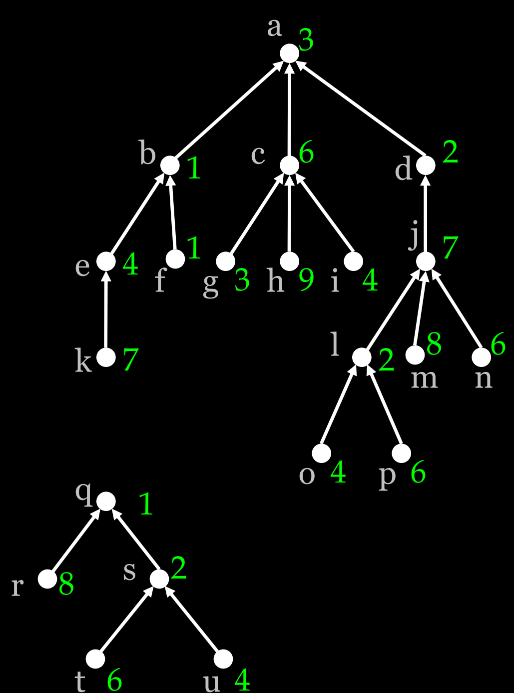
Dynamic Trees

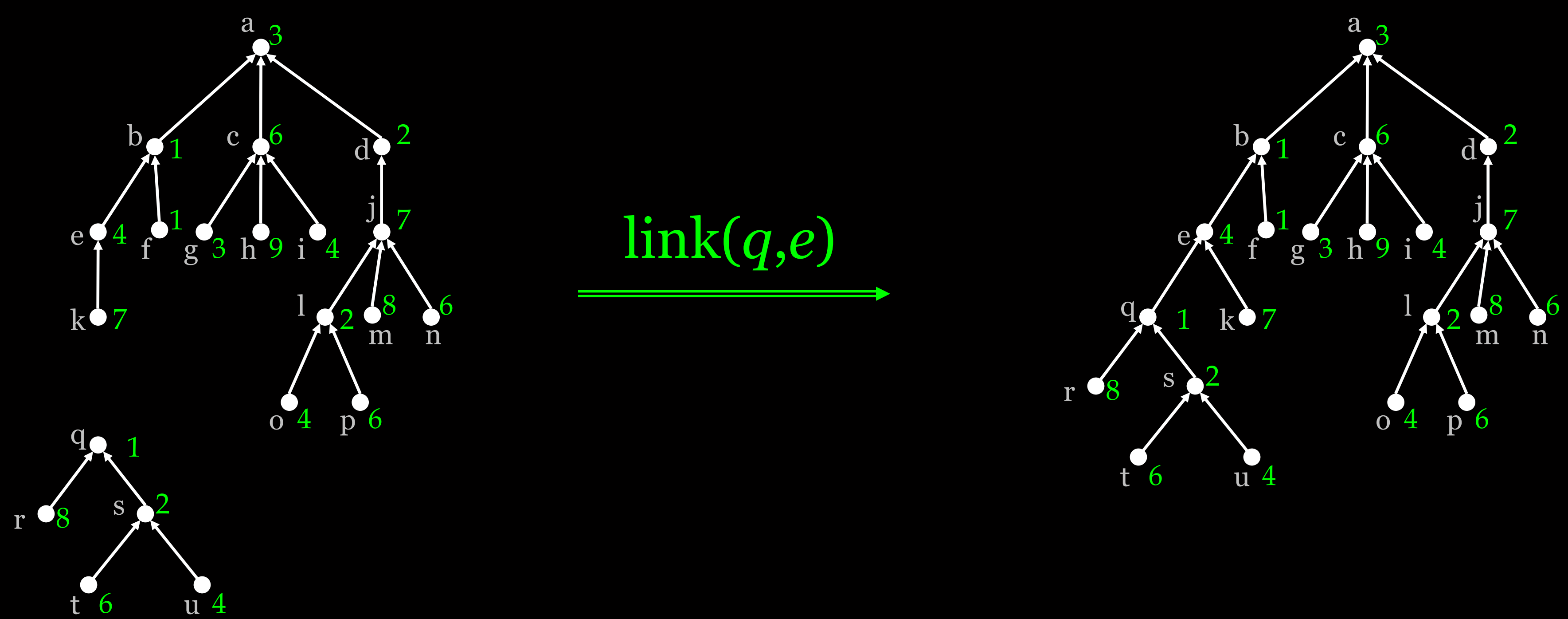
- Goal: maintain a forest of rooted trees with costs on vertices.
  - Each tree has a root, every edge directed towards the root.
- Operations allowed:
  - link(v,w): creates an edge between v (a root) and w.
  - cut(v,w): deletes edge (v,w).
  - findcost(v): returns the cost of vertex v.
  - findroot(v): returns the root of the tree containing v.
  - findmin(v): returns the vertex w of minimum cost on the path from v to the root (if there is a tie, choose the closest to the root).
  - addcost(v,x): adds x to the cost every vertex from v to root.

Dynamic Trees

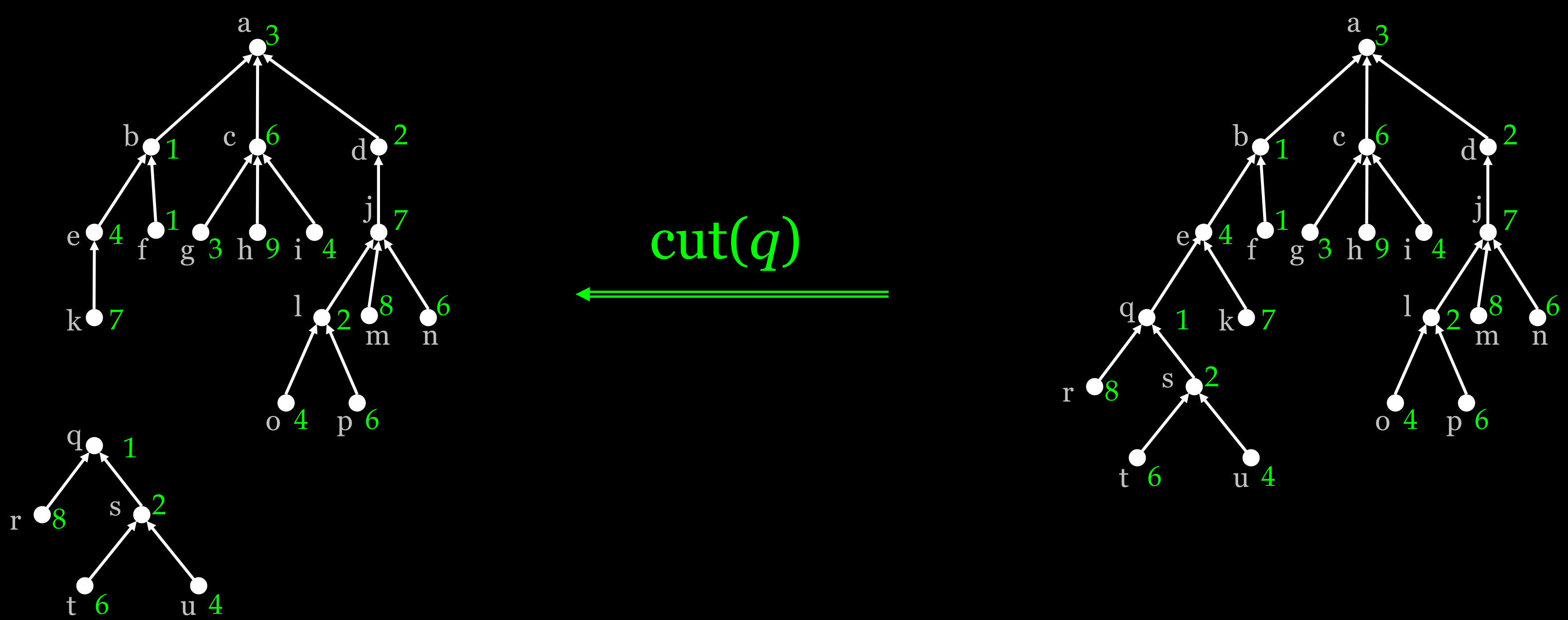
- An example (two trees):



Dynamic Trees

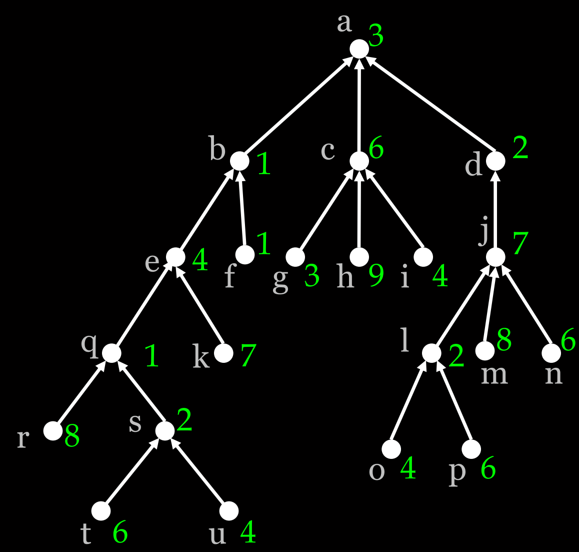


Dynamic Trees

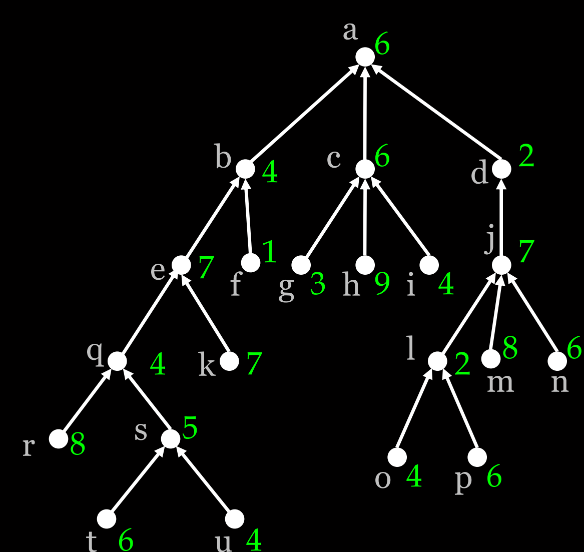


Dynamic Trees

- findmin(s) = b
- findroot(s) = a
- findcost(s) = 2



- addcost(s,3)

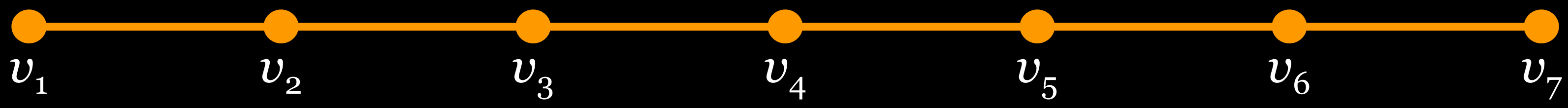


Obvious Implementation

- A node represents each vertex;
- Each node x points to its parent $p(x)$:
  - cut, split, findcost: constant time.
  - findroot, findmin, addcost: linear time on the size of the path.
- Acceptable if paths are small, but O(n) in the worst case.
- Cleverer data structures achieve O(log n) for all operations.

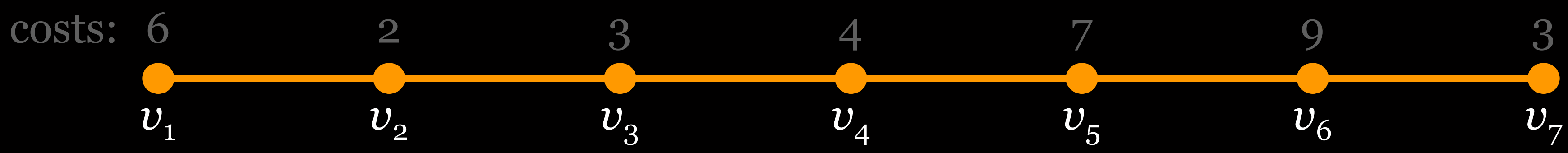
Simple Paths

- We start with a simpler problem:
  - Maintain set of paths subject to:
    - split: cuts a path in two;
    - concatenate: links endpoints of two paths, creating a new path.
  - Operations allowed:
    - findcost(v): returns the cost of vertex v;
    - addcost(v,x): adds x to the cost of vertices in path containing v;
    - findmin(v): returns minimum-cost vertex path containing v.



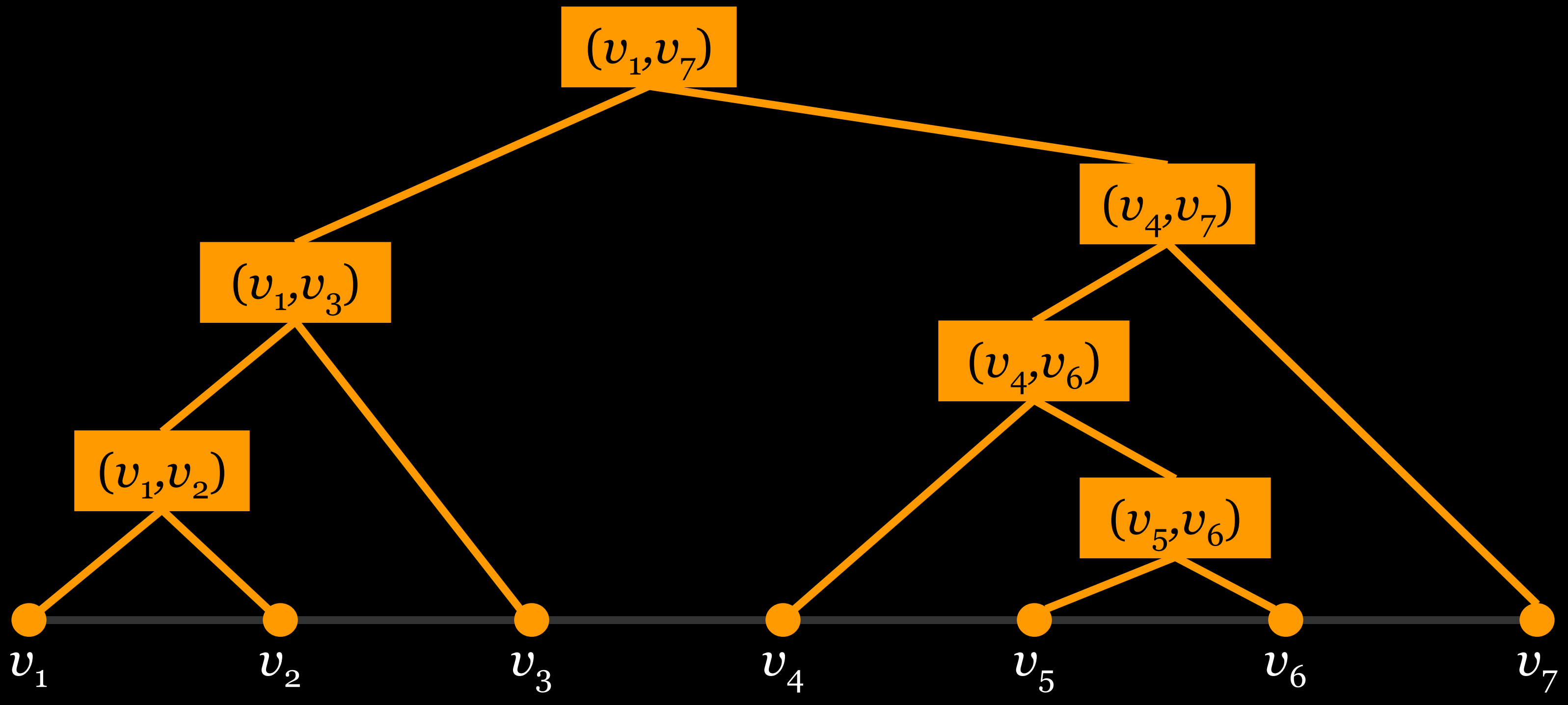
Simple Paths as Lists

- Natural representation: doubly linked list.
  - Constant time for findcost.
  - Constant time for concatenate and split if endpoints given, linear time otherwise.
  - Linear time for findmin and addcost.
- Can we do it O(log n) time?



