DATA MINING
LECTURE 12

Link Analysis Ranking

Random walks

GRAPHS AND LINK ANALYSIS RANKING

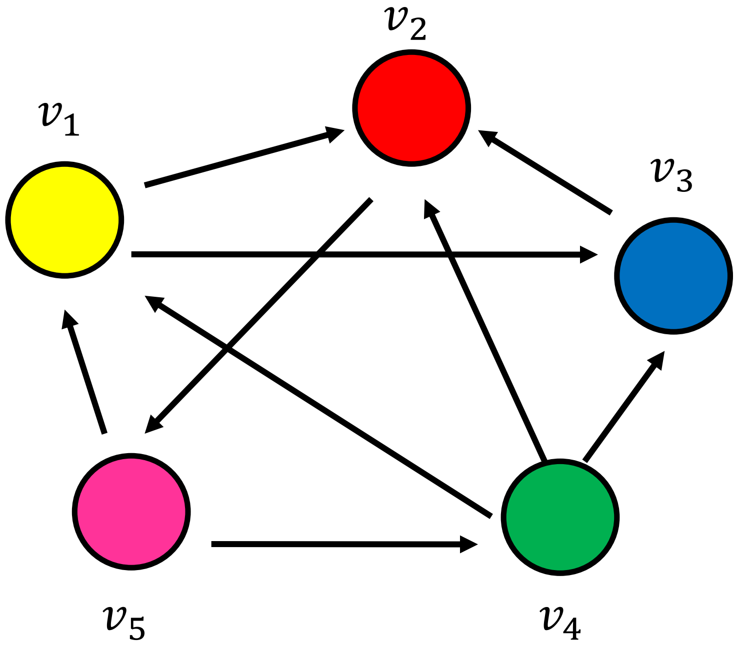
Link Analysis Ranking

- Use the graph structure in order to determine the relative importance of the nodes
  - Applications: Ranking on graphs (Web, Twitter, FB, etc)

- Intuition: An edge from node p to node q denotes endorsement
  - Node p endorses/recommends/confirms the authority/centrality/importance of node q
  - Use the graph of recommendations to assign an authority value to every node

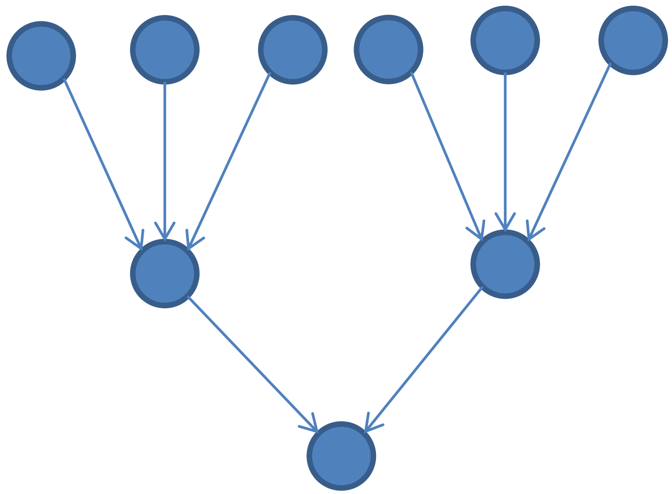
Rank by Popularity

- Rank pages according to the number of incoming edges (in-degree, degree centrality)



1. **Red Page**
2. **Yellow Page**
3. **Blue Page**
4. **Purple Page**
5. **Green Page**

Popularity

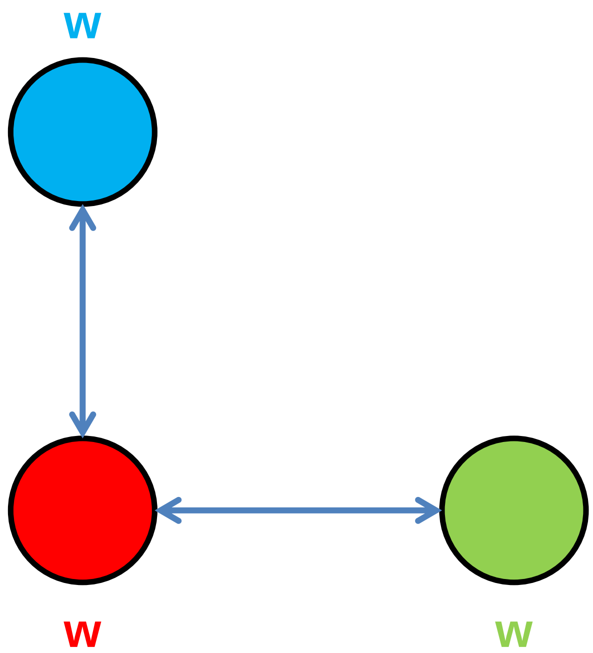


- It is not important only how many link to you, but how important are the people that link to you.
- Good authorities are pointed by good authorities
  - Recursive definition of importance

PageRank

- Good authorities should be pointed by good authorities
  - The value of a node is the value of the nodes that point to it.

- How do we implement that?
  - Assume that we have a unit of authority to distribute to all nodes.
  - Each node distributes the authority value they have to their neighbors
  - The authority value of each node is the sum of the authority fractions it collects from its neighbors.
  - Solving the system of equations we get the authority values for the nodes
    - w = ½ , w = ¼ , w = ¼



w + w + w = 1

w = w + w

w = ½ w

w = ½ w

A more complex example

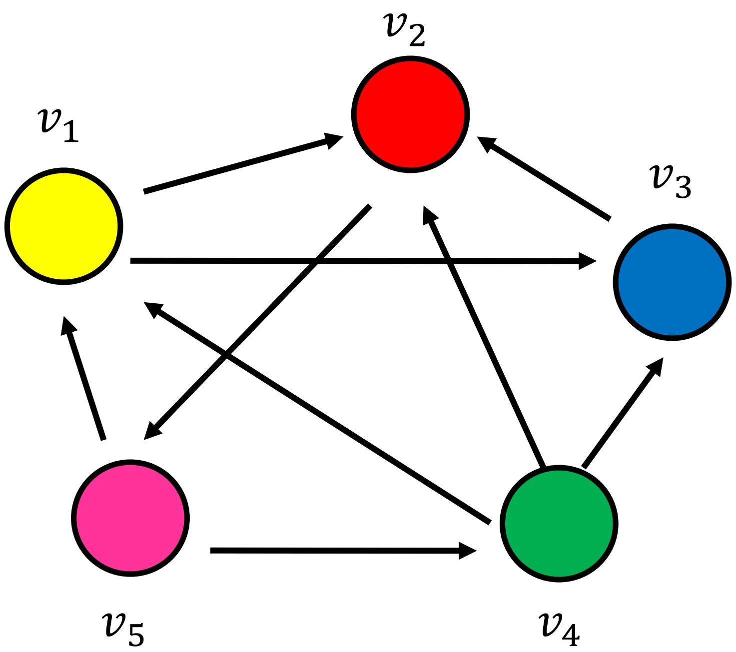
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

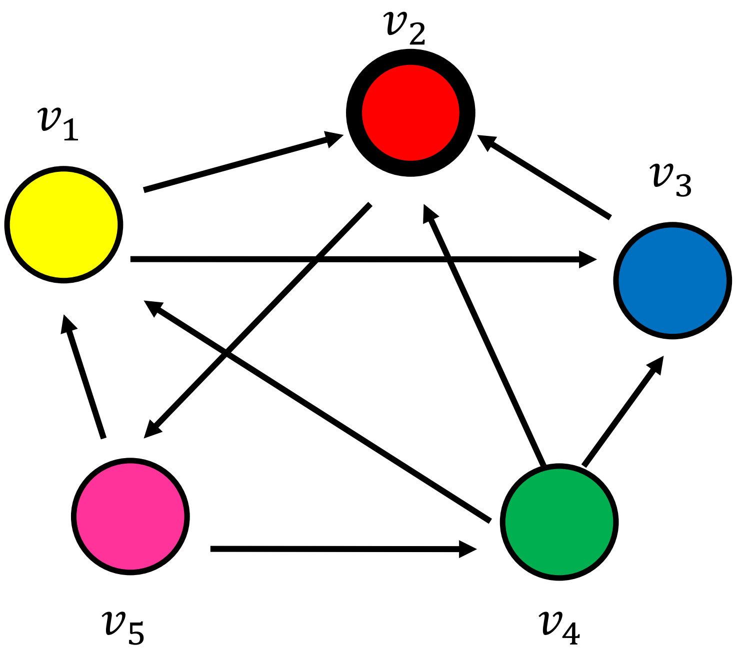
Random Walks on Graphs

- What we described is equivalent to a random walk on the graph

- Random walk:
  - Start from a node uniformly at random
  - Pick one of the outgoing edges uniformly at random
  - Move to the destination of the edge
  - Repeat.

