$$
$$p^2 = \alpha p^1 P' + (1-\alpha)v$$
$$= \alpha^2 v P'^2 + (1-\alpha)v\alpha P' + (1-\alpha)v$$
$$\vdots$$
$$p^\infty = (1-\alpha)v + (1-\alpha)v\alpha P' + (1-\alpha)v\alpha^2 P'^2 + \cdots$$
$$= (1-\alpha)(I - \alpha P')^{-1}$$

- With the random jump the shorter paths are more important, since the weight decreases exponentially
  - makes sense when thought of as a restart

Random walks with restarts

- If v is not uniform, we can bias the random walk towards the nodes that are close to v
- Personalized PageRank:
  - Restart the random walk from a specific node x
  - All nodes are ranked according to their closeness to x
- Topic-Specific Pagerank.
  - Restart the random walk from a specific set of nodes (e.g., nodes about a topic)
  - All nodes are ranked according to their closeness to the topic.
- Random Walks with restarts is a general technique for measuring closeness on graphs.

Effects of random jump

- Guarantees convergence to unique distribution
- Motivated by the concept of random surfer
- Offers additional flexibility
  - personalization
  - anti-spam
- Controls the rate of convergence
  - the second eigenvalue of matrix P'' is α

Random walks on undirected graphs

- For undirected graphs, the stationary distribution of a random walk is proportional to the degrees of the nodes
  - Thus in this case a random walk is the same as degree popularity
- This is not longer true if we do random jumps
  - Now the short paths play a greater role, and the previous distribution does not hold.
  - Random walks with restarts to a single node are commonly used on undirected graphs for measuring similarity between nodes

PageRank implementation

- Store the graph as a list of edges
- Keep current pagerank values and new pagerank values
- Go through edges and update the values of the destination nodes.
- Repeat until the difference between the pagerank vectors (L_1 or L_∞ difference) is below some small value ε.

A (Matlab-friendly) PageRank algorithm

- Performing vanilla power method is now too expensive – the matrix is not sparse

$q^0 = v$

$t = 1$

repeat

$q^t = (P'')^T q^{t-1}$

$\delta = \|q^t - q^{t-1}\|$

$t = t + 1$

until $\delta < \varepsilon$

Efficient computation of $y = (P'')^T x$

$y = \alpha P^T x$

$\beta = \|x\|_1 - \|y\|_1$

$y = y + \beta v$

P = normalized adjacency matrix

$P' = P + dv^T$, where d_i is 1 if i is sink and 0 o.w.

$P'' = \alpha P' + (1-\alpha)uv^T$, where u is the vector of all 1s

PageRank history

- Huge advantage for Google in the early days
  - It gave a way to get an idea for the value of a page, which was useful in many different ways
    - Put an order to the web.
  - After a while it became clear that the anchor text was probably more important for ranking
  - Also, link spam became a new (dark) art
- Flood of research
  - Numerical analysis got rejuvenated
  - Huge number of variations
  - Efficiency became a great issue.
  - Huge number of applications in different fields
    - Random walk is often referred to as PageRank.

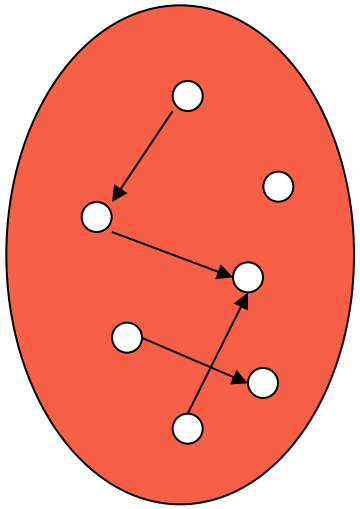
THE HITS ALGORITHM

The HITS algorithm

- Another algorithm proposed around the same time as PageRank for using the hyperlinks to rank pages
  - Kleinberg: then an intern at IBM Almaden
  - IBM never made anything out of it