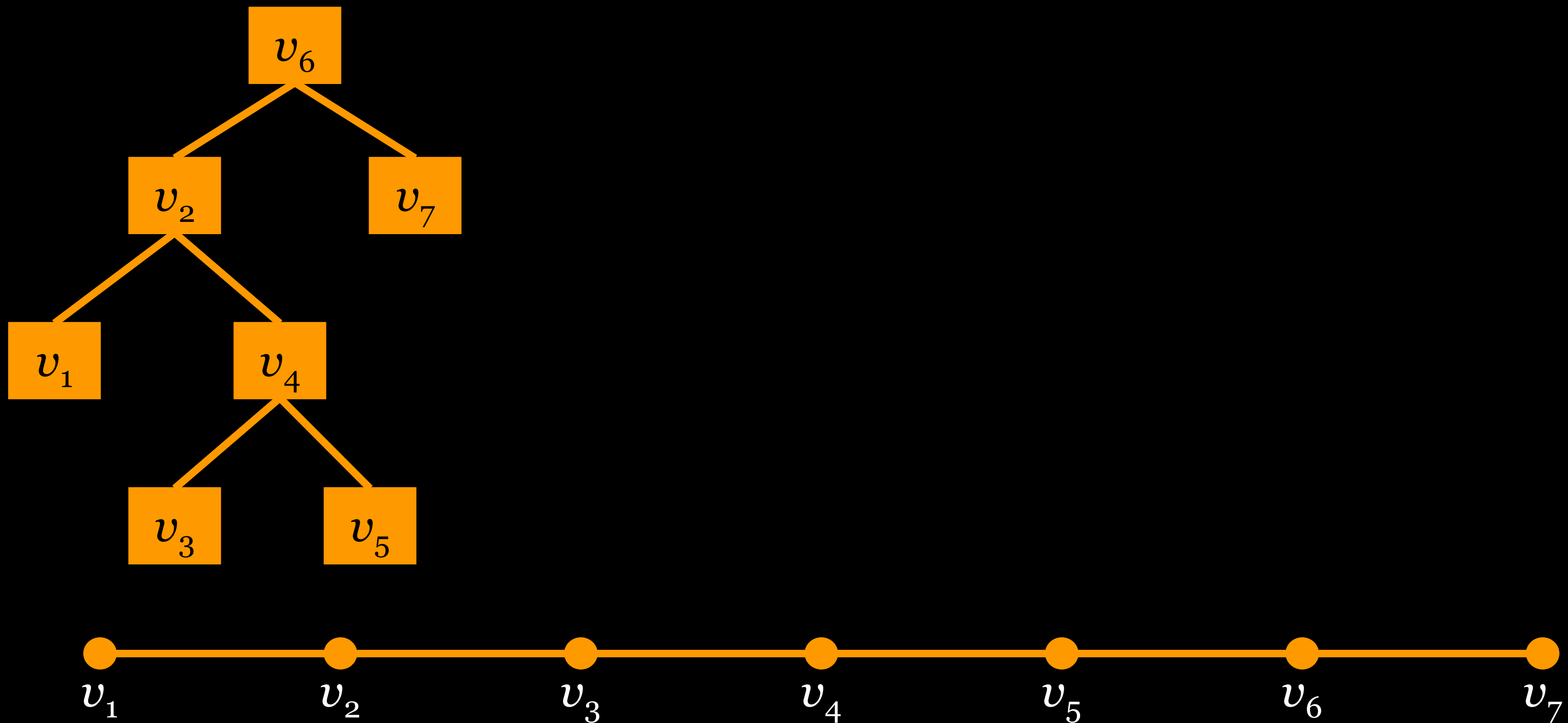
Simple Paths as Binary Trees

- Alternative representation: balanced binary trees.
  - Leaves: vertices in symmetric order.
  - Internal nodes: subpaths between extreme descendants.



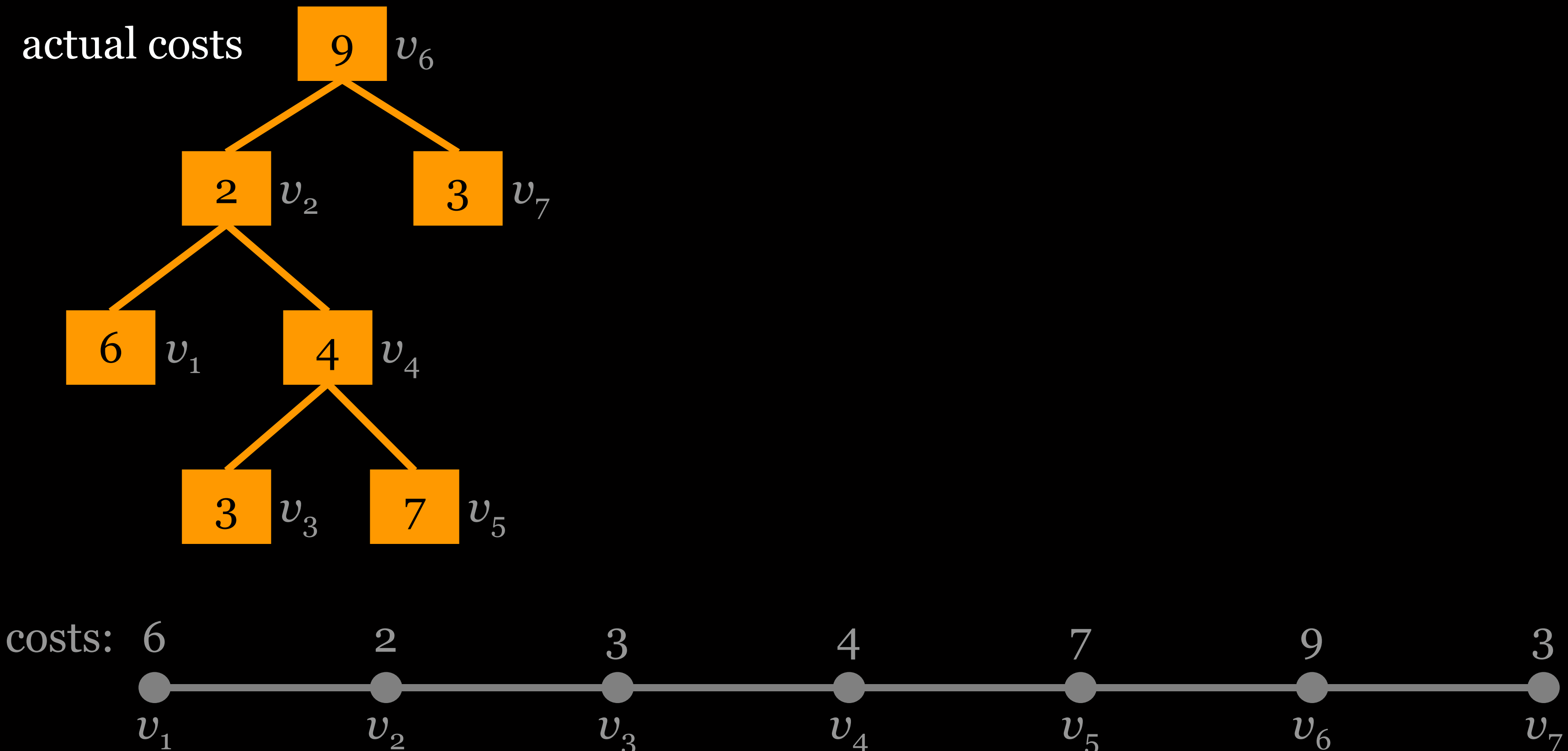
Simple Paths as Binary Trees

- Compact alternative:
  - Each internal node represents both a vertex and a subpath:
    - subpath from leftmost to rightmost descendant.



Simple Paths: Maintaining Costs

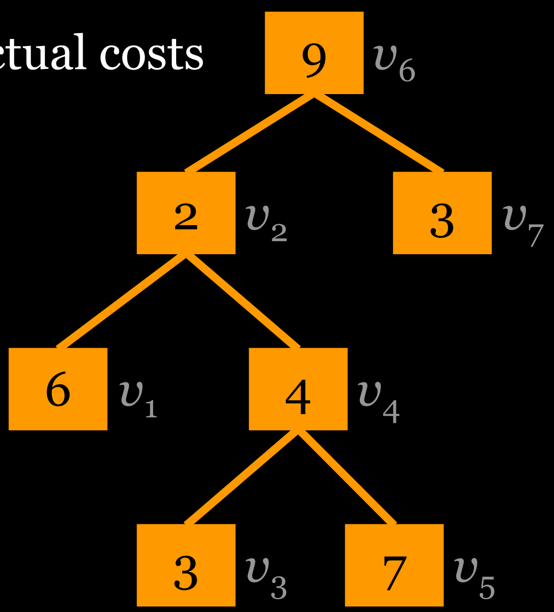
- Keeping costs:
  - First idea: store cost(x) directly on each vertex;
  - Problem: addcost takes linear time (must update all vertices).



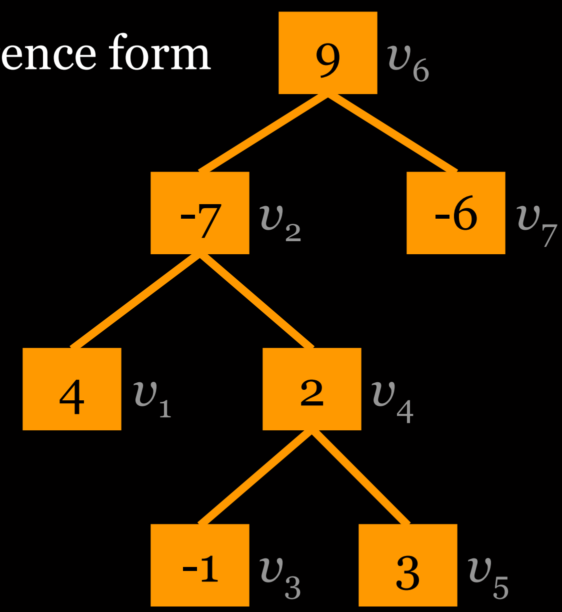
Simple Paths: Maintaining Costs

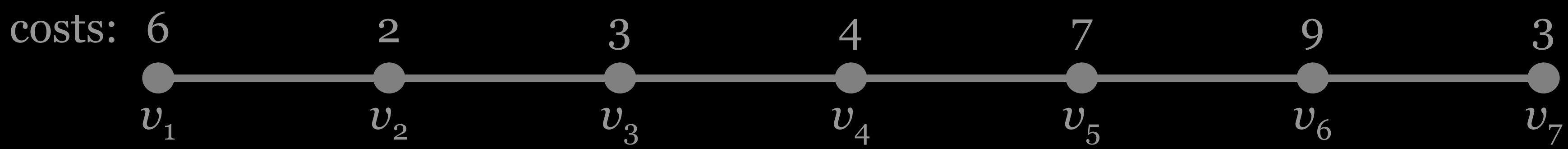
- Better approach: store $\Delta cost(x)$ instead:
  - Root: $\Delta cost(x) = cost(x)$
  - Other nodes: $\Delta cost(x) = cost(x) - cost(p(x))$

