Hellenic Robotics Forum
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Unlocking the Matrix Form of the Quaternion Fourier Transform and Quaternion Convolution: Properties, connections, and application to Lipschitz constant bounding

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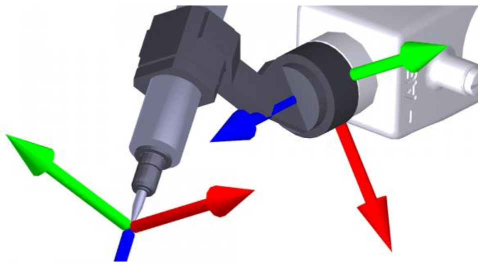

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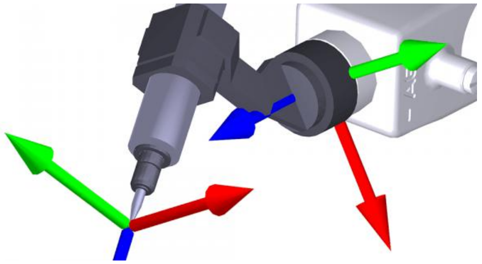
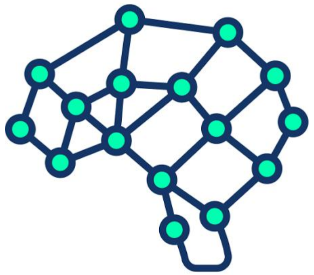


- 🧠 Hypercomplex or SO3-equivariant Neural Networks (e.g. “Fast Quaternion Product Units for Learning Disentangled Representations in SO(3), IEEE TPAMI 2023)



Quaternions: elements

Basic form

Quaternions $q \in \mathbb{H}$ share the basic form:

$$q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$$

where $a, b, c, d \in \mathbb{R}$ and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are independent imaginary units.

Generalizations of complex numbers

- Two ways to see this:
  - 2 extra imaginary units ($\boldsymbol{j}, \boldsymbol{k}$). Real and complex numbers can be regarded as quaternions with $b, c, d = 0$ or $c, d = 0$ respectively.
  - OR: $a + b\boldsymbol{j}$ with $a, b \in \mathbb{C}$ (Cayley-Dickson form)

