DATA MINING LECTURE 6

Sketching,

Min-Hashing,

Locality Sensitive Hashing

MIN-HASHING AND LOCALITY SENSITIVE HASHING

Thanks to:

Rajaraman, Ullman, Lekovec “Mining Massive Datasets”

Evimaria Terzi, slides for Data Mining Course.

Motivating problem

- Find duplicate and near-duplicate documents from a web crawl.

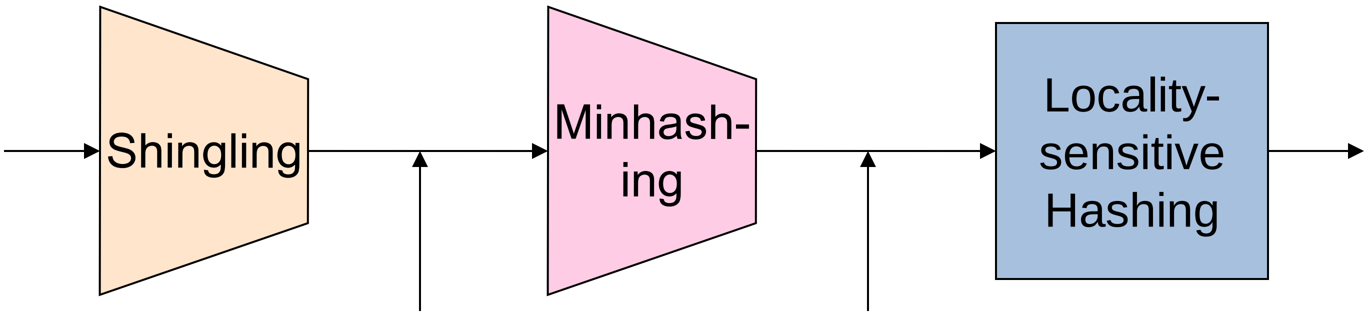
- If we wanted exact duplicates we could do this by hashing
  - We will see how to adapt this technique for near duplicate documents

Main issues

- What is the right representation of the document when we check for similarity?
  - E.g., representing a document as a set of characters will not do (why?)
- When we have billions of documents, keeping the full text in memory is not an option.
  - We need to find a shorter representation
- How do we do pairwise comparisons of billions of documents?
  - If exact match was the issue it would be ok, can we replicate this idea?

The Big Picture

Docu-
ment



Candidate pairs : those pairs of signatures that we need to test for similarity.

The set of strings of length *k* that appear in the doc- ument

Signatures : short integer vectors that represent the sets, and reflect their similarity

Shingling

- Shingle: a sequence of k contiguous characters

Set of Shingles	Hash function (Rabin's fingerprints)	Set of 64-bit integers
a rose is	→	1111
rose is a	→	2222
rose is a	→	3333
ose is a r	→	4444
se is a ro	→	5555
e is a ros	→	6666
is a rose	→	7777
is a rose	→	8888
s a rose i	→	9999
a rose is	→	0000

Basic Data Model: Sets

- Document: A document is represented as a set of shingles (more accurately, hashes of shingles)

- Document similarity: Jaccard similarity of the sets of shingles.
  - Common shingles over the union of shingles
  - *Sim* $(C_1, C_2) = |C_1 \cap C_2| / |C_1 \cup C_2|$.

- Applicable to any kind of sets.
  - E.g., similar customers or items.

Signatures

- Key idea: "hash" each set S to a small signature Sig (S), such that:

  1. Sig (S) is small enough that we can fit all signatures in main memory.
  2. Sim (S_1, S_2) is (almost) the same as the "similarity" of Sig (S_1) and Sig (S_2). (signature preserves similarity).

- Warning: This method can produce false negatives, and false positives (if an additional check is not made).
  - False negatives: Similar items deemed as non-similar
  - False positives: Non-similar items deemed as similar

From Sets to Boolean Matrices

- Represent the data as a boolean matrix M
  - Rows = the universe of all possible set elements
    - In our case, shingle fingerprints take values in $[0\ldots2^{64}-1]$
  - Columns = the sets
    - In our case, documents, sets of shingle fingerprints
  - $M(r,S) = 1$ in row r and column S if and only if r is a member of S.

- Typical matrix is sparse.
  - We do not really materialize the matrix

Example

- Universe: **U = {A,B,C,D,E,F,G}**

- X = {A,B,F,G}
- Y = {A,E,F,G}

- Sim(X,Y) = $\frac{3}{5}$

	X	Y
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

Example

- Universise: **U = {A,B,C,D,E,F,G}**

- X = {A,B,F,G}
- Y = {A,E,F,G}

- Sim(X,Y) = $\frac{3}{5}$

At least one of the columns has value 1

	X	Y
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

Example

- Universise: **U = {A,B,C,D,E,F,G}**

- X = {A,B,F,G}
- Y = {A,E,F,G}

- Sim(X,Y) = $\frac{3}{5}$

Both columns have value 1

	X	Y
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

Minhashing

- Pick a random permutation of the rows (the universe U).
- Define "hash" function for set S
  - h(S) = the index of the first row (in the permuted order) in which column S has 1.

  or equivalently

  - h(S) = the index of the first element of S in the permuted order.
- Use k (e.g., k = 100) independent random permutations to create a signature.

