DATA MINING LECTURE 14

Absorbing Random walks

Coverage

Random Walks on Graphs

- Random walk:
  - Start from a node chosen uniformly at random with probability $\frac{1}{n}$.
    - Pick one of the outgoing edges uniformly at random
    - Move to the destination of the edge
    - Repeat.

Random walk

- Question: what is the probability p_i^t of being at node i after t steps?

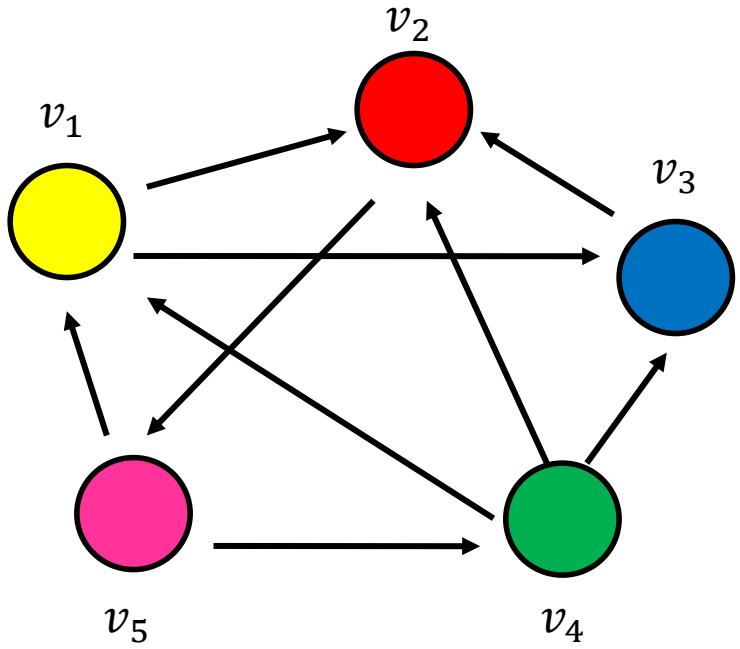
$$p_1^0 = \frac{1}{5} \qquad p_1^t = \frac{1}{3}p_4^{t-1} + \frac{1}{2}p_5^{t-1}$$

$$p_2^0 = \frac{1}{5} \qquad p_2^t = \frac{1}{2}p_1^{t-1} + p_3^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_3^0 = \frac{1}{5} \qquad p_3^t = \frac{1}{2}p_1^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_4^0 = \frac{1}{5} \qquad p_4^t = \frac{1}{2}p_5^{t-1}$$

$$p_5^0 = \frac{1}{5} \qquad p_5^t = p_2^{t-1}$$



Stationary distribution

- After many many steps ($t \rightarrow \infty$) the probabilities converge (updating the probabilities does not change the numbers)
- The converged probabilities define the stationary distribution of a random walk π
- The probability π_i is the fraction of times that we visited state i as $t \rightarrow \infty$
- Markov Chain Theory: The random walk converges to a unique stationary distribution independent of the initial vector if the graph is strongly connected, and not bipartite.

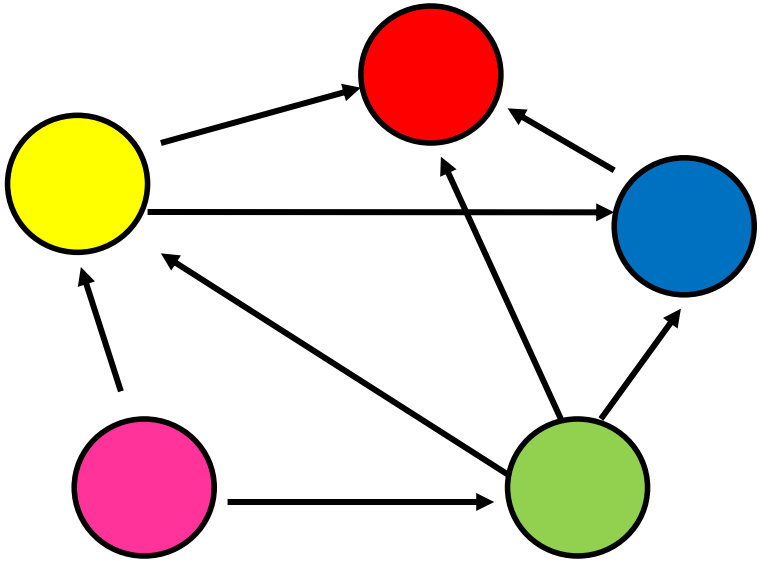
Random walk with Restarts

- This is the random walk used by the PageRank algorithm
  - At every step with probability 1-α do a step of the random walk (follow a random link)
  - With probability α restart the random walk from a randomly selected node.
- The effect of the restart is that paths followed are never too long.
  - In expectation paths have length 1/α
- Restarts can also be from a specific node in the graph (always start the random walk from there)
- What is the effect of that?
  - The nodes that are close to the starting node have higher probability to be visited.
  - The probability defines a notion of proximity between the starting node and all the other nodes in the graph

ABSORBING RANDOM WALKS

Random walk with absorbing nodes

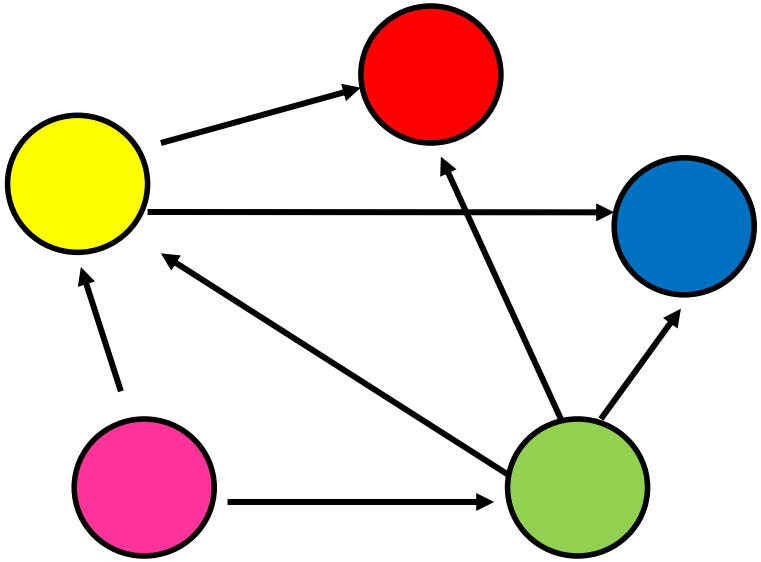
- What happens if we do a random walk on this graph? What is the stationary distribution?



- All the probability mass on the red sink node:
  - The red node is an absorbing node

Random walk with absorbing nodes

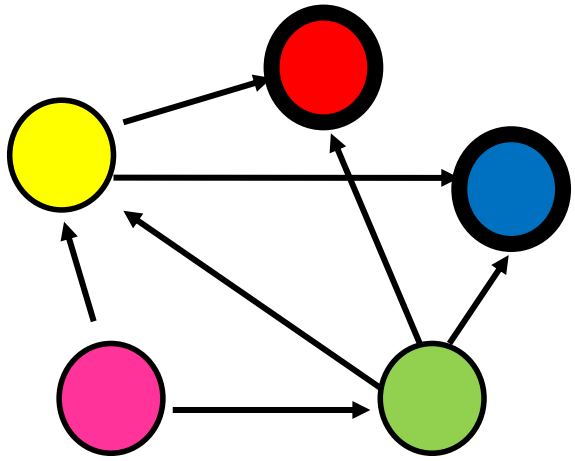
- What happens if we do a random walk on this graph? What is the stationary distribution?