Quaternions: elements

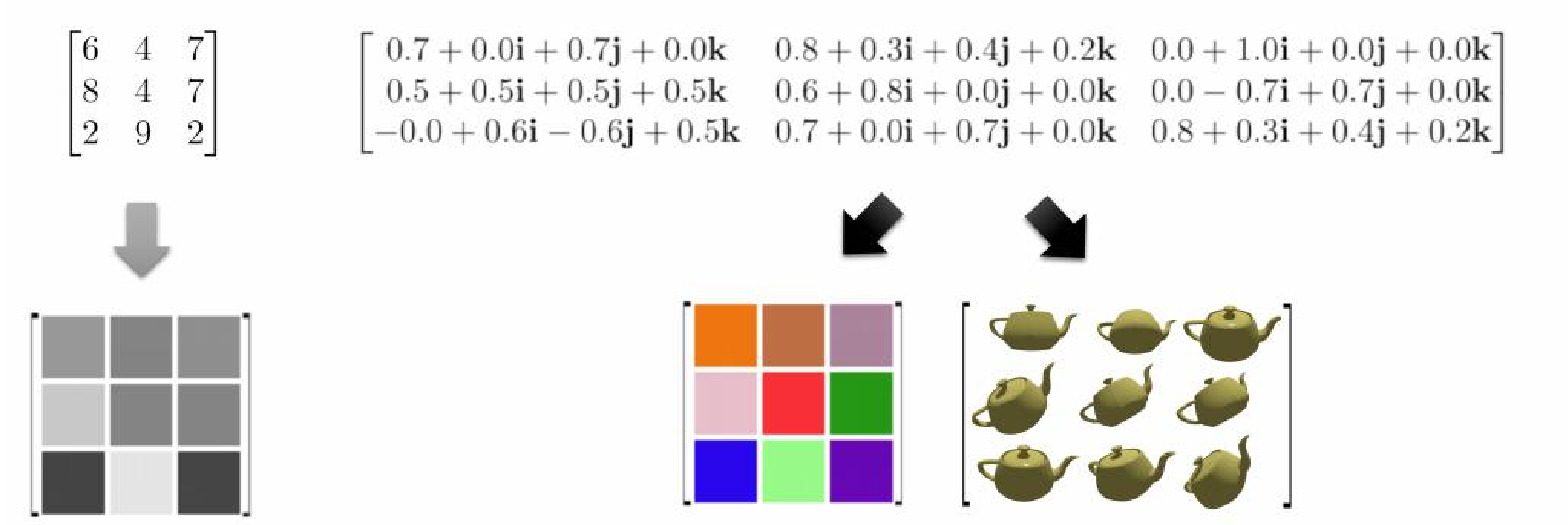


Figure 1: Matrices of real-valued elements (top left) form a standard representational unit for unidimensional data, such as grayscale pixels (bottom left), neural network weights or other unidimensional signals. Matrices of quaternion-valued elements (top right) represent an arrangement of multidimensional data such as color images (Ell & Sangwine, 2007) or 3D spatial rotation (Kerbl et al., 2023) (bottom right).

Motivation and scope of current work

[Sfikas & Retsinas, TMLR 2025]

$$g = Cf \implies Ag = AA^{-1}\Lambda_C Af \implies G = \Lambda_C F$$

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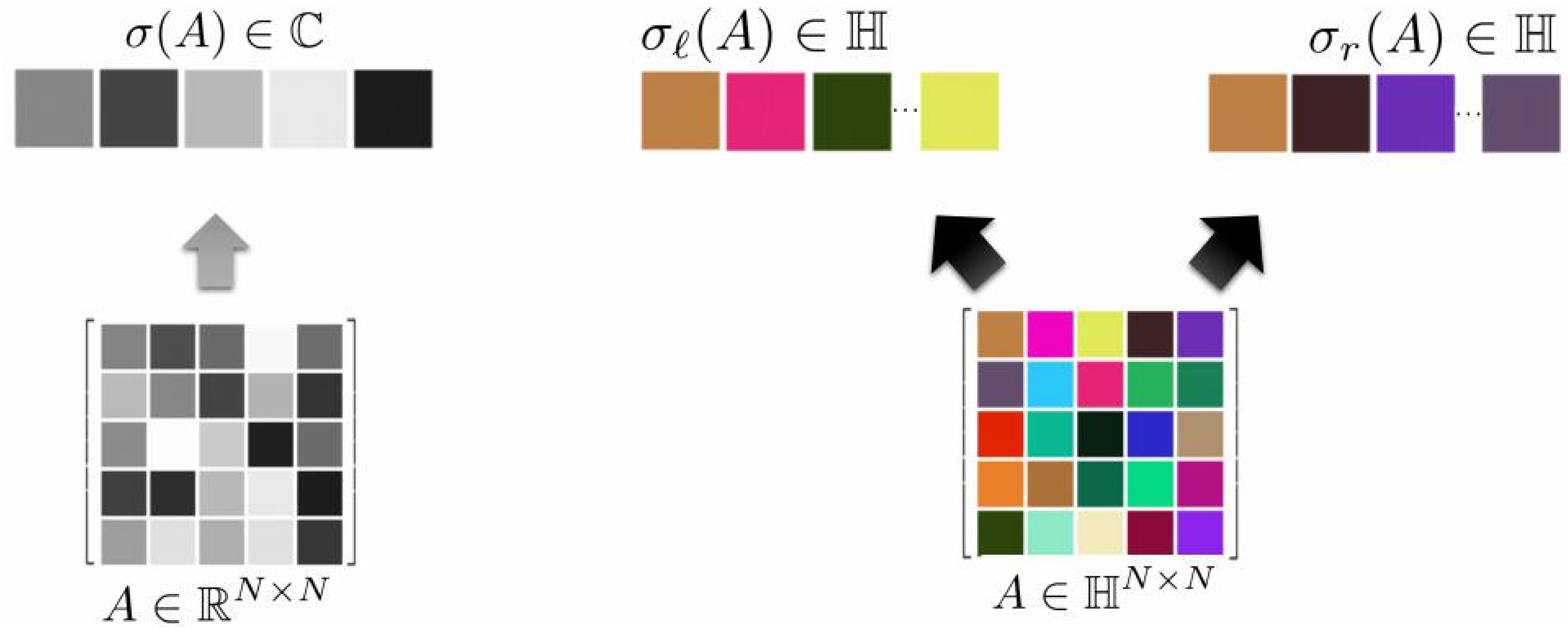
🔄 Simplify quaternionic operations into standard linear algebra.
🧠 Connect quaternion convolution and Fourier matrix forms
📈 Show how properties of the eigenstructure of convolution and Fourier matrices generalize into the significantly more complex structure of quaternion matrices.

