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Unlocking the Matrix Form of the Quaternion Fourier Transform and Quaternion Convolution

Properties, Connections, and Application to Lipschitz constant bounding

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The Quaternion ($\mathbb{H}$)

ROBOTICS GRAPHICS HYPERCOMPLEX NNs

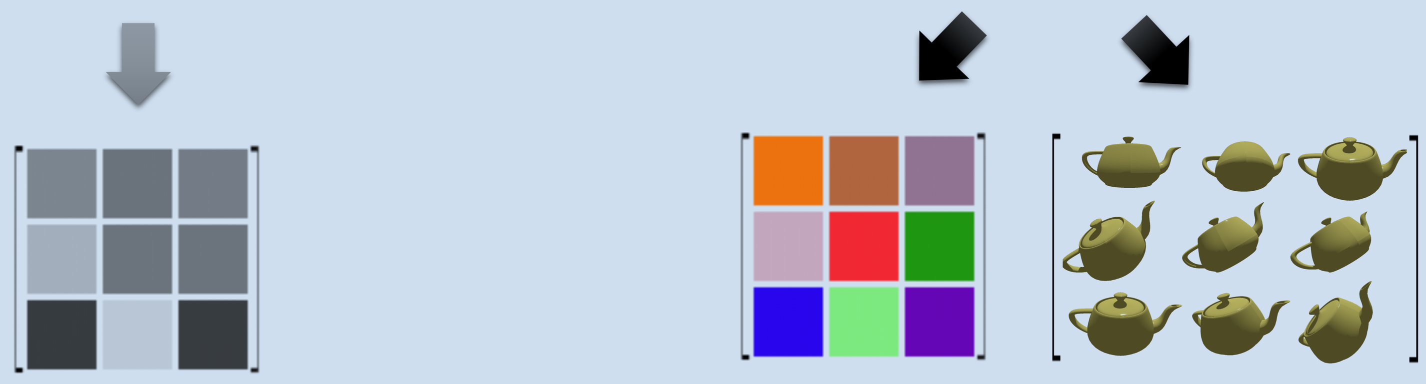
$$q = r + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

Quaternionic Matrices ($\mathbb{H}^{M \times N}$)

$$\begin{bmatrix} 6 & 4 & 7 \\ 8 & 4 & 7 \\ 2 & 9 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.7+0.0\mathbf{i}+0.7\mathbf{j}+0.0\mathbf{k} & 0.8+0.3\mathbf{i}+0.4\mathbf{j}+0.2\mathbf{k} & 0.0+1.0\mathbf{i}+0.0\mathbf{j}+0.0\mathbf{k} \\ 0.5+0.5\mathbf{i}+0.5\mathbf{j}+0.5\mathbf{k} & 0.6+0.8\mathbf{i}+0.0\mathbf{j}+0.0\mathbf{k} & 0.0-0.7\mathbf{i}+0.7\mathbf{j}+0.0\mathbf{k} \\ -0.0+0.6\mathbf{i}-0.6\mathbf{j}+0.5\mathbf{k} & 0.7+0.0\mathbf{i}+0.7\mathbf{j}+0.0\mathbf{k} & 0.8+0.3\mathbf{i}+0.4\mathbf{j}+0.2\mathbf{k} \end{bmatrix}$$



Matrix Forms in $\mathbb{H}$

$$C^\top = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

$$Q_N^\mu = \frac{1}{\sqrt{N}} \begin{pmatrix} w_{N\mu}^{0\cdot 0} & w_{N\mu}^{1\cdot 0} & \cdots & w_{N\mu}^{(N-1)\cdot 0} \\ w_{N\mu}^{0\cdot 1} & w_{N\mu}^{1\cdot 1} & \cdots & w_{N\mu}^{(N-1)\cdot 1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N\mu}^{0\cdot(N-1)} & w_{N\mu}^{1\cdot(N-1)} & \cdots & w_{N\mu}^{(N-1)\cdot(N-1)} \end{pmatrix}$$