- There are two absorbing nodes: the red and the blue.
- The probability mass will be divided between the two

Absorption probability

- If there are more than one absorbing nodes in the graph a random walk that starts from a non-absorbing node will be absorbed in one of them with some probability
  - The probability of absorption gives an estimate of how close the node is to red or blue



Absorption probability

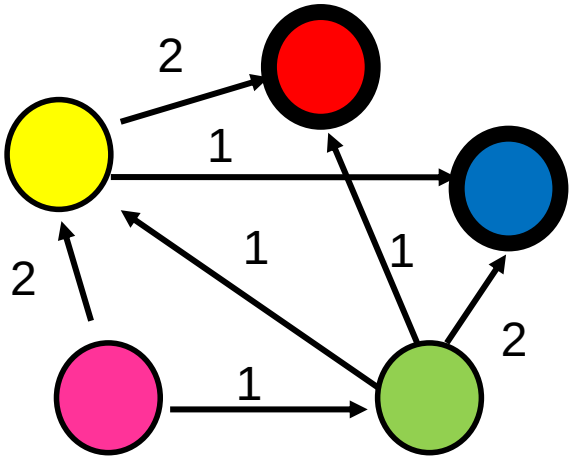
- Computing the probability of being absorbed:
  - The absorbing nodes have probability 1 of being absorbed in themselves and zero of being absorbed in another node.
  - For the non-absorbing nodes, take the (weighted) average of the absorption probabilities of your neighbors
    - if one of the neighbors is the absorbing node, it has probability 1
  - Repeat until convergence (= very small change in probs)

$$P(Red|Pink) = \frac{2}{3}P(Red|Yellow) + \frac{1}{3}P(Red|Green)$$

$$P(Red|Green) = \frac{1}{4}P(Red|Yellow) + \frac{1}{4}$$

$$P(Red|Yellow) = \frac{2}{3}$$

$$P(Red|Red) = 1\,, P(Red|Blue) = 0$$



Absorption probability

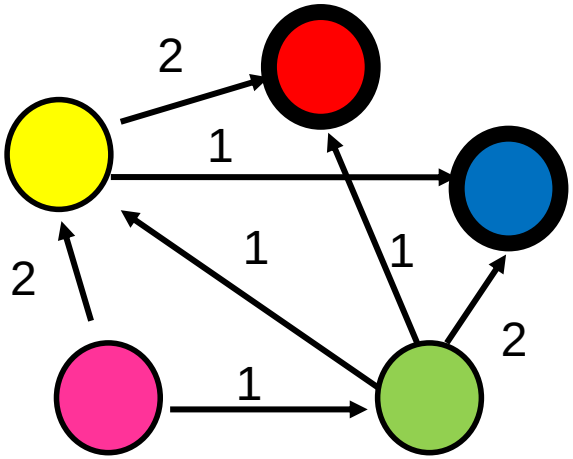
- Computing the probability of being absorbed:
  - The absorbing nodes have probability 1 of being absorbed in themselves and zero of being absorbed in another node.
  - For the non-absorbing nodes, take the (weighted) average of the absorption probabilities of your neighbors
    - if one of the neighbors is the absorbing node, it has probability 1
  - Repeat until convergence (= very small change in probs)

$$P(Blue|Pink) = \frac{2}{3}P(Blue|Yellow) + \frac{1}{3}P(Blue|Green)$$

$$P(Blue|Green) = \frac{1}{4}P(Blue|Yellow) + \frac{1}{2}$$

$$P(Blue|Yellow) = \frac{1}{3}$$

$$P(Blue|Blue) = 1\,, P(Blue|Red) = 0$$

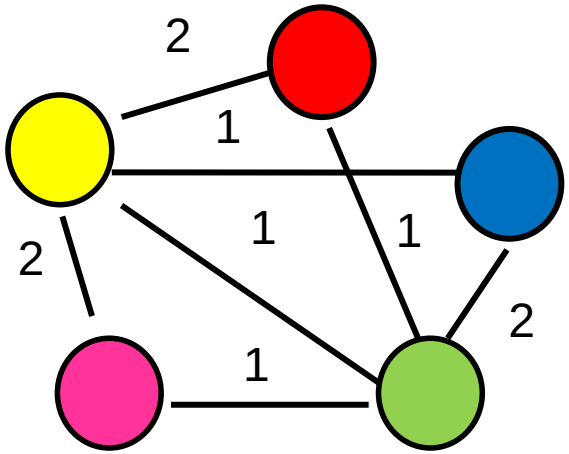


Why do we care?

- Why do we care to compute the absorption probability to sink nodes?
- Given a graph (directed or undirected) we can choose to make some nodes absorbing.
  - Simply direct all edges incident on the chosen nodes towards them and remove outgoing edges.
- The absorbing random walk provides a measure of proximity of non-absorbing nodes to the chosen nodes.
  - Useful for understanding proximity in graphs
  - Useful for propagation in the graph
    - E.g, some nodes have positive opinions for an issue, some have negative, to which opinion is a non-absorbing node closer?

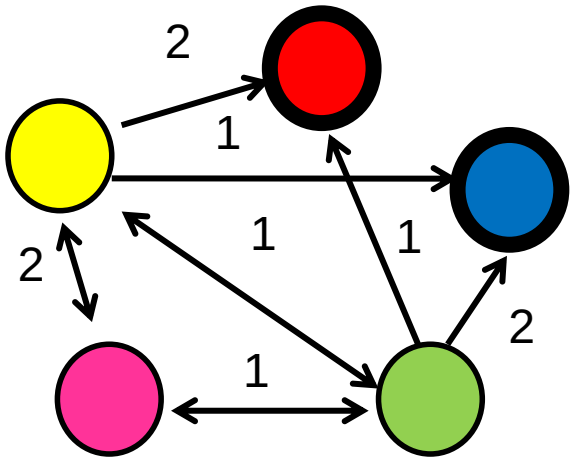
Example

- In this undirected graph we want to learn the proximity of nodes to the red and blue nodes



Example

- Make the nodes absorbing



Absorption probability

- Compute the absorbtion probabilities for red and blue

$$P(Red|Pink) = \frac{2}{3}P(Red|Yellow) + \frac{1}{3}P(Red|Green)$$

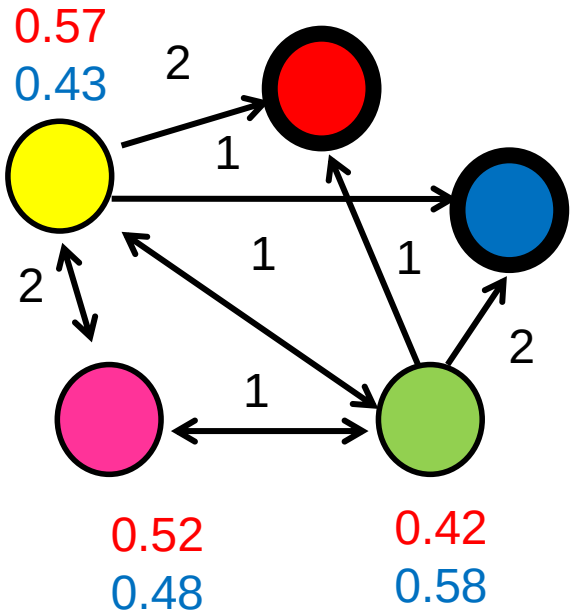
$$P(Red|Green) = \frac{1}{5}P(Red|Yellow) + \frac{1}{5}P(Red|Pink) + \frac{1}{5}$$

$$P(Red|Yellow) = \frac{1}{6}P(Red|Green) + \frac{1}{3}P(Red|Pink) + \frac{1}{3}$$

$$P(Blue|Pink) = 1 - P(Red|Pink)$$

$$P(Blue|Green) = 1 - P(Red|Green)$$

$$P(Blue|Yellow) = 1 - P(Red|Yellow)$$

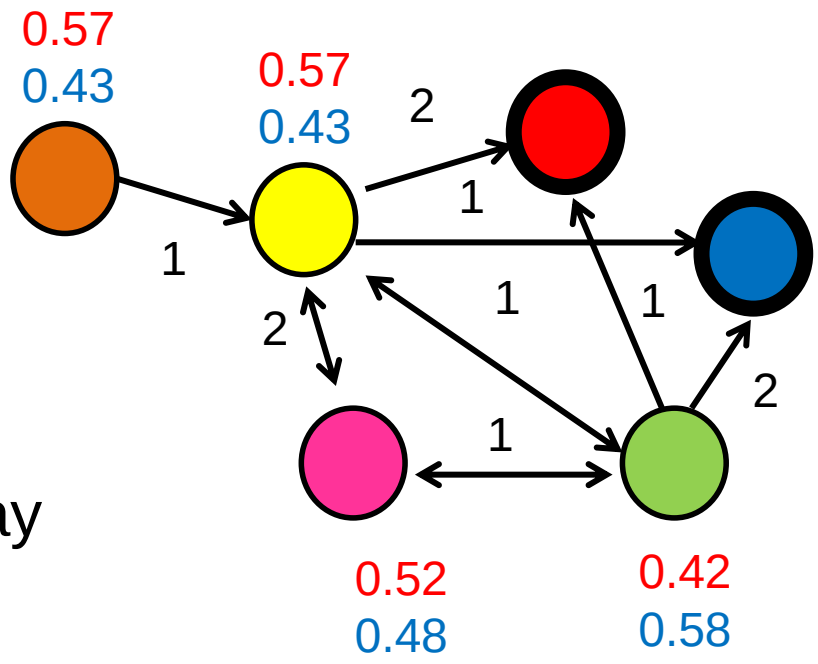


Penalizing long paths

- The orange node has the same probability of reaching red and blue as the yellow one