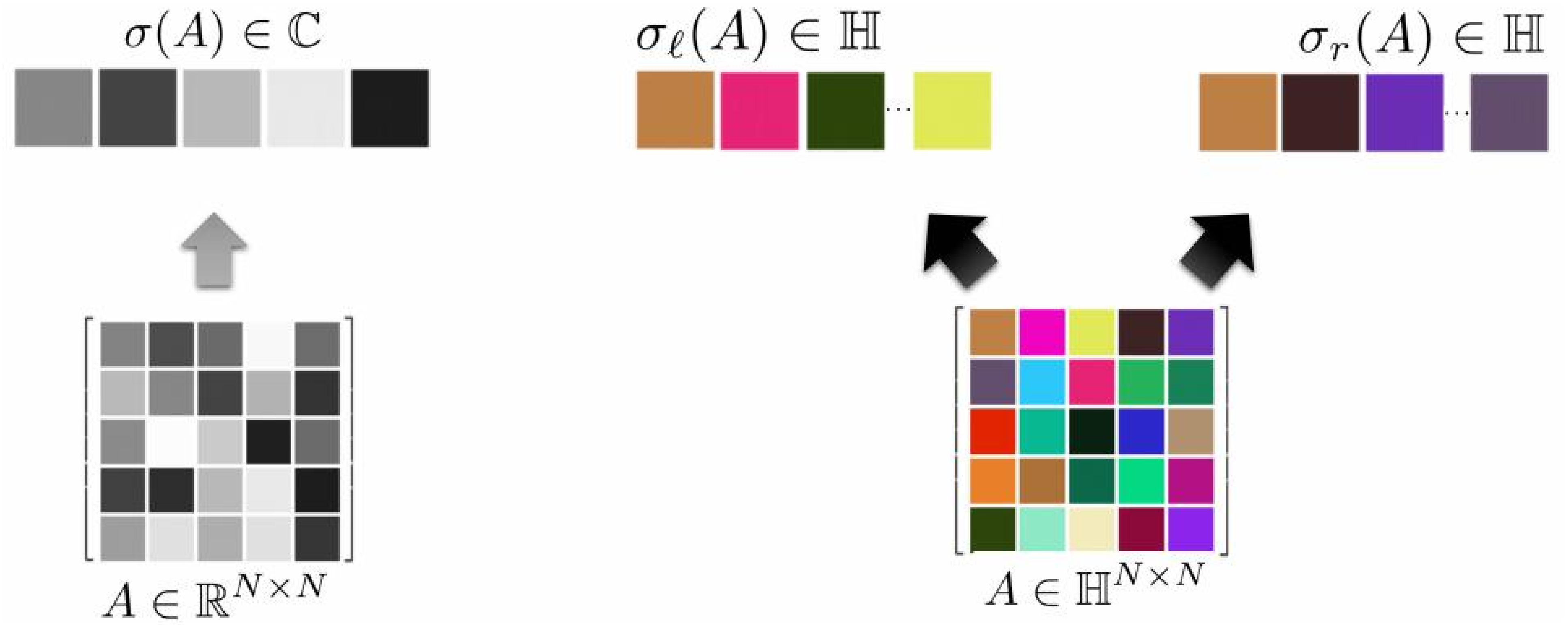
Challenges



The eigenstructure of Quaternion matrices is much more complex than R or C!