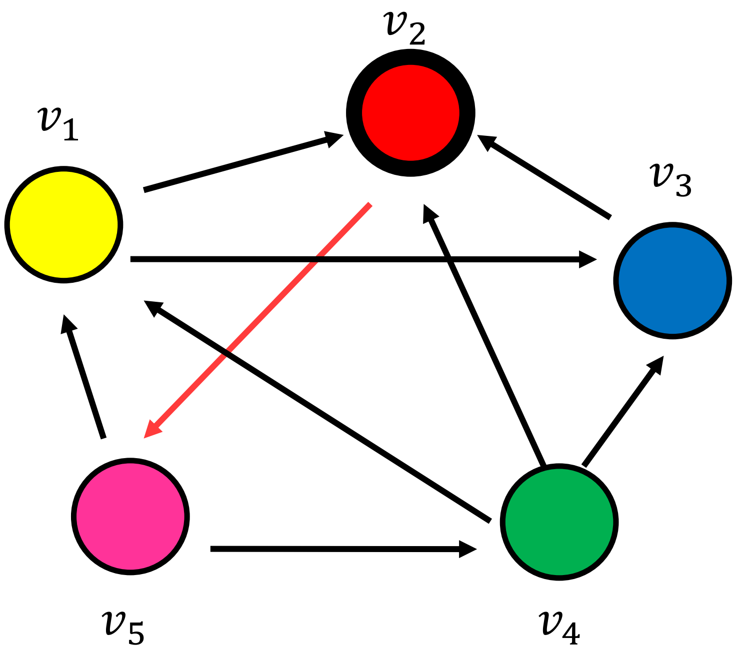
Example

- Step 0



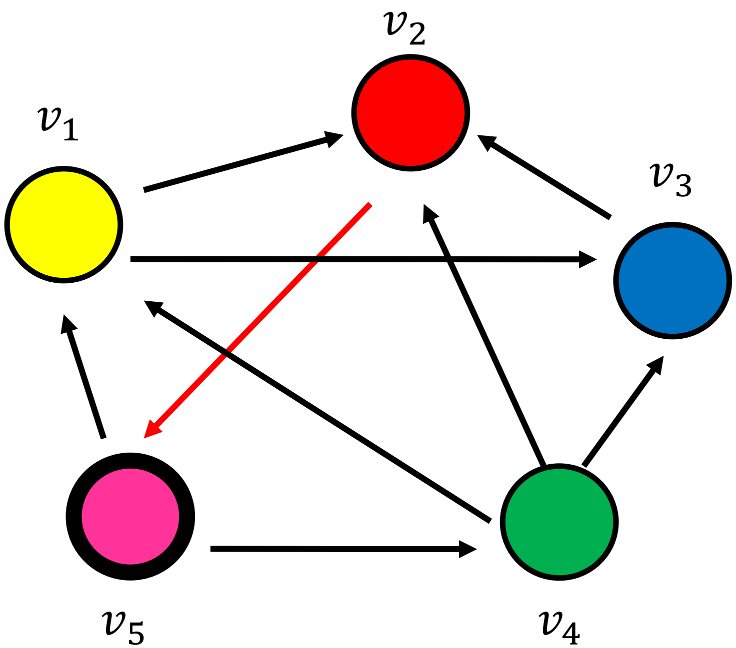
Example

- Step 0



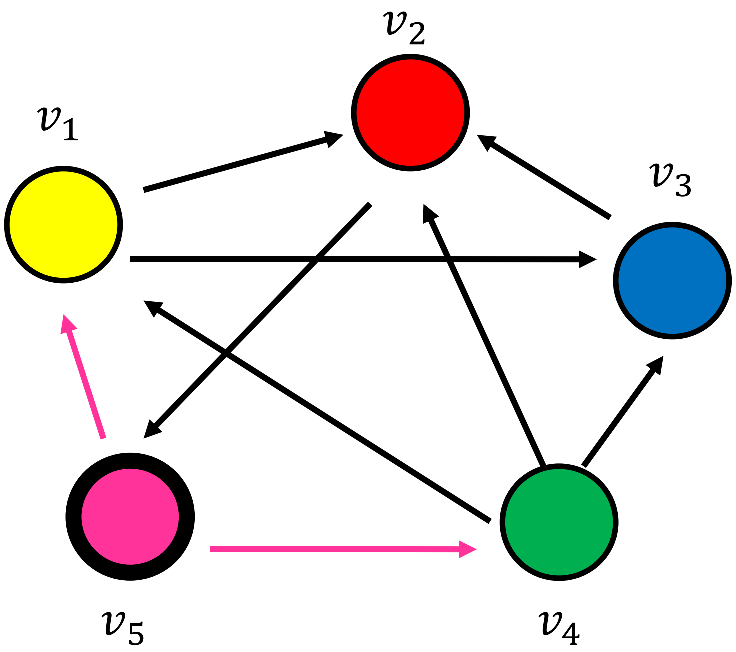
Example

- Step 1



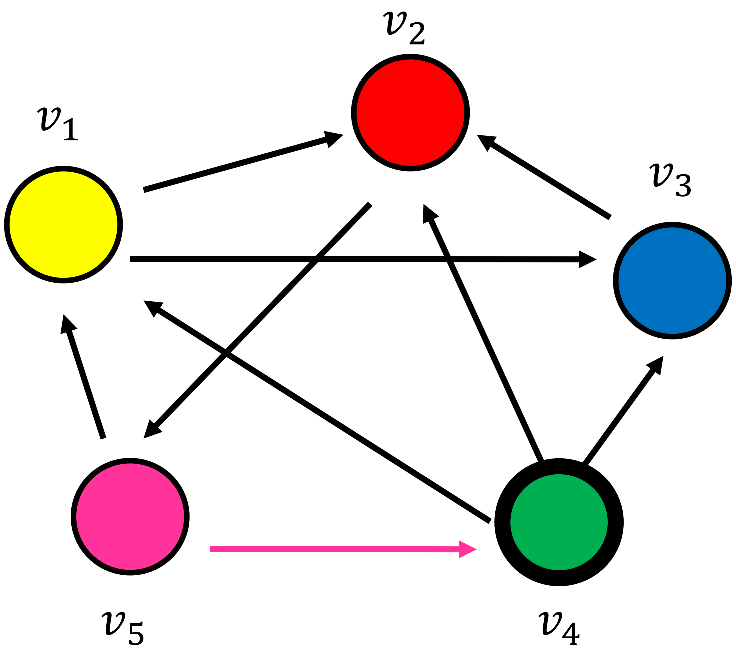
Example

- Step 1



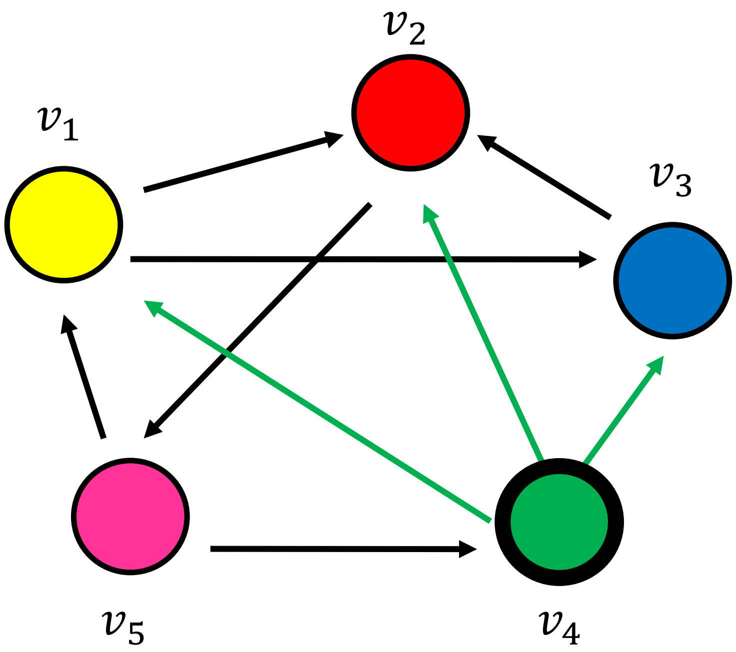
Example

- Step 2



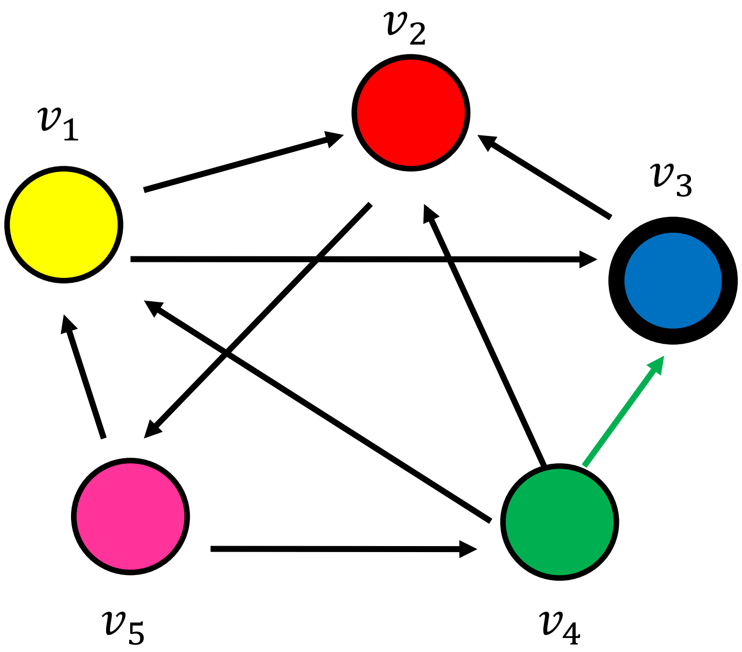
Example

- Step 2



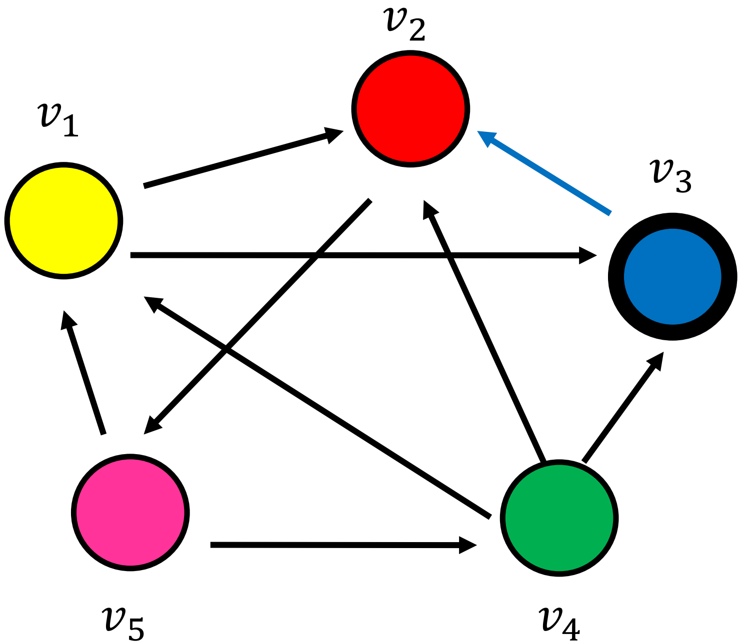
Example

- Step 3



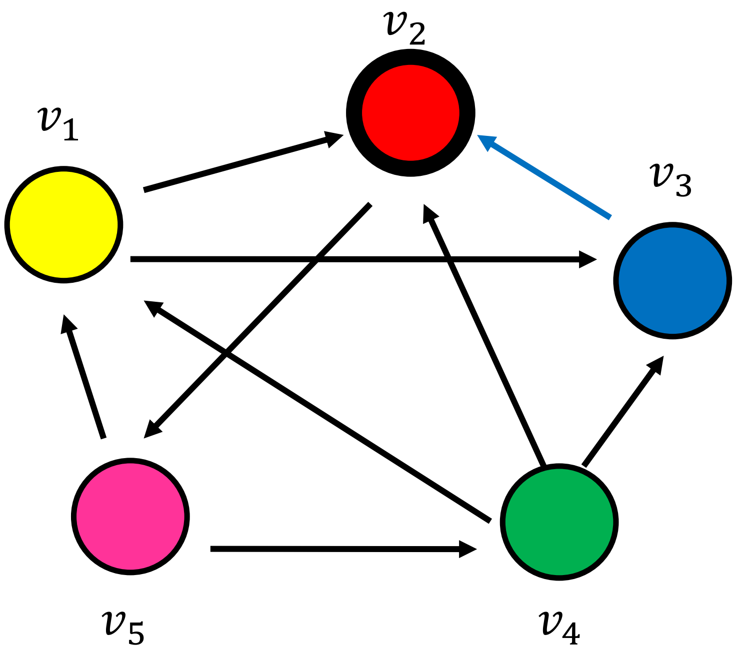
Example

- Step 3



Example

- Step 4…



Memorylessness

- Question: what is the probability p_i^t of being at node i after t steps?

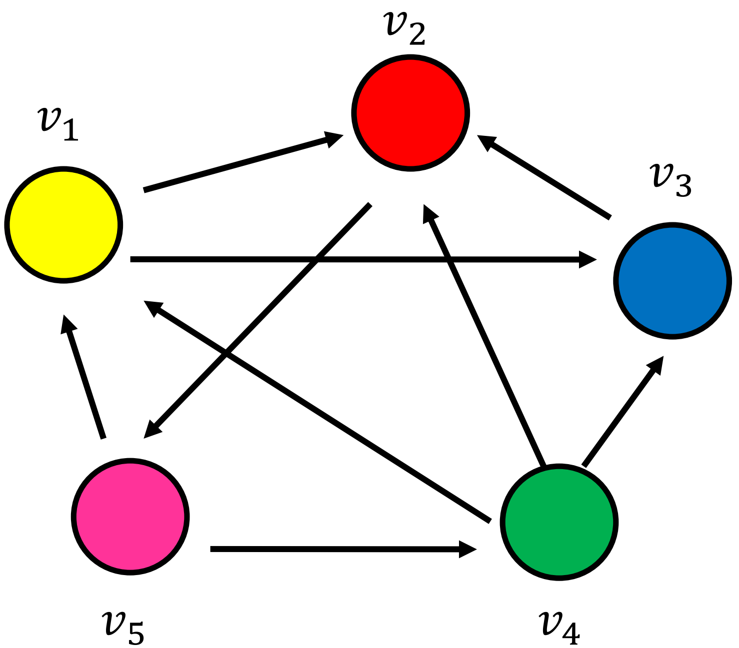
$$p_1^t = \frac{1}{3}p_4^{t-1} + \frac{1}{2}p_5^{t-1}$$

$$p_2^t = \frac{1}{2}p_1^{t-1} + p_3^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_3^t = \frac{1}{2}p_1^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_4^t = \frac{1}{2}p_5^{t-1}$$