Example of minhash signatures

- Input matrix

	S_1	S_2	S_3	S_4
A	1	0	1	0
B	1	0	0	1
C	0	1	0	1
D	0	1	0	1
E	0	1	1	1
F	1	0	1	0
G	1	0	1	0

Random Permutation

A
C
G
F
B
E
D

		S_1	S_2	S_3	S_4
1	**A**	1	0	1	0
2	**C**	0	1	0	1
3	**G**	1	0	1	0
4	**F**	1	0	1	0
5	**B**	1	0	0	1
6	**E**	0	1	1	1
7	**D**	0	1	0	1
		1	**2**	**1**	**2**

Example of minhash signatures

- Input matrix

	S_1	S_2	S_3	S_4
A	1	0	1	0
B	1	0	0	1
C	0	1	0	1
D	0	1	0	1
E	0	1	1	1
F	1	0	1	0
G	1	0	1	0

Random Permutation

D
B
A
C
F
G
E

		S_1	S_2	S_3	S_4
1	**D**	0	1	0	1
2	**B**	1	0	0	1
3	**A**	1	0	1	0
4	**C**	0	1	0	1
5	**F**	1	0	1	0
6	**G**	1	0	1	0
7	**E**	0	1	1	1
		2	**1**	**3**	**1**

Example of minhash signatures

- Input matrix

	S_1	S_2	S_3	S_4
A	1	0	1	0
B	1	0	0	1
C	0	1	0	1
D	0	1	0	1
E	0	1	1	1
F	1	0	1	0
G	1	0	1	0

Random Permutation

C
D
G
F
A
B
E

		S_1	S_2	S_3	S_4
1	**C**	0	1	0	1
2	**D**	0	1	0	1
3	**G**	1	0	1	0
4	**F**	1	0	1	0
5	**A**	1	0	1	0
6	**B**	1	0	0	1
7	**E**	0	1	1	1
		3	**1**	**3**	**1**

Example of minhash signatures

- Input matrix

	S_1	S_2	S_3	S_4
A	1	0	1	0
B	1	0	0	1
C	0	1	0	1
D	0	1	0	1
E	0	1	0	1
F	1	0	1	0
G	1	0	1	0

$\approx$

Signature matrix

	S_1	S_2	S_3	S_4
h_1	1	2	1	2
h_2	2	1	3	1
h_3	3	1	3	1

- Sig(S) = vector of hash values
  - e.g., $Sig(S_2) = [2,1,1]$
- Sig(S,i) = value of the i-th hash function for set S
  - E.g., $Sig(S_2,3) = 1$

Hash function Property

$$\Pr(h(S_1) = h(S_2)) = Sim(S_1,S_2)$$

- where the probability is over all choices of permutations.

- Why?
  - The first row where one of the two sets has value 1 belongs to the union.
    - Recall that union contains rows with at least one 1.
  - We have equality if both sets have value 1, and this row belongs to the intersection

Example

- Universe: **U = {A,B,C,D,E,F,G}**
- X = {A,B,F,G}
- Y = {A,E,F,G}

Rows C,D could be anywhere they do not affect the probability

- Union = {A,B,E,F,G}
- Intersection = {A,F,G}

	X	Y
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

D
*
*
C
*
*
*

	X	Y
D	0	0
C	0	0

Example

- Universe: **U = {A,B,C,D,E,F,G}**
- X = {A,B,F,G}
- Y = {A,E,F,G}

The * rows belong to the union

- Union = {A,B,E,F,G}
- Intersection = {A,F,G}

	X	Y
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

D
*
*
C
*
*
*

	X	Y
D	0	0
C	0	0

Example

- Universe: **U = {A,B,C,D,E,F,G}**
- X = {A,B,F,G}
- Y = {A,E,F,G}

The question is what is the value of the **first *** element

- Union = {A,B,E,F,G}
- Intersection = {A,F,G}

	X	Y
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

D
*
*
C
*
*
*

	X	Y
D	0	0
C	0	0

Example

- Universe: **U = {A,B,C,D,E,F,G}**
- X = {A,B,F,G}
- Y = {A,E,F,G}

If it belongs to the intersection then h(X) = h(Y)

- Union = {A,B,E,F,G}
- Intersection = {A,F,G}

	X	Y
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

D
*
*
C
*
*
*

	X	Y
D	0	0
C	0	0

Example

- Universe: **U = {A,B,C,D,E,F,G}**
- X = {A,B,F,G}
- Y = {A,E,F,G}

Every element of the union is equally likely to be the * element

$$\Pr(h(X) = h(Y)) = \frac{|\{A,F,G\}|}{|\{A,B,E,F,G\}|} = \frac{3}{5} = \mathrm{Sim}(X,Y)$$

- Union = {A,B,E,F,G}
- Intersection = {A,F,G}

	X	**Y**
A	1	1
B	1	0
C	0	0
D	0	0
E	0	1
F	1	1
G	1	1

D
*
*
C
*
*
*

	X	**Y**
D	0	0
C	0	0

Similarity for Signatures

- The similarity of signatures is the fraction of the hash functions in which they agree.

	S_1	S_2	S_3	S_4
A	1	0	1	0
B	1	0	0	1
C	0	1	0	1
D	0	1	0	1
E	0	1	0	1
F	1	0	1	0
G	1	0	1	0

$\approx$

Signature matrix

S_1	S_2	S_3	S_4
1	2	1	2
2	1	3	1
3	1	3	1

	Actual	Sig
(S_1, S_2)	0	0
(S_1, S_3)	3/5	2/3
(S_1, S_4)	1/7	0
(S_2, S_3)	0	0
(S_2, S_4)	3/4	1
(S_3, S_4)	0	0

Zero similarity is preserved
High similarity is well approximated

- With multiple signatures we get a good approximation

Is it now feasible?