Challenges

$$Ax = \lambda x$$



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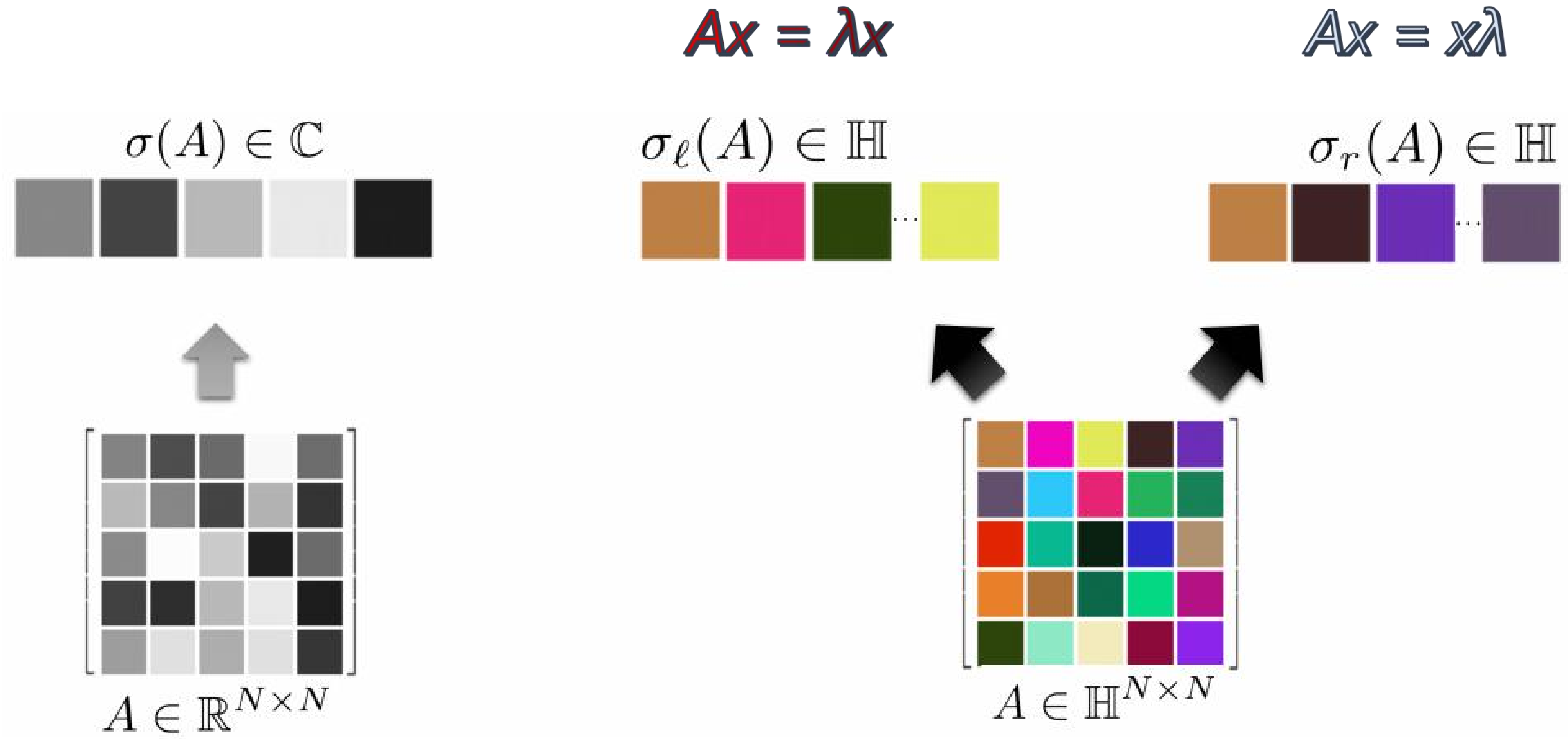

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Our «protagonists»

$$C^\top = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

Quaternion Circulant matrix

$$Q_N^\mu = \frac{1}{\sqrt{N}} \begin{pmatrix} w_{N\mu}^{0\cdot 0} & w_{N\mu}^{1\cdot 0} & \cdots & w_{N\mu}^{(N-1)\cdot 0} \\ w_{N\mu}^{0\cdot 1} & w_{N\mu}^{1\cdot 1} & \cdots & w_{N\mu}^{(N-1)\cdot 1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N\mu}^{0\cdot (N-1)} & w_{N\mu}^{1\cdot (N-1)} & \cdots & w_{N\mu}^{(N-1)\cdot (N-1)} \end{pmatrix}$$