$$p_5^t = p_2^{t-1}$$



- Memorylessness property: The next node on the walk depends only at the current node and not on the past of the process

Transition probability matrix

- Since the random walk process is memoryless we can describe it with the transition probability matrix
- Transition probability matrix: A matrix P, where $P[i,j]$ is the probability of transitioning from node i to node j

$$P[i,j] = 1/\deg_{out}(i)$$

- Matrix P has the property that the entries of all rows sum to 1

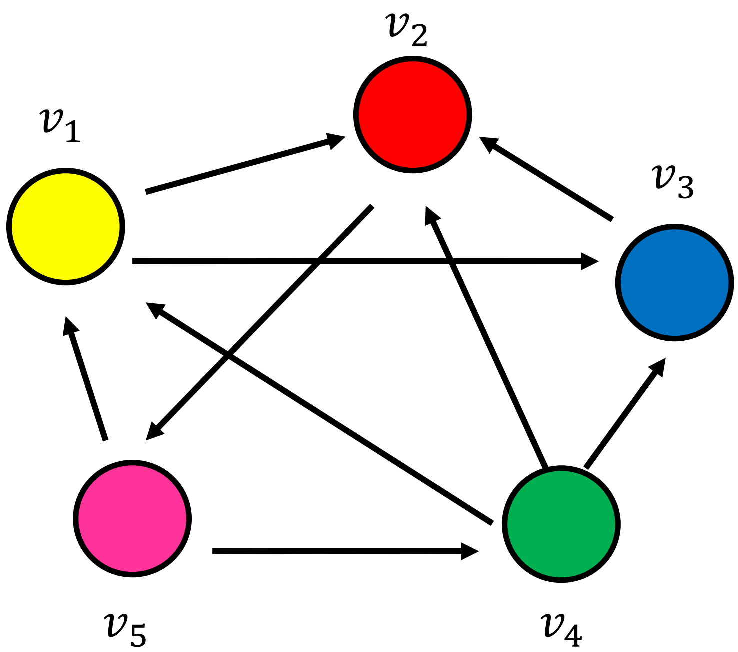
$$\sum_j P[i,j] = 1$$

- A matrix with this property is called stochastic

An example

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



Node Probability vector

- The vector $p^t = (p_i^t, p_2^t, \ldots, p_n^t)$ that stores the probability of being at node v_i at step t