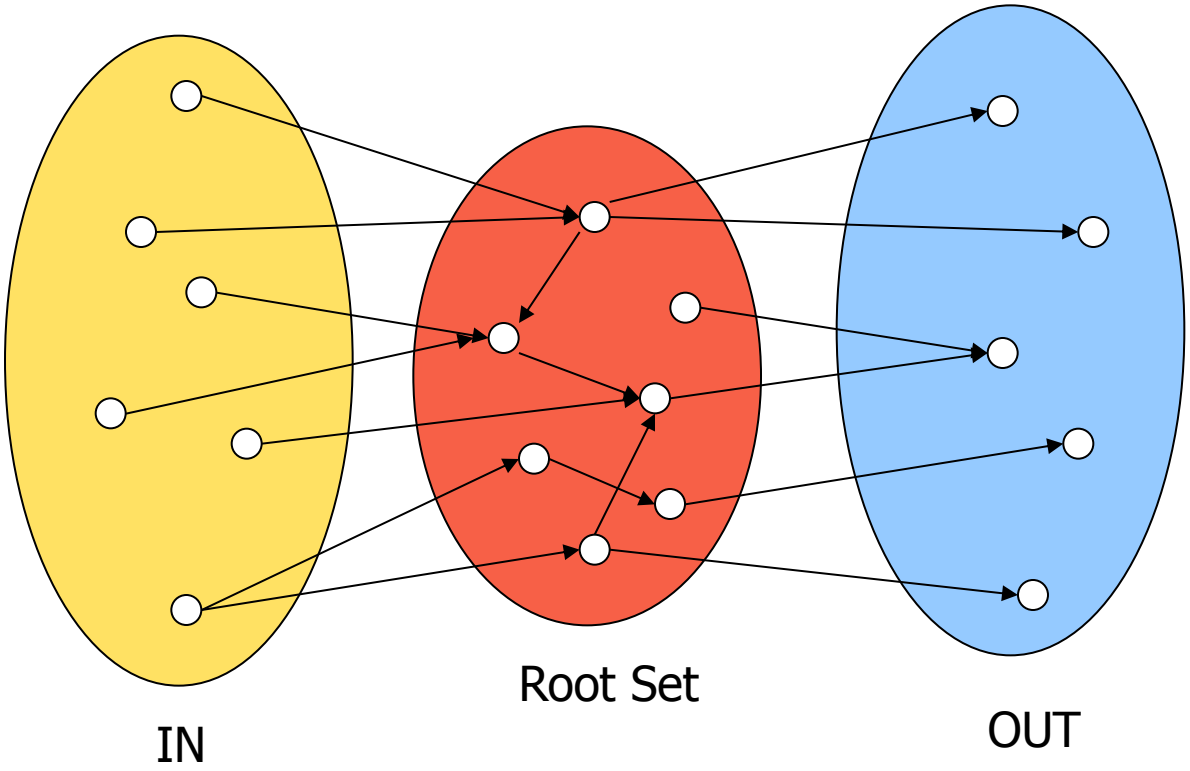
Query dependent input

Root set obtained from a text-only search engine

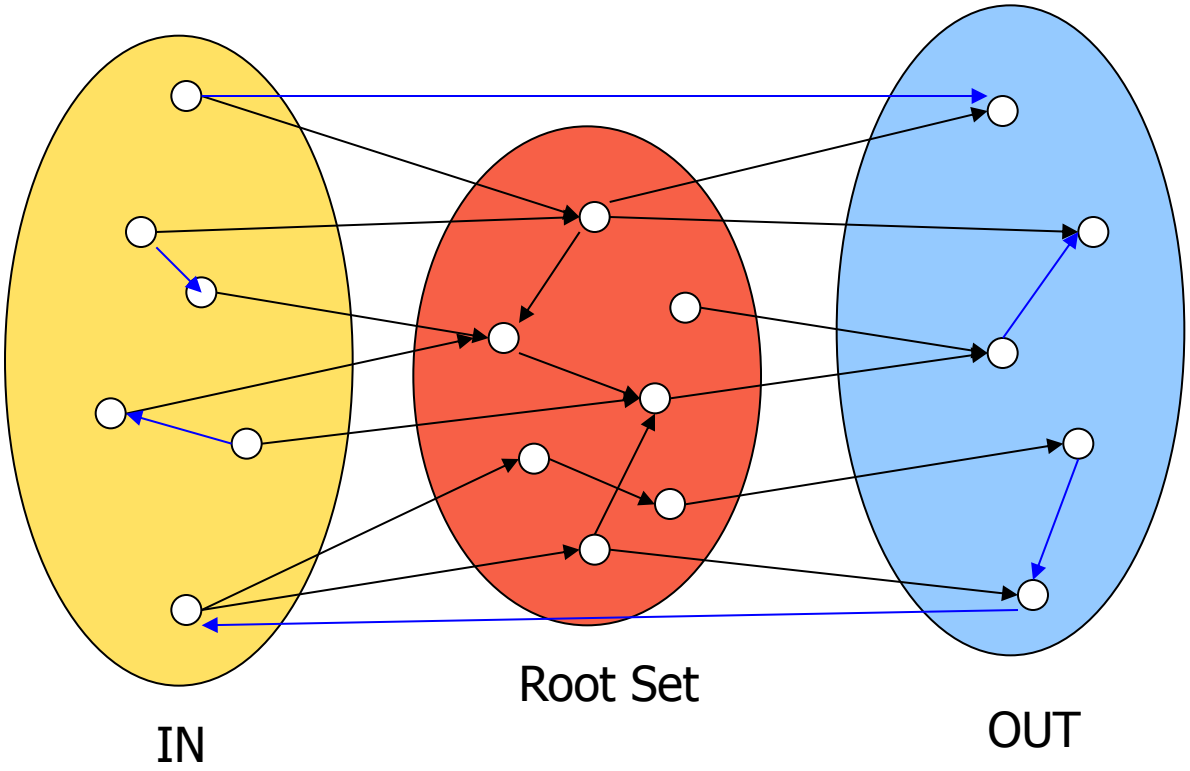


Root Set

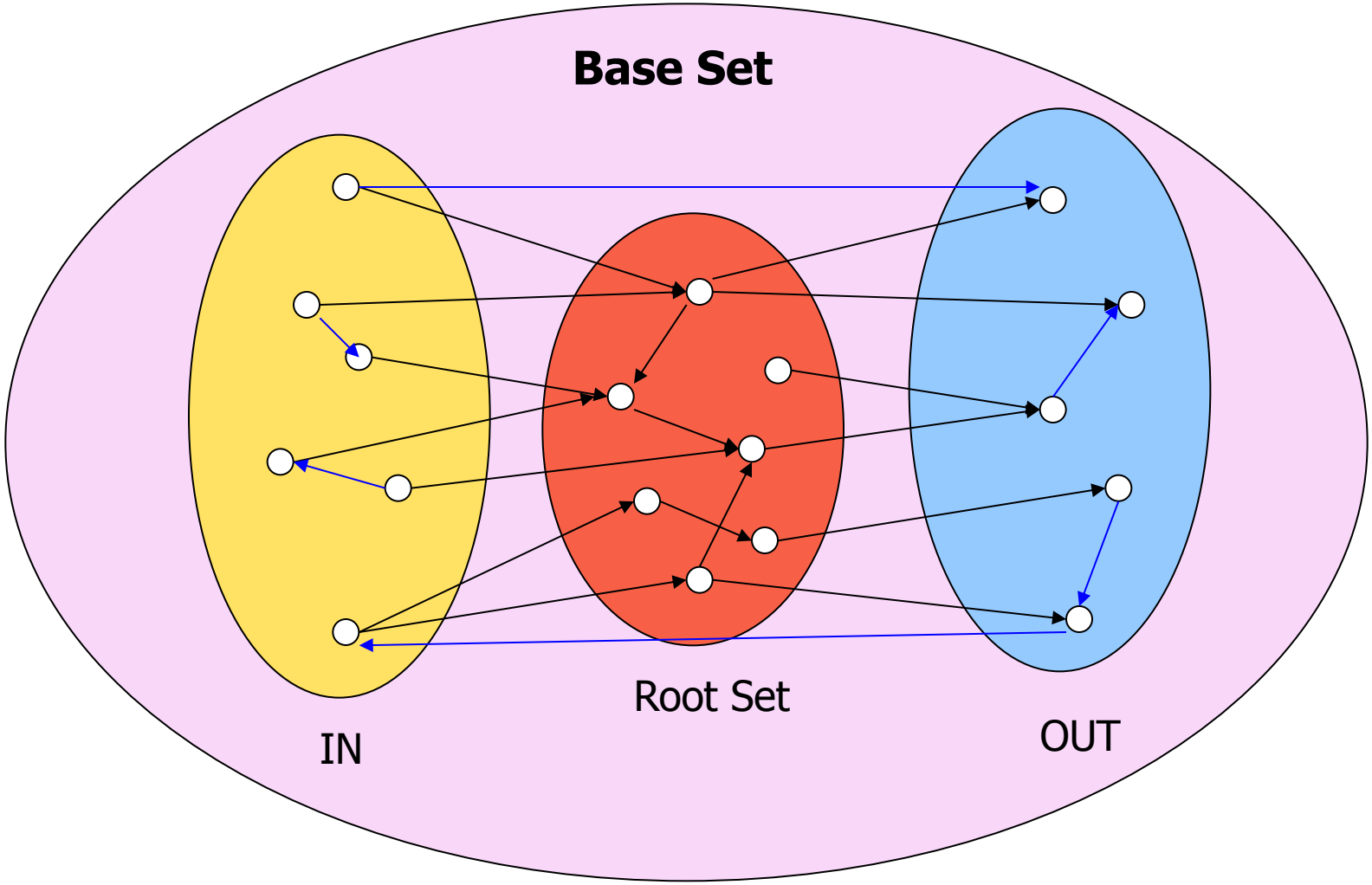
Query dependent input



Query dependent input

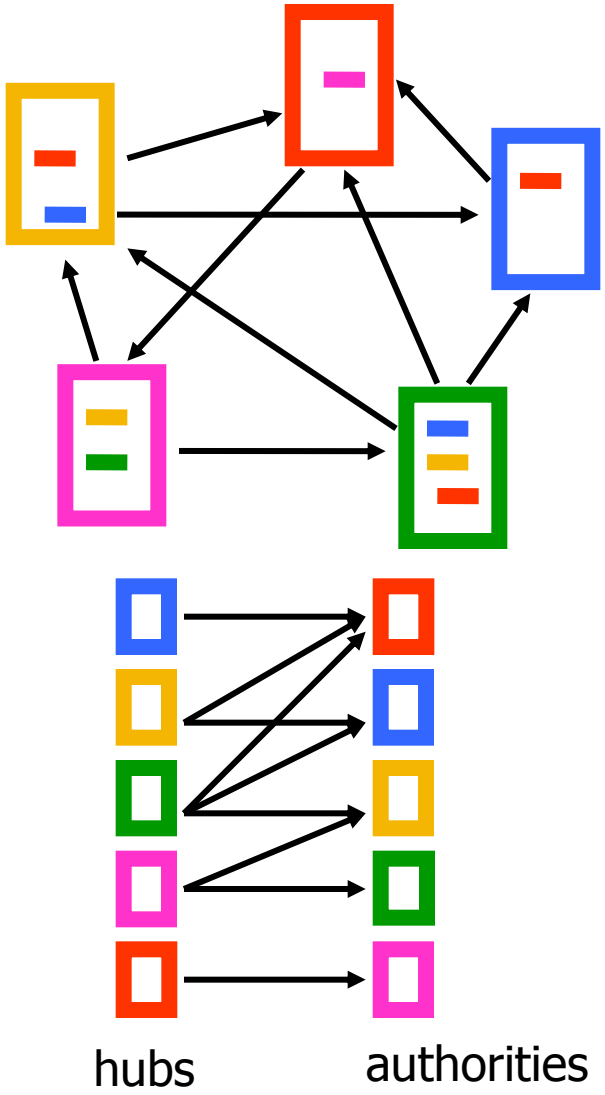


Query dependent input



Hubs and Authorities [K98]

- Authority is not necessarily transferred directly between authorities
- Pages have double identity
  - hub identity
  - authority identity
- Good hubs point to good authorities
- Good authorities are pointed by good hubs



Hubs and Authorities

- Two kind of weights:
  - Hub weight
  - Authority weight

- The hub weight is the sum of the authority weights of the authorities pointed to by the hub

- The authority weight is the sum of the hub weights that point to this authority.

HITS Algorithm

- Initialize all weights to 1.
- Repeat until convergence
  - O operation : hubs collect the weight of the authorities

$$h_i = \sum_{j:i \to j} a_j$$

  - I operation: authorities collect the weight of the hubs

$$a_i = \sum_{j:j \to i} h_j$$

  - Normalize weights under some norm

HITS and eigenvectors

- The HITS algorithm is a power-method eigenvector computation
- In vector terms
  - $a^t = A^T h^{t-1}$ and $h^t = A a^{t-1}$
  - $a^t = A^T A a^{t-1}$ and $h^t = A A^T h^{t-1}$
  - Repeated iterations will converge to the eigenvectors