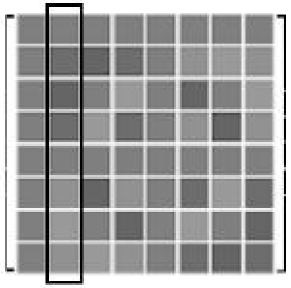
$$w_{N\mu} = e^{-\mu 2\pi N^{-1}}$$

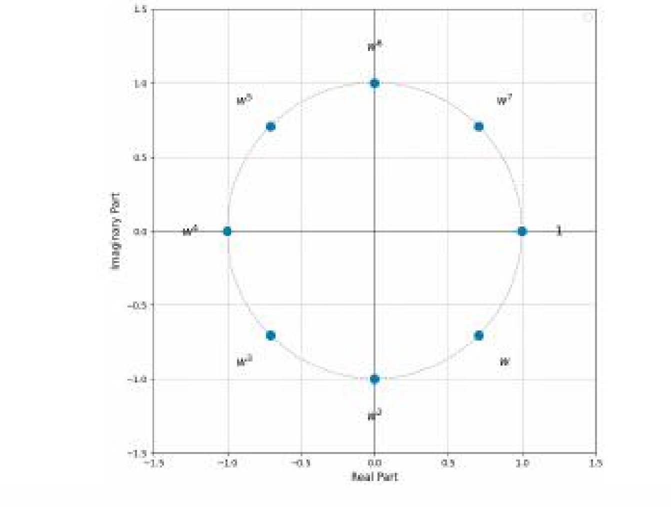
Quaternion Fourier matrix

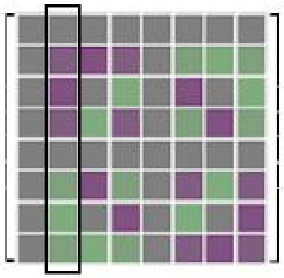
Basic theoretical results

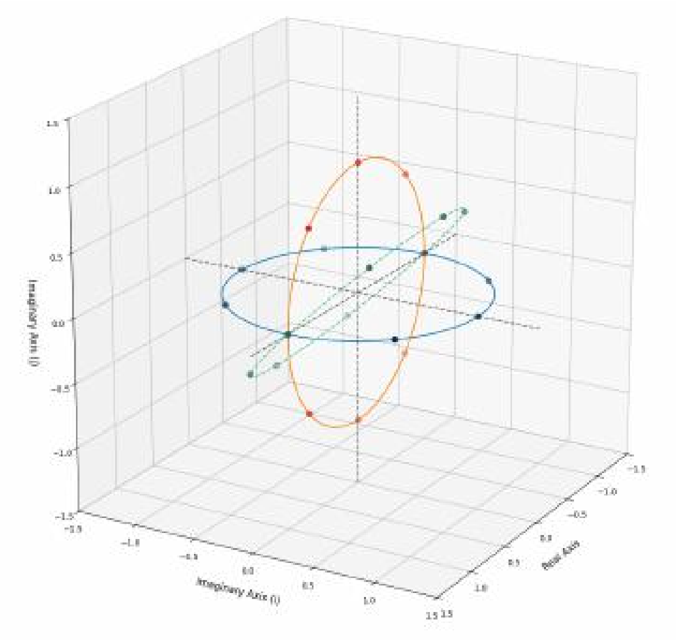
Proposition 3.4. *Let Q_N^{μ}, Q_N^{ν} Quaternionic Fourier matrices with non-collinear axes μ, ν. We can always find unit $p \in \mathbb{H}$ such that*

$$Q_N^{\mu} = pQ_N^{\nu}\bar{p} \tag{20}$$









Basic theoretical results

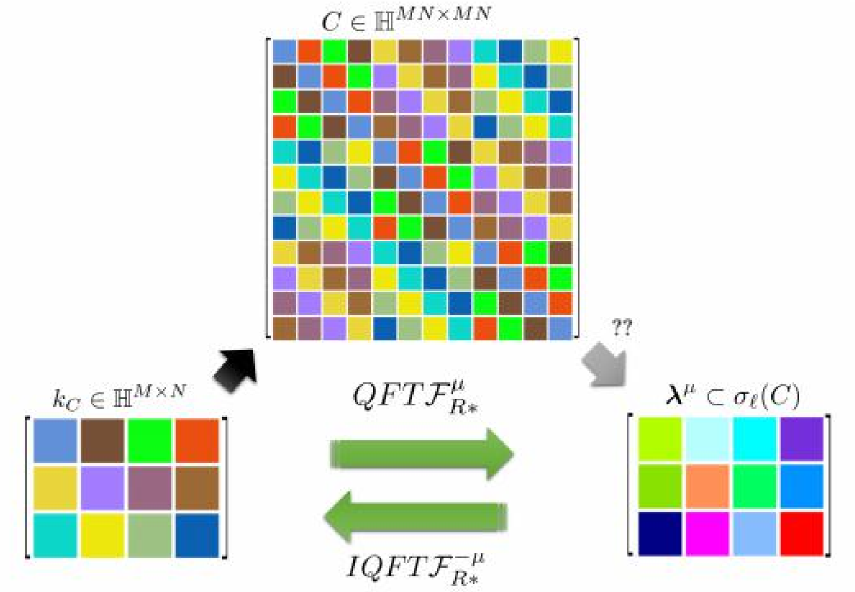
3.3 Connection of Circulant and Fourier matrices