$$P(Red|Orange) = P(Red|Yellow)$$

$$P(Blue|Orange) = P(Blue|Yellow)$$



- Intuitively though it is further away

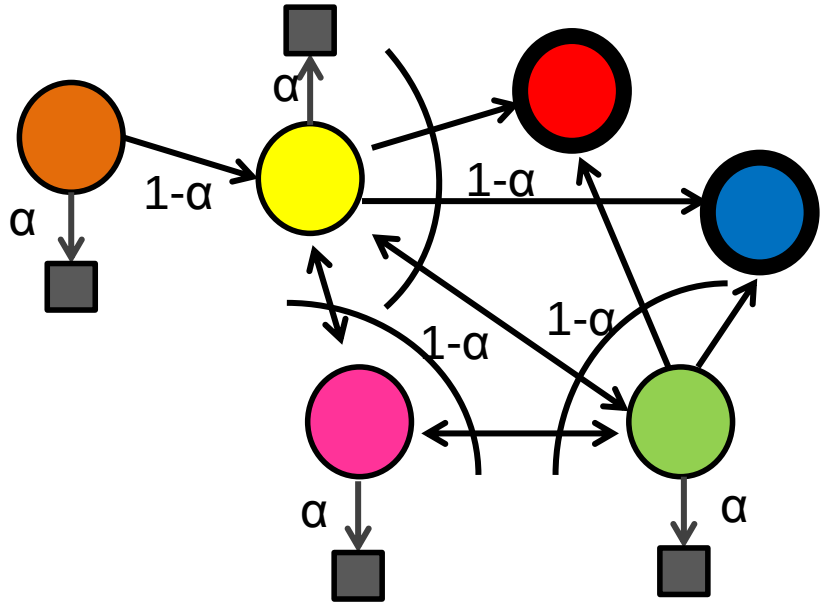
Penalizing long paths

- Add an universal absorbing node to which each node gets absorbed with probability α.

With probability α the random walk dies

With probability (1-α) the random walk continues as before

The longer the path from a node to an absorbing node the more likely the random walk dies along the way, the lower the absorbtion probability



e.g.

$$P(Red|Green) = (1-\alpha)\left(\frac{1}{5}P(Red|Yellow) + \frac{1}{5}P(Red|Pink) + \frac{1}{5}\right)$$

Random walk with restarts

- Adding a jump with probability α to a universal absorbing node seems similar to Pagerank

- Random walk with restart:
  - Start a random walk from node u
  - At every step with probability α, jump back to u
  - The probability of being at node v after large number of steps defines again a similarity between nodes u,v

- The Random Walk With Restarts (RWS) and Absorbing Random Walk (ARW) are similar but not the same
  - RWS computes the probability of paths from the starting node u to a node v, while AWR the probability of paths from a node v, to the absorbing node u.
  - RWS defines a distribution over all nodes, while AWR defines a probability for each node
  - An absorbing node blocks the random walk, while restarts simply bias towards starting nodes
    - Makes a difference when having multiple (and possibly competing) absorbing nodes

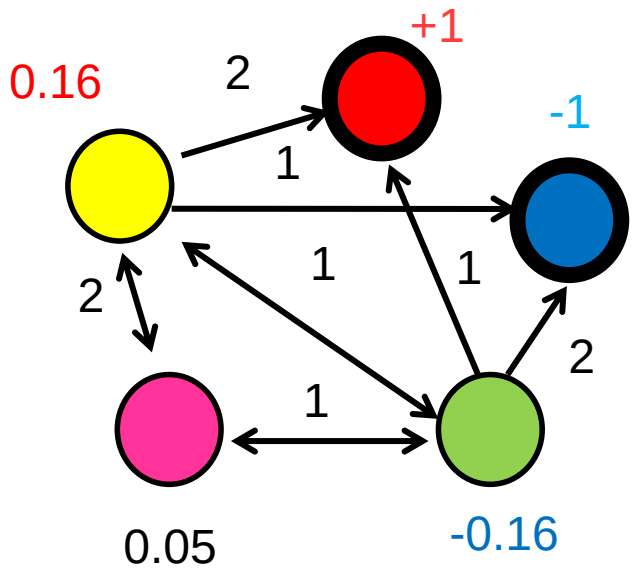
Propagating values

- Assume that Red has a positive value and Blue a negative value
  - Positive/Negative class, Positive/Negative opinion
- We can compute a value for all the other nodes by repeatedly averaging the values of the neighbors
  - The value of node u is the expected value at the point of absorption for a random walk that starts from u

$$V(Pink) = \frac{2}{3}V(Yellow) + \frac{1}{3}V(Green)$$

$$V(Green) = \frac{1}{5}V(Yellow) + \frac{1}{5}V(Pink) + \frac{1}{5} - \frac{2}{5}$$

$$V(Yellow) = \frac{1}{6}V(Green) + \frac{1}{3}V(Pink) + \frac{1}{3} - \frac{1}{6}$$



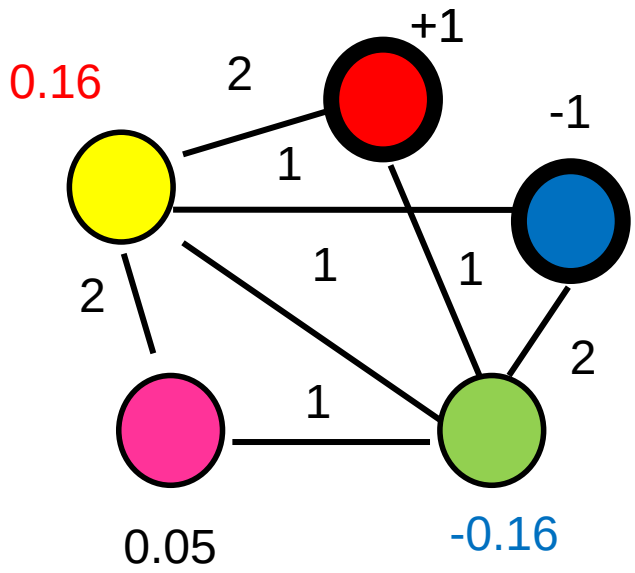
Electrical networks and random walks

- Our graph corresponds to an electrical network
- There is a positive voltage of +1 at the Red node, and a negative voltage -1 at the Blue node
- There are resistances on the edges inversely proportional to the weights (or conductance proportional to the weights)
- The computed values are the voltages at the nodes

$$V(Pink) = \frac{2}{3}V(Yellow) + \frac{1}{3}V(Green)$$

$$V(Green) = \frac{1}{5}V(Yellow) + \frac{1}{5}V(Pink) + \frac{1}{5} - \frac{2}{5}$$

$$V(Yellow) = \frac{1}{6}V(Green) + \frac{1}{3}V(Pink) + \frac{1}{3} - \frac{1}{6}$$



Opinion formation

- The value propagation can be used as a model of opinion formation.
- Model:
  - Opinions are values in [-1,1]
  - Every user u has an internal opinion s_u, and expressed opinion z_u.
  - The expressed opinion minimizes the personal cost of user u:

$$c(z_u) = (s_u - z_u)^2 + \sum_{v:v \text{ is a friend of } u} w_u (z_u - z_v)^2$$

    - Minimize deviation from your beliefs and conflicts with the society

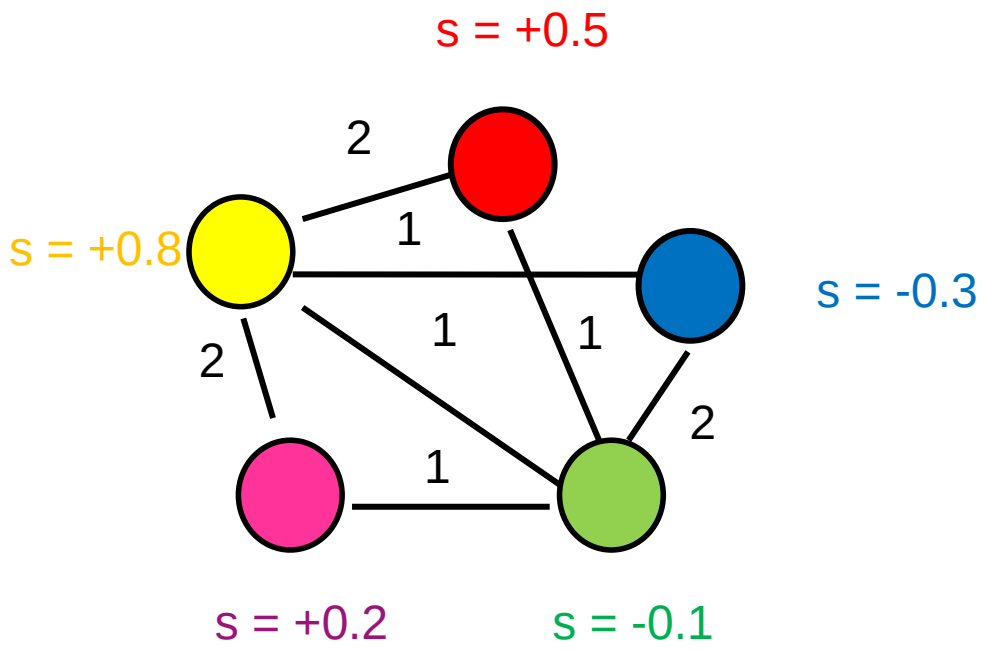
- If every user tries independently (selfishly) to minimize their personal cost then the best thing to do is to set z_u to the average of all opinions:

$$z_u = \frac{s_u + \sum_{v:v \text{ is a friend of } u} w_u z_u}{1 + \sum_{v:v \text{ is a friend of } u} w_u}$$

- This is the same as the value propagation we described before!