actual costs



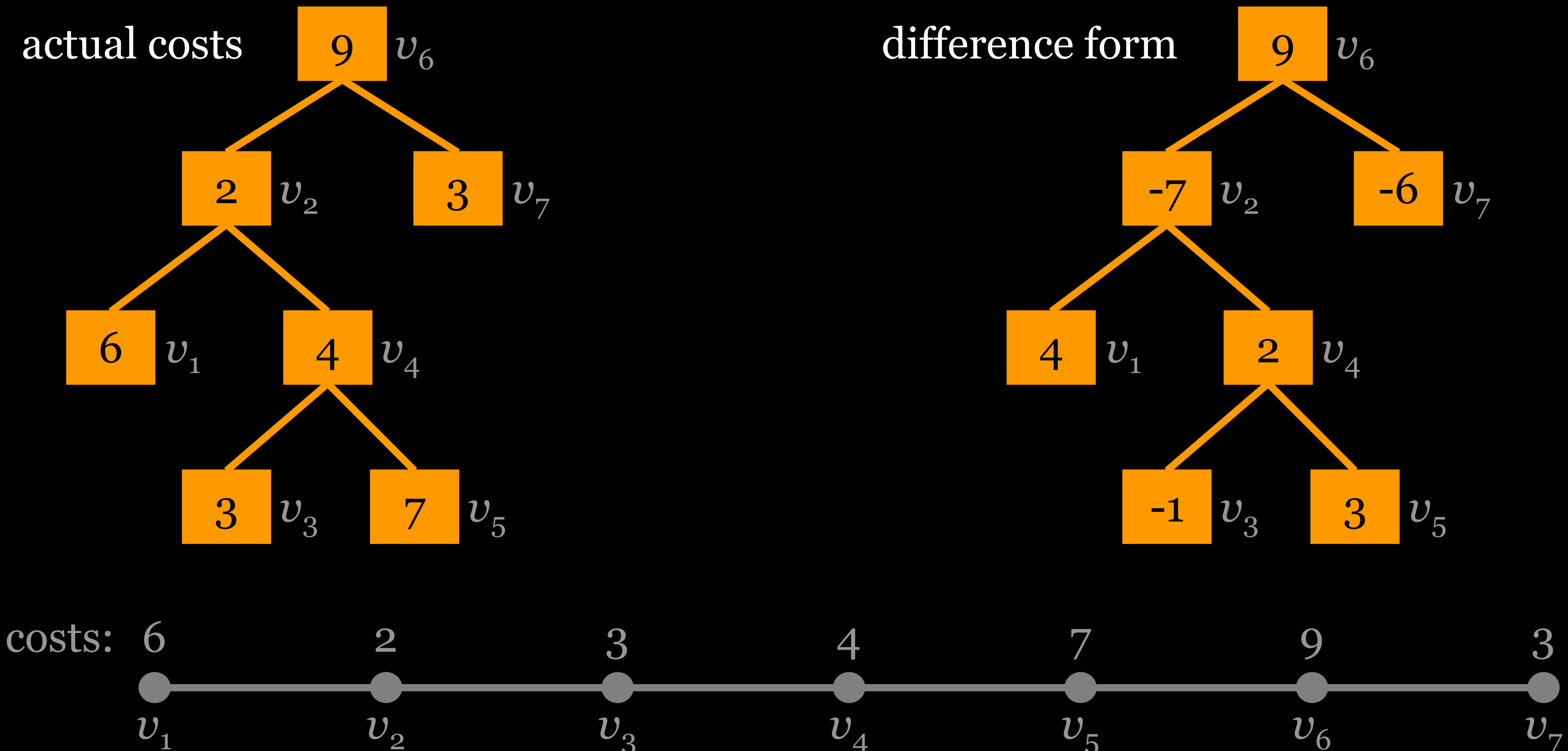
difference form





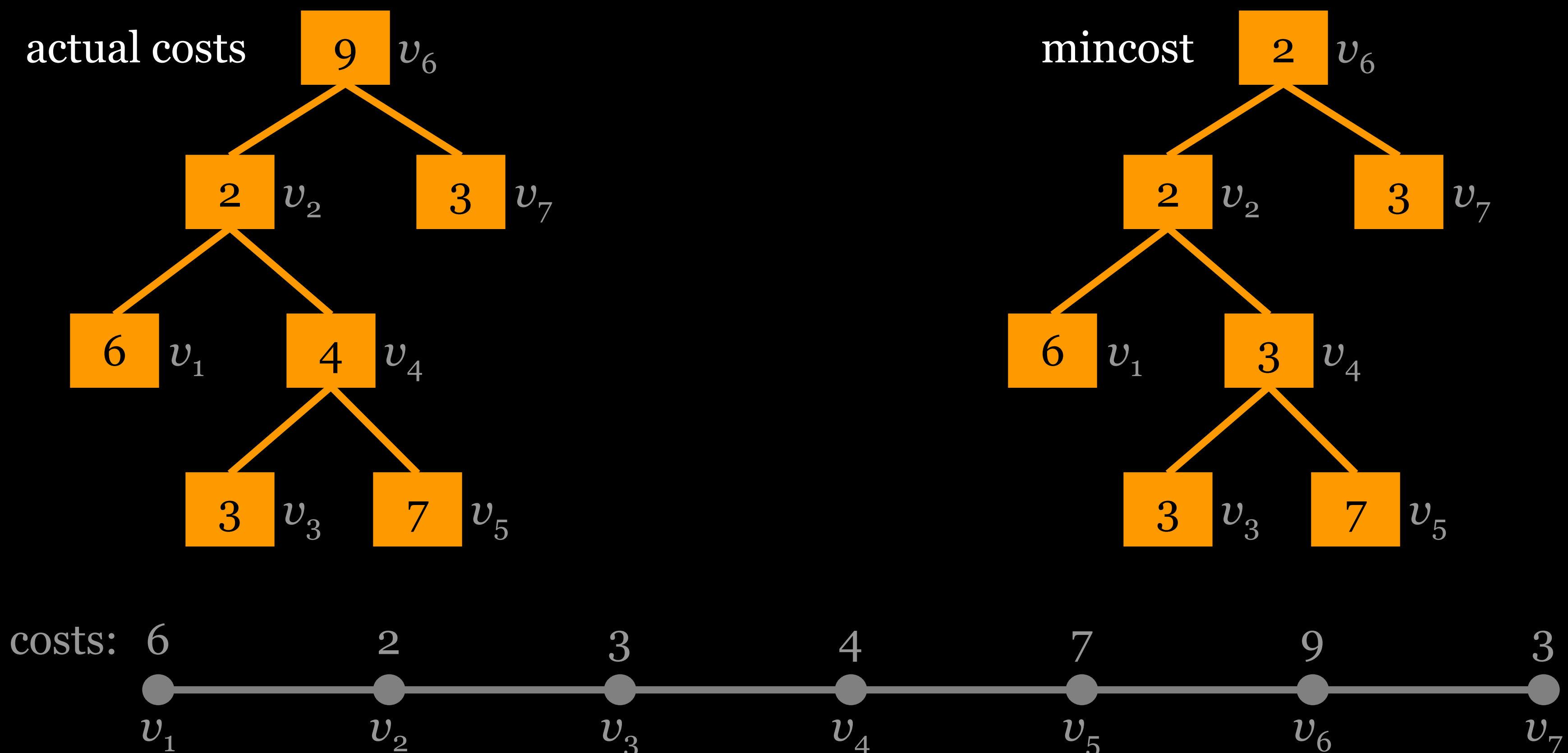
Simple Paths: Maintaining Costs

- Costs:
  - addcost: constant time (just add to root)
  - Finding cost(x) is slightly harder: O(depth(x)).



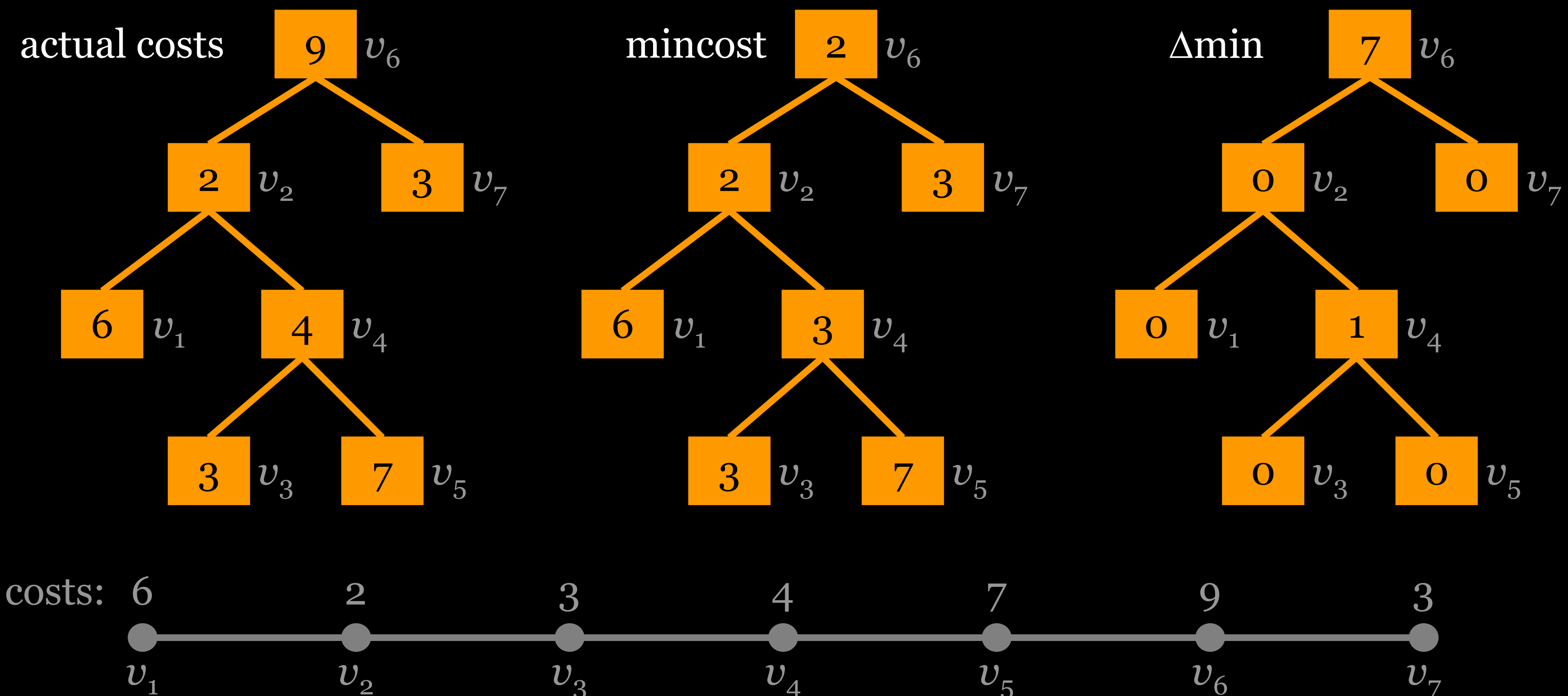
Simple Paths: Finding Minima

- Still have to implement findmin:
  - Store mincost(x), the minimum cost on subpath with root x.
    - findmin runs in O(log n) time, but addcost is linear.



Simple Paths: Finding Minima

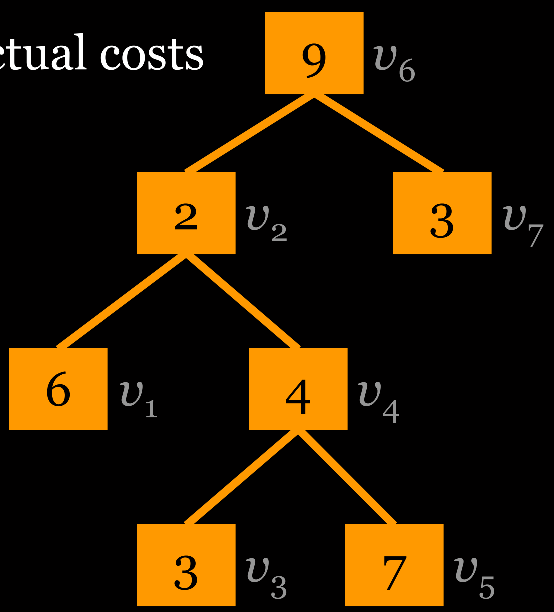
- Store $\Delta\min(x) = \text{cost}(x) - \text{mincost}(x)$ instead.
  - findmin still runs in O(log n) time, addcost now constant.



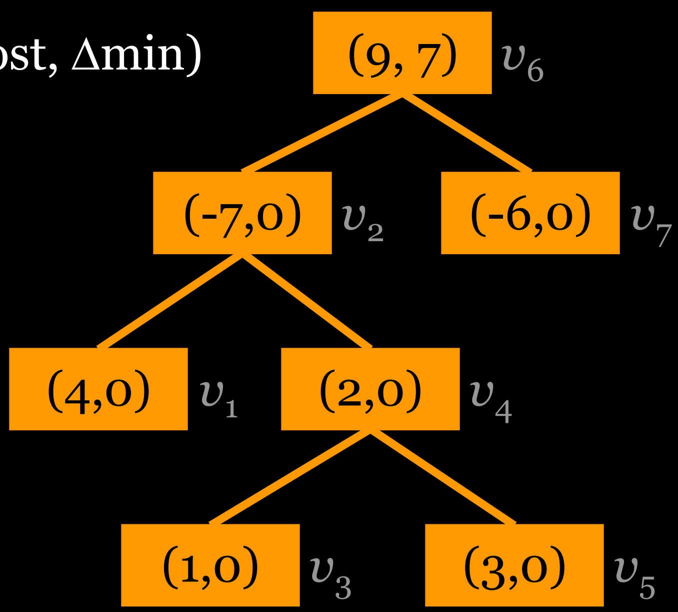
Simple Paths: Data Fields

- Final version:
  - Stores $\Delta\min(x)$ and $\Delta\text{cost}(x)$ for every vertex

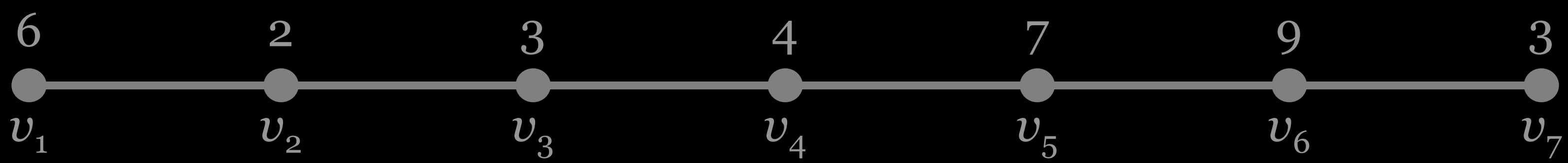
actual costs



(Δcost, Δmin)



costs:

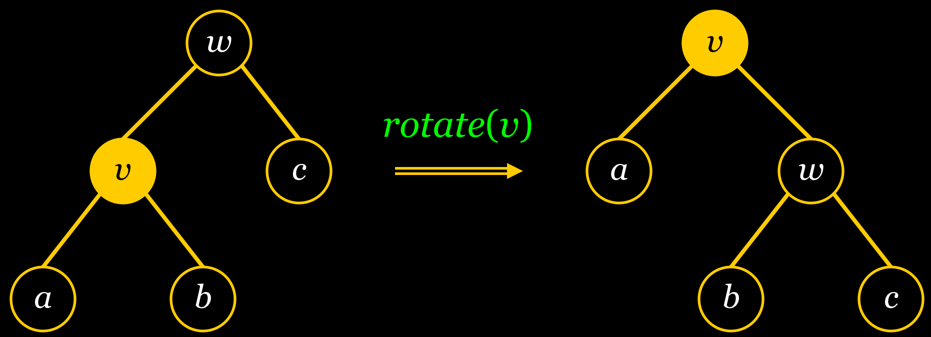


Simple Paths: Structural Changes

- Concatenating and splitting paths:
  - Join or split the corresponding binary trees;
  - Time proportional to tree height.
  - For balanced trees, this is $O(\log n)$.
    - Rotations must be supported in constant time.
    - We must be able to update Δmin and Δcost.