The following results can then be proved, underpinning the relation between Quaternionic Circulant and Quaternionic Fourier matrices, in particular with respect to the left spectrum of the former:

Proposition 3.5 (Circulant & Fourier Matrices). *For any $C \in \mathbb{H}^{N\times N}$ that is circulant, and any pure unit $\mu \in \mathbb{H}$,*

(1) *Any column $k = 0, 1, \ldots, N-1$ of the inverse QFT matrix $Q_N^{-\mu}$ is an eigenvector of C. Column k corresponds to the k^{th} component of the vector of* left *eigenvalues $\boldsymbol{\lambda}^\mu = [\lambda_1^\mu \lambda_2^\mu \cdots \lambda_N^\mu]^T \in \mathbb{H}^N$. Vector $\boldsymbol{\lambda}^\mu$ is equal to the* right *QFT $\mathcal{F}_{R*}^\mu$ of the kernel of C.*

(2) *Any column $k = 0, 1, \ldots, N-1$ of the inverse QFT matrix $Q_N^{-\mu}$ is an eigenvector of C^H. Column k corresponds to the k^{th} component of the vector of* left *eigenvalues $\boldsymbol{\kappa}^\mu = [\kappa_1^\mu \kappa_2^\mu \cdots \kappa_N^\mu]^T \in \mathbb{H}^N$. The conjugate of the vector $\boldsymbol{\kappa}^\mu$ is equal to the* left *QFT $\mathcal{F}_{L*}^\mu$ of the kernel of C.*



Basic theoretical results

Proposition 3.7 (Eigenstructure of sums, products and inverse of circulant matrices)**.**

Let $L, K \in \mathbb{H}^{N\times N}$ circulant matrices, then the following propositions hold. (Analogous results hold for L, K doubly-block circulant).

(a) The sums $L+K$ and products LK, KL are also circulant. Any scalar product pL or Lp where $p \in \mathbb{H}$ also results to a circulant matrix.

(b) Given some pure unit axis $\mu \in \mathbb{H}$, let w be the i^{th} column of the inverse QFT matrix $Q_N^{-\mu}$, and $\lambda_i^\mu, \kappa_i^\mu$ are left eigenvalues of L, K with respect to the shared eigenvector w. Then, w is an eigenvector of...

i. $L+K$ with left eigenvalue equal to $\lambda_i^\mu + \kappa_i^\mu$.

ii. pL with left eigenvalue equal to $p\lambda_i^\mu$.