$$w_{N\mu} = e^{-\mu 2\pi N^{-1}}$$

Convolution Theorem in $\mathbb{R}$,$\mathbb{C}$

$$g = Cf \implies Ag = AA^{-1}\Lambda_C Af \implies G = \Lambda_C F$$

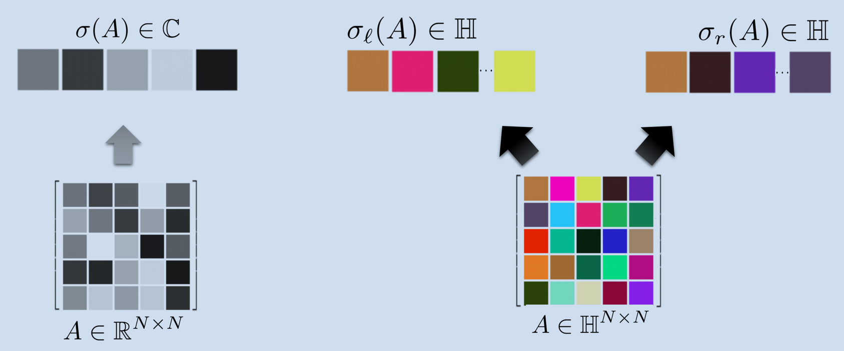
Circulant matrix (Complex) Fourier Matrix Kernel Eigenvalues

How does this transfer to H, if at all?

TL;DR

We show how the relation between *Convolution* and the *Fourier transform* generalizes to the quaternionic domain, vis-à-vis their respective matrix forms and eigenstructure.

Quaternionic eigenstructure: The whole is more than the sum of its parts



- In the real and complex domains, spectrum $\sigma(\cdot)$ is a subset of $\mathbb{C}$;
- In $\mathbb{H}$, we have *two distinct spectra* $\sigma_\ell(\cdot)$ and $\sigma_r(\cdot)$, corresponding to forms $Ax = \lambda x$ and $Ax = x\lambda$ respectively;
- No known well-posed connection exists that relates the two spectra, and in general they can both be infinite sets;
- With our results we shed light on this complex landscape!

Key Results

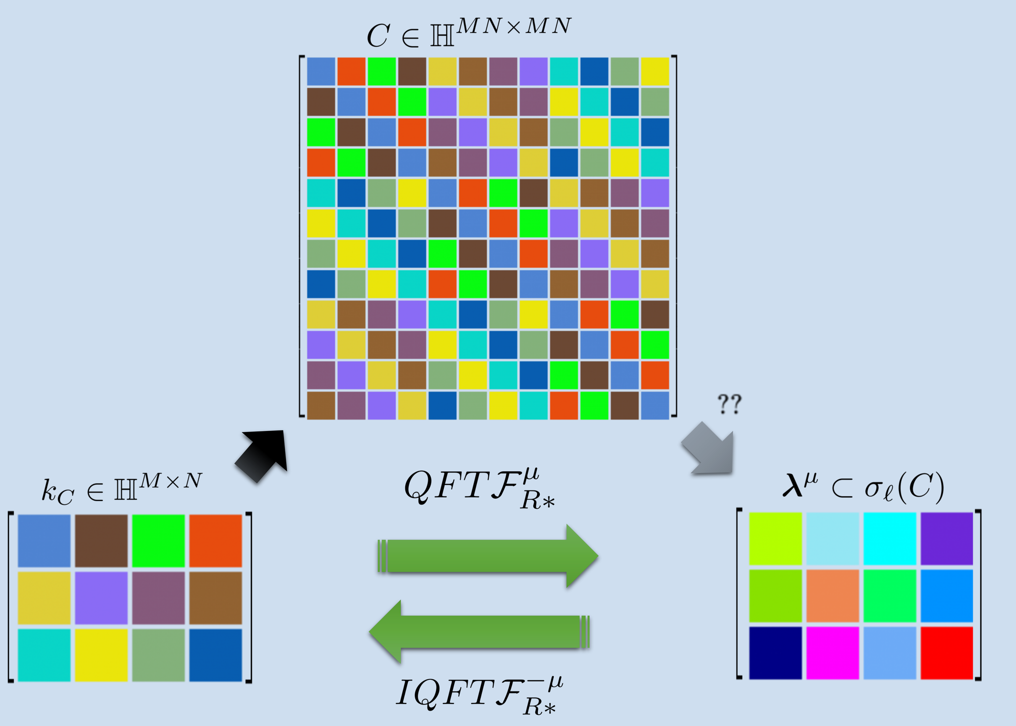
Proposition 3.5 (Circulant & Fourier Matrices)**.** *For any $C \in \mathbb{H}^{N\times N}$ that is circulant, and any pure unit $\mu \in \mathbb{H}$,*