Simple Paths: Structural Changes

- Restructuring primitive: *rotation*.



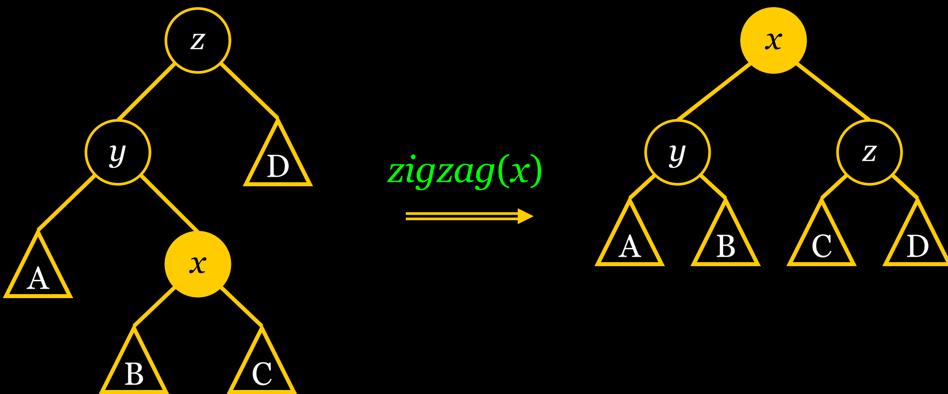
- Fields are updated as follows (for left and right rotations):
  - $\Delta cost'(v) = \Delta cost(v) + \Delta cost(w)$
  - $\Delta cost'(w) = -\Delta cost(v)$
  - $\Delta cost'(b) = \Delta cost(v) + \Delta cost(b)$
  - $\Delta min'(w) = \max\{0, \Delta min(b) - \Delta cost'(b), \Delta min(c) - \Delta cost(c)\}$
  - $\Delta min'(v) = \max\{0, \Delta min(a) - \Delta cost(a), \Delta min'(w) - \Delta cost'(w)\}$

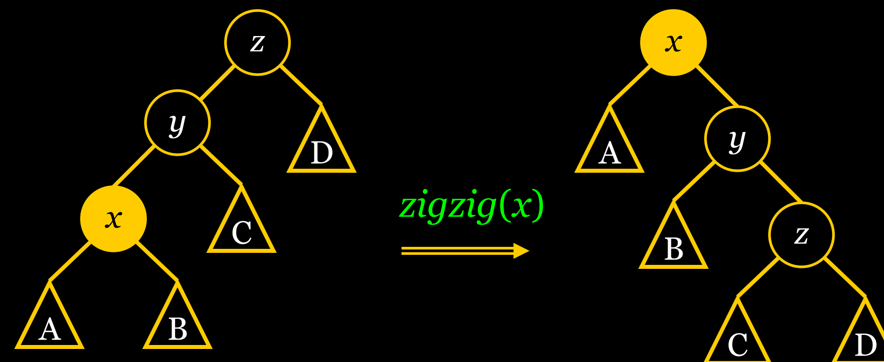
Splaying

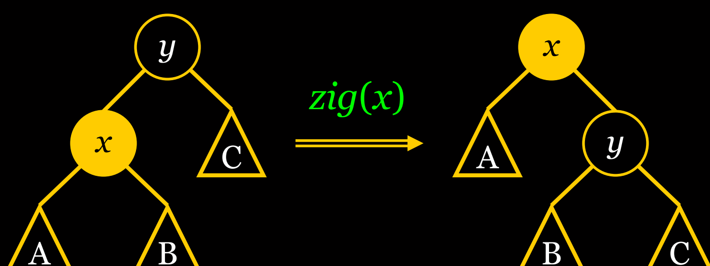
- Simpler alternative to balanced binary trees: splaying.
  - Does not guarantee that trees are balanced in the worst case.
  - Guarantees $O(\log n)$ access in the amortized sense.
  - Makes the data structure much simpler to implement.
- Basic characteristics:
  - Does not require any balancing information;
  - On an access to v, splay on v:
    - Moves v to the root;
    - Roughly halves the depth of other nodes in the access path.
  - Based entirely on rotations.
- Other operations (insert, delete, join, split) use splay.

Splaying

- Three restructuring operations:

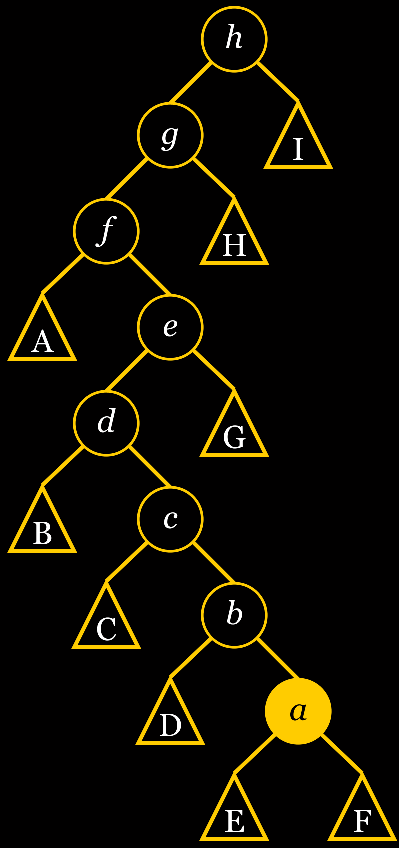




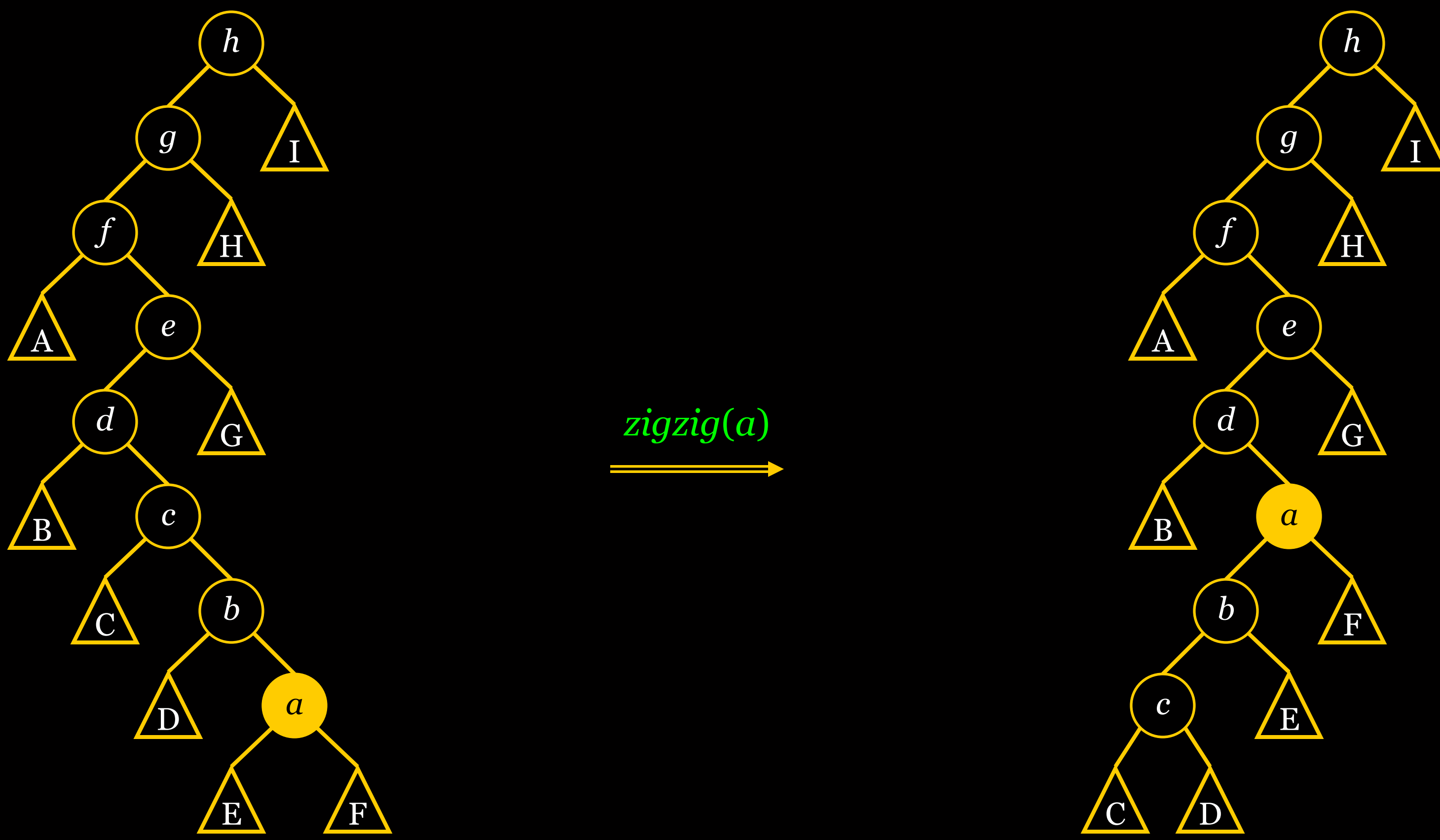


(only happens if y is root)