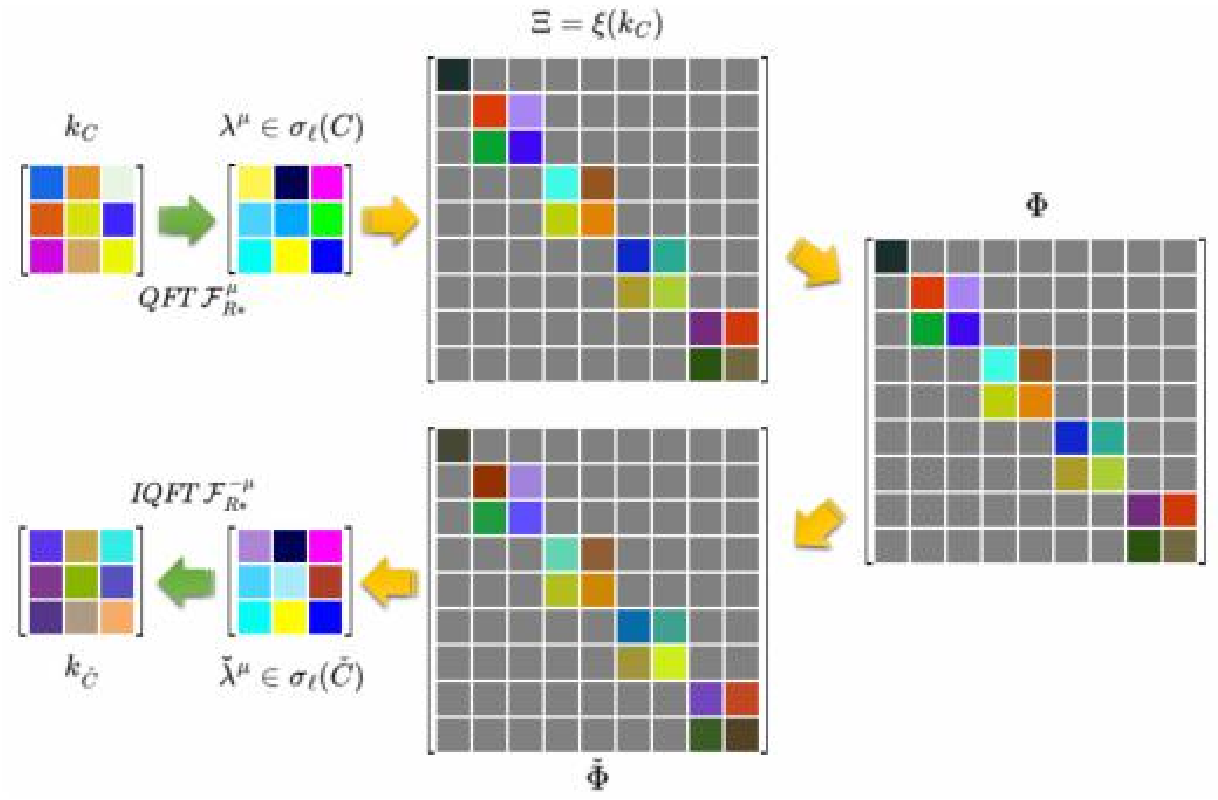
iii. Lp with left eigenvalue equal to $\lambda_i^{p\mu p^{-1}} p$.

iv. LK with left eigenvalue equal to $\lambda_i^{\kappa_i^\mu \mu [\kappa_i^\mu]^{-1}} \kappa_i^\mu$ if $\kappa_i^\mu \neq 0$ and equal to 0 if $\kappa_i^\mu = 0$.

v. Provided L^{-1} exists, $\lambda_i^\mu w (\lambda_i^\mu)^{-1}$ is an eigenvector of L^{-1} with left eigenvalue equal to $(\lambda_i^\mu)^{-1}$.