Example

- Social network with internal opinions



Example

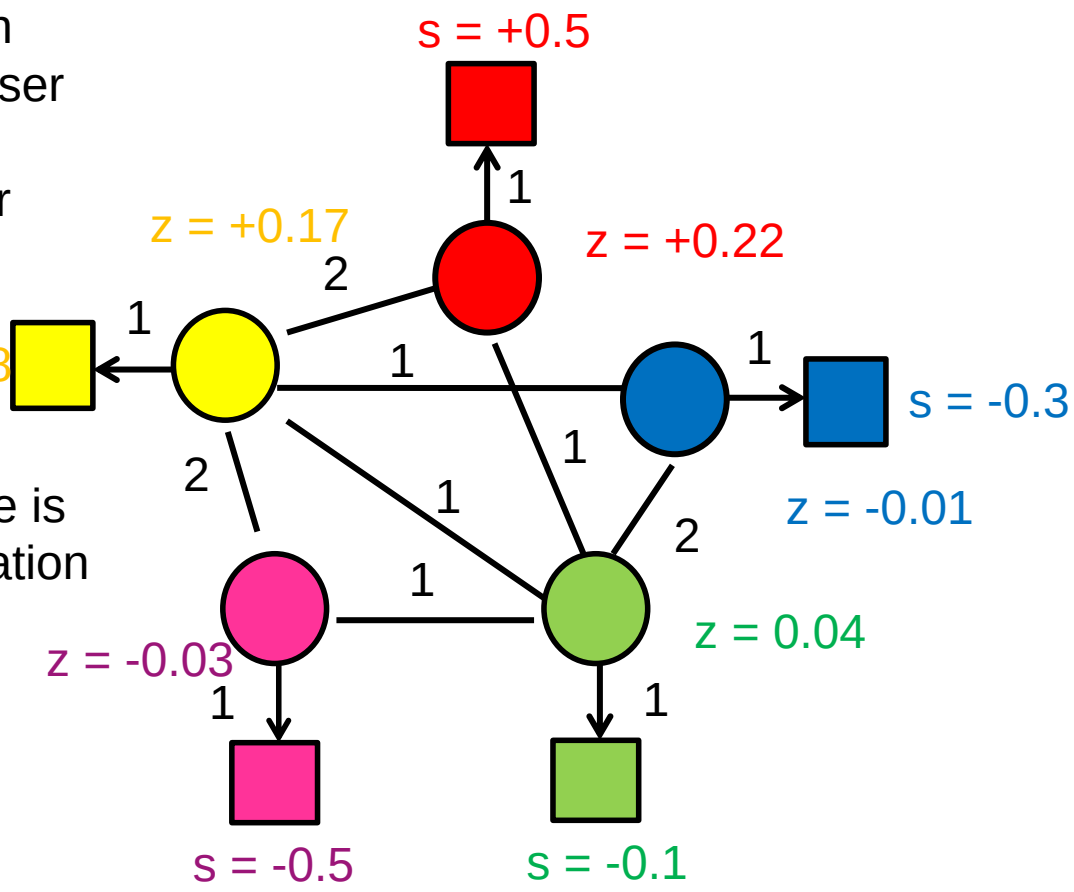
One absorbing node per user with value the internal opinion of the user

One non-absorbing node per user that links to the corresponding absorbing node

The external opinion for each node is computed using the value propagation we described before

- Repeated averaging

Intuitive model: my opinion is a combination of what I believe and what my social network believes.



Hitting time

- A related quantity: Hitting time H(u,v)
  - The expected number of steps for a random walk starting from node u to end up in v for the first time
    - Make node v absorbing and compute the expected number of steps to reach v
    - Assumes that the graph is strongly connected, and there are no other absorbing nodes.
- Commute time H(u,v) + H(v,u): often used as a distance metric
  - Proportional to the total resistance between nodes u, and v

Transductive learning

- If we have a graph of relationships and some labels on some nodes we can propagate them to the remaining nodes
  - Make the labeled nodes to be absorbing and compute the probability for the rest of the graph
  - E.g., a social network where some people are tagged as spammers
  - E.g., the movie-actor graph where some movies are tagged as action or comedy.
- This is a form of semi-supervised learning
  - We make use of the unlabeled data, and the relationships
- It is also called transductive learning because it does not produce a model, but just labels the unlabeled data that is at hand.
  - Contrast to inductive learning that learns a model and can label any new example

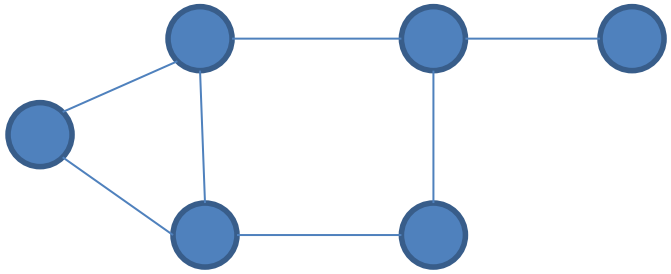
Implementation details

- Implementation is in many ways similar to the PageRank implementation
  - For an edge (u, v)instead of updating the value of v we update the value of u.
    - The value of a node is the average of its neighbors
  - We need to check for the case that a node u is absorbing, in which case the value of the node is not updated.
  - Repeat the updates until the change in values is very small.

COVERAGE

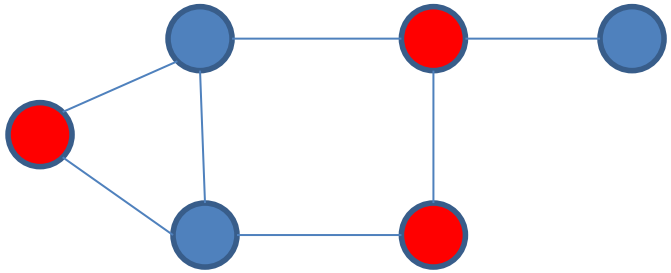
Example

- Promotion campaign on a social network
  - We have a social network as a graph.
  - People are more likely to buy a product if they have a friend who has the product.
  - We want to offer the product for free to some people such that every person in the graph is covered: they have a friend who has the product.
  - We want the number of free products to be as small as possible



Example

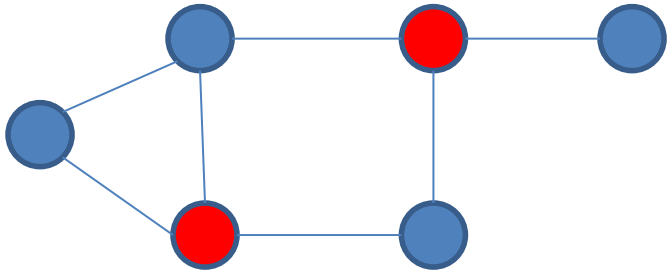
- Promotion campaign on a social network
  - We have a social network as a graph.
  - People are more likely to buy a product if they have a friend who has the product.
  - We want to offer the product for free to some people such that every person in the graph is covered: they have a friend who has the product.
  - We want the number of free products to be as small as possible



One possible selection

Example

- Promotion campaign on a social network
  - We have a social network as a graph.
  - People are more likely to buy a product if they have a friend who has the product.
  - We want to offer the product for free to some people such that every person in the graph is covered: they have a friend who has the product.
  - We want the number of free products to be as small as possible



A better selection

Dominating set

- Our problem is an instance of the dominating set problem

- Dominating Set: Given a graph $G = (V, E)$, a set of vertices $D \subseteq V$ is a dominating set if for each node u in V, either u is in D, or u has a neighbor in D.

- The Dominating Set Problem: Given a graph $G = (V, E)$ find a dominating set of minimum size.