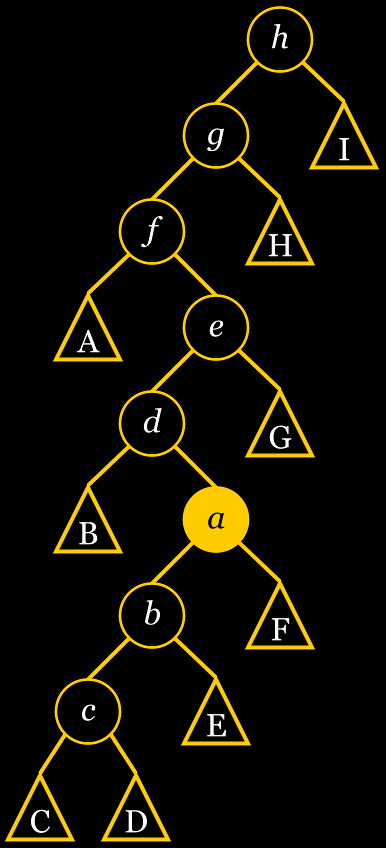
An Example of Splaying



An Example of Splaying



An Example of Splaying



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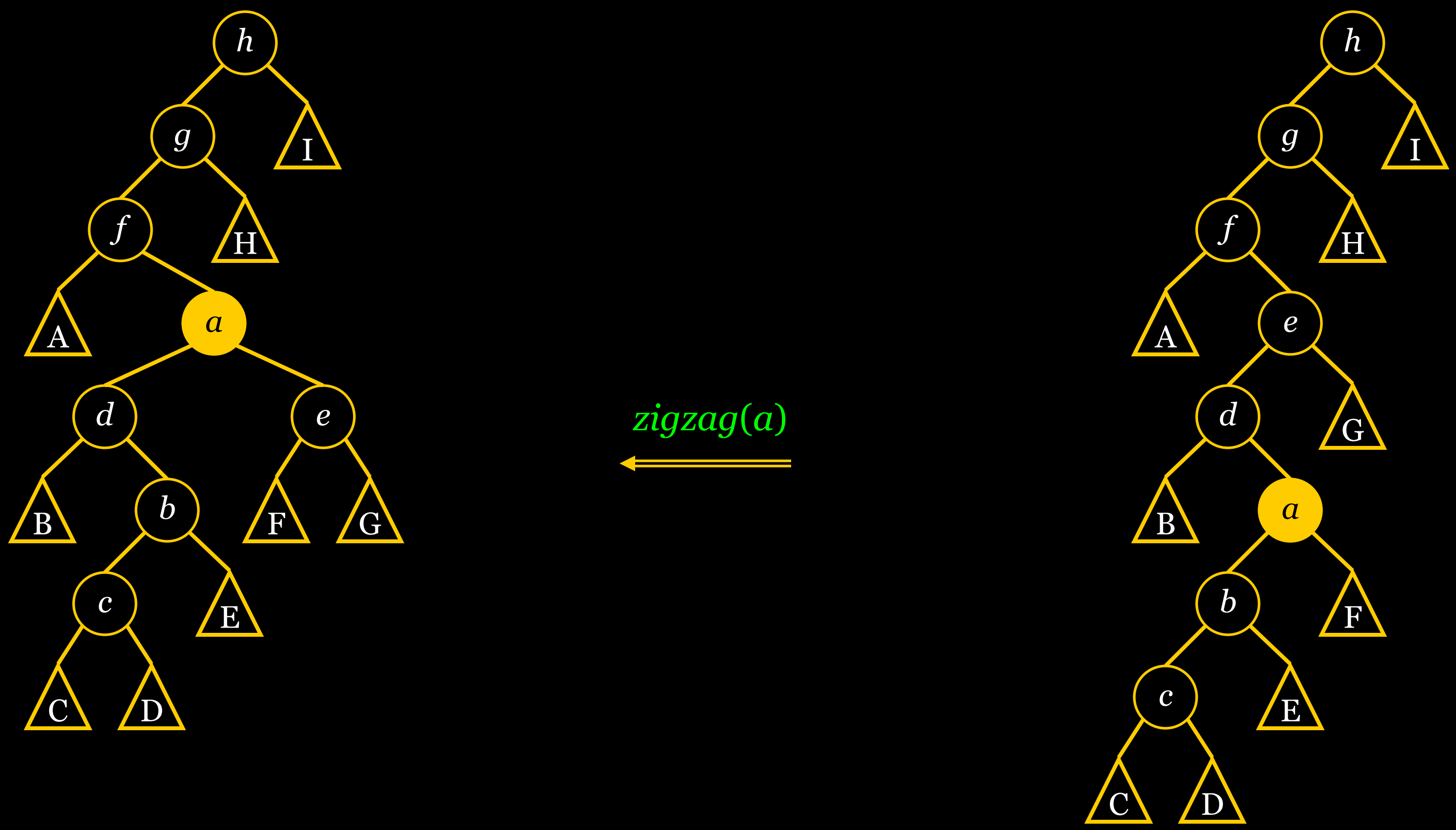
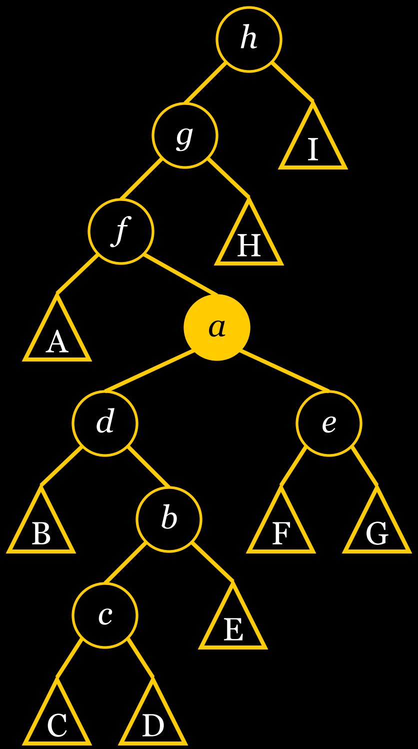
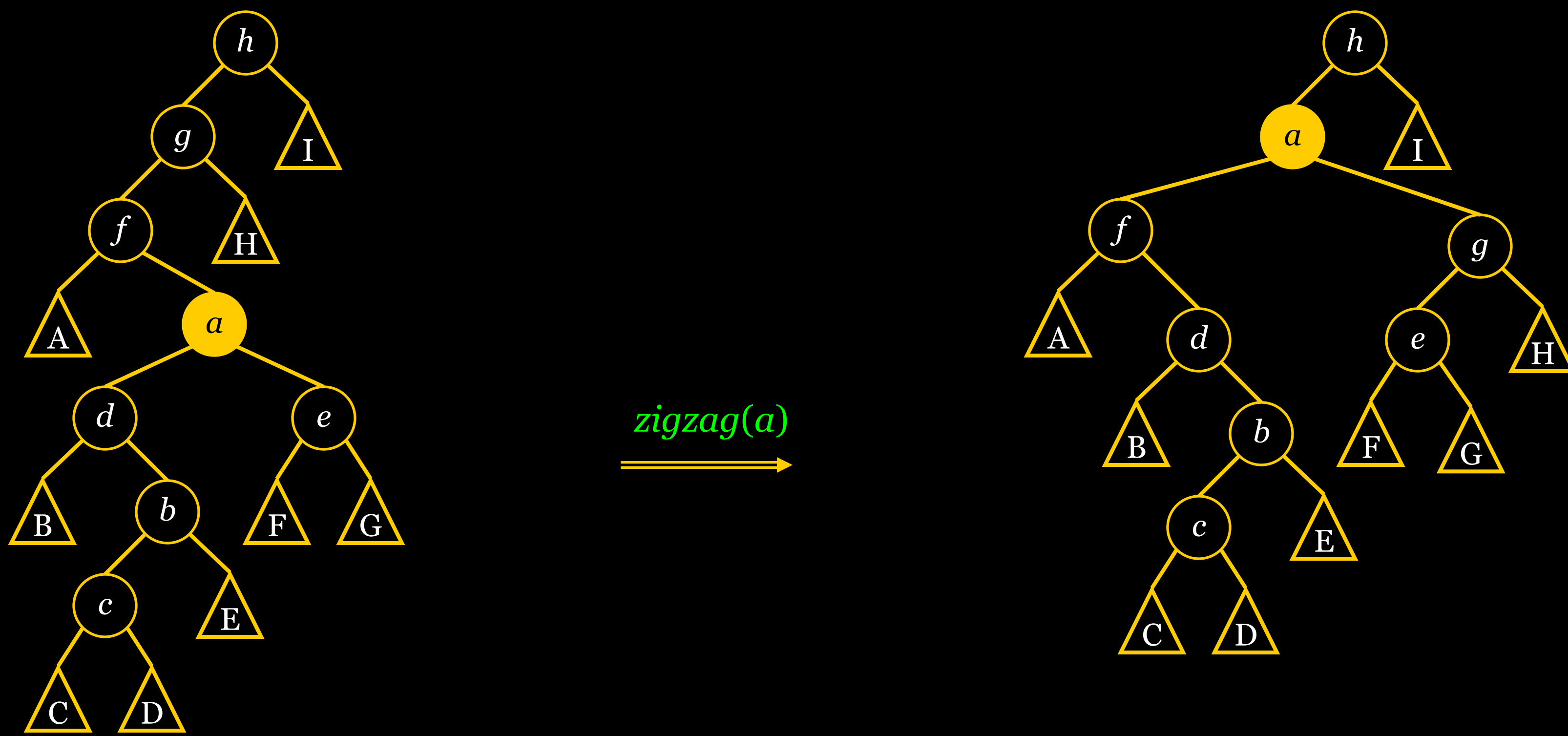


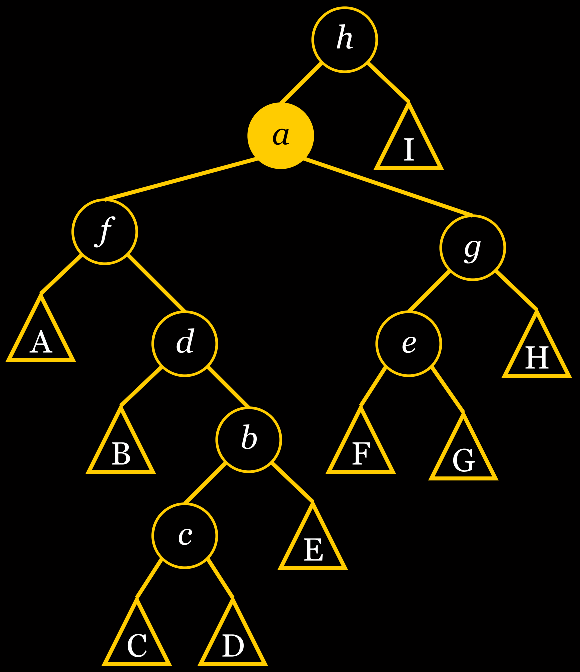
An Example of Splaying



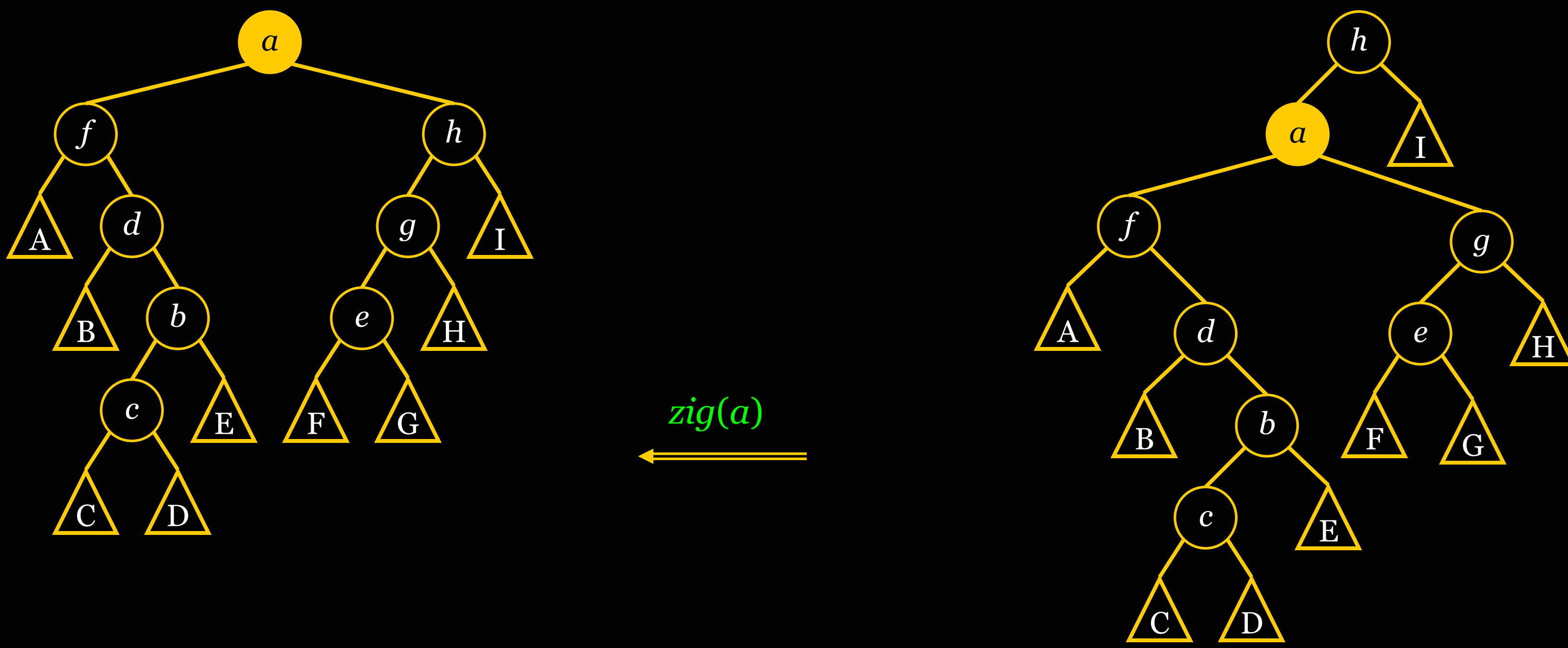
An Example of Splaying



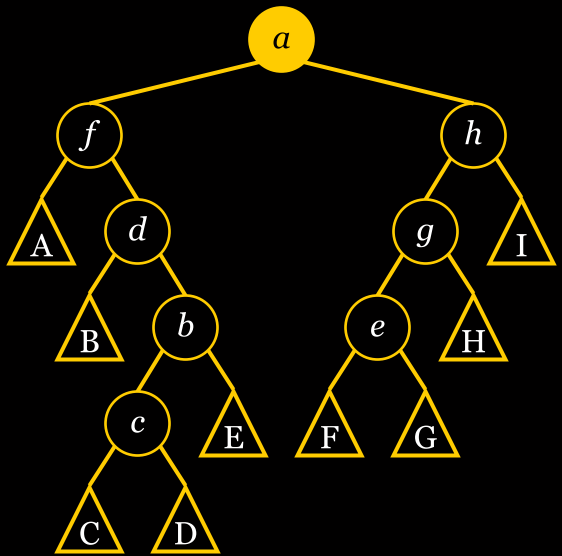
An Example of Splaying



An Example of Splaying

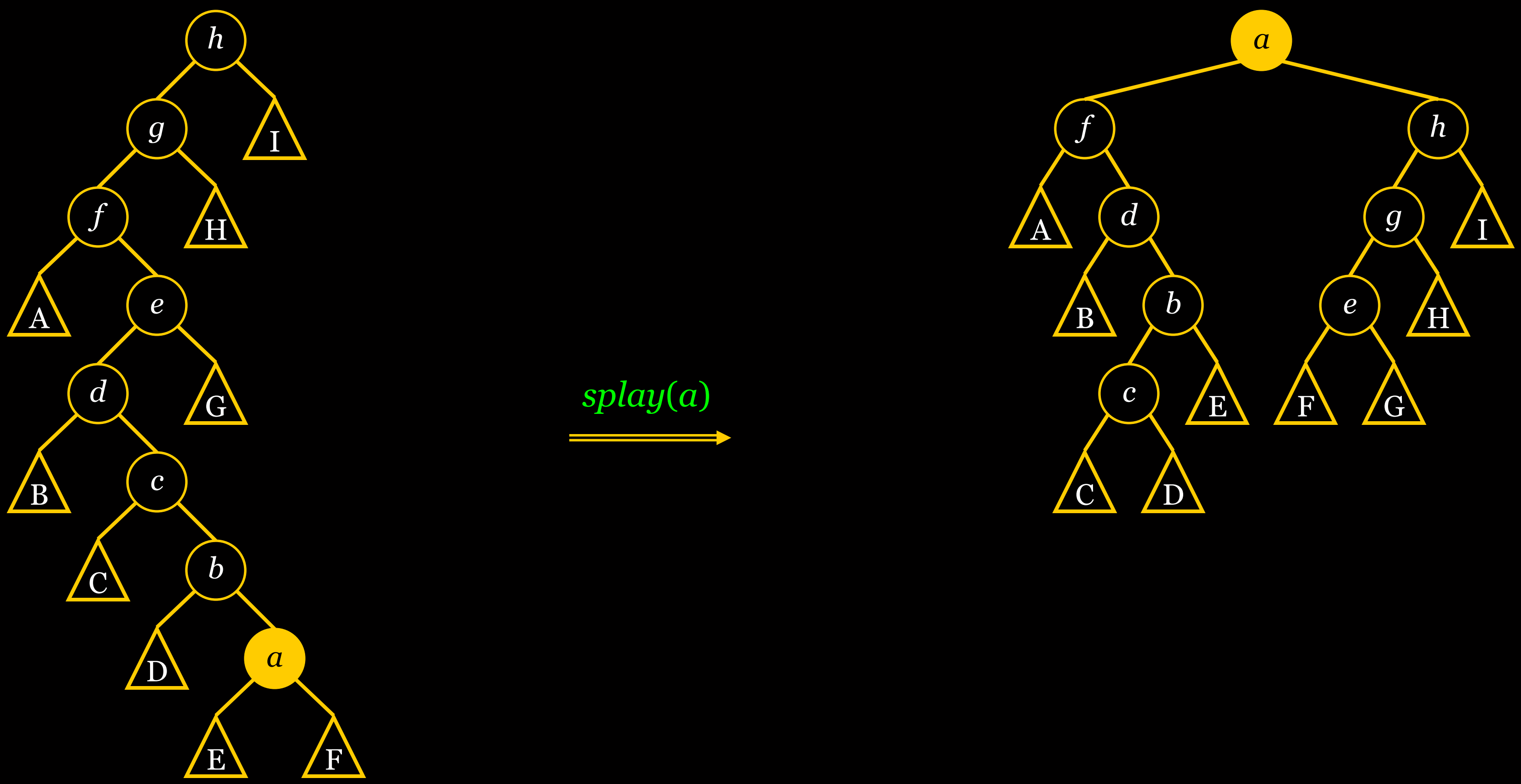


An Example of Splaying



An Example of Splaying

- End result:



Amortized Analysis

- Bounds the running time of a sequence of operations.
- Potential function Φ maps each configuration to real number.
- Amortized time to execute each operation:
  - $a_i = t_i + \Phi_i - \Phi_{i-1}$
    - a_i: amortized time to execute i-th operation;
    - t_i: actual time to execute the operation;
    - Φ_i: potential after the i-th operation.
- Total time for m operations:

$$\Sigma_{i=1..m}\, t_i = \Sigma_{i=1..m}(a_i + \Phi_{i-1} - \Phi_i) = \Phi_0 - \Phi_m + \Sigma_{i=1..m}\, a_i$$