Proof of concept application

Fast computation of singular values, Lipschitz constant bounding



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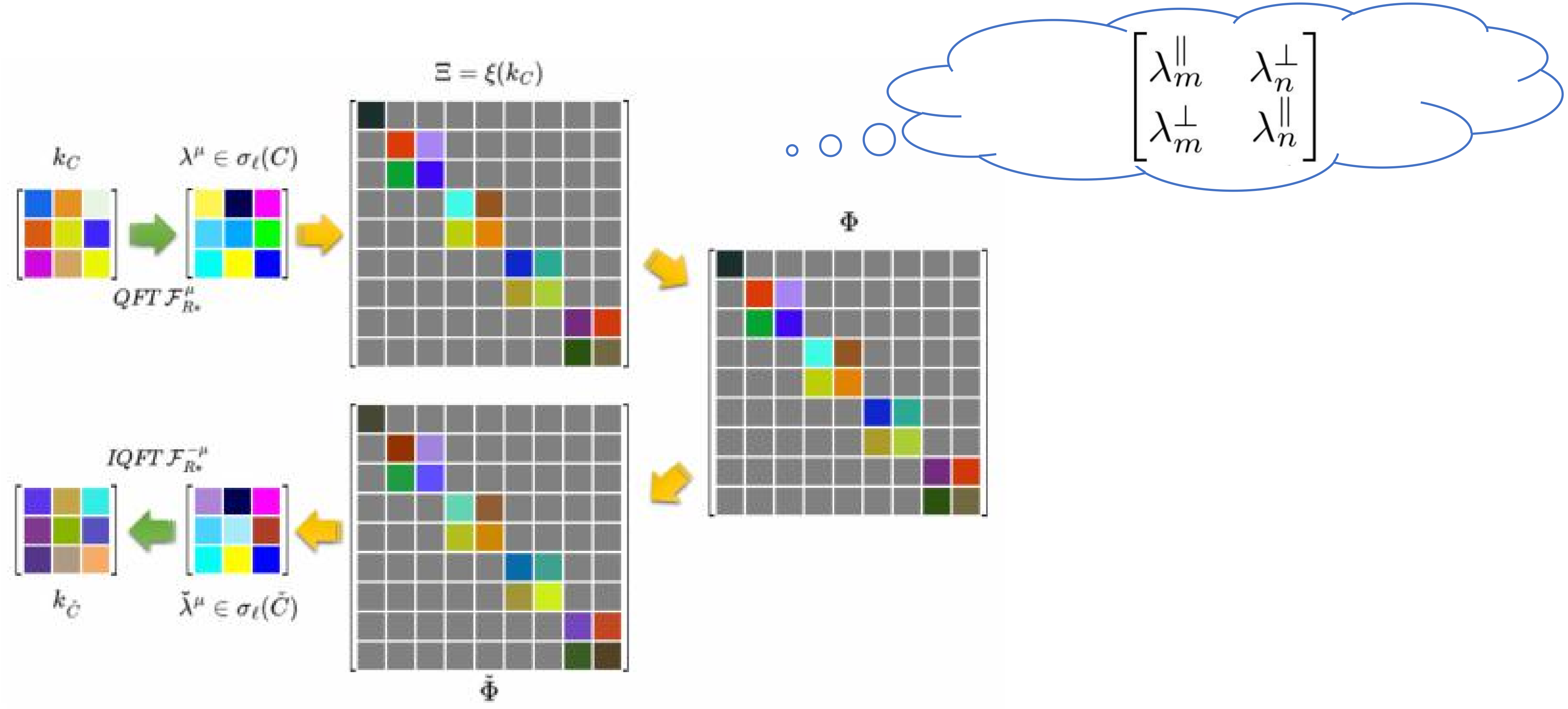

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Table 1: Runtime comparison for computation of Quaternionic Convolution Singular Values and Spectral Norm Clipping & Reconstruction. Convolution kernel size refers to $2D$ kernel side length. Average CPU time in milliseconds is reported; an exception are the fields containing an asterisk ($\star$), which denote that required runtime is impractical (measured in hours). Our results enable fast and exact carrying out of both tasks.

		Required time (in ms) ↓					
Convolution kernel size		4	8	16	32	64	128
Singular Value Computation	ours	4	16	80	210	708	1,795
	brute-force	5	95	1,244	69,319	$\star$	$\star$
Clipping & Reconstruction	ours	5	20	113	280	1,119	2,590
	brute-force	8	121	1,866	150,000	$\star$	$\star$

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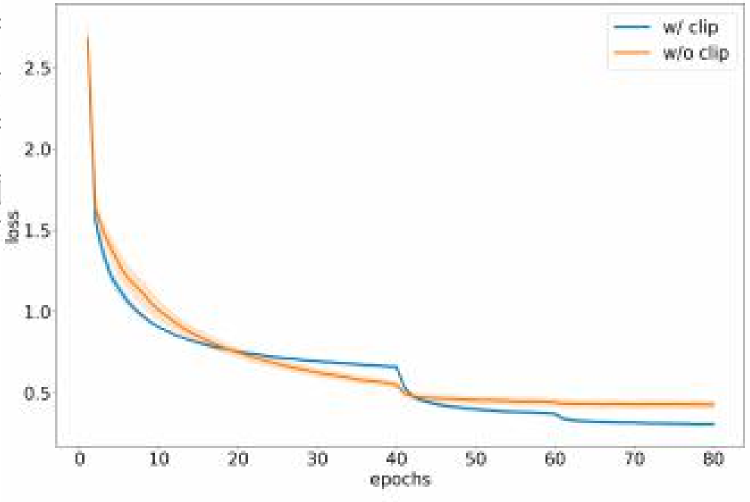

With our results, runs on $O(N^2 \log N)$
(vs previous best time of $O(N^6)$!!)

Thank you for your attention!

Come chat with us by our poster!

-- Poster Session 1 (Wednesday, 11.30 - 12.30, id#3)

paper

Open Review

code

GitHub

Questions ?