- The authority weight vector a is the eigenvector of $A^T A$ and the hub weight vector h is the eigenvector of $A A^T$

- The vectors a and h are called the singular vectors of the matrix A

Singular Value Decomposition

$$A = U \quad \Sigma \quad V^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_r \end{bmatrix}$$

$$[n \times r]\ [r \times r]\ [r \times n]$$

- **r** : rank of matrix A
- $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$: singular values (square roots of eig-vals AA^T, A^TA)
- $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r$: left singular vectors (eig-vectors of AA^T)
- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$: right singular vectors (eig-vectors of A^TA)

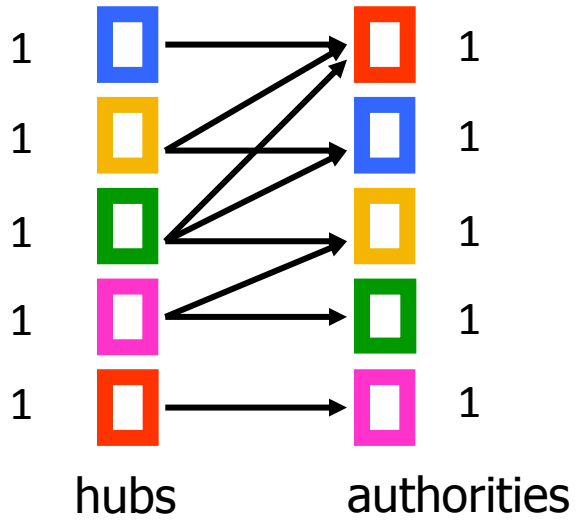
$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

Why does the Power Method work?

- If a matrix R is real and symmetric, it has real eigenvalues and eigenvectors: $(\lambda_1, w_1), (\lambda_2, w_2), \ldots, (\lambda_r, w_r)$
  - r is the rank of the matrix
  - $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|$
- For any matrix R, the eigenvectors $w_1, w_2, \ldots, w_r$ of R define a basis of the vector space
  - For any vector x, $Rx = a_1 w_1 + a_2 w_2 + \cdots + a_r w_r$
- After t multiplications we have:
  - $R^t x = \lambda_1^{t-1} a_1 w_1 + \lambda_2^{t-1} a_2 w_2 + \cdots + \lambda_r^{t-1} a_r w_r$
- Normalizing (divide by λ_1^{t-1}) leaves only the term w_1.