Amortized Analysis of Splaying

- Definitions:
  - $s(x)$: size of node x (number of descendants, including x);
    - At most n, by definition.
  - $r(x)$: rank of node x, defined as $\log s(x)$;
    - At most $\log n$, by definition.
  - Φ_i: potential of the data structure (twice the sum of all ranks).
    - At most $O(n \log n)$, by definition.
- Access Lemma [ST85]: *The amortized time to splay a tree with root t at a node x is at most*

$$6(r(t)-r(x)) + 1 = O(\log(s(t)/s(x))).$$

Proof of Access Lemma

- Access Lemma [ST85]: *The amortized time to splay a tree with root t at a node x is at most*

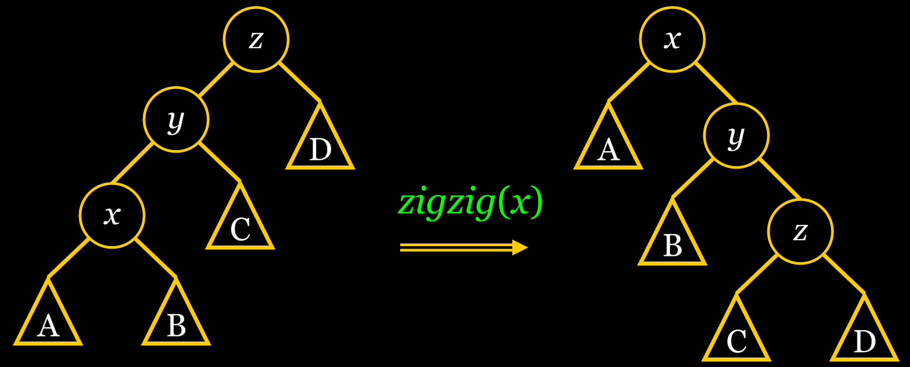
$$6(r(t)-r(x)) + 1 = O(\log(s(t)/s(x))).$$

- Proof idea:
  - $r_i(x)$ = rank of x after the i-th splay step;
  - a_i = amortized cost of the i-th splay step;
  - $a_i \leq 6(r_i(x)-r_{i-1}(x)) + 1$ (for the zig step, if any)
  - $a_i \leq 6(r_i(x)-r_{i-1}(x))$ (for any zig-zig and zig-zag steps)
  - Total amortized time for all k steps:

$$\begin{aligned}\textstyle\sum_{i=1..k} a_i &\leq \textstyle\sum_{i=1..k\text{-}1} [6(r_i(x)-r_{i-1}(x))] + [6(r_k(x)-r_{k-1}(x)) + 1] \\ &= 6r_k(x) - 6r_0(x) + 1\end{aligned}$$

Proof of Access Lemma: Splaying Step

- Zig-zig:



Claim: $a \leq 6\,(r'(x) - r(x))$

$t + \Phi' - \Phi \leq 6\,(r'(x) - r(x))$

$2 + 2(r'(x)+r'(y)+r'(z)) - 2(r(x)+r(y)+r(z)) \leq 6\,(r'(x) - r(x))$

$1 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \leq 3\,(r'(x) - r(x))$

$1 + r'(y) + r'(z) - r(x) - r(y) \leq 3\,(r'(x) - r(x))$ since $r'(x) = r(z)$

$1 + r'(y) + r'(z) - 2r(x) \leq 3\,(r'(x) - r(x))$ since $r(y) \geq r(x)$

$1 + r'(x) + r'(z) - 2r(x) \leq 3\,(r'(x) - r(x))$ since $r'(x) \geq r'(y)$

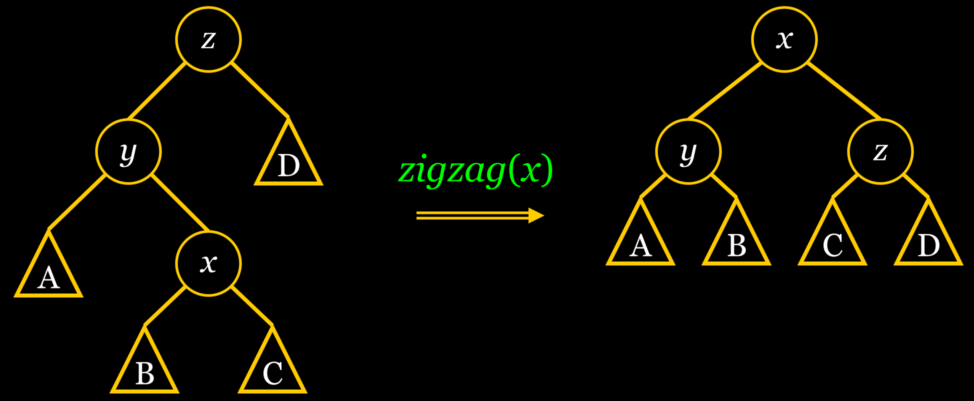
$(r(x) - r'(x)) + (r'(z) - r'(x)) \leq -1$ rearranging

$\log(s(x)/s'(x)) + \log(s'(z)/s'(x)) \leq -1$ definition of rank

TRUE because $s(x)+s'(z)<s'(x)$: both ratios are smaller than 1, at least one is at most 1/2.

Proof of Access Lemma: Splaying Step

- Zig-zag:



Claim: $a \leq 4\,(r'(x) - r(x))$

$t + \Phi' - \Phi \leq 4\,(r'(x) - r(x))$

$2 + (2r'(x)+2r'(y)+2r'(z)) - (2r(x)+2r(y)+2r(z)) \leq 4\,(r'(x) - r(x))$

$2 + 2r'(y) + 2r'(z) - 2r(x) - 2r(y) \leq 4\,(r'(x) - r(x))$, since $r'(x) = r(z)$

$2 + 2r'(y) + 2r'(z) - 4r(x) \leq 4\,(r'(x) - r(x))$, since $r(y) \geq r(x)$

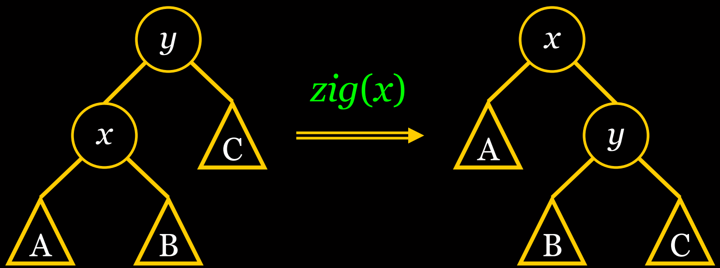
$(r'(y) - r'(x)) + (r'(z) - r'(x)) \leq -1$, rearranging

$\log(s'(y)/s'(x)) + \log(s'(z)/s'(x)) \leq -1$ definition of rank

TRUE because $s'(y)+s'(z)<s'(x)$: both ratios are smaller than 1, at least one is at most 1/2.

Proof of Access Lemma: Splaying Step

- Zig:



(only happens if y is root)

Claim: $a \le 1 + 6\,(r'(x) - r(x))$

$t + \Phi' - \Phi \le 1 + 6\,(r'(x) - r(x))$

$1 + (2r'(x) + 2r'(y)) - (2r(x) + 2r(y)) \le 1 + 6\,(r'(x) - r(x))$

$1 + 2\,(r'(x) - r(x)) \le 1 + 6\,(r'(x) - r(x))$, since $r(y) \ge r'(y)$

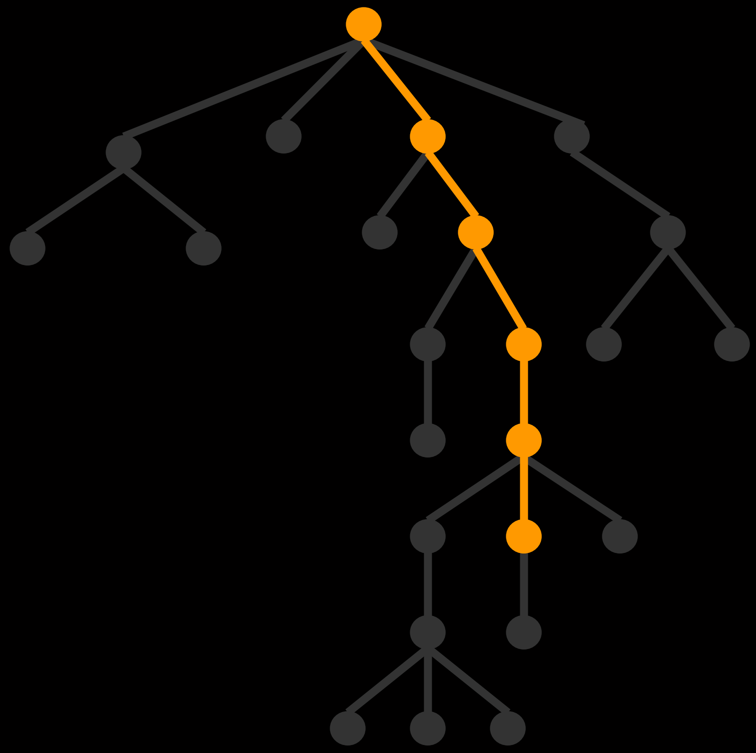
TRUE because $r'(x) \ge r(x)$.

Splaying

- To sum up:
  - No rotation: $a = 1$
  - Zig: $a \leq 6\ (r'(x) - r(x)) + 1$
  - Zig-zig: $a \leq 6\ (r'(x) - r(x))$
  - Zig-zag: $a \leq 4\ (r'(x) - r(x))$
  - Total amortized time at most $6\ (r(t) - r(x)) + 1 = O(\log n)$
- Since accesses bring the relevant element to the root, other operations (insert, delete, join, split) become trivial.

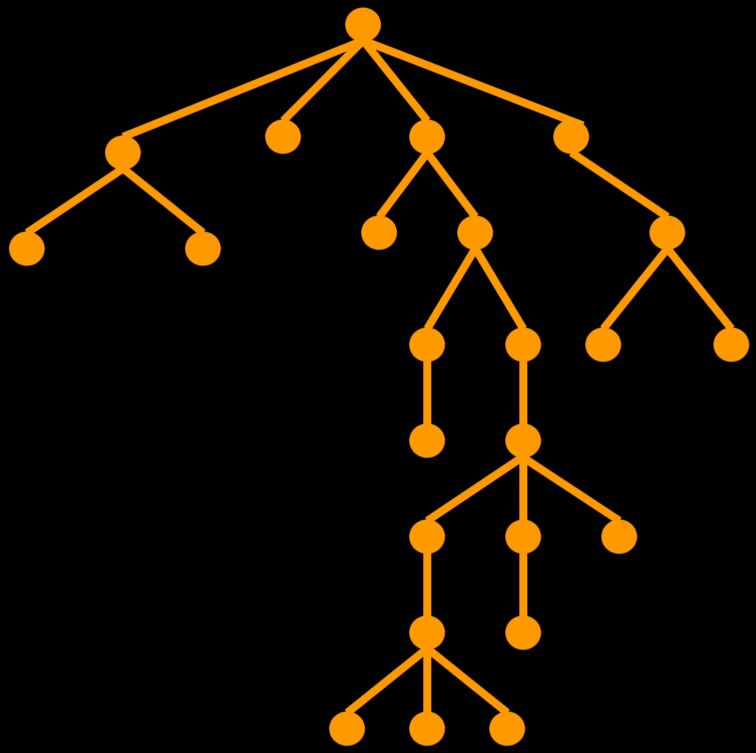
Dynamic Trees

- We know how to deal with isolated paths.
- How to deal with paths within a tree?



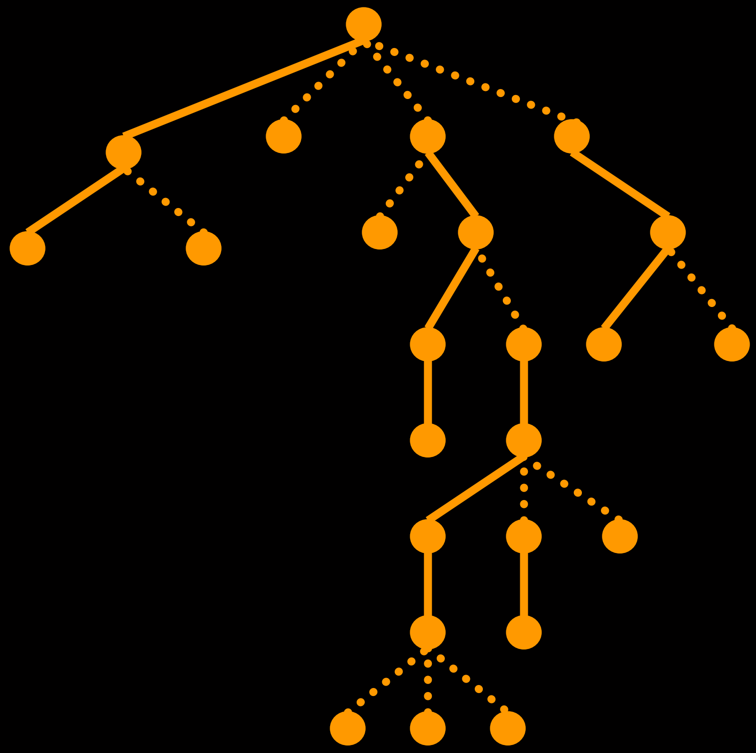
Dynamic Trees

- Main idea: partition the vertices in a tree into disjoint solid paths connected by dashed edges.