- Assume a billion rows
- Hard to pick a random permutation of 1…billion
- **Even representing a random permutation requires 1 billion entries!!!**
- How about accessing rows in permuted order? ☹

Being more practical

- Instead of permuting the rows we will apply a hash function that maps the rows to a new (possibly larger) space
  - The value of the hash function is the position of the row in the new order (permutation).
  - Each set is represented by the smallest hash value among the elements in the set

- The space of the hash functions should be such that if we select one at random each element (row) has equal probability to have the smallest value
  - Min-wise independent hash functions

Algorithm – One set, one hash function

Computing **Sig(S,i)** for a single column S and single hash function h_i

In practice only the rows (shingles) that appear in the data

for each row **r**

compute $\mathbf{h_i\ (r\ )}$

$\mathbf{h_i\ (r)}$ = index of row r in permutation

if column **S** that has **1** in row **r**

S contains row r

if $\mathbf{h_i\ (r\ )}$ is a smaller value than **Sig(S,i) then**

Sig(S,i) = $\mathbf{h_i\ (r)}$**;**

Find the row r with minimum index

Sig(S,i) will become the smallest value of $\mathbf{h_i(r)}$ among all rows (shingles) for which column **S** has value **1** (shingle belongs in S)*;* *i*.e., $\mathbf{h_i\ (r)}$ gives the min index for the **i-**th permutation

Algorithm – All sets, k hash functions

Pick $k=100$ hash functions $(h_1,\ldots,h_k)$

In practice this means selecting the hash function parameters

for each row r

for each hash function h_i

compute $h_i(r)$

Compute $h_i(r)$ only once for all sets

for each column S that has 1 in row r

if $h_i(r)$ is a smaller value than $Sig(S,i)$ **then**

$Sig(S,i) = h_i(r);$

Example

x	Row	S1	S2	h(x)	g(x)
0	A	1	0	1	3
1	B	0	1	2	0
2	C	1	1	3	2
3	D	1	0	4	4
4	E	0	1	0	1

$h(x) = x+1 \text{ mod } 5$
$g(x) = 2x+3 \text{ mod } 5$

h(Row)	Row	S1	S2
0	E	0	1
1	A	1	0
2	B	0	1
3	C	1	1
4	D	1	0

g(Row)	Row	S1	S2
0	B	0	1
1	E	0	1
2	C	1	0
3	A	1	1
4	D	1	0

	Sig1	Sig2
$h(0) = 1$	1	-
$g(0) = 3$	3	-
$h(1) = 2$	1	2
$g(1) = 0$	3	0
$h(2) = 3$	1	2
$g(2) = 2$	2	0
$h(3) = 4$	1	2
$g(3) = 4$	2	0
$h(4) = 0$	1	0
$g(4) = 1$	2	0

Implementation

- Often, data is given by column, not row.
  - E.g., columns = documents, rows = shingles.
- If so, sort matrix once so it is by row.
- And always compute $h_i(r)$ only once for each row.

Finding similar pairs

- Problem: Find all pairs of documents with similarity at least $t = 0.8$
- While the signatures of all columns may fit in main memory, comparing the signatures of all pairs of columns is quadratic in the number of columns.
- Example: 10^6 columns implies $5*10^{11}$ column-comparisons.
- At 1 microsecond/comparison: 6 days.

Locality-Sensitive Hashing

- What we want: a function f(X,Y) that tells whether or not X and Y is a candidate pair: a pair of elements whose similarity must be evaluated.

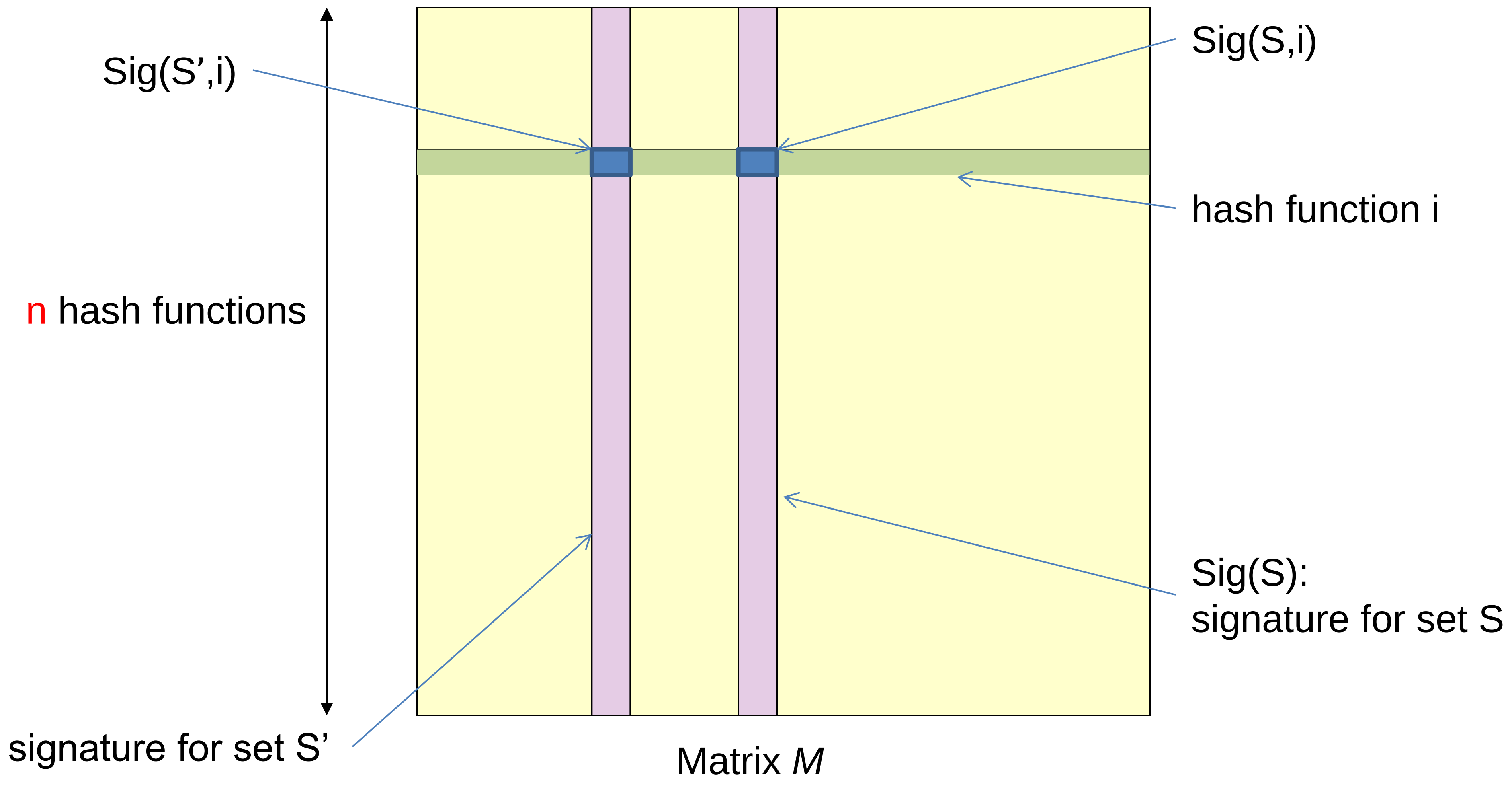
- A simple idea: X and Y are a candidate pair if they have the same min-hash signature.
  - Easy to test by hashing the signatures.
  - Similar sets are more likely to have the same signature.
  - Likely to produce many false negatives.
    - Requiring full match of signature is strict, some similar sets will be lost.