(1) Any column $k = 0, 1, \dots, N-1$ of the inverse QFT matrix $Q_N^{-\mu}$ is an eigenvector of C. Column k corresponds to the k^{th} component of the vector of left *eigenvalues $\boldsymbol{\lambda}^\mu = [\lambda_1^\mu \lambda_2^\mu \cdots \lambda_N^\mu]^T \in \mathbb{H}^N$. Vector $\boldsymbol{\lambda}^\mu$ is equal to the* right *QFT $\mathcal{F}_{R*}^\mu$ of the kernel of C.*

(2) Any column $k = 0, 1, \dots, N-1$ of the inverse QFT matrix $Q_N^{-\mu}$ is an eigenvector of C^H. Column k corresponds to the k^{th} component of the vector of left *eigenvalues $\boldsymbol{\kappa}^\mu = [\kappa_1^\mu \kappa_2^\mu \cdots \kappa_N^\mu]^T \in \mathbb{H}^N$. The conjugate of the vector $\boldsymbol{\kappa}^\mu$ is equal to the* left *QFT $\mathcal{F}_{L*}^\mu$ of the kernel of C.*

where transforms denoted with an asterisk (*) refer to using a unitary coefficient instead of $1/\sqrt{N}$:

$$F_{L*}^\mu[u] = \sum_{n=0}^{N-1} e^{-\mu 2\pi N^{-1} nu} f[n], \quad F_{R*}^\mu[u] = \sum_{n=0}^{N-1} f[n] e^{-\mu 2\pi N^{-1} nu}.$$



Proposition 3.7 (Eigenstructure of sums, products and inverse of circulant matrices)**.**

Let $L, K \in \mathbb{H}^{N\times N}$ circulant matrices, then the following propositions hold. (Analogous results hold for L, K doubly-block circulant).

(a) The sums $L + K$ and products LK, KL are also circulant. Any scalar product pL or Lp where $p \in \mathbb{H}$ also results to a circulant matrix.

(b) Given some pure unit axis $\mu \in \mathbb{H}$, let w be the i^{th} column of the inverse QFT matrix $Q_N^{-\mu}$, and $\lambda_i^\mu, \kappa_i^\mu$ are left eigenvalues of L, K with respect to the shared eigenvector w. Then, w is an eigenvector of…

i. $L + K$ with left eigenvalue equal to $\lambda_i^\mu + \kappa_i^\mu$.