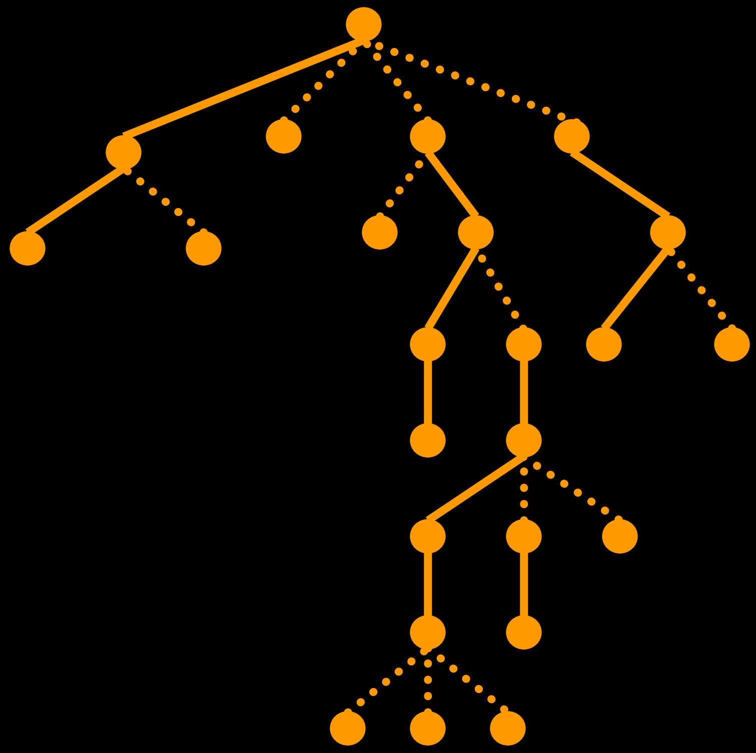
Dynamic Trees

- Main idea: partition the vertices in a tree into disjoint solid paths connected by dashed edges.



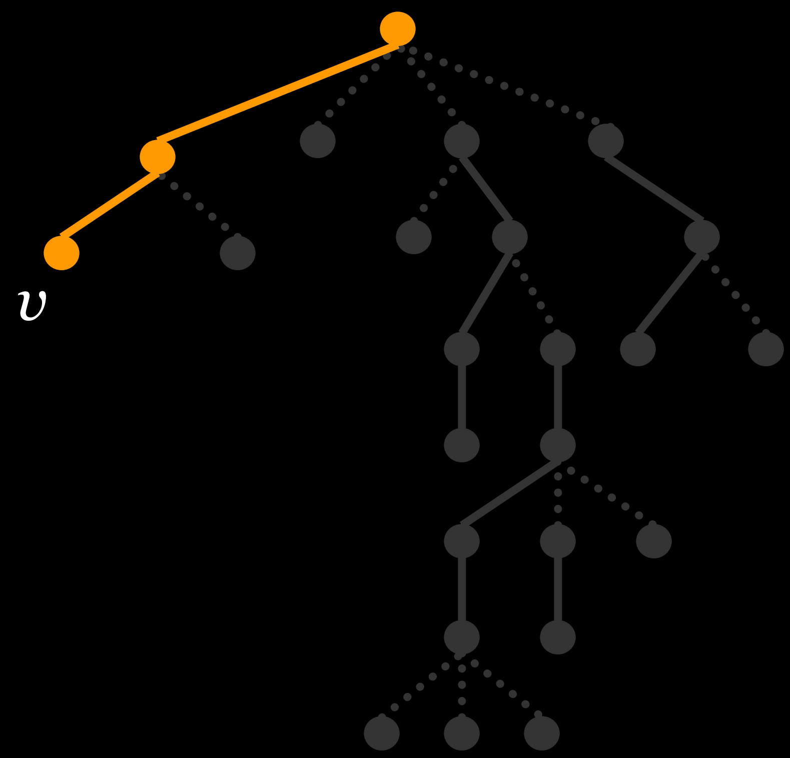
Dynamic Trees

- A vertex v is exposed if:
  - There is a solid path from v to the root;
  - No solid edge enters v.



Dynamic Trees

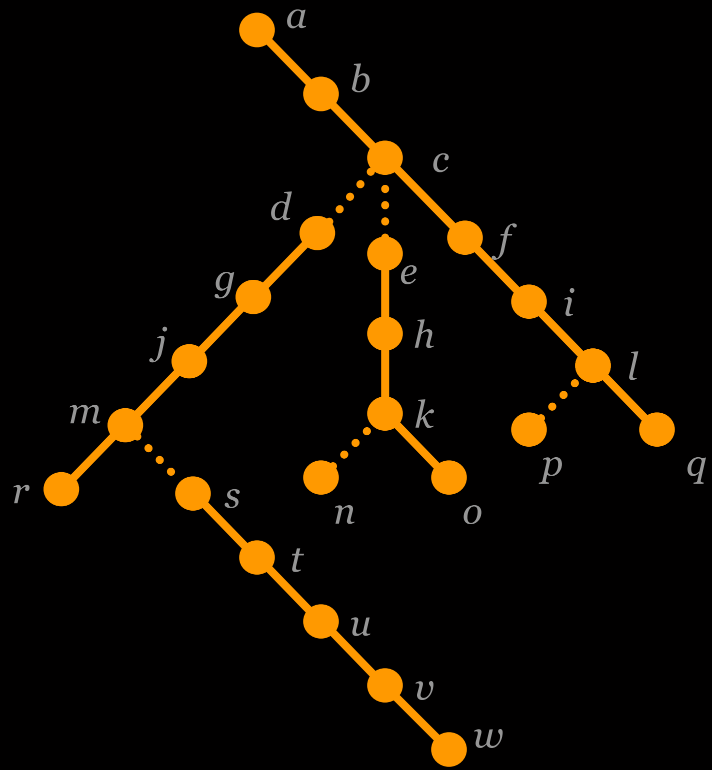
- A vertex v is exposed if:
  - There is a solid path from v to the root;
  - No solid edge enters v.
- It is unique.



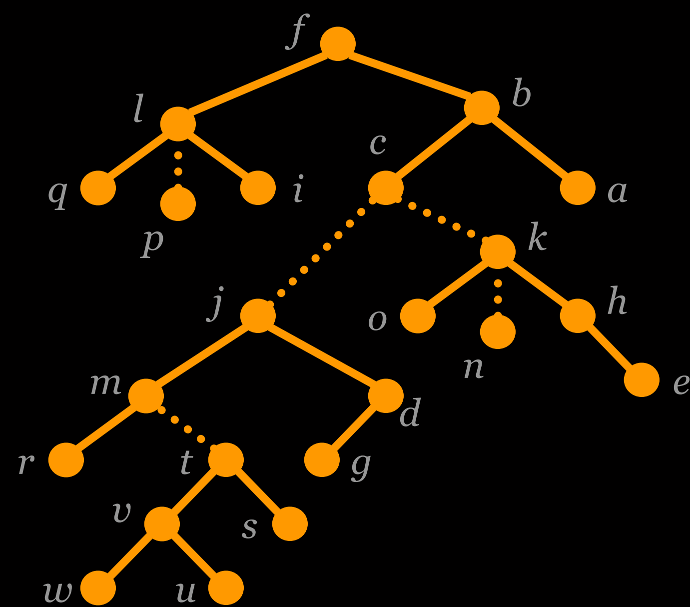
Dynamic Trees

- Solid paths:
  - Represented as binary trees (as seen before).
  - Parent pointer of root is the outgoing dashed edge.
  - Hierarchy of solid binary trees linked by dashed edges: "virtual tree".
- "Isolated path" operations handle the exposed path.
  - The solid path entering the root.
  - Dashed pointers go up, so the solid path does not "know" it has dashed children.
- If a different path is needed:
  - expose(v): make entire path from v to the root solid.

Virtual Tree: An Example



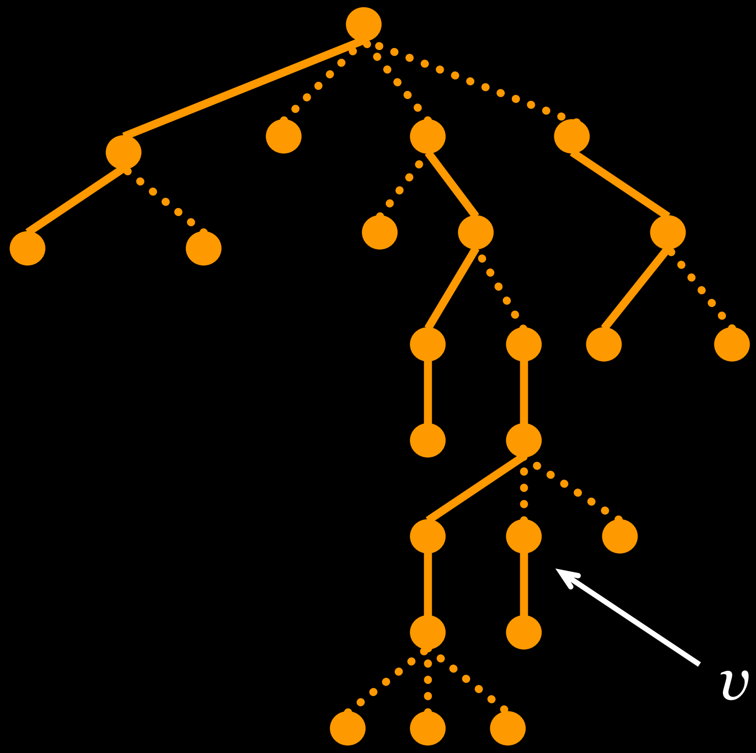
actual tree



virtual tree

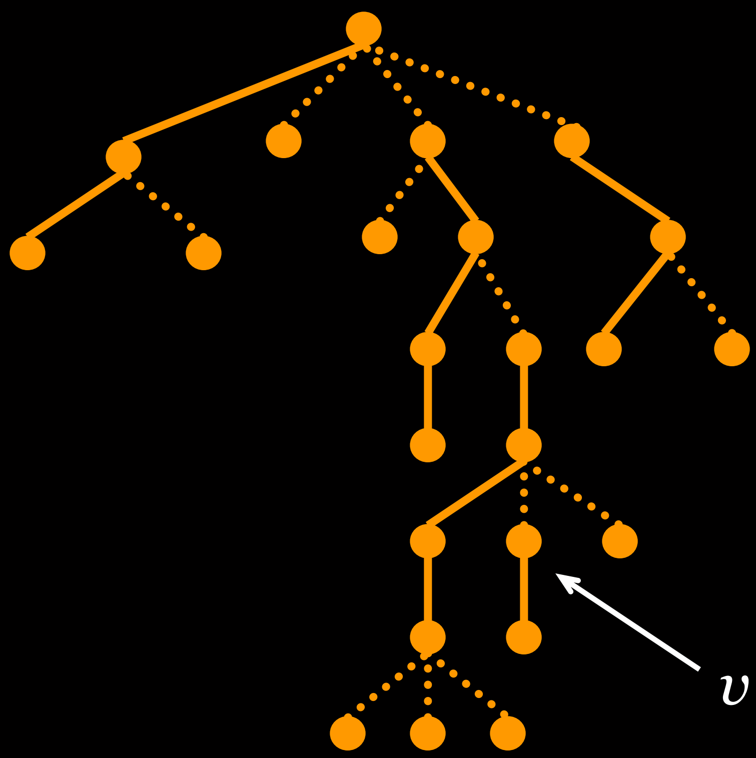
Dynamic Trees

- Example: expose(v)



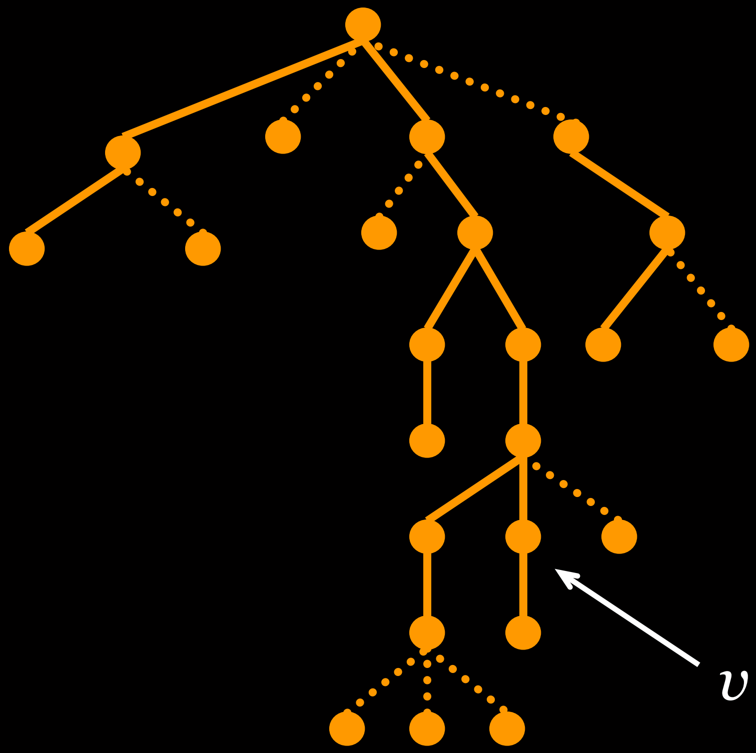
Dynamic Trees

- Example: expose(v)
  - Take all edges in the path to the root, ...



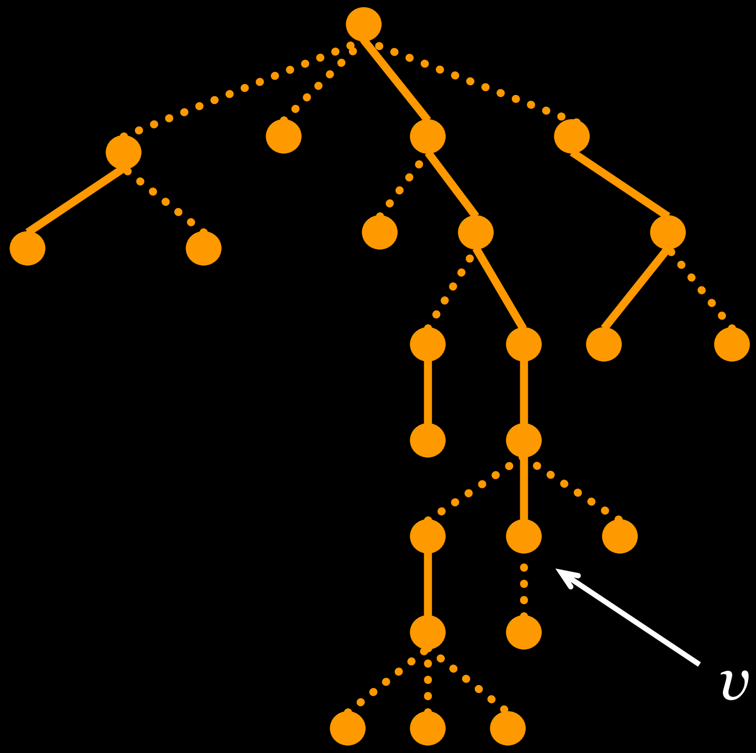
Dynamic Trees

- Example: expose(v)
  - ..., make them solid, ...