- p_i^0= the probability of starting from state i (usually set to uniform)

- We can compute the vector p^t at step t using a vector-matrix multiplication

$$p^t = p^{t-1}\, P$$

An example

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

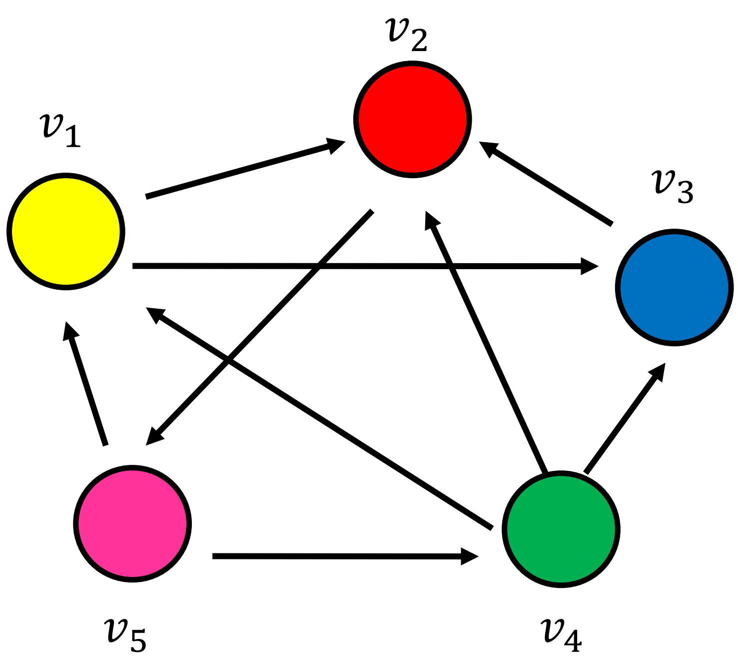
$$p_1^t = \frac{1}{3} p_4^{t-1} + \frac{1}{2} p_5^{t-1}$$

$$p_2^t = \frac{1}{2} p_1^{t-1} + p_3^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_3^t = \frac{1}{2} p_1^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_4^t = \frac{1}{2} p_5^{t-1}$$

$$p_5^t = p_2^{t-1}$$



Stationary distribution

- The stationary distribution of a random walk with transition matrix P, is a probability distribution π, such that $\pi = \pi P$

- The stationary distribution is an eigenvector of matrix P
  - the principal left eigenvector of P – stochastic matrices have maximum eigenvalue 1

- The probability π_i is the fraction of times that we visited state i as $t \to \infty$

Computing the stationary distribution

- The Power Method
  - Initialize to some distribution q^0
  - Iteratively compute $q^t = q^{t-1}P$
  - After many iterations $q^t \approx \pi$ regardless of the initial vector q^0
  - Power method because it computes $q^t = q^0P^t$
- Rate of convergence
  - determined by the second eigenvalue λ_2^t

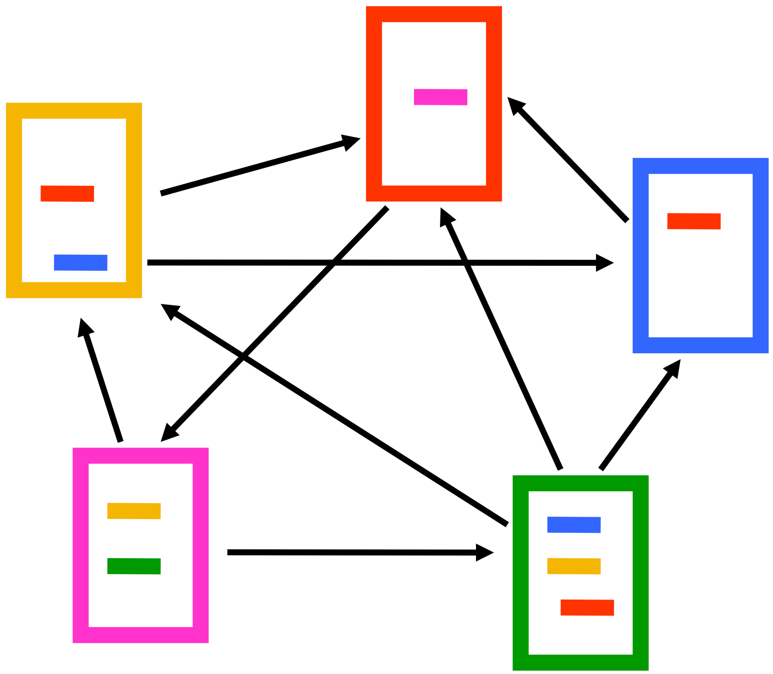
The stationary distribution

- What is the meaning of the stationary distribution π of a random walk?
- $\pi(i)$: the probability of being at node i after very large (infinite) number of steps
- $\pi = p_0 P^\infty$, where P is the transition matrix, p_0 the original vector
  - $P(i,j)$: probability of going from i to j in one step
  - $P^2(i,j)$: probability of going from i to j in two steps (probability of all paths of length 2)
  - $P^\infty(i,j) = \pi(j)$: probability of going from i to j in infinite steps – starting point does not matter.

The PageRank random walk

- Vanilla random walk
  - make the adjacency matrix stochastic and run a random walk

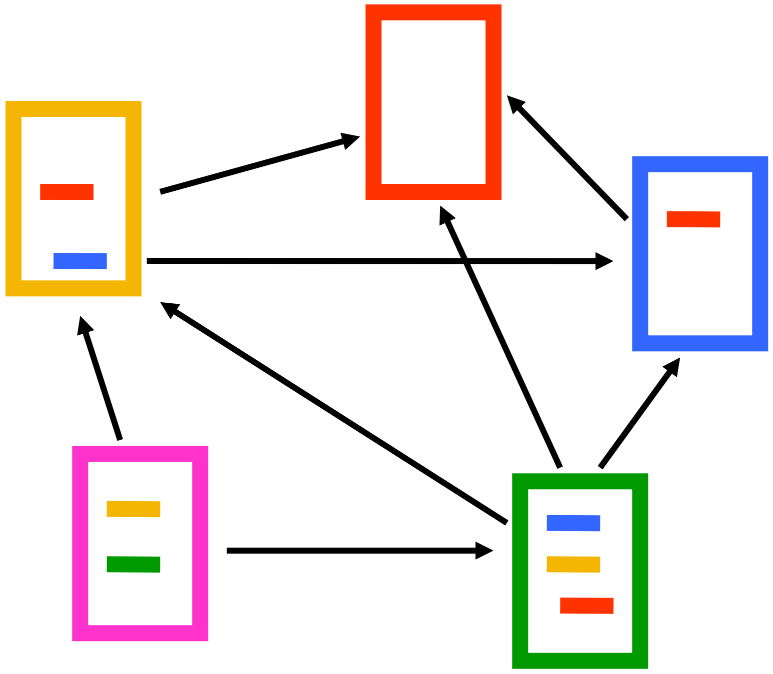
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- What about sink nodes?
  - what happens when the random walk moves to a node without any outgoing inks?