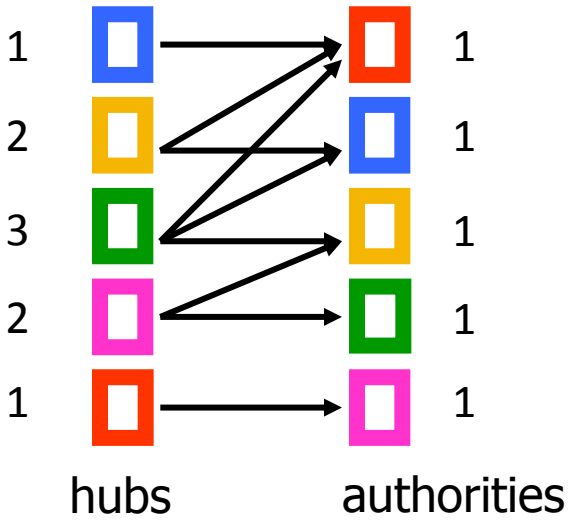
Example

Initialize



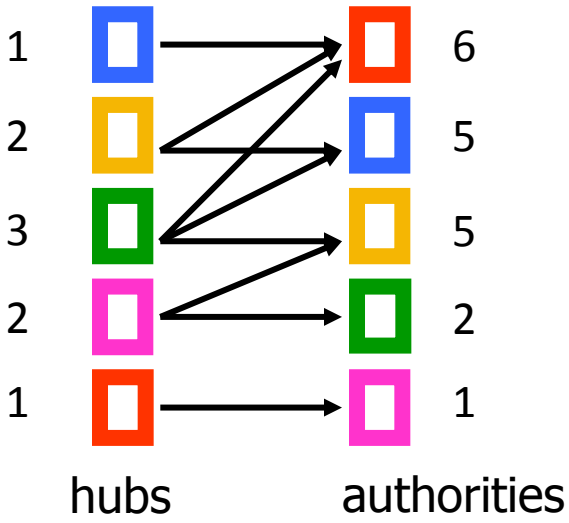
Example

Step 1: O operation



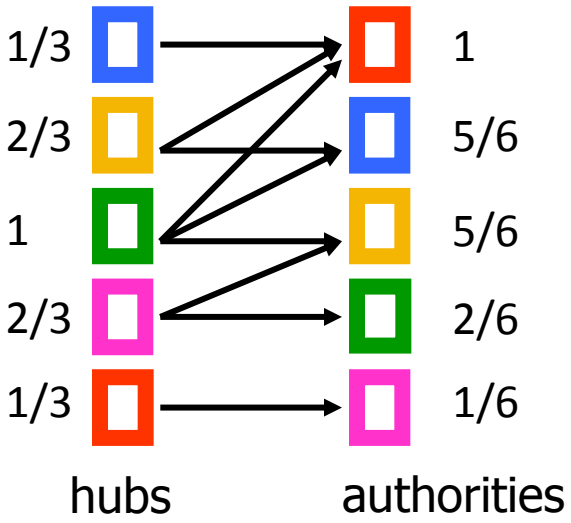
Example

Step 1: I operation



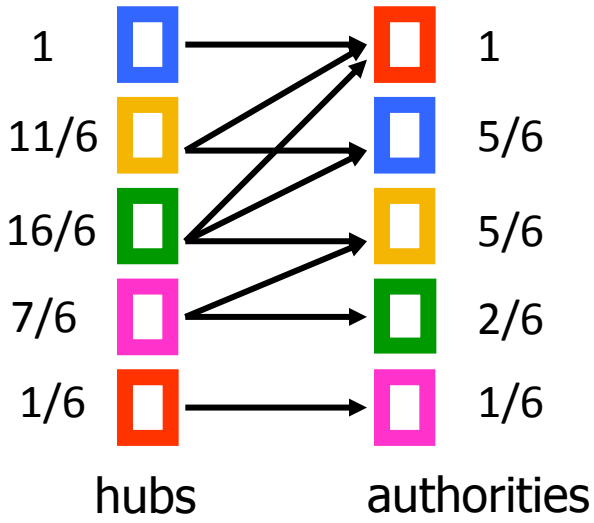
Example

Step 1: Normalization (Max norm)



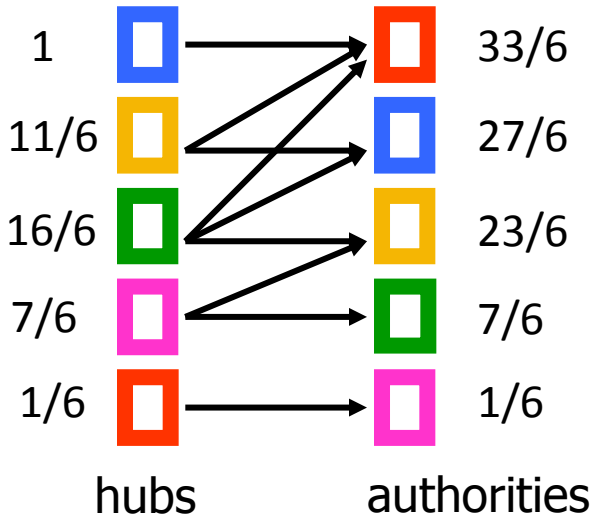
Example

Step 2: O step



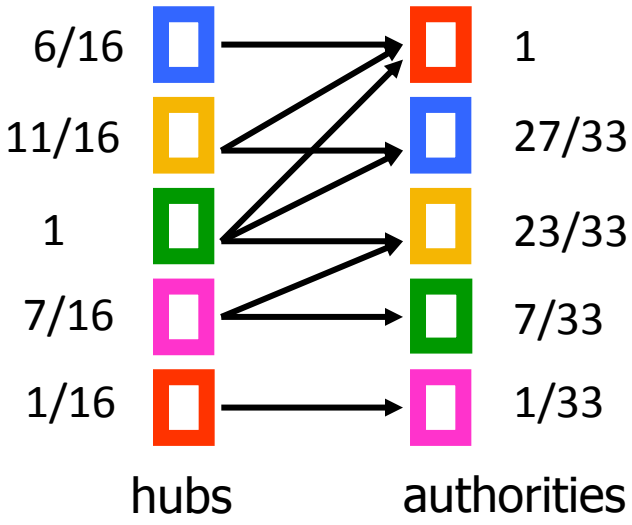
Example

Step 2: I step



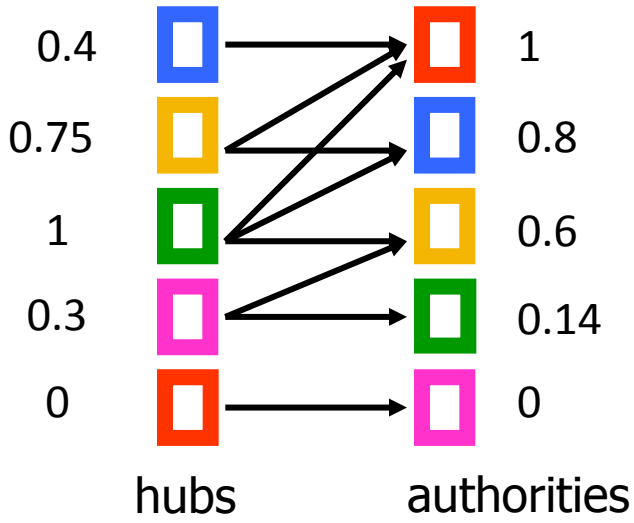
Example

Step 2: Normalization



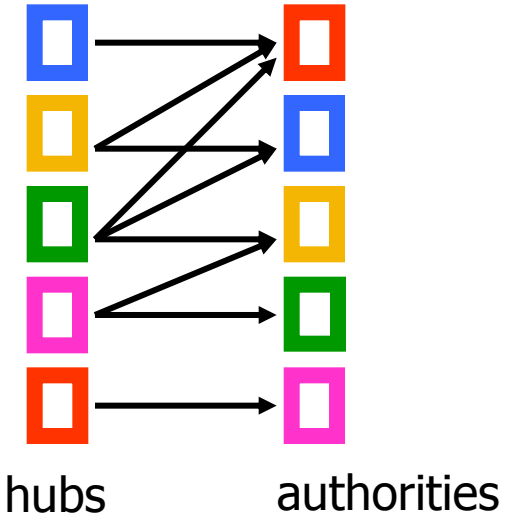
Example

Convergence



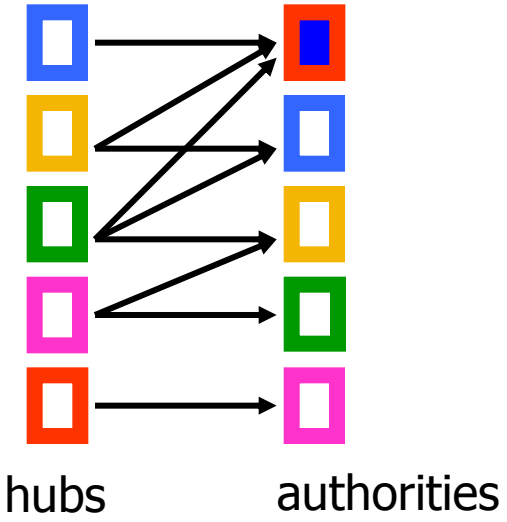
The SALSA algorithm

- Perform a random walk on the bipartite graph of hubs and authorities alternating between the two



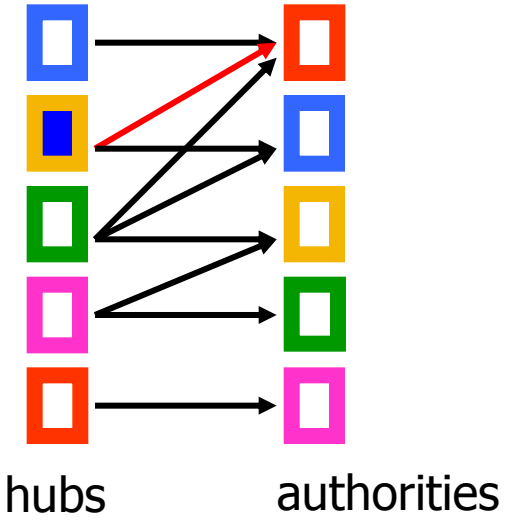
The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority



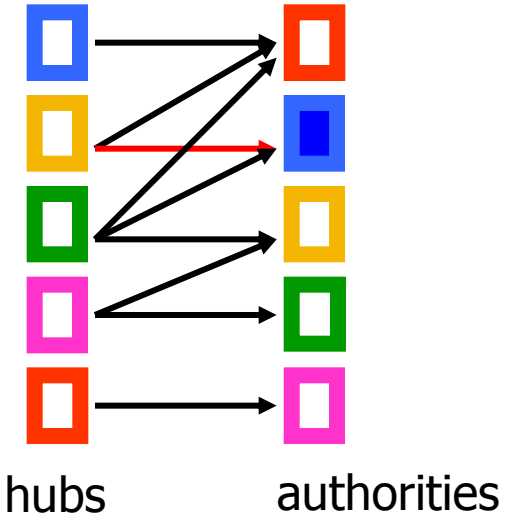
The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority
- Choose one of the in-coming links uniformly at random and move to a hub
  - e.g. move to the yellow authority with probability 1/3



The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority
- Choose one of the in-coming links uniformly at random and move to a hub
  - e.g. move to the yellow authority with probability 1/3
- Choose one of the out-going links uniformly at random and move to an authority
  - e.g. move to the blue authority with probability 1/2



The SALSA algorithm

- Formally we have probabilities:
  - a_i: probability of being at authority i
  - h_j: probability of being at hub j
- The probability of being at authority i is computed as:

$$a_i = \sum_{j \in N_{in}(i)} \frac{1}{d_{out}(j)} h_j$$

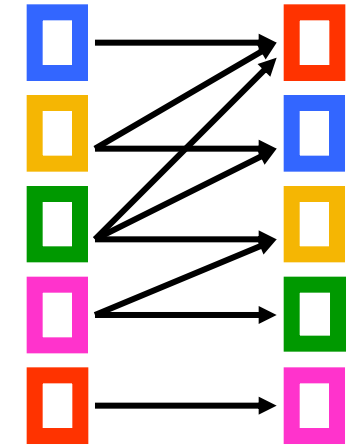
- The probability of being at hub j is computed as

$$h_j = \sum_{i \in N_{out}(j)} \frac{1}{d_{in}(i)} a_i$$

- Repeated computation converges

The SALSA algorithm [LM00]

- In matrix terms
  - A_c = the matrix A where columns are normalized to sum to 1
  - A_r = the matrix A where rows are normalized to sum to 1
- The hub computation
  - $h = A_c\, a$
- The authority computation
  - $a = A_r^T\, h = A_r^T\, Ac\, a$
- In MC terms the transition matrix
  - $P = A_r\, A_c^T$



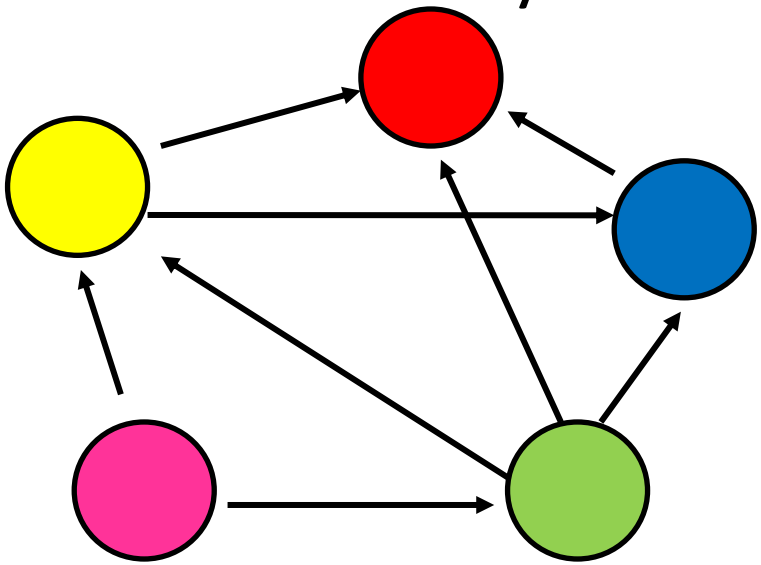
$$h_2 = 1/3\, a_1 + 1/2\, a_2$$

$$a_1 = h_1 + 1/2\, h_2 + 1/3\, h_3$$

ABSORBING RANDOM WALKS
LABEL PROPAGATION
OPINION FORMATION ON SOCIAL NETWORKS

Random walk with absorbing nodes

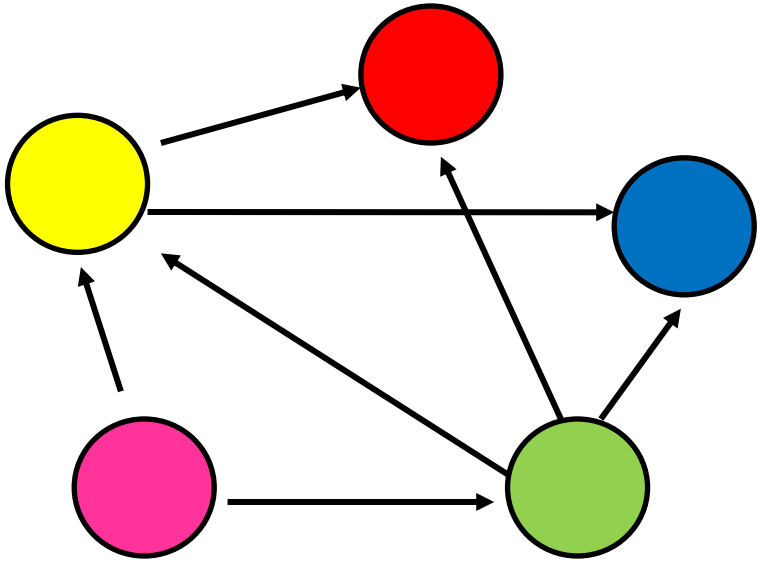
- What happens if we do a random walk on this graph? What is the stationary distribution?



- All the probability mass on the red sink node:
  - The red node is an absorbing node

Random walk with absorbing nodes

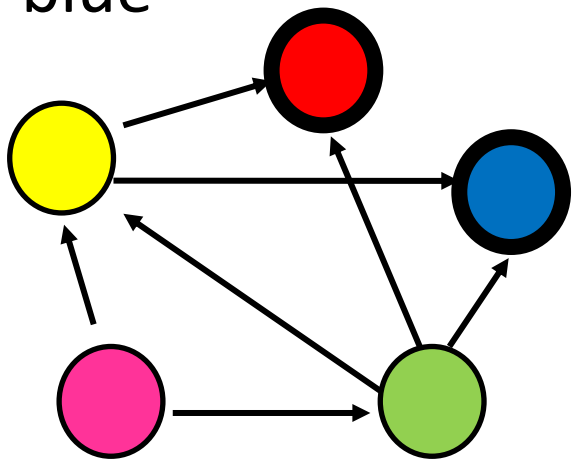
- What happens if we do a random walk on this graph? What is the stationary distribution?