! Multiple levels of Hashing!

- Improvement: Compute multiple signatures; candidate pairs should have at least one common signature.
  - Reduce the probability for false negatives.

Signature matrix reminder

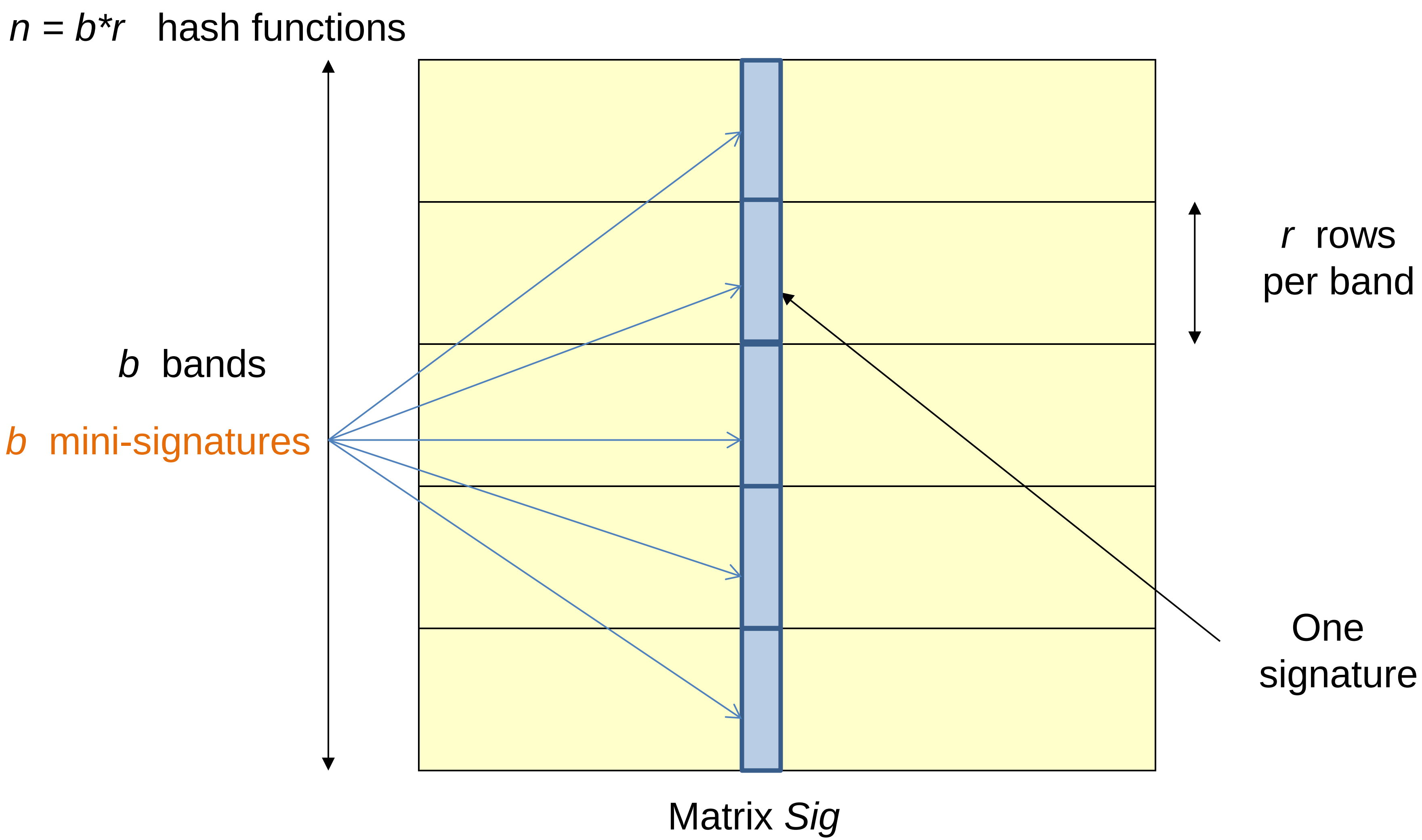
Prob(Sig(S,i) == Sig(S',i)) = sim(S,S')



Partition into Bands – (1)