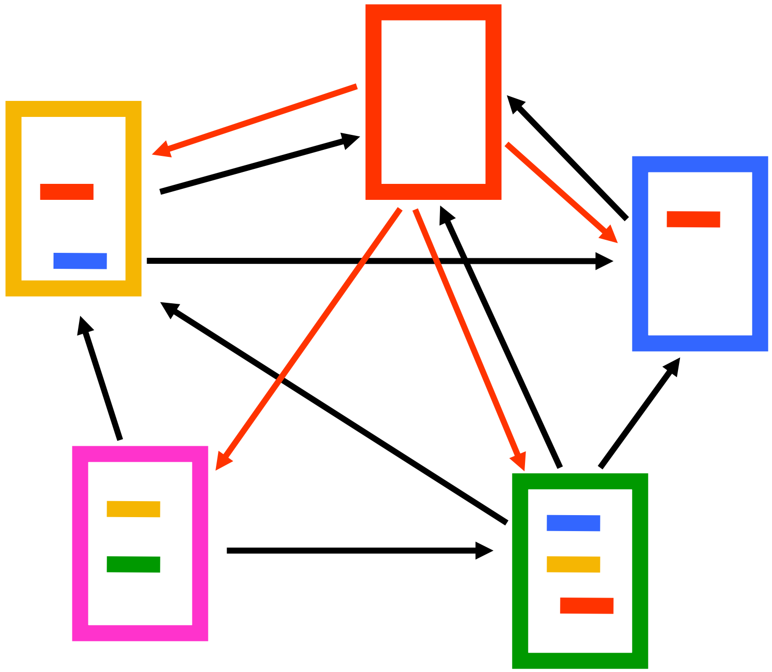
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0 \quad 0} \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- Replace these row vectors with a vector v
  - typically, the uniform vector

$$P' = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

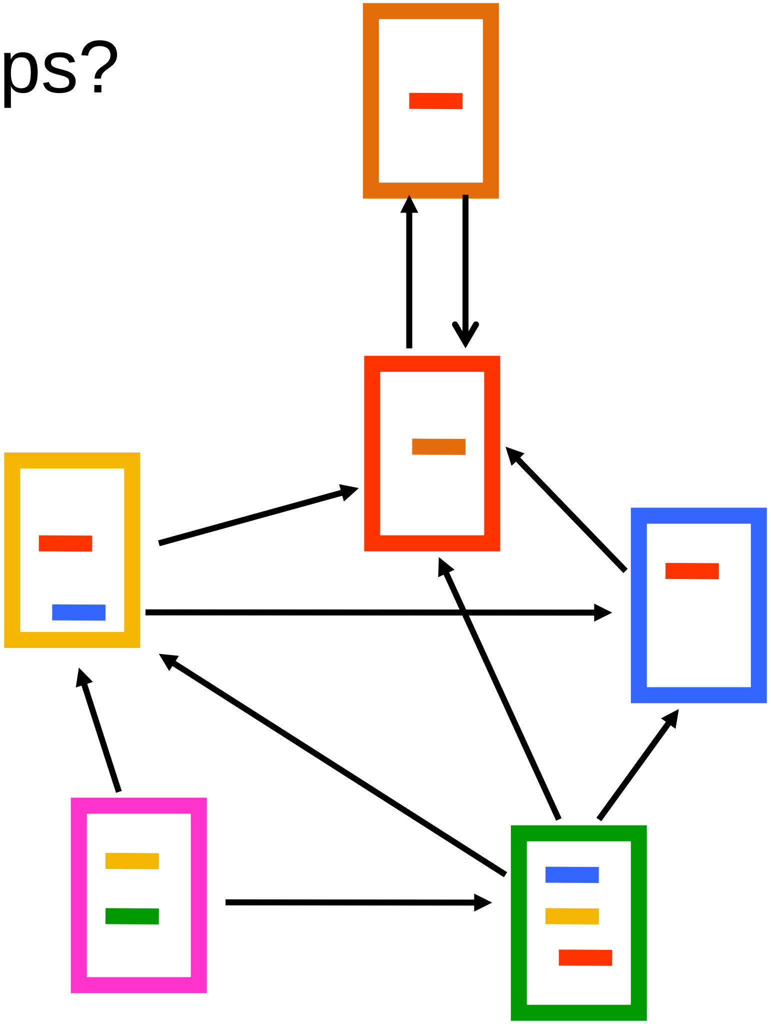


$$P' = P + dv^T$$

$$d = \begin{cases} 1 & \text{if i is sink} \\ 0 & \text{otherwise} \end{cases}$$

The PageRank random walk

- What about loops?
  - Spider traps



The PageRank random walk

- Add a random jump to vector v with prob 1-α
  - typically, to a uniform vector
- Restarts after 1/(1-α) steps in expectation
  - Guarantees irreducibility, convergence

$$P'' = \alpha \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

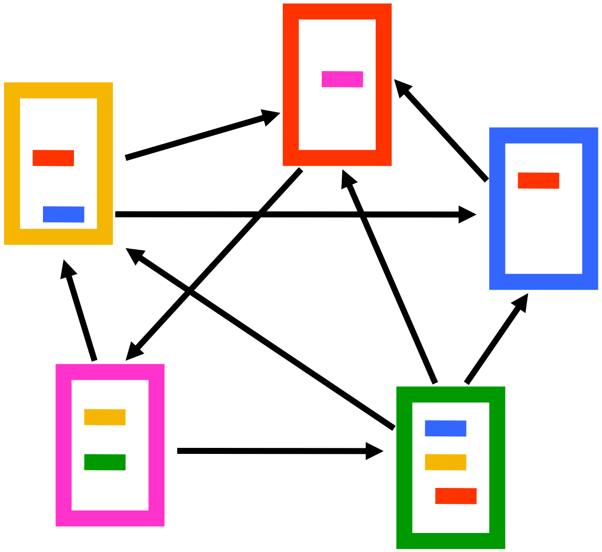
$P'' = \alpha P' + (1-\alpha)uv^T$, where u is the vector of all 1s

Random walk with restarts

PageRank algorithm [BP98]

- The Random Surfer model
  - pick a page at random
  - with probability $1-\alpha$ jump to a random page
  - with probability ɑ follow a random outgoing link
- Rank according to the stationary distribution
- $$PR(p) = \alpha \sum_{q \to p} \frac{PR(q)}{|Out(q)|} + (1-\alpha)\frac{1}{n}$$

$\alpha = 0.85$ in most cases



1. Red Page
2. Purple Page
3. Yellow Page
4. Blue Page
5. Green Page

Stationary distribution with random jump

- If v is the jump vector

$$p^0 = v$$
$$p^1 = \alpha p^0 P + (1-\alpha)v = \alpha v P + (1-\alpha)v$$
$$p^2 = \alpha p^1 P + (1-\alpha)v = \alpha^2 v P^2 + (1-\alpha)v\alpha P + (1-\alpha)v$$
$$\vdots$$
$$p^\infty = (1-\alpha)v + (1-\alpha)v\alpha P + (1-\alpha)v\alpha^2 P^2 + \cdots$$
$$= (1-\alpha)(I - \alpha P)^{-1}$$

- With the random jump the shorter paths are more important, since the weight decreases exponentially
  - makes sense when thought of as a restart

- If v is not uniform, we can bias the random walk towards the nodes that are close to v
  - Personalized and Topic-Specific Pagerank.

Effects of random jump

- Guarantees convergence to unique distribution
- Motivated by the concept of random surfer
- Offers additional flexibility
  - personalization
  - anti-spam
- Controls the rate of convergence
  - the second eigenvalue of matrix P'' is α

Random walks on undirected graphs

- For undirected graphs, the stationary distribution is proportional to the degrees of the nodes
  - Thus in this case a random walk is the same as degree popularity
- This is not longer true if we do random jumps
  - Now the short paths play a greater role, and the previous distribution does not hold.

Pagerank implementation

- Store the graph in adjacency list, or list of edges
- Keep current pagerank values and new pagerank values
- Go through edges and update the values of the destination nodes.
- Repeat until the difference (L_1 or L_∞ difference) is below some small value ε.

A (Matlab-friendly) PageRank algorithm

- Performing vanilla power method is now too expensive – the matrix is not sparse

$$q^0 = v$$
$$t = 1$$
repeat
$$q^t = (P'')^T q^{t-1}$$
$$\delta = \|q^t - q^{t-1}\|$$
$$t = t + 1$$
until $\delta < \varepsilon$

Efficient computation of $y = (P'')^T x$

$$y = \alpha P^T x$$
$$\beta = \|x\|_1 - \|y\|_1$$
$$y = y + \beta v$$

P = normalized adjacency matrix

$P' = P + dv^T$, where d_i is 1 if i is sink and 0 o.w.