- There are two absorbing nodes: the red and the blue.
- The probability mass will be divided between the two

Absorption probability

- If there are more than one absorbing nodes in the graph a random walk that starts from a non-absorbing node will be absorbed in one of them with some probability
  - The probability of absorption gives an estimate of how close the node is to red or blue



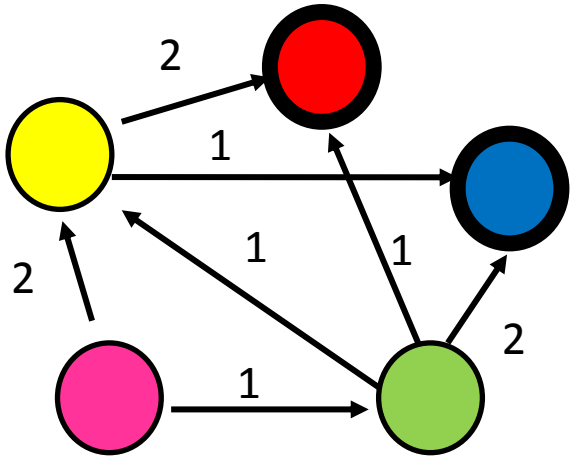
Absorption probability

- Computing the probability of being absorbed:
  - The absorbing nodes have probability 1 of being absorbed in themselves and zero of being absorbed in another node.
  - For the non-absorbing nodes, take the (weighted) average of the absorption probabilities of your neighbors
    - if one of the neighbors is the absorbing node, it has probability 1
  - Repeat until convergence (= very small change in probs)

$$P(Red|Pink) = \frac{2}{3}P(Red|Yellow) + \frac{1}{3}P(Red|Green)$$

$$P(Red|Green) = \frac{1}{4}P(Red|Yellow) + \frac{1}{4}$$

$$P(Red|Yellow) = \frac{2}{3}$$



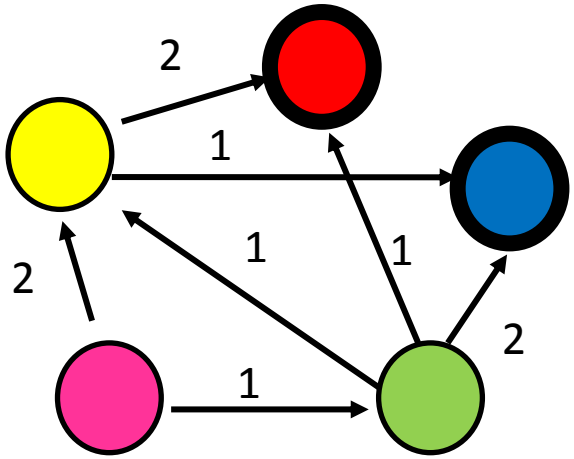
Absorption probability

- Computing the probability of being absorbed:
  - The absorbing nodes have probability 1 of being absorbed in themselves and zero of being absorbed in another node.
  - For the non-absorbing nodes, take the (weighted) average of the absorption probabilities of your neighbors
    - if one of the neighbors is the absorbing node, it has probability 1
  - Repeat until convergence (= very small change in probs)

$$P(Blue|Pink) = \frac{2}{3}P(Blue|Yellow) + \frac{1}{3}P(Blue|Green)$$

$$P(Blue|Green) = \frac{1}{4}P(Blue|Yellow) + \frac{1}{2}$$

$$P(Blue|Yellow) = \frac{1}{3}$$

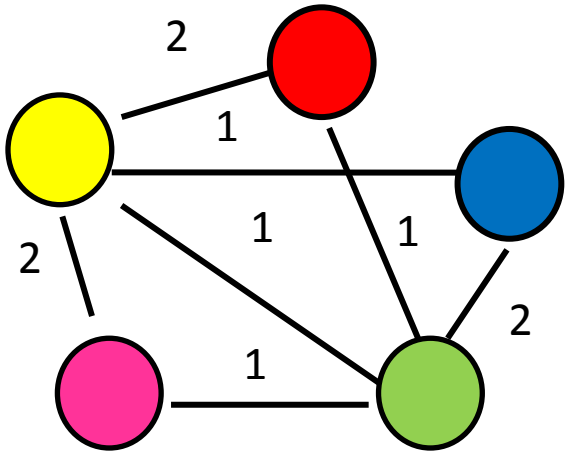


Why do we care?

- Why do we care to compute the absorbtion probability to sink nodes?
- Given a graph (directed or undirected) we can choose to make some nodes absorbing.
  - Simply direct all edges incident on the chosen nodes towards them.
- The absorbing random walk provides a measure of proximity of non-absorbing nodes to the chosen nodes.
  - Useful for understanding proximity in graphs
  - Useful for propagation in the graph
    - E.g, on a social network some nodes have high income, some have low income, to which income class is a non-absorbing node closer?

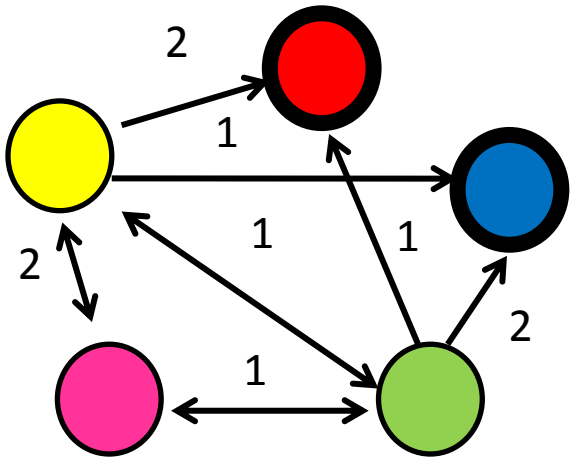
Example

- In this undirected graph we want to learn the proximity of nodes to the red and blue nodes



Example

- Make the nodes absorbing



Absorption probability

- Compute the absorbtion probabilities for red and blue

$$P(Red|Pink) = \frac{2}{3}P(Red|Yellow) + \frac{1}{3}P(Red|Green)$$

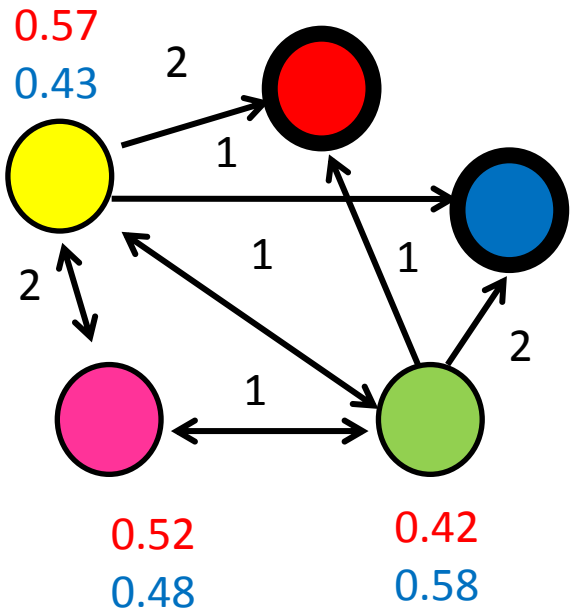
$$P(Red|Green) = \frac{1}{5}P(Red|Yellow) + \frac{1}{5}P(Red|Pink) + \frac{1}{5}$$

$$P(Red|Yellow) = \frac{1}{6}P(Red|Green) + \frac{1}{3}P(Red|Pink) + \frac{1}{3}$$