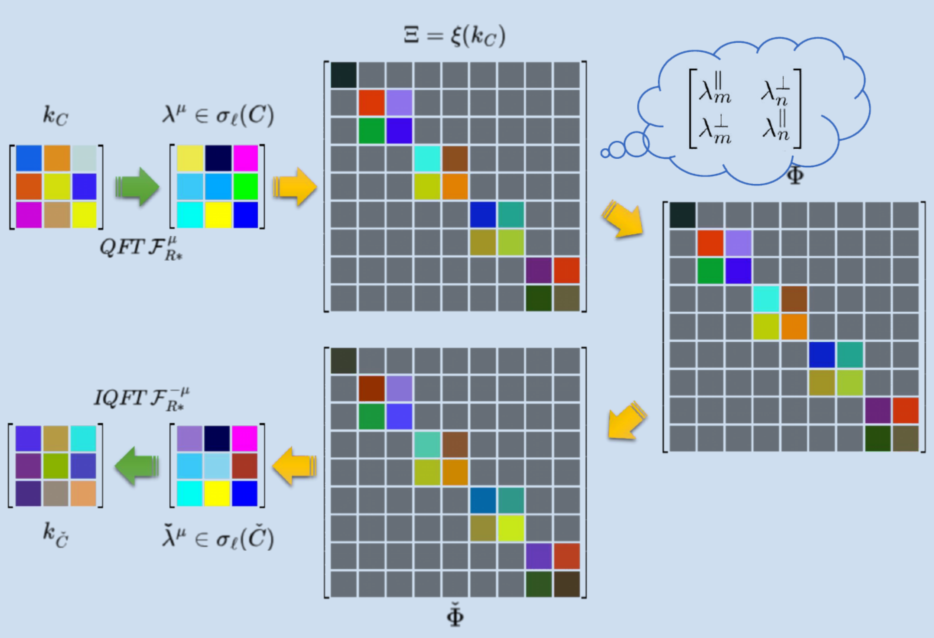
ii. pL with left eigenvalue equal to $p\lambda_i^\mu$.

iii. Lp with left eigenvalue equal to $\lambda_i^{p\mu p^{-1}} p$.

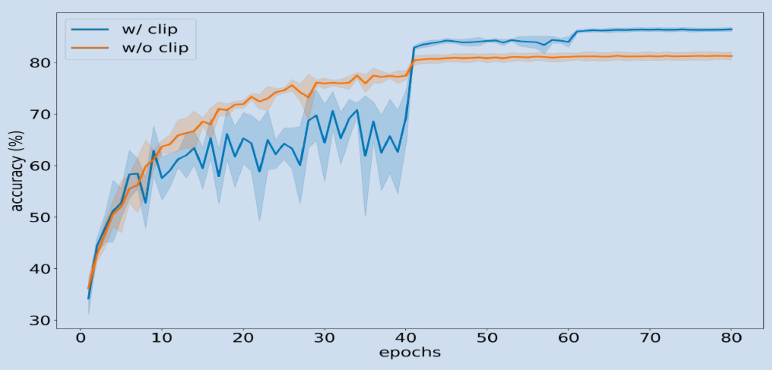
iv. LK with left eigenvalue equal to $\lambda_i^{\kappa_i^\mu \mu [\kappa_i^\mu]^{-1}} \kappa_i^\mu$ if $\kappa_i^\mu \neq 0$ and equal to 0 if $\kappa_i^\mu = 0$.

v. Provided L^{-1} exists, $\lambda_i^\mu w (\lambda_i^\mu)^{-1}$ is an eigenvector of L^{-1} with left eigenvalue equal to $(\lambda_i^\mu)^{-1}$.

Proof of concept: Lipschitz constant bounding



		Required time (in ms) ↓					
Convolution kernel size		4	8	16	32	64	128
Singular Value Computation	ours	4	16	80	210	708	1,795
	brute-force	5	95	1,244	69,319	⋆	⋆
Clipping & Reconstruction	ours	5	20	113	280	1,119	2,590
	brute-force	8	121	1,866	150,000	⋆	⋆



Enables bounding the constant in ***O(N²logN)*** (vs ***O(N⁶)*** for 'brute force'!)

Perspectives

- "Quaternionized" models that include convolution (Conformer, Fast Fourier Convolution, …)
- Quaternion convolution can be used as an operator in richer observation models (cf. $p(g|f) \propto \exp\{-\frac{\beta}{2}\|g - Hf\|^2\}$)
- Enables lightweight versions of models where Lipschitz constant determination is important (Wasserstein GANs)
- …

References & Links

A.K.Jain, "Fundamentals of Digital Image Processing" (1988); F.Zhang, "Quaternion Matrices" (1998); F.Dyson, "Quaternion Determinants" (1972); T.Ell et al., "Quaternion Fourier Transforms for Signal Image Processing" (2014).

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