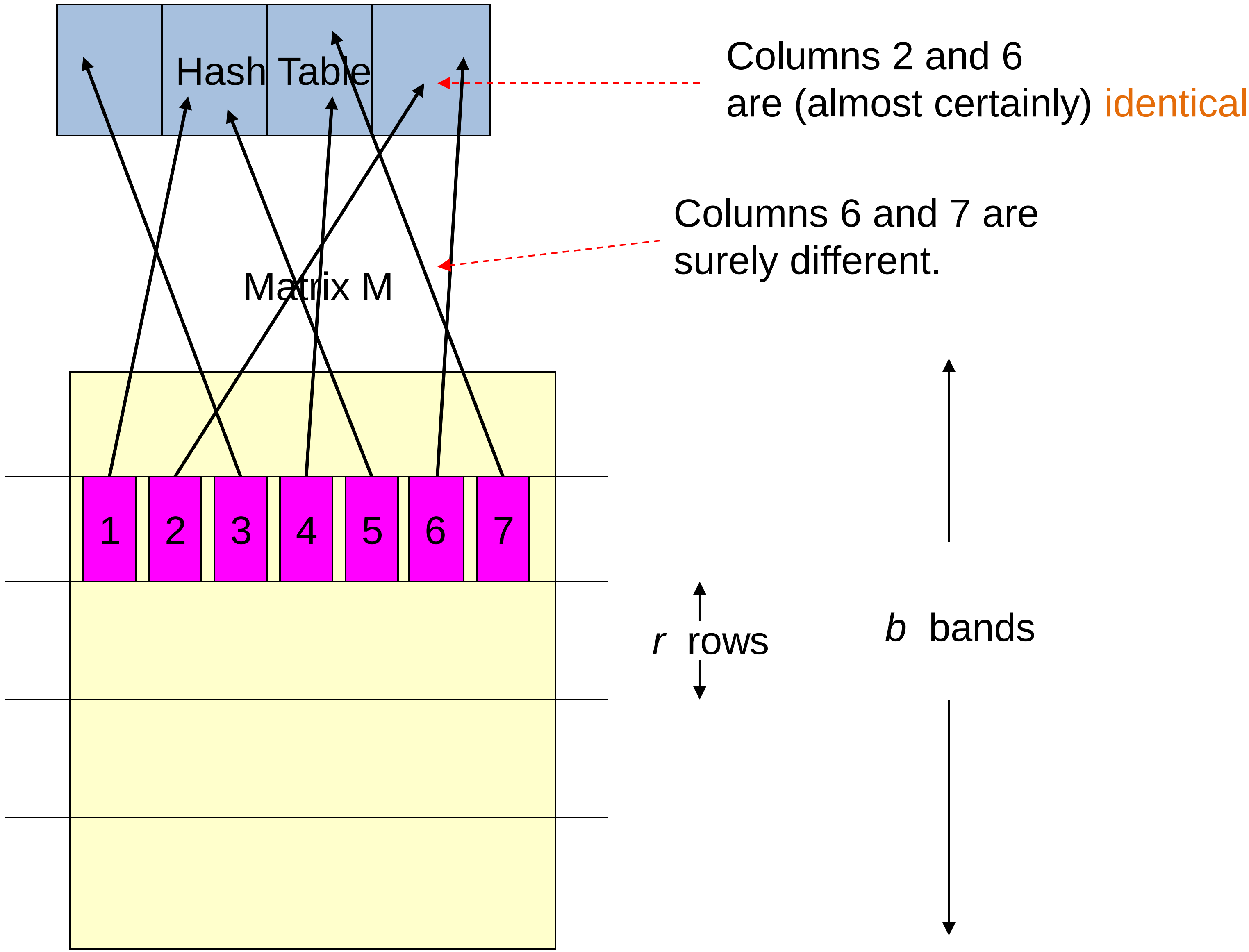
- Divide the signature matrix Sig into b bands of r rows.
  - Each band is a mini-signature with r hash functions.

Partitioning into bands



Partition into Bands – (2)