$$P(Blue|Pink) = 1 - P(Red|Pink)$$

$$P(Blue|Green) = 1 - P(Red|Green)$$

$$P(Blue|Yellow) = 1 - P(Red|Yellow)$$

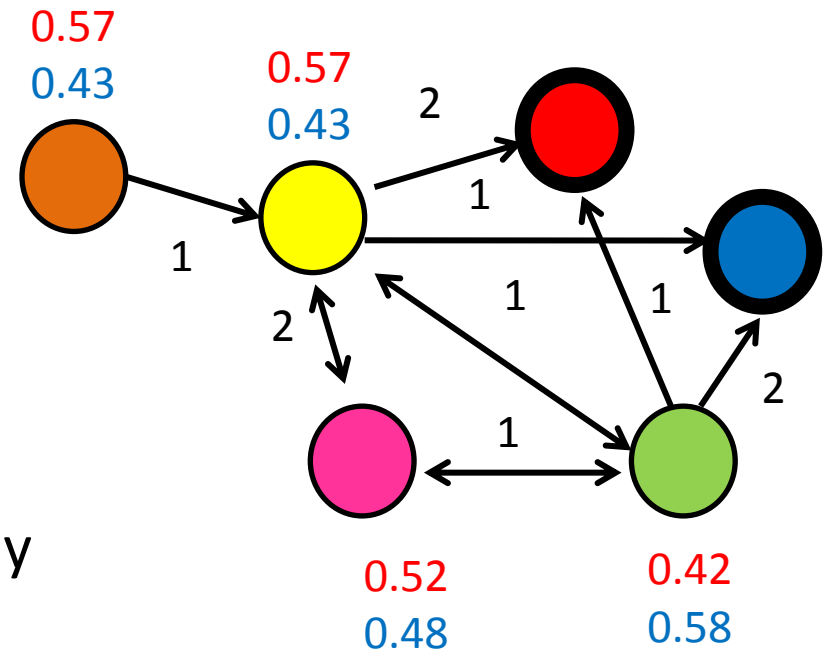


Penalizing long paths

- The orange node has the same probability of reaching red and blue as the yellow one

$$P(Red|Orange) = P(Red|Yellow)$$

$$P(Blue|Orange) = P(Blue|Yellow)$$



- Intuitively though it is further away

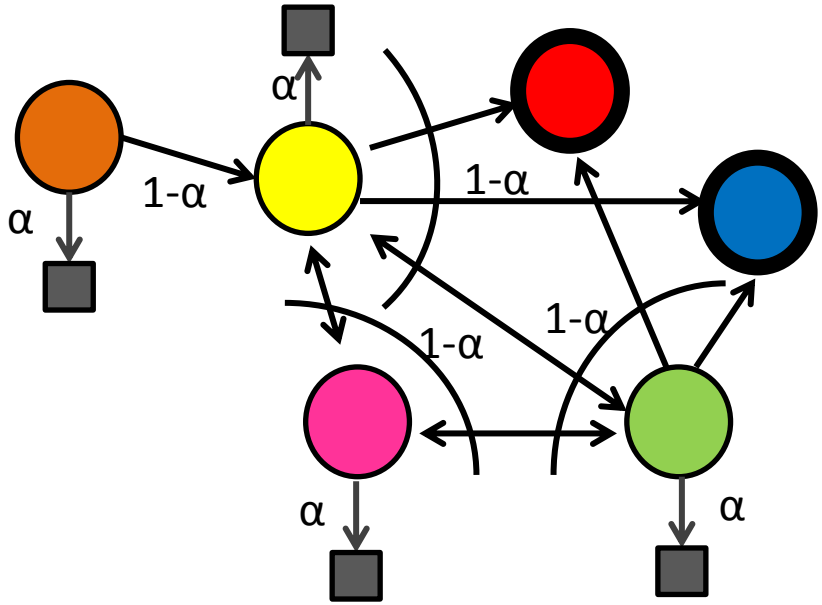
Penalizing long paths

- Add an universal absorbing node to which each node gets absorbed with probability α.

With probability α the random walk dies

With probability (1-α) the random walk continues as before

The longer the path from a node to an absorbing node the more likely the random walk dies along the way, the lower the absorbtion probability



$$P(Red|Green) = (1-\alpha)\left(\frac{1}{5}P(Red|Yellow) + \frac{1}{5}P(Red|Pink) + \frac{1}{5}\right)$$

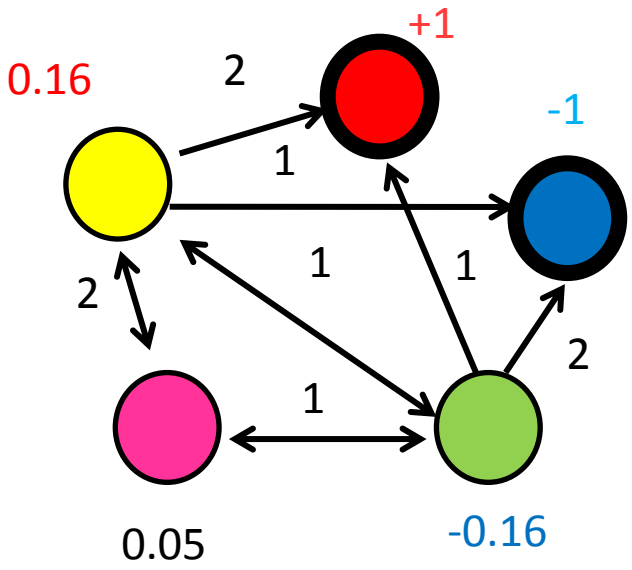
Propagating values

- Assume that Red has a positive value and Blue a negative value
  - Positive/Negative class, Positive/Negative opinion
- We can compute a value for all the other nodes in the same way
  - This is the expected value for the node

$$V(Pink) = \frac{2}{3}V(Yellow) + \frac{1}{3}V(Green)$$

$$V(Green) = \frac{1}{5}V(Yellow) + \frac{1}{5}V(Pink) + \frac{1}{5} - \frac{2}{5}$$

$$V(Yellow) = \frac{1}{6}V(Green) + \frac{1}{3}V(Pink) + \frac{1}{3} - \frac{1}{6}$$



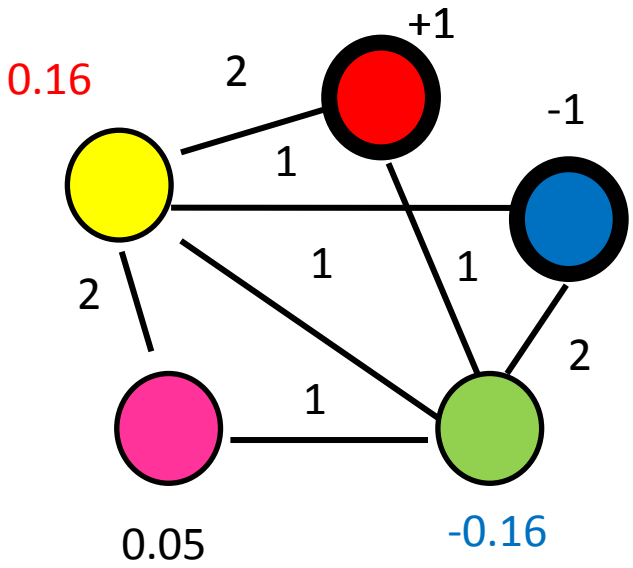
Electrical networks and random walks

- Our graph corresponds to an electrical network
- There is a positive voltage of +1 at the Red node, and a negative voltage -1 at the Blue node
- There are resistances on the edges inversely proportional to the weights (or conductance proportional to the weights)
- The computed values are the voltages at the nodes

$$V(Pink) = \frac{2}{3}V(Yellow) + \frac{1}{3}V(Green)$$

$$V(Green) = \frac{1}{5}V(Yellow) + \frac{1}{5}V(Pink) + \frac{1}{5} - \frac{2}{5}$$

$$V(Yellow) = \frac{1}{6}V(Green) + \frac{1}{3}V(Pink) + \frac{1}{3} - \frac{1}{6}$$