Set Cover

- The dominating set problem is a special case of the Set Cover problem

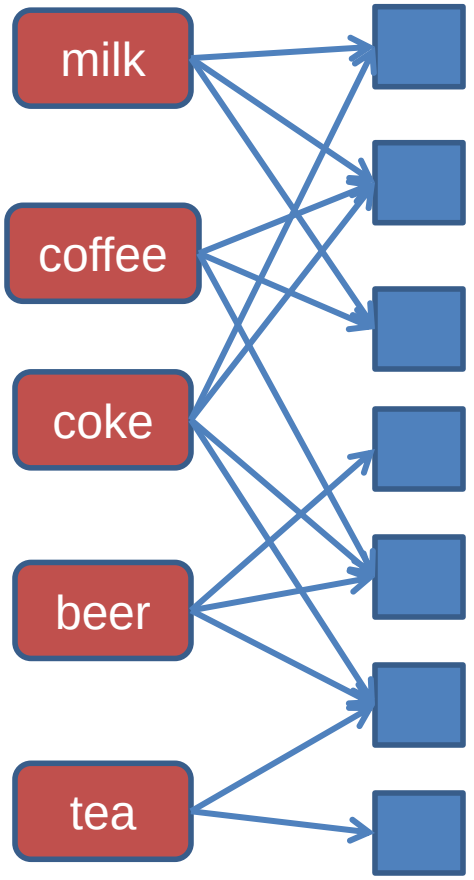
- The Set Cover problem:
  - We have a universe of elements $U = \{x_1, \ldots, x_N\}$
  - We have a collection of subsets of U, $\boldsymbol{S} = \{S_1, \ldots, S_n\}$, such that $\bigcup_i S_i = U$
  - We want to find the smallest sub-collection $\boldsymbol{C} \subseteq \boldsymbol{S}$ of $\boldsymbol{S}$, such that $\bigcup_{S_i \in \boldsymbol{C}} S_i = U$
    - The sets in $\boldsymbol{C}$ cover the elements of U

Applications

- Suppose that we want to create a catalog (with coupons) to give to customers of a store:
  - We want for every customer, the catalog to contain a product bought by the customer (this is a small store)
- How can we model this as a set cover problem?

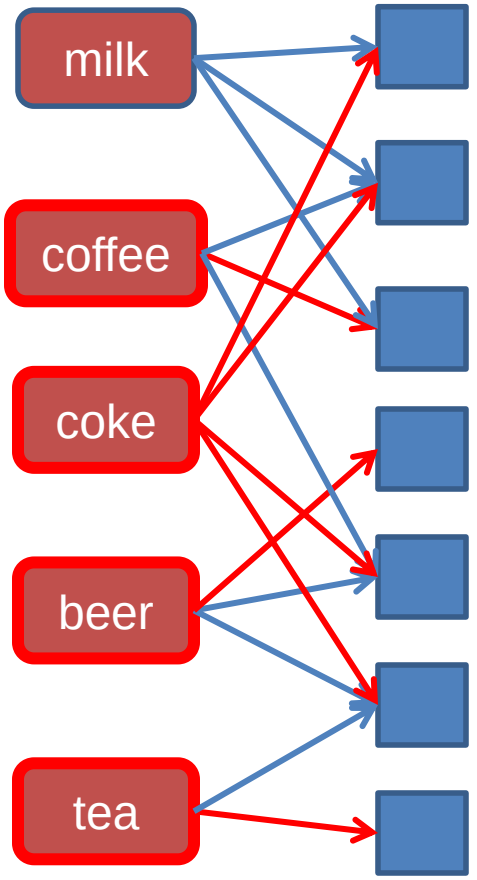
Applications

- The universe U of elements is the set of customers of a store.
- Each set corresponds to a product p sold in the store:
  $S_p = \{customers\ that\ bought\ p\}$
- Set cover: Find the minimum number of products (sets) that cover all the customers (elements of the universe)



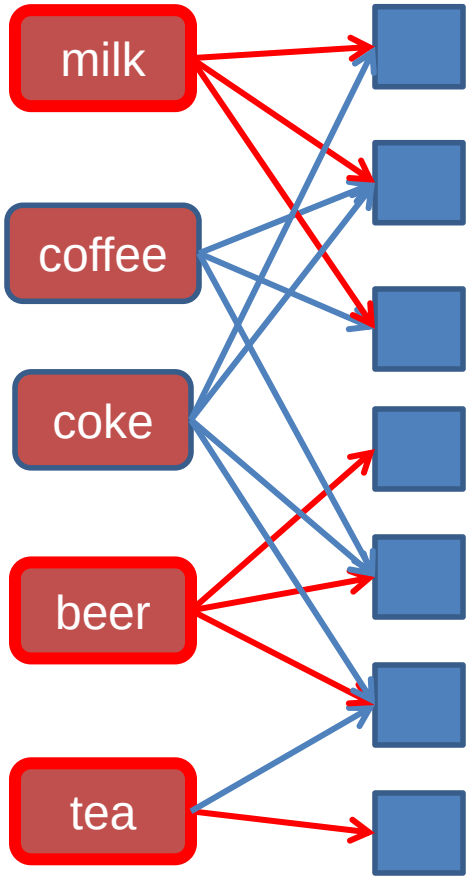
Applications

- The universe U of elements is the set of customers of a store.
- Each set corresponds to a product p sold in the store:
  $S_p = \{customers\ that\ bought\ p\}$
- Set cover: Find the minimum number of products (sets) that cover all the customers (elements of the universe)



Applications

- The universe U of elements is the set of customers of a store.
- Each set corresponds to a product p sold in the store:
  $S_p = \{customers\ that\ bought\ p\}$
- Set cover: Find the minimum number of products (sets) that cover all the customers (elements of the universe)



Applications

- Dominating Set (or Promotion Campaign) as Set Cover:
  - The universe U is the set of nodes V
  - Each node u defines a set S_u consisting of the node u and all of its neighbors
  - We want the minimum number of sets S_u (nodes) that cover all the nodes in the graph.
- Many more…

Best selection variant

- Suppose that we have a budget K of how big our set cover can be
  - We only have K products to give out for free.
  - We want to cover as many customers as possible.

- Maximum-Coverage Problem: Given a universe of elements U, a collection $\boldsymbol{S}$ of subsets of U, and a budget K, find a sub-collection $\boldsymbol{C} \subseteq \boldsymbol{S}$ of size at most K, such that the number of covered elements $\cup_{S_i \in \boldsymbol{C}} S_i$ is maximized.

Complexity

- Both the Set Cover and the Maximum Coverage problems are NP-complete
  - What does this mean?
  - Why do we care?

- There is no algorithm that can guarantee to find the best solution in polynomial time
  - Can we find an algorithm that can guarantee to find a solution that is close to the optimal?
  - Approximation Algorithms.

Approximation Algorithms

- For an (combinatorial) optimization problem, where:
  - X is an instance of the problem,
  - OPT(X) is the value of the optimal solution for X,
  - ALG(X) is the value of the solution of an algorithm ALG for X

ALG is a good approximation algorithm if the ratio of OPT(X) and ALG(X) is bounded for all input instances X

- Minimum set cover: X = G is the input graph, OPT(G) is the size of minimum set cover, ALG(G) is the size of the set cover found by an algorithm ALG.

- Maximum coverage: X = (G,k) is the input instance, OPT(G,k) is the coverage of the optimal algorithm, ALG(G,k) is the coverage of the set found by an algorithm ALG.

Approximation Algorithms

- For a minimization problem, the algorithm ALG is an α-approximation algorithm, for $\alpha > 1$, if for all input instances X,

$$ALG(X) \leq \alpha OPT(X)$$

- α is the approximation ratio of the algorithm – we want α to be as close to 1 as possible
  - Best case: $\alpha = 1 + \epsilon$ and $\epsilon \to 0$, as $n \to \infty$ (e.g., $\epsilon = \frac{1}{n}$)
  - Good case: $\alpha = O(1)$ is a constant
  - OK case: $\alpha = O(\log n)$
  - Bad case $\alpha = O(n^{\epsilon})$

Approximation Algorithms

- For a maximization problem, the algorithm ALG is an α-approximation algorithm, for $\alpha < 1$, if for all input instances X,

$$ALG(X) \geq \alpha OPT(X)$$

- α is the approximation ratio of the algorithm – we want α to be as close to 1 as possible
  - Best case: $\alpha = 1 - \epsilon$ and $\epsilon \to 0$, as $n \to \infty$(e.g., $\epsilon = \frac{1}{n}$)
  - Good case: $\alpha = O(1)$ is a constant
  - OK case: $\alpha = O(\frac{1}{\log n})$
  - Bad case $\alpha = O(n^{-\epsilon})$

A simple approximation ratio for set cover

- Any algorithm for set cover has approximation ratio $\alpha = |S_{max}|$, where S_{max} is the set in **S** with the largest cardinality

- Proof:
  - $OPT(X) \geq N/|S_{max}| \Rightarrow N \leq |S_{max}|OPT(X)$
  - $ALG(X) \leq N \leq |S_{max}|OPT(X)$

- This is true for any algorithm.
- Not a good bound since it can be that $|S_{max}|=O(N)$

An algorithm for Set Cover

- What is the most natural algorithm for Set Cover?

- Greedy: each time add to the collection C the set S_i from S that covers the most of the remaining elements.