$P'' = \alpha P' + (1-\alpha)uv^T$, where u is the vector of all 1s

Pagerank history

- Huge advantage for Google in the early days
  - It gave a way to get an idea for the value of a page, which was useful in many different ways
    - Put an order to the web.
  - After a while it became clear that the anchor text was probably more important for ranking
  - Also, link spam became a new (dark) art
- Flood of research
  - Numerical analysis got rejuvenated
  - Huge number of variations
  - Efficiency became a great issue.
  - Huge number of applications in different fields
    - Random walk is often referred to as PageRank.

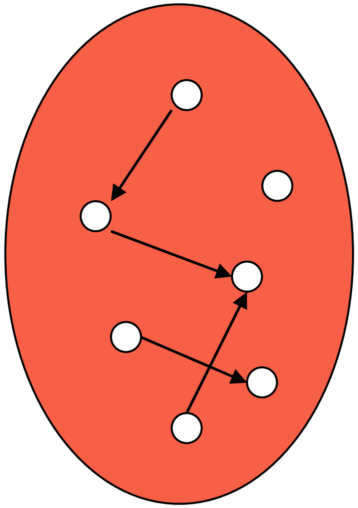
THE HITS ALGORITHM

The HITS algorithm

- Another algorithm proposed around the same time as Pagerank for using the hyperlinks to rank pages
  - Kleinberg: then an intern at IBM Almaden
  - IBM never made anything out of it