Opinion formation

- The value propagation can be used as a model of opinion formation.
- Model:
  - Opinions are values in [-1,1]
  - Every user u has an internal opinion s_u, and expressed opinion z_u.
  - The expressed opinion minimizes the personal cost of user u:

$$c(z_u) = (s_u - z_u)^2 + \sum_{v:v \text{ is a friend of } u} w_u(z_u - z_v)^2$$

    - Minimize deviation from your beliefs and conflicts with the society

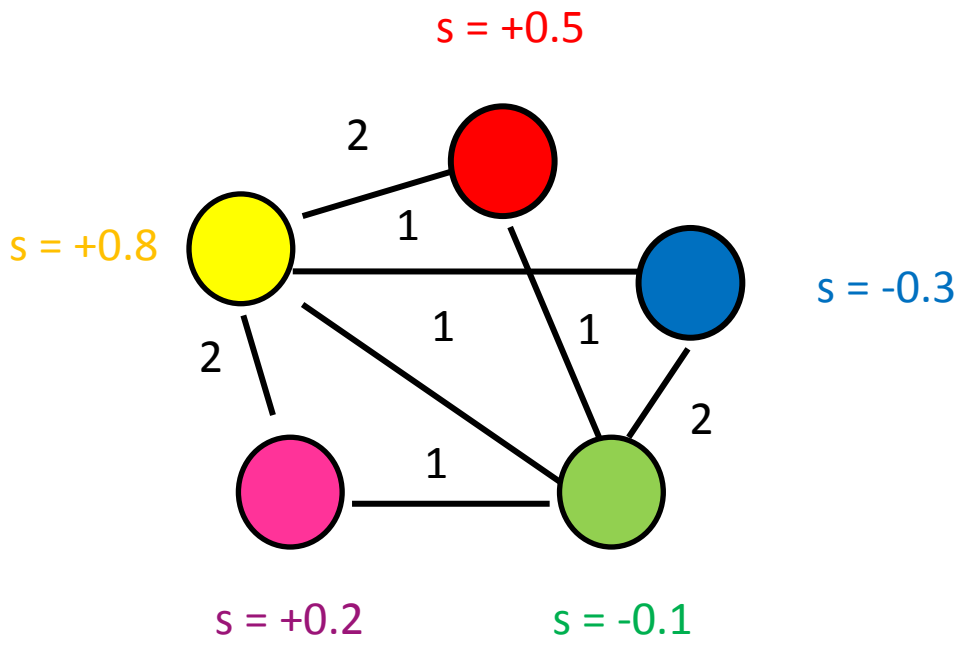
- If every user tries independently (selfishly) to minimize their personal cost then the best thing to do is to set z_u to the average of all opinions:

$$z_u = \frac{s_u + \sum_{v:v \text{ is a friend of } u} w_u z_u}{1 + \sum_{v:v \text{ is a friend of } u} w_u}$$

- This is the same as the value propagation we described before!

Example

- Social network with internal opinions



Example

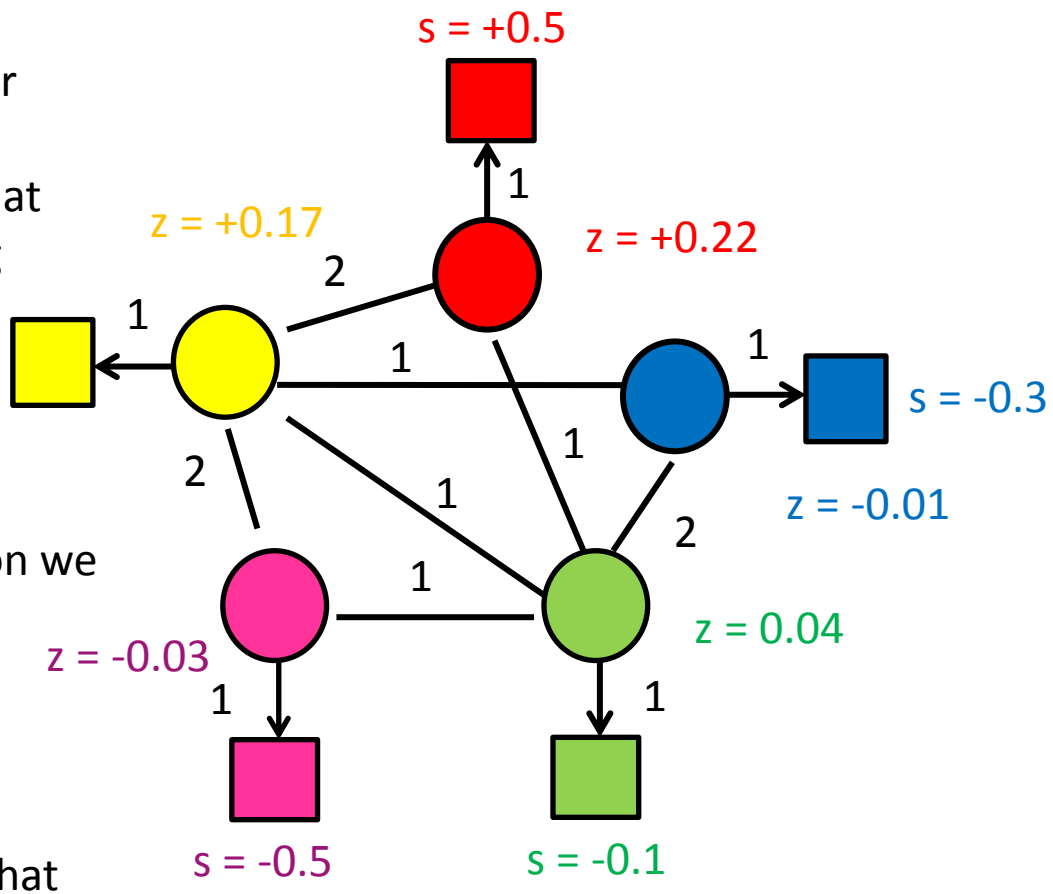
One absorbing node per user with value the internal opinion of the user

One non-absorbing node per user that links to the corresponding absorbing node

The external opinion for each node is computed using the value propagation we described before

- Repeated averaging

Intuitive model: my opinion is a combination of what I believe and what my social network believes.



Transductive learning

- If we have a graph of relationships and some labels on some nodes we can propagate them to the remaining nodes
  - Make the labeled nodes to be absorbing and compute the probability for the rest of the graph
  - E.g., a social network where some people are tagged as spammers
  - E.g., the movie-actor graph where some movies are tagged as action or comedy.

- This is a form of semi-supervised learning
  - We make use of the unlabeled data, and the relationships

- It is also called transductive learning because it does not produce a model, but just labels the unlabeled data that is at hand.
  - Contrast to inductive learning that learns a model and can label any new example

Implementation details

- Implementation is in many ways similar to the PageRank implementation
  - For an edge (u, v)instead of updating the value of v we update the value of u.
    - The value of a node is the average of its neighbors
  - We need to check for the case that a node u is absorbing, in which case the value of the node is not updated.
  - Repeat the updates until the change in values is very small.