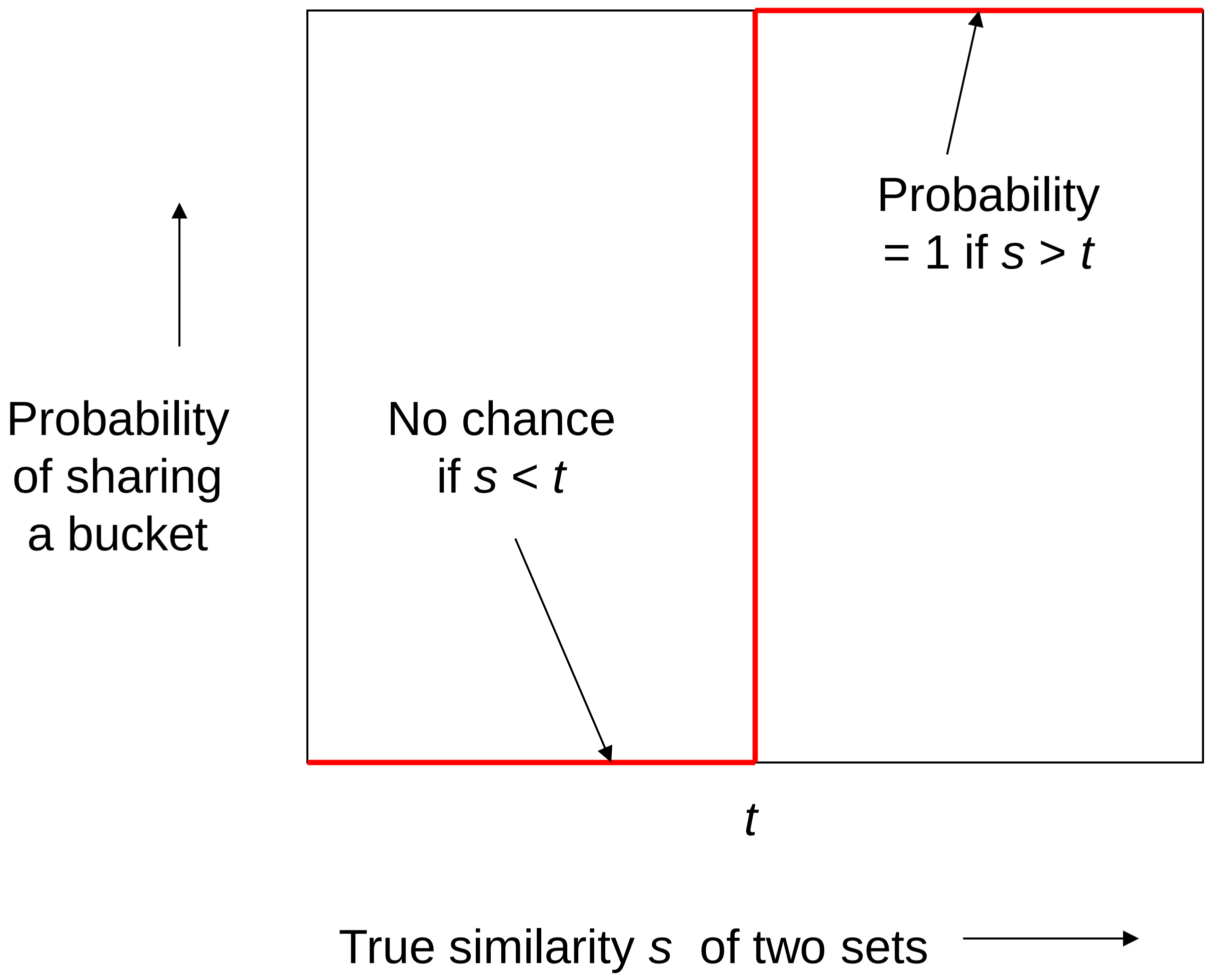
- Divide the signature matrix Sig into *b* bands of *r* rows.
  - Each band is a mini-signature with r hash functions.
- For each band, hash the mini-signature to a hash table with *k* buckets.
  - Make *k* as large as possible so that mini-signatures that hash to the same bucket are almost certainly identical.


Hash Table
Columns 2 and 6
are (almost certainly) identical.
Columns 6 and 7 are
surely different.
Matrix M
1
2
3
4
5
6
7
r rows
b bands

Partition into Bands – (3)

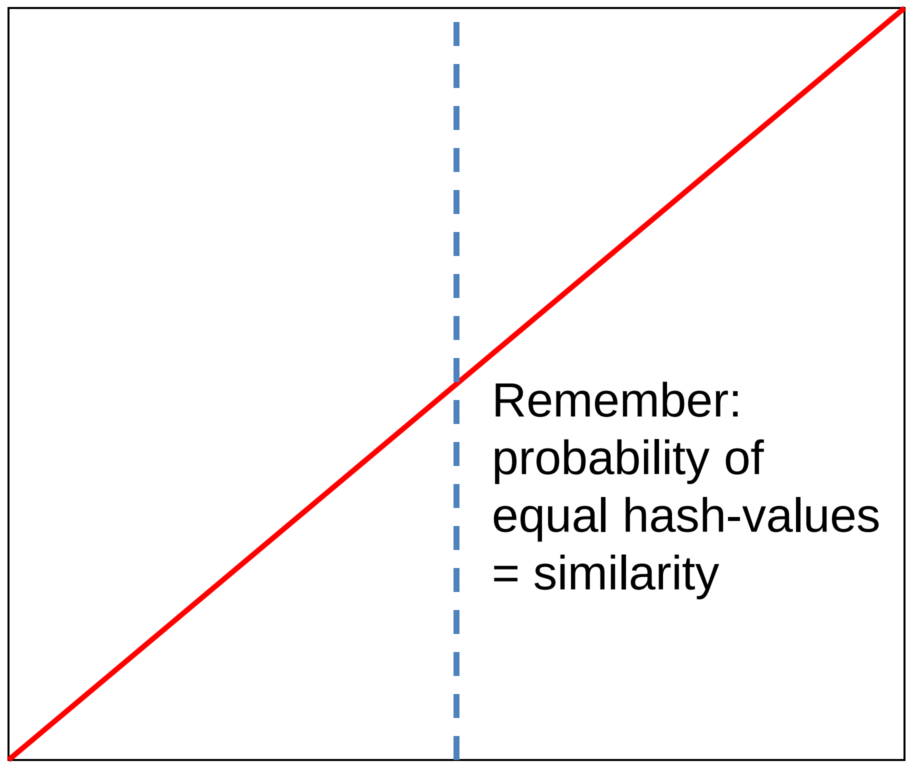
- Divide the signature matrix Sig into *b* bands of *r* rows.
  - Each band is a mini-signature with r hash functions.
- For each band, hash the mini-signature to a hash table with *k* buckets.
  - Make *k* as large as possible so that mini-signatures that hash to the same bucket are almost certainly identical.
- Candidate column pairs are those that hash to the same bucket for at least 1 band.
- Tune *b* and *r* to catch most similar pairs, but few non-similar pairs.