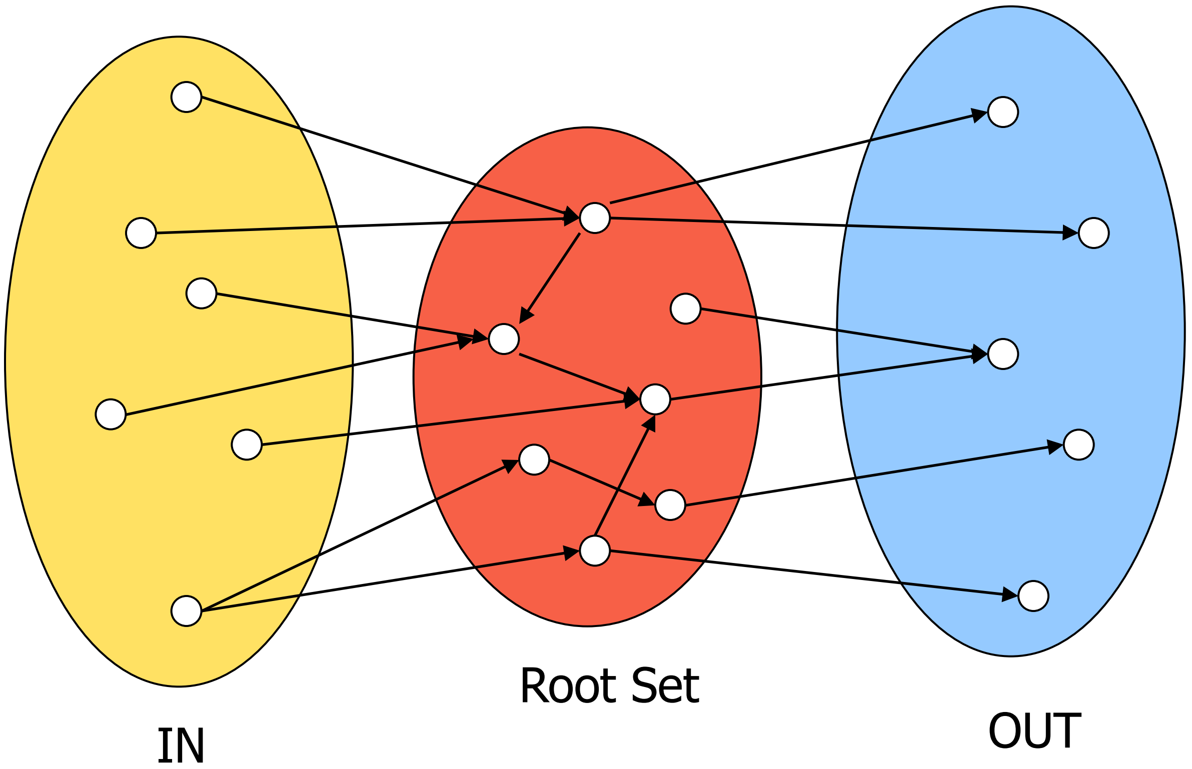
Query dependent input

Root set obtained from a text-only search engine

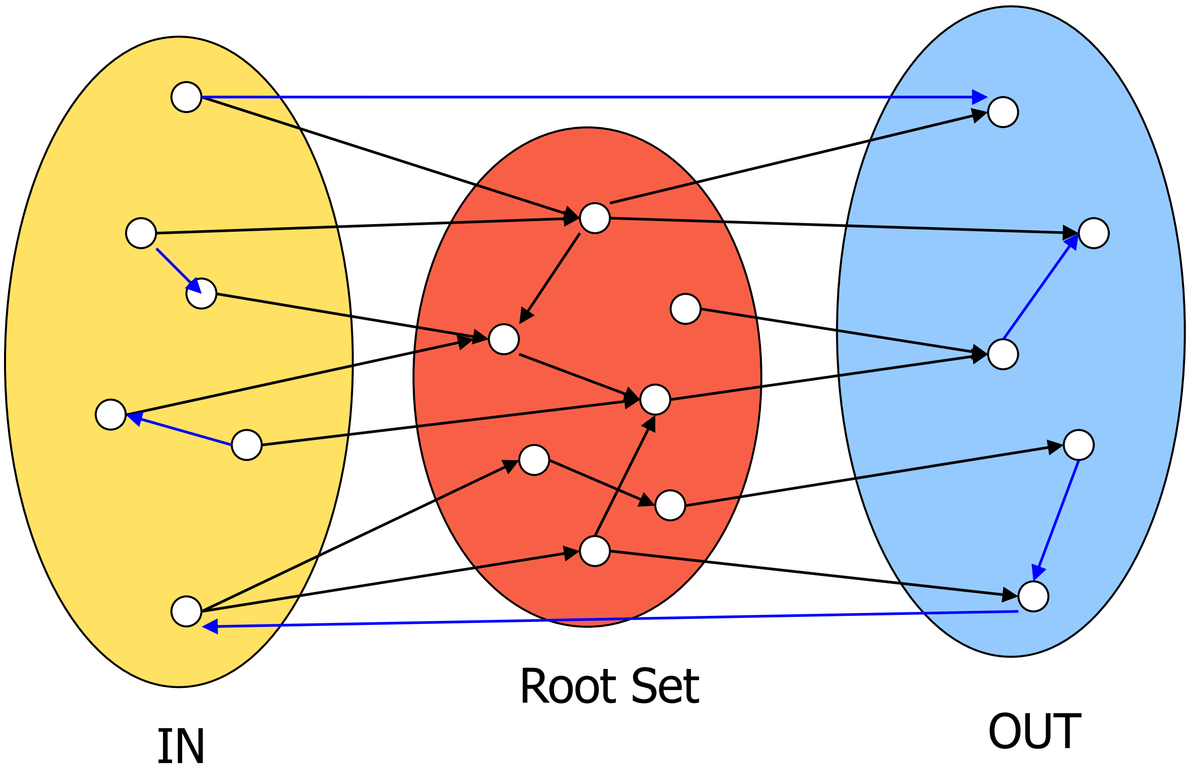


Root Set

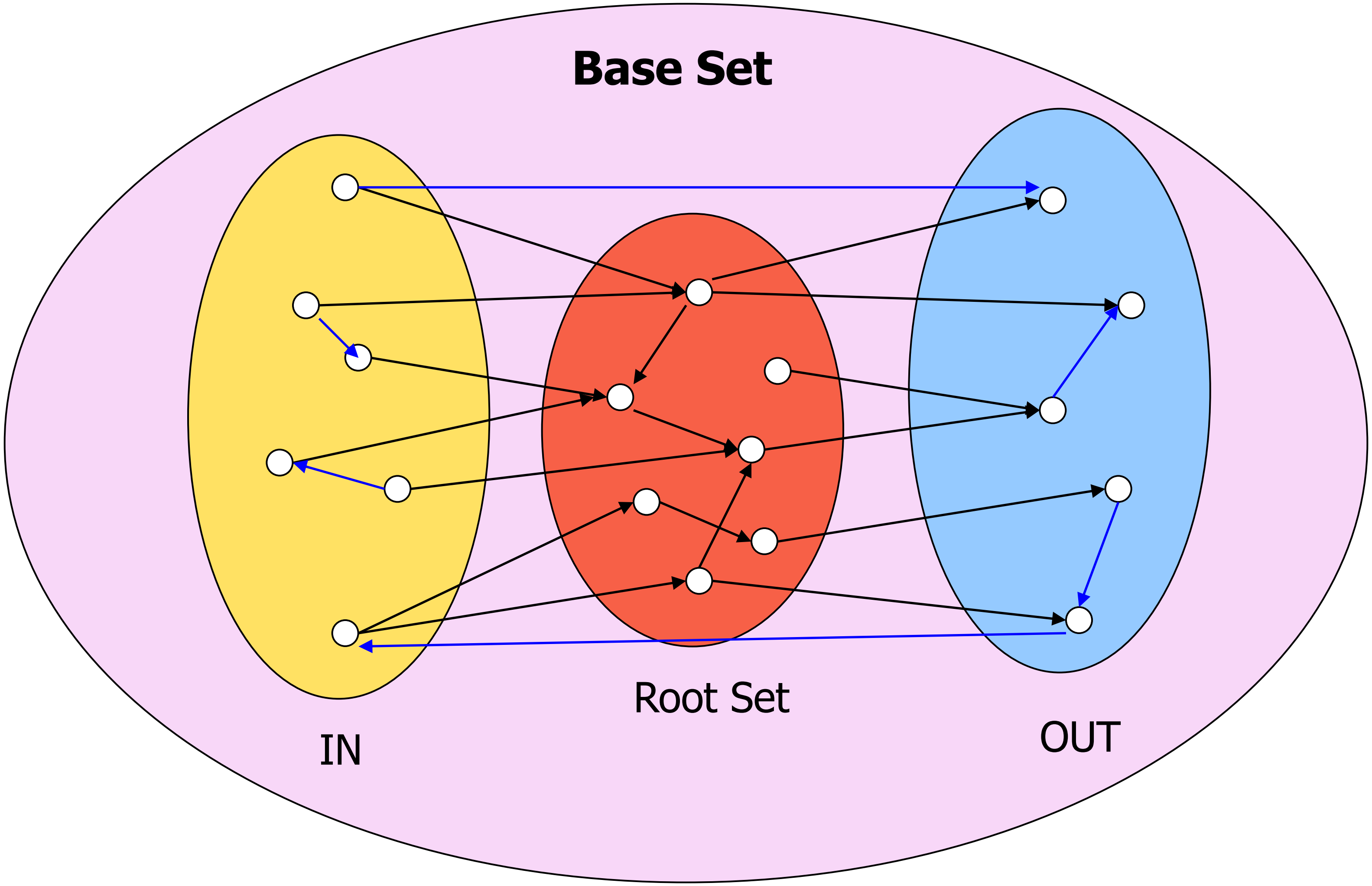
Query dependent input



Query dependent input

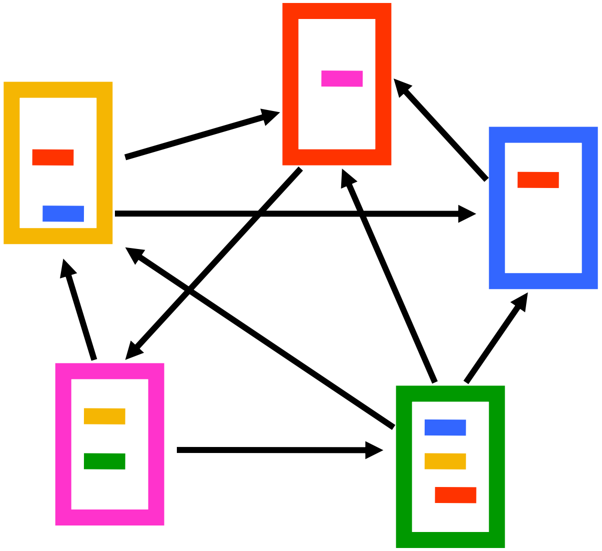


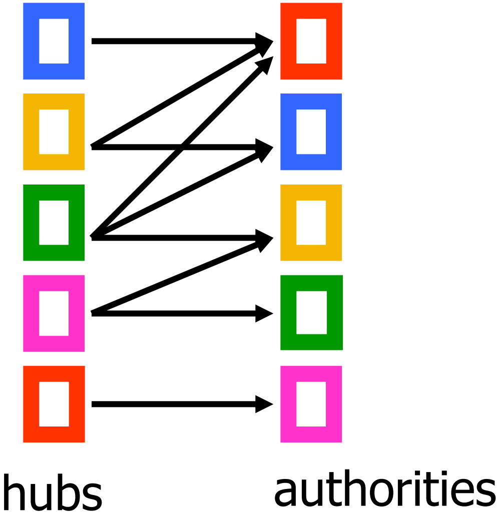
Query dependent input



Hubs and Authorities [K98]

- Authority is not necessarily transferred directly between authorities
- Pages have double identity
  - hub identity
  - authority identity
- Good hubs point to good authorities
- Good authorities are pointed by good hubs





Hubs and Authorities

- Two kind of weights:
  - Hub weight
  - Authority weight

- The hub weight is the sum of the authority weights of the authorities pointed to by the hub

- The authority weight is the sum of the hub weights that point to this authority.

HITS Algorithm

- Initialize all weights to 1.
- Repeat until convergence
  - O operation : hubs collect the weight of the authorities

$$h_i = \sum_{j:i \to j} a_j$$

  - I operation: authorities collect the weight of the hubs

$$a_i = \sum_{j:j \to i} h_j$$

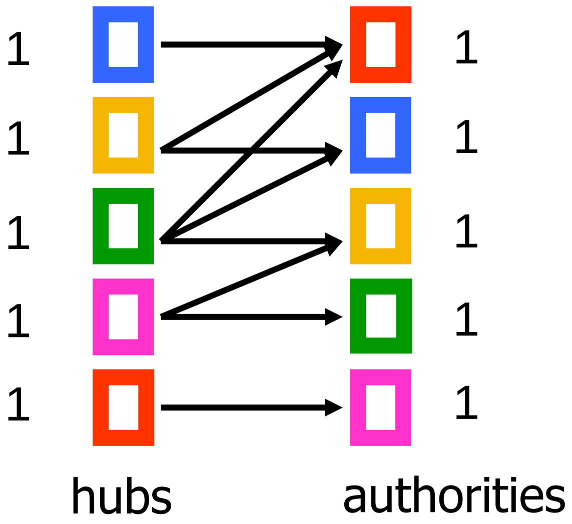
  - Normalize weights under some norm

HITS and eigenvectors

- The HITS algorithm is a power-method eigenvector computation
  - in vector terms $a^t = A^T h^{t-1}$ and $h^t = A a^{t-1}$
  - so $a^t = A^T A a^{t-1}$ and $h^t = A A^T h^{t-1}$
  - The authority weight vector a is the eigenvector of $A^T A$ and the hub weight vector h is the eigenvector of $A A^T$
- The vectors a and h are singular vectors of the matrix A