The GREEDY algorithm

GREEDY(U,S)

X= U

C = {}

while X is not empty do

For all $S_i \in \boldsymbol{S}$ let $\text{gain}(S_i) = |S_i \cap X|$

Let S_* be such that $gain(S_*)$ is maximum

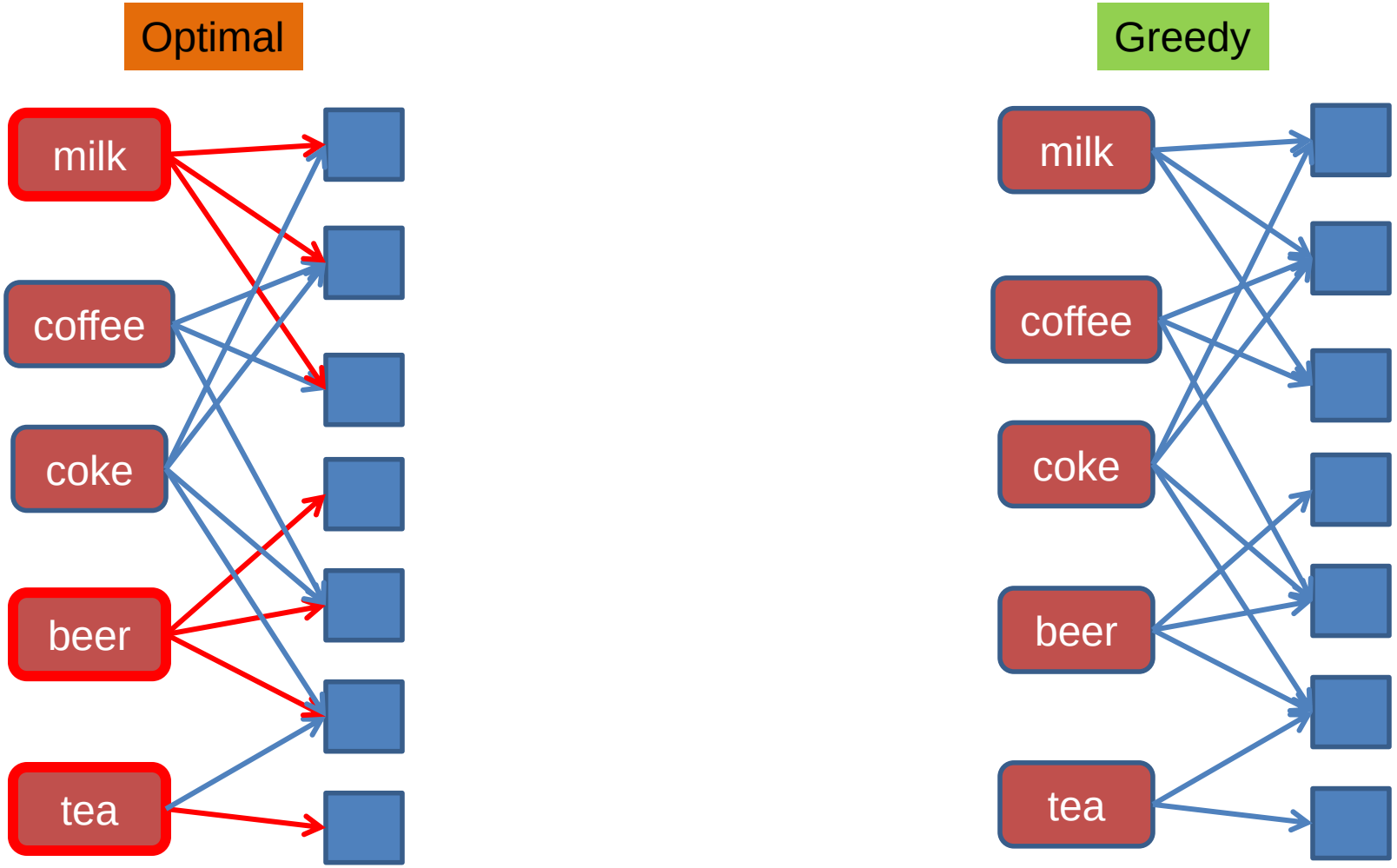
C = C U $\{S_*\}$

X = X\ S_*

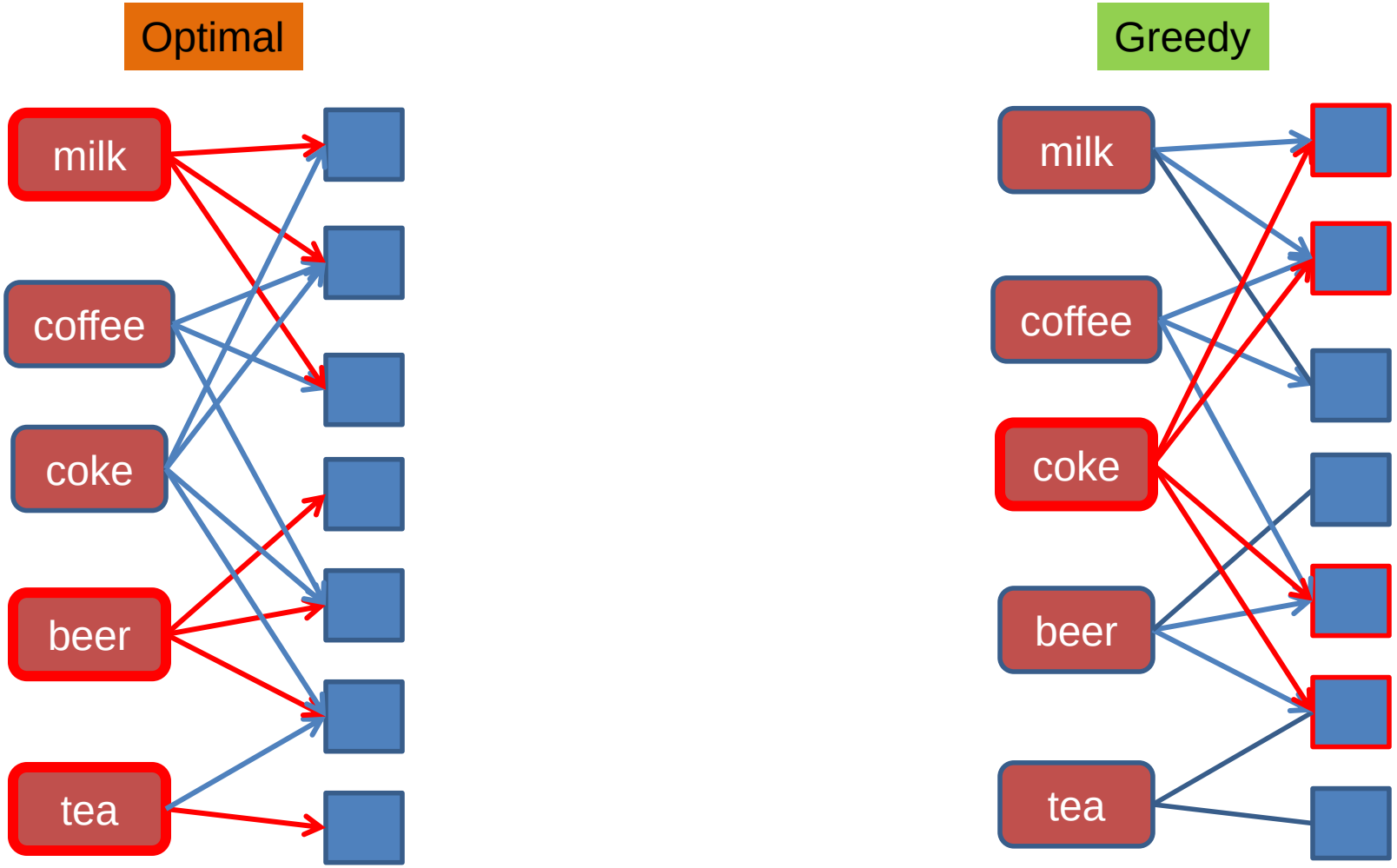
S = ***S***\ S_*

The number of elements covered by S_i not already covered by C.