Analysis of LSH – What We Want



What One Band of One Row Gives You

Probability of sharing a bucket



Single hash signature

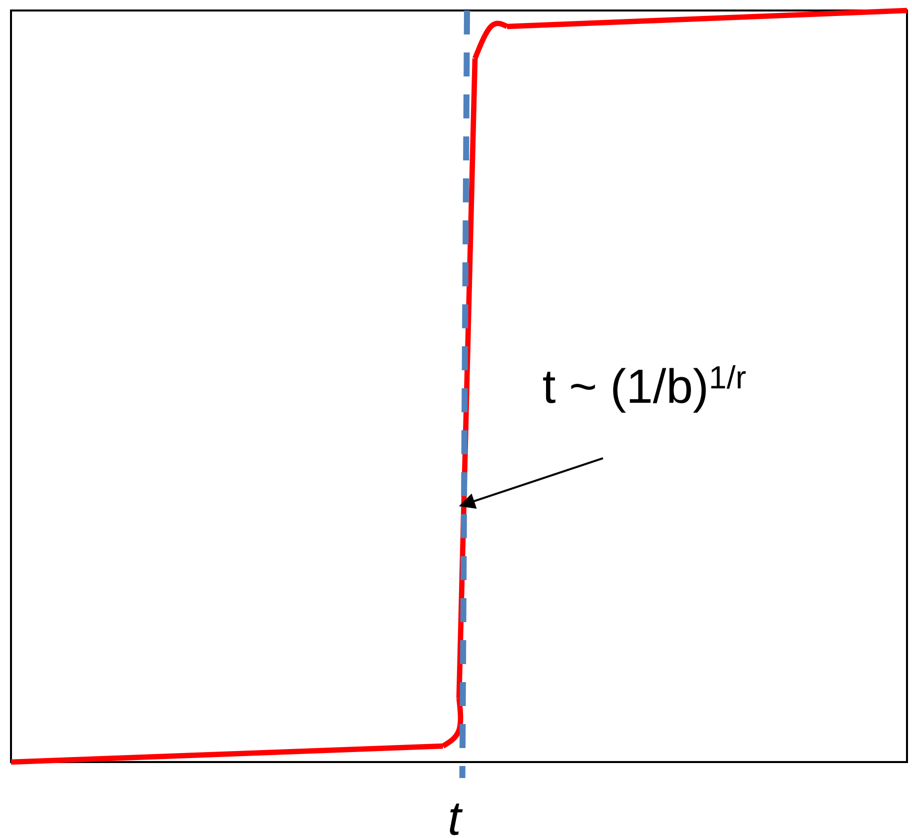
t

Prob(Sig(S,i) == Sig(S',i)) = sim(S,S')

Similarity s of two sets

What *b* Bands of *r* Rows Gives You

Probability of sharing a bucket



At least one band identical

No bands identical

$$1 - (1 - s^{r})^{b}$$

Some row of a band unequal

All rows of a band are equal

Similarity *s* of two sets

Example: $b = 20$; $r = 5$

s	$1-(1-s^r)^b$
.2	.006
.3	.047
.4	.186
.5	.470
.6	.802
.7	.975
.8	.9996

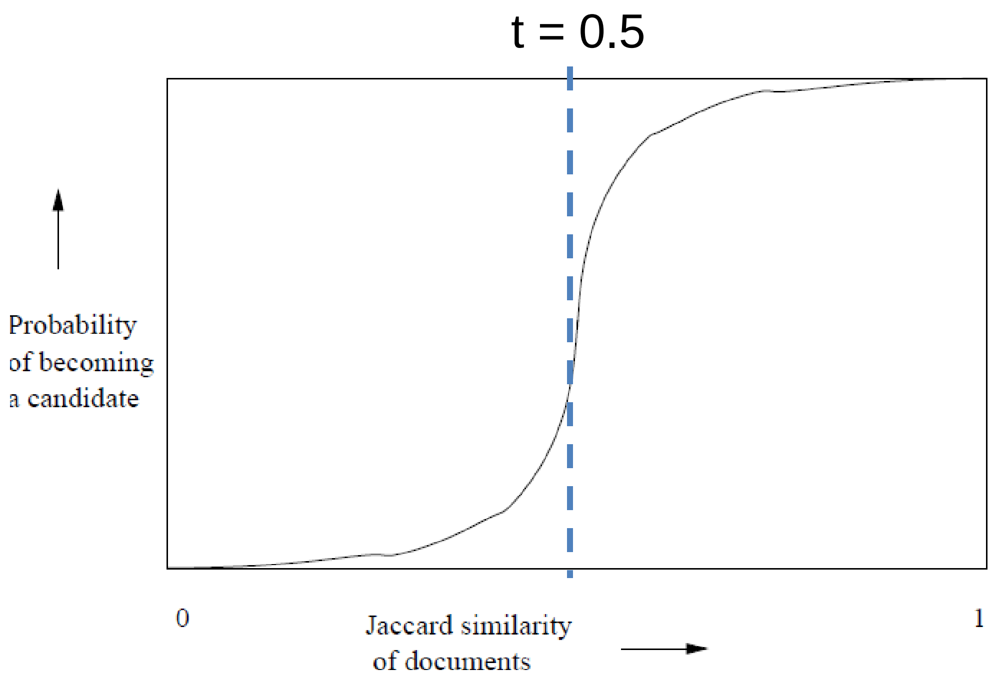


Figure 3.7: The S-curve

Suppose S_1, S_2 are 80% Similar

- We want all 80%-similar pairs. Choose 20 bands of 5 integers/band.

- Probability S_1, S_2 identical in one particular band:

$$(0.8)^5 = 0.328.$$

- Probability S_1, S_2 are not similar in any of the 20 bands:

$$(1-0.328)^{20} = 0.00035$$

  - i.e., about 1/3000-th of the 80%-similar column pairs are false negatives.

- Probability S_1, S_2 are similar in at least one of the 20 bands:

$$1-0.00035 = 0.999$$

Suppose S_1, S_2 Only 40% Similar

- Probability S_1, S_2 identical in any one particular band:

  $$(0.4)^5 = 0.01 .$$

- Probability S_1, S_2 identical in at least 1 of 20 bands:

  $$\leq 20 * 0.01 = 0.2 .$$

- But false positives much lower for similarities << 40%.