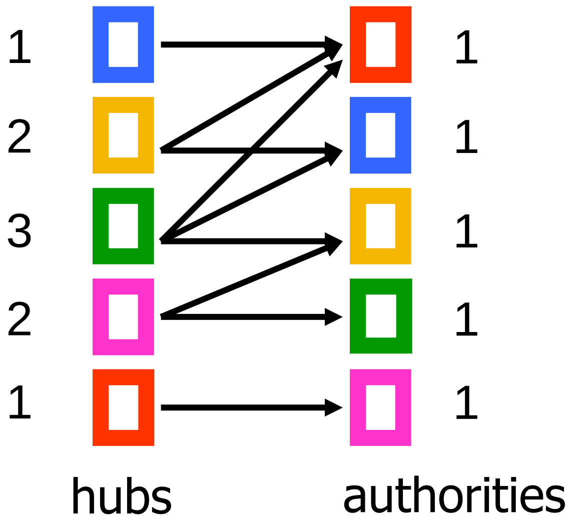
Example

Initialize



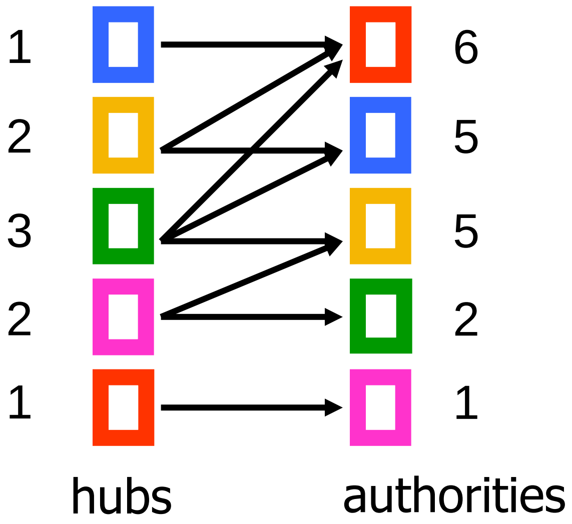
Example

Step 1: O operation



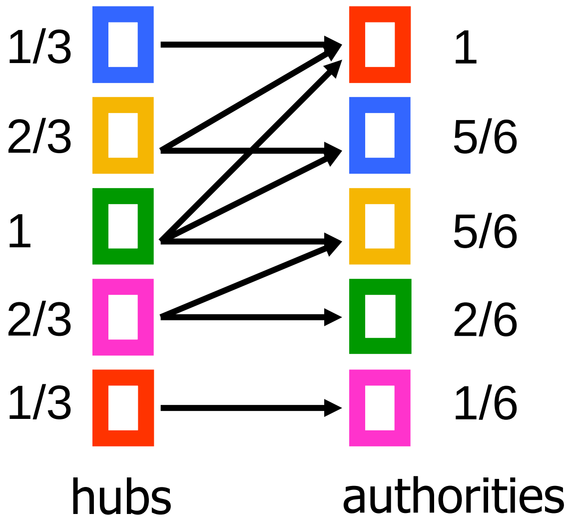
Example

Step 1: I operation



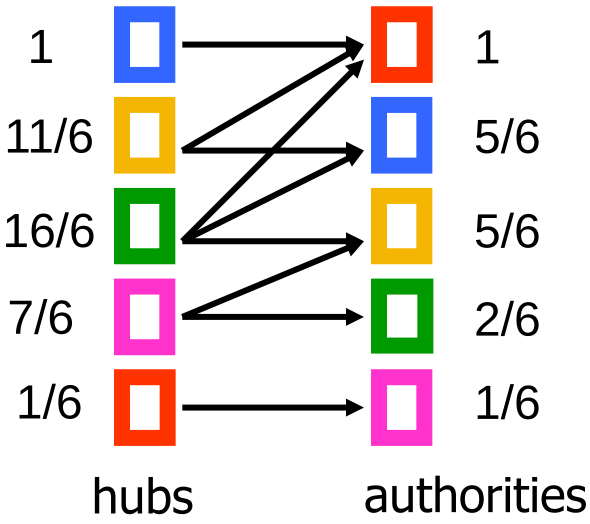
Example

Step 1: Normalization (Max norm)



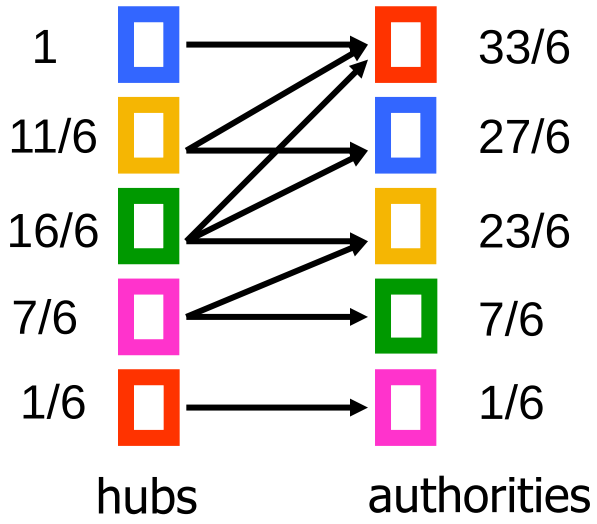
Example

Step 2: O step



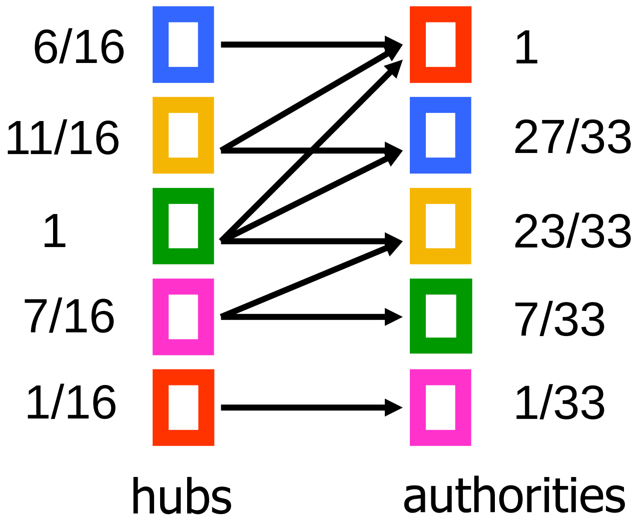
Example

Step 2: I step



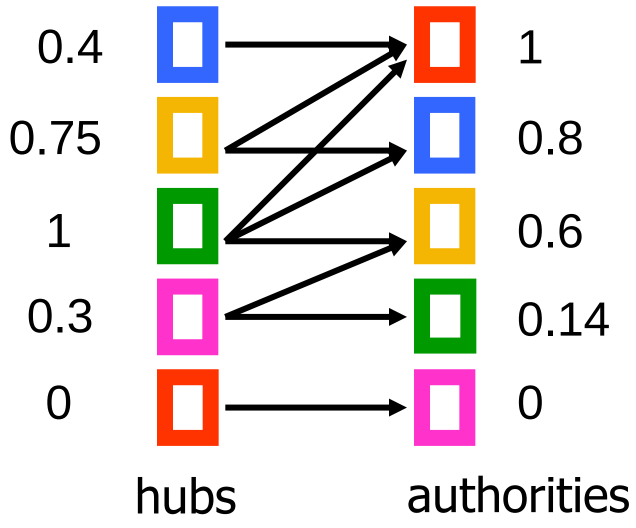
Example

Step 2: Normalization



Example

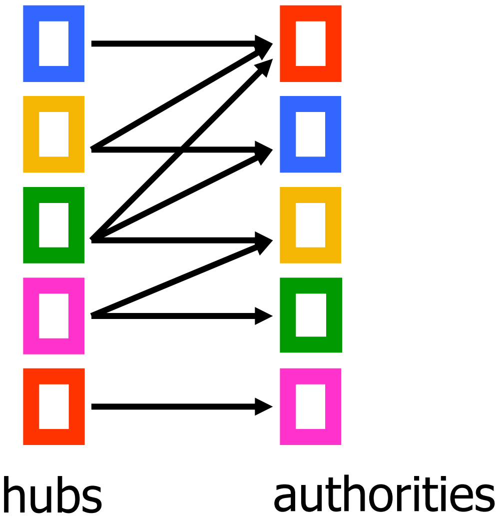
Convergence



OTHER ALGORITHMS

The SALSA algorithm [LM00]

- Perform a random walk alternating between hubs and authorities



- What does this random walk converge to?

- The graph is essentially undirected, so it will be proportional to the degree.

Social network analysis

- Evaluate the centrality of individuals in social networks
  - degree centrality
    - the (weighted) degree of a node
  - distance centrality
    - the average (weighted) distance of a node to the rest in the graph

$$D_c(v) = \frac{1}{\sum_{u \neq v} d(v,u)}$$

  - betweenness centrality
    - the average number of (weighted) shortest paths that use node v

$$B_c(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

Counting paths – Katz 53

- The importance of a node is measured by the weighted sum of paths that lead to this node
- $A^m[i,j]$ = number of paths of length m from i to j
- Compute

$$P = bA + b^2A^2 + \cdots + b^mA^m + \cdots = (I - bA)^{-1} - I$$

- converges when $b < \lambda_1(A)$
- Rank nodes according to the column sums of the matrix P

Bibliometrics

- Impact factor (E. Garfield 72)
  - counts the number of citations received for papers of the journal in the previous two years
- Pinsky-Narin 76
  - perform a random walk on the set of journals
  - P_{ij} = the fraction of citations from journal i that are directed to journal j