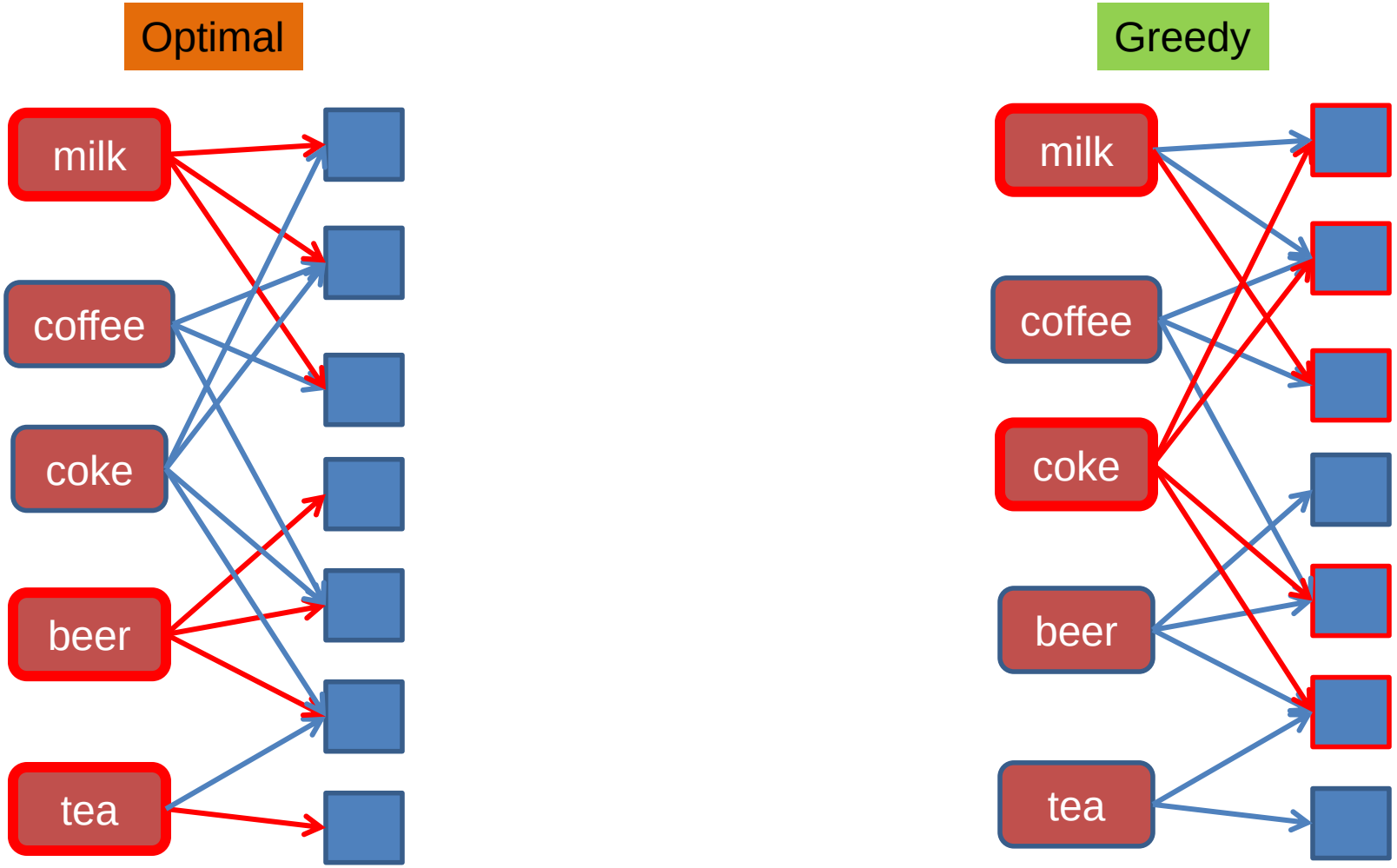
Greedy is not always optimal



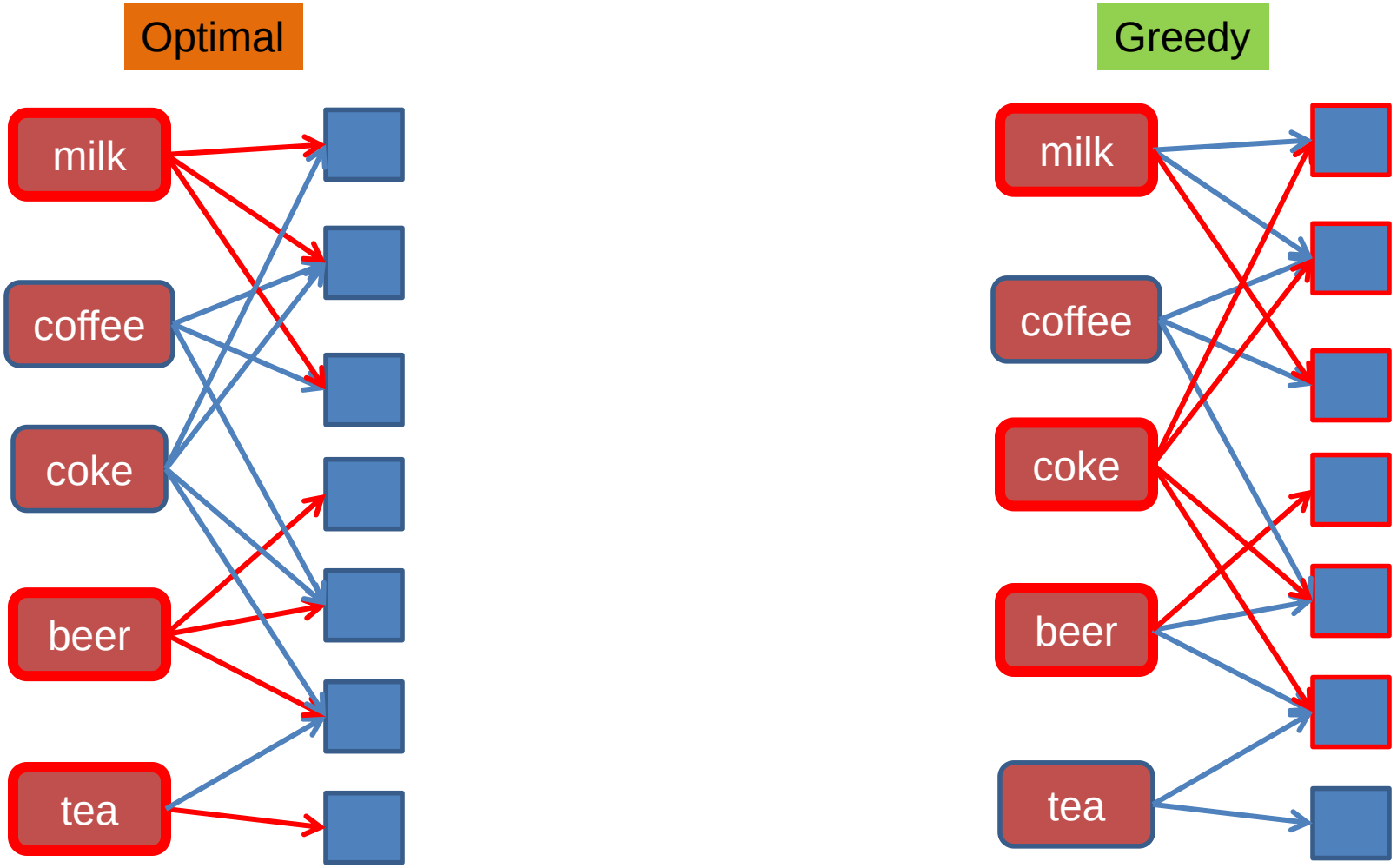
Greedy is not always optimal



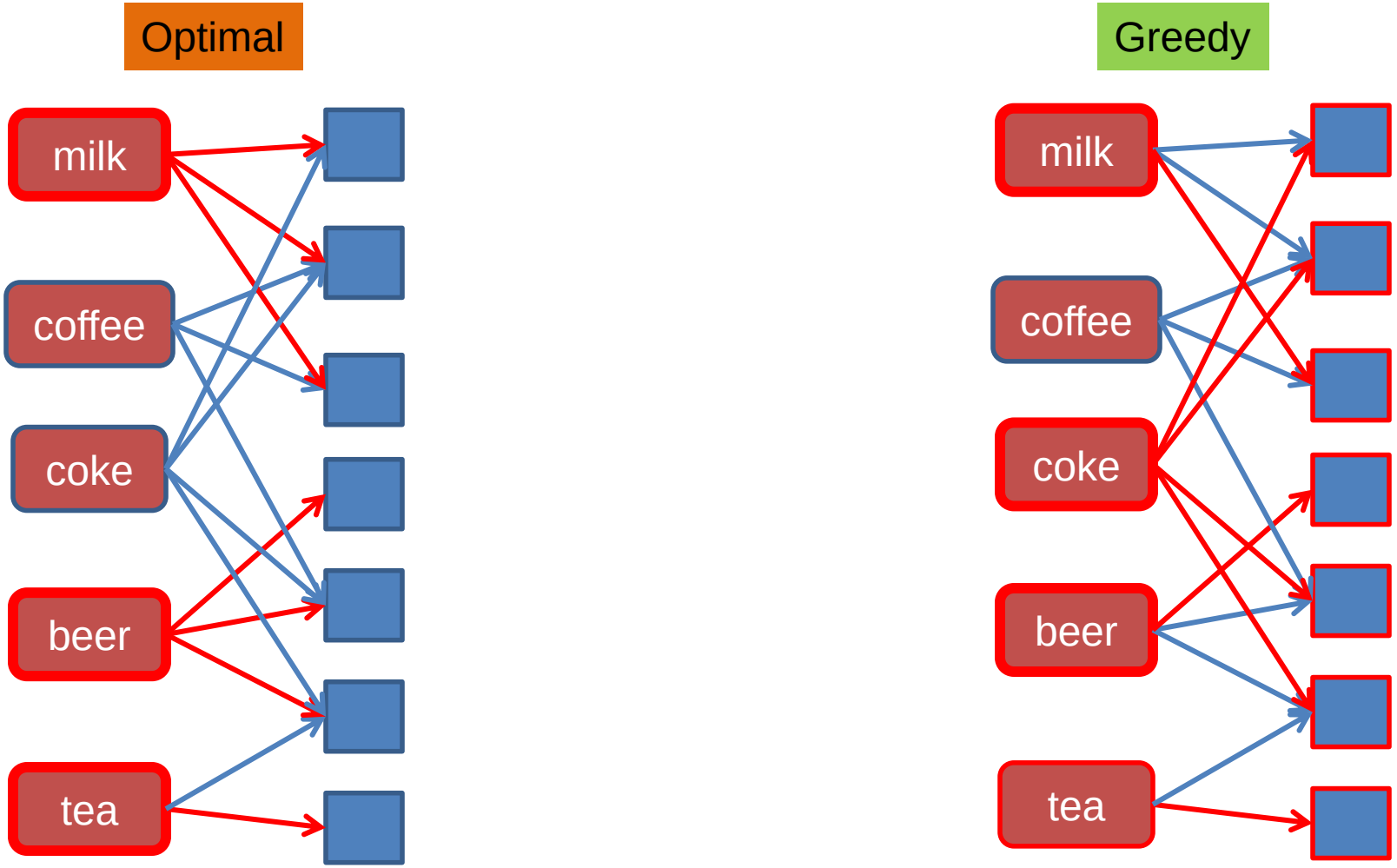
Greedy is not always optimal



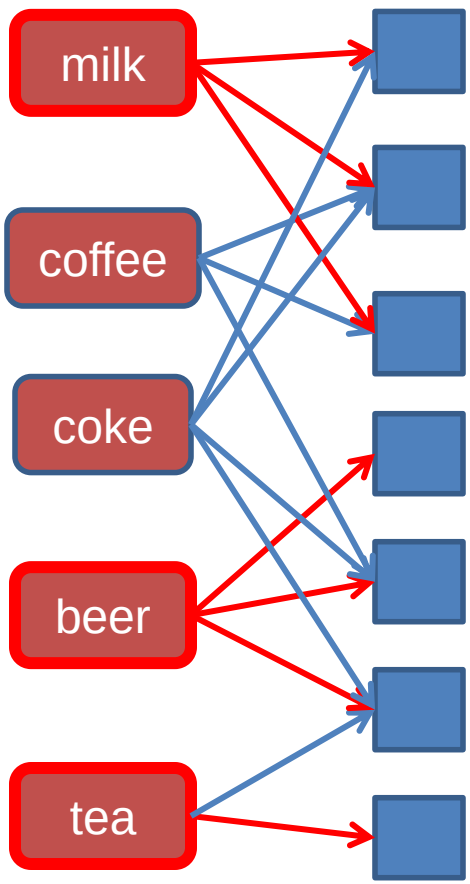
Greedy is not always optimal



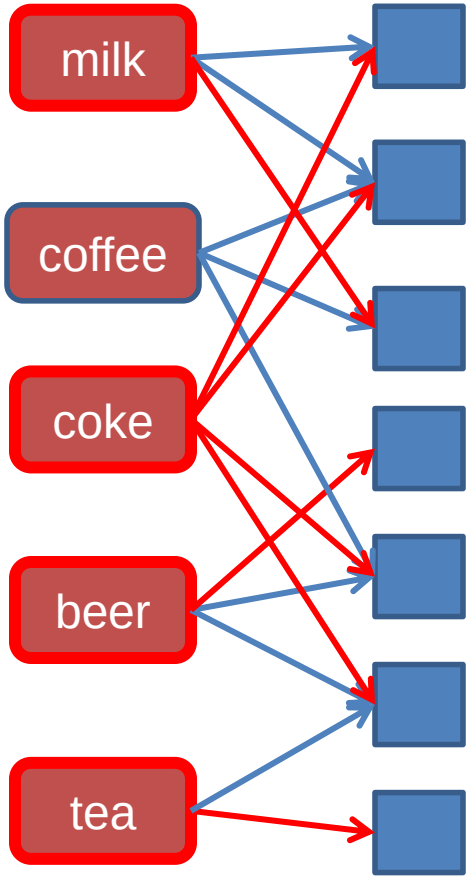
Greedy is not always optimal



Greedy is not always optimal



- Selecting Coke first forces us to pick coffee as well
- Milk and Beer cover more customers together



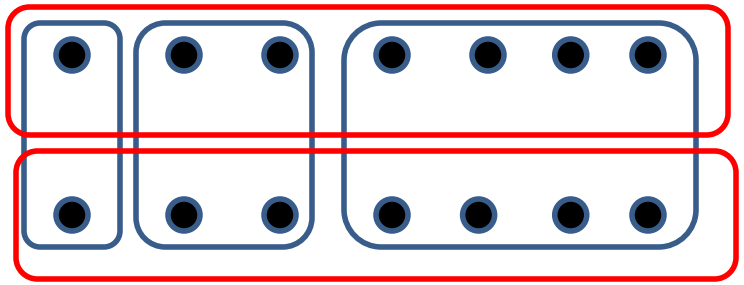
Approximation ratio of GREEDY

- Good news: GREEDY has approximation ratio:

$$\alpha = H(|S_{\max}|) = 1 + \ln|S_{\max}|, \qquad H(n) = \sum_{k=1}^{n} \frac{1}{k}$$

$$GREEDY(X) \leq (1 + \ln|S_{\max}|)OPT(X), \text{ for all X}$$

- The approximation ratio is tight up to a constant
  - Tight means that we can find a counter example with this ratio



OPT(X) = 2
GREEDY(X) = logN
α=½logN

Maximum Coverage

- What is a reasonable algorithm?

GREEDY(U,S,K)

X = U

C = {}

while |***C***| < K

For all $S_i \in \boldsymbol{S}$ let $\text{gain}(S_i) = |S_i \cap X|$

Let S_* be such that $gain(S_*)$ is maximum

C = C U $\{S_*\}$

X = X\ S_*

S= ***S***\ S_*

The number of elements covered by S_i not already covered by C.

Approximation Ratio for Max-K Coverage

- Better news! The GREEDY algorithm has approximation ratio $\alpha = 1 - \frac{1}{e}$

$$GREEDY(X) \geq \left(1 - \frac{1}{e}\right) OPT(X), \text{ for all X}$$

- The coverage of the Greedy solution is at least 63% that of the optimal

Proof of approximation ratio

- For a collection C, let $F(C) = \left|\bigcup_{S_i \in \boldsymbol{C}} S_i\right|$ be the number of elements that are covered.