LSH Summary

- Tune to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures.
- Check in main memory that candidate pairs really do have similar signatures.
- Optional: In another pass through data, check that the remaining candidate pairs really represent similar *sets* .

Locality-sensitive hashing (LSH)

- Big Picture: Construct hash functions **$h: R^d \rightarrow U$** such that for any pair of points **p,q**, for distance function **D** we have:
  - If **$D(p,q) \leq r$**, then **$Pr[h(p)=h(q)] \geq \alpha$** is high
  - If **$D(p,q) \geq cr$**, then **$Pr[h(p)=h(q)] \leq \beta$** is small
- Then, we can find close pairs by hashing

- LSH is a general framework: for a given distance function **D** we need to find the right **h**
  - **h** is **(r,cr, α, β)-**sensitive

LSH for Cosine Distance

- For cosine distance, there is a technique analogous to minhashing for generating a $\left(d_1,d_2,(1-d_1/180),(1-d_2/180)\right)$- sensitive family for any d_1 and d_2.
- Called *random hyperplanes*.

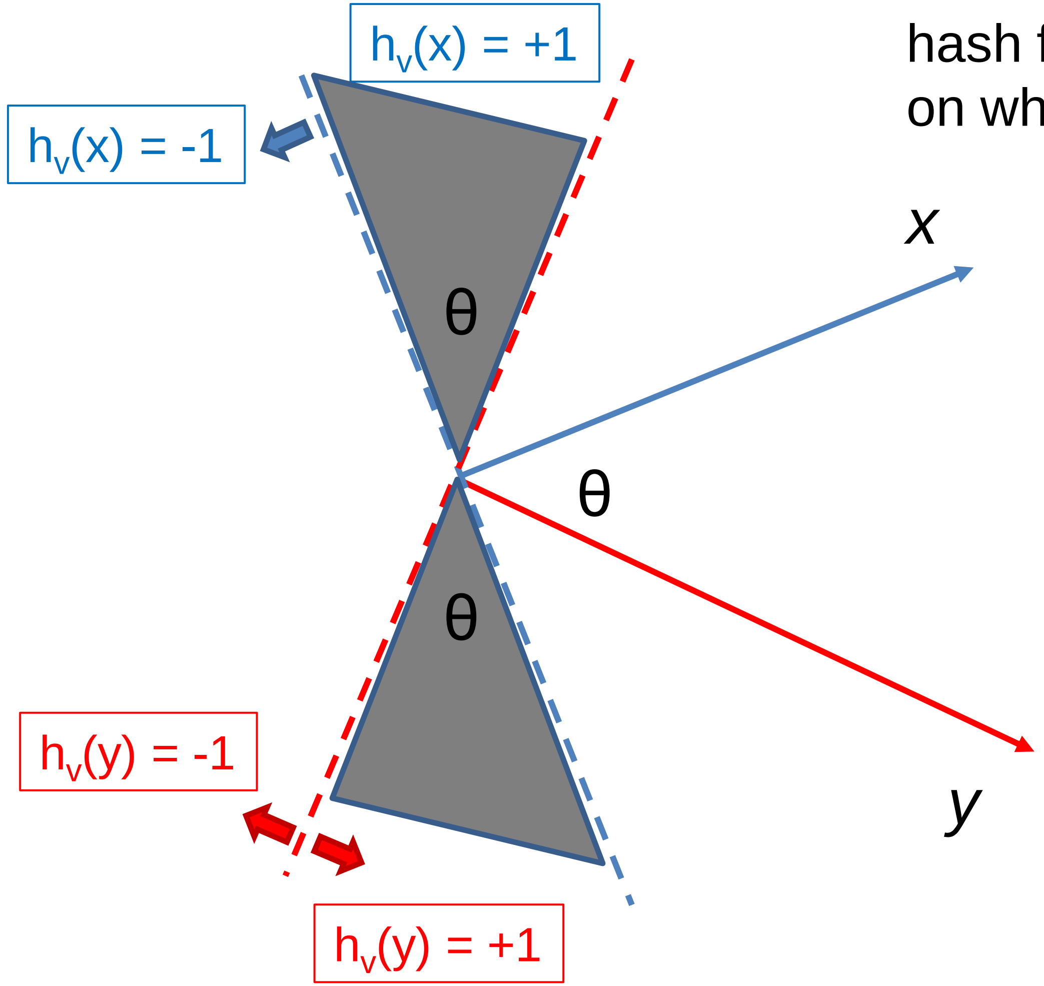
Random Hyperplanes

- Pick a random vector *v*, which determines a hash function h_v with two buckets.
  - $h_v(x) = +1$ if $v.x > 0$; $= -1$ if $v.x < 0$.
- LS-family **H** = set of all functions derived from any vector.
- Claim:
  - Prob[h(x)=h(y)] = 1 – (angle between *x* and *y*)/180

Proof of Claim

Look in the plane of x and y.

For a random vector v the values of the hash functions $h_v(x)$ and $h_v(y)$ depend on where the vector v falls



$h_v(x) \neq h_v(y)$ when v falls into the shaded area.
What is the probability of this for a randomly chosen vector v?

$P[h_v(x) \neq h_v(y)] = 2\theta/360 = \theta/180$

$P[h_v(x) = h_v(y)] = 1 - \theta/180$

Signatures for Cosine Distance

- Pick some number of vectors, and hash your data for each vector.
- The result is a signature (sketch) of +1's and –1's that can be used for LSH like the minhash signatures for Jaccard distance.

Simplification

- We need not pick from among all possible vectors *v* to form a component of a sketch.
- It suffices to consider only vectors *v* consisting of +1 and –1 components.