- The function F has two properties:

- F is monotone:

$$F(A) \leq F(B) \; if \; A \subseteq B$$

- F is submodular:

$$F(A \cup \{S\}) - F(A) \geq F(B \cup \{S\}) - F(B) \;\; if \; A \subseteq B$$

- The addition of set S to a set of nodes has greater effect (more new covered items) for a smaller set.
  - The diminishing returns property

Optimizing submodular functions

- Theorem:

If we want to optimizes a monotone and submodular function F under cardinality constraints (size of set at most K),

then

A greedy algorithm that each time adds to the solution C, the set S that maximizes the gain $F(C \cup \{S\}) - F(C)$ has approximation ratio $\alpha = \left(1 - \frac{1}{e}\right)$

True for any monotone and submodular set function!

Other variants of Set Cover

- Hitting Set: select a set of elements so that you hit all the sets (the same as the set cover, reversing the roles)
- Vertex Cover: Select a subset of vertices such that you cover all edges (an endpoint of each edge is in the set)
  - There is a 2-approximation algorithm
- Edge Cover: Select a set of edges that cover all vertices (there is one edge that has endpoint the vertex)
  - There is a polynomial algorithm

OVERVIEW

Class Overview

- In this class you saw a set of tools for analyzing data
  - Frequent Itemsets, Association Rules
  - Sketching
  - Recommendation Systems
  - Clustering
  - Minimum Description Length
  - Singular Value Decomposition
  - Classification
  - Link Analysis Ranking
  - Random Walks
  - Coverage
- All these are useful when trying to make sense of the data. A lot more tools exist.
- I hope that you found this interesting, useful and fun.