Dynamic Trees

- Example: expose(v)
  - …make sure there is no other solid edge incident into the path.
    - Uses splice operation.

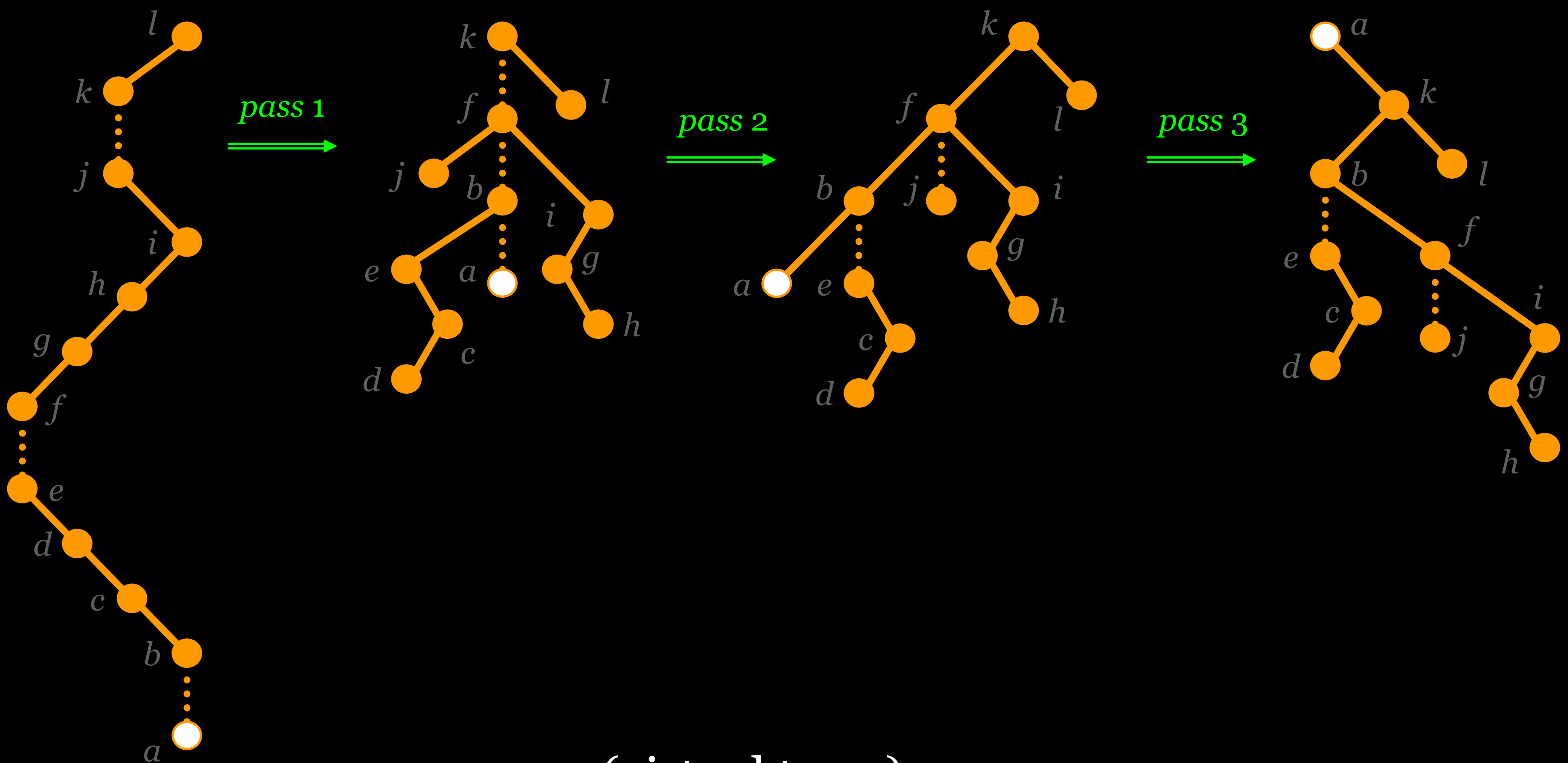


Exposing a Vertex

- expose(x): makes the path from x to the root solid.
- Implemented in three steps:
  1. Splay within each solid tree in the path from x to root.
  2. Splice each dashed edge from x to the root.
     - splice makes a dashed become the left solid child;
     - If there is an original left solid child, it becomes dashed.
  3. Splay on x, which will become the root.

Exposing a Vertex: An Example

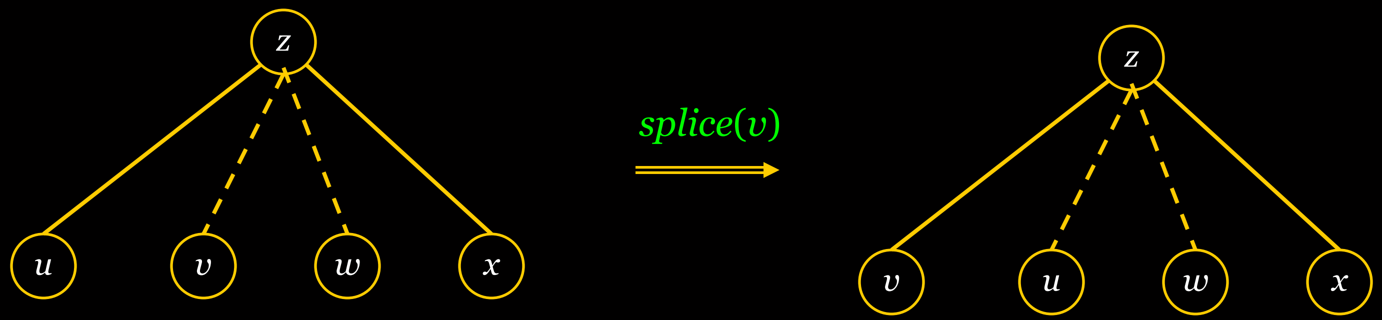
- expose(a)



(virtual trees)

Dynamic Trees: Splice

- Additional restructuring primitive: *splice*.



  - Will only occur when z is the root of a tree.

- Updates:
  - $\Delta cost'(v) = \Delta cost(v) - \Delta cost(z)$
  - $\Delta cost'(u) = \Delta cost(u) + \Delta cost(z)$
  - $\Delta min'(z) = \max\{0, \Delta min(v) - \Delta cost'(v), \Delta min(x) - \Delta cost(x)\}$

Exposing a Vertex: Running Time

- Running time of expose(x):
  - proportional to initial depth of x;
  - x is rotated all the way to the root;
  - we just need to count the number of rotations;
    - will actually find amortized number of rotations: $O(\log n)$.
  - proof uses the Access Lemma.
    - $s(x)$, $r(x)$ and potential are defined as before;
    - In particular, $s(x)$ is the size of the whole subtree rooted at x.
      - Includes both solid and dashed edges.

Exposing a Vertex: Running Time (Proof)

- k: number of dashed edges from x to the root t.
- Amortized costs of each pass:
  1. Splay within each solid tree:
     - x_i: vertex splayed on the i-th solid tree.
     - amortized cost of i-th splay: $6\,(r'(x_i) - r(x_i)) + 1$.
     - $r(x_{i+1}) \geq r'(x_i)$, so the sum over all steps telescopes;
     - Amortized cost first of pass: $6(r'(x_k) - r(x_1)) + k \leq 6 \log n + k$.
  2. Splice dashed edges:
     - no rotations, no potential changes: amortized cost is zero.
  3. Splay on x:
     - amortized cost is at most $6 \log n + 1$.
     - x ends up in root, so exactly k rotations happen;
     - each rotation costs one credit, but is charged two;
     - they pay for the extra k rotations in the first pass.
- Amortized number of rotations = $O(\log n)$.

Implementing Dynamic Tree Operations

- findcost(v):
  - expose v, return $cost(v)$.
- findroot(v):
  - expose v;
  - find w, the rightmost vertex in the solid subtree containing v;
  - splay at w and return w.
- findmin(v):
  - expose v;
  - use $\Delta cost$ and Δmin to walk down from v to w, the last minimum-cost node in the solid subtree;
  - splay at w and return w.

Implementing Dynamic Tree Operations

- addcost(v, x):
  - expose v;
  - add x to $\Delta cost(v)$;
- link(v,w):
  - expose v and w (they are in different trees);
  - set $p(v)=w$ (that is, make v a middle child of w).
- cut(v):
  - expose v;
  - add $\Delta cost(v)$ to $\Delta cost(right(v))$;
  - make $p(right(v))$=**null** and $right(v)$=**null**.

Extensions and Variants

- Simple extensions:
  - Associate values with edges:
    - just interpret cost(v) as cost(v,p(v)).
  - other path queries (such as length):
    - change values stored in each node and update operations.
  - free (unrooted) trees.
    - implement evert operation, which changes the root.
- Not-so-simple extension:
  - subtree-related operations:
    - requires that vertices have bounded degree;
    - Approach for arbitrary trees: "ternarize" them:
      - [Goldberg, Grigoriadis and Tarjan, 1991]

Alternative Implementation

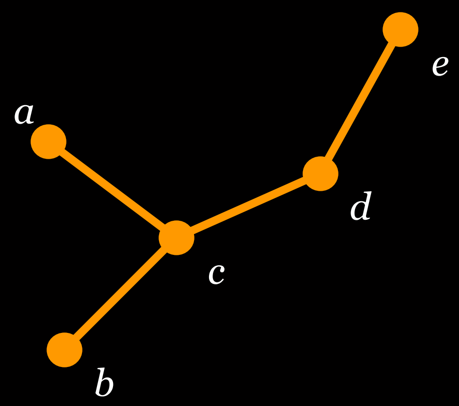
- Total time per operation depends on the data structure used to represent paths:
  - Splay trees: O(log n) amortized [ST85].
  - Balanced search tree: O($\log^2 n$) amortized [ST83].
  - Locally biased search tree: O(log n) amortized [ST83].
  - Globally biased search trees: O(log n) worst-case [ST83].
- Biased search trees:
  - Support leaves with different "weights".
  - Some solid leaves are "heavier" because they also represent subtrees dangling from it from dashed edges.
  - Much more complicated than splay trees.

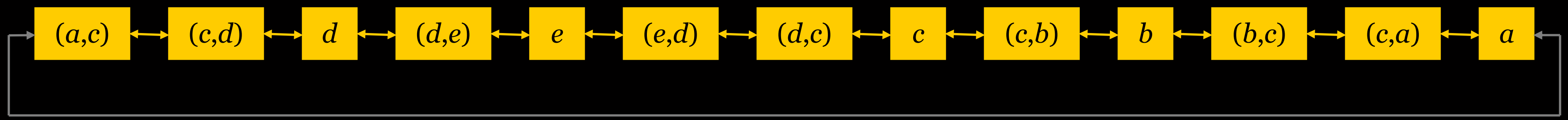
Other Data Structures

- Some applications require tree-related information:
  - minimum vertex in a tree;
  - add value to all elements in the tree;
  - link and cut as usual.
- ET-Trees can do that:
  - Henzinger and King (1995);
  - Tarjan (1997).

ET-Trees

- Each tree represented by its *Euler tour*.
  - Edge $\{v,w\}$:
    - appears as arcs (v,w) and (w,v)
  - Vertex v:
    - appears once as a self-loop (v,v):
    - used as an “anchor” for new links.
    - stores vertex-related information.
  - Representation is not circular: tour broken at arbitrary place.





ET-Trees

- Consider link(v,w):
  - Create elements representing arcs (v,w) and (w,v):

    $(v,w) \quad (w,v)$

  - Split and concatenate tours appropriately:
    - Original tours:

      $L_v \leftrightarrow v \leftrightarrow L'_v \qquad L_w \leftrightarrow w \leftrightarrow L'_w$

    - Final tour:

      $L_v \leftrightarrow v \leftrightarrow (v,w) \leftrightarrow L'_w \leftrightarrow L_w \leftrightarrow w \leftrightarrow (w,v) \leftrightarrow L'_v$

- The cut operation is similar.

ET-Trees

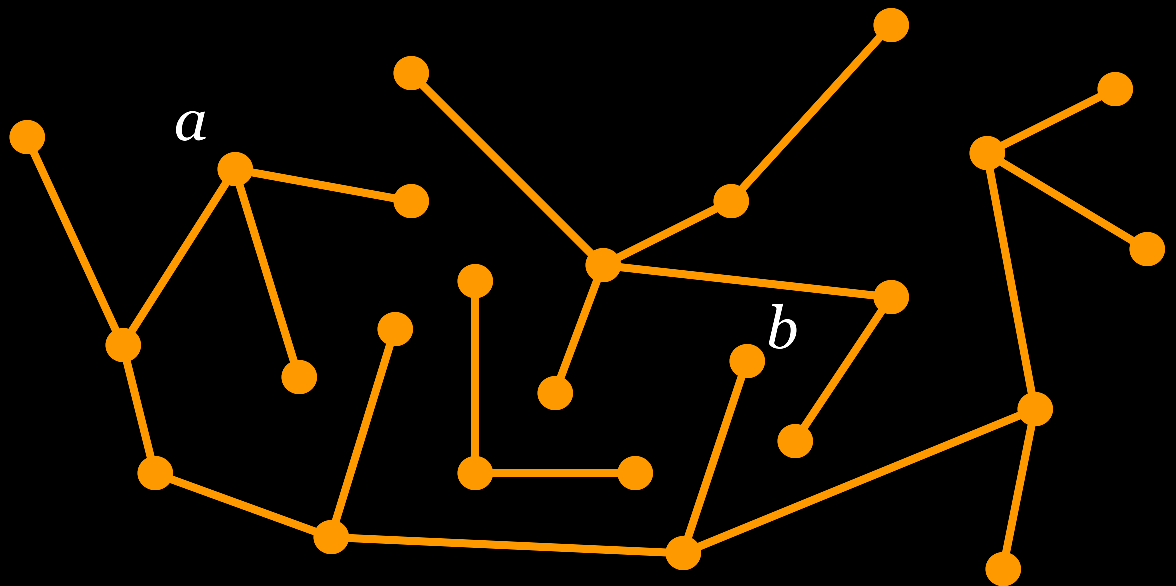
- Tours as doubly-linked lists:
  - Natural representation.
  - link/cut: O(1) time.
  - addcost/findmin: $O(n)$ time.
- Tours as balanced binary search trees:
  - link/cut: O(log n) time (binary tree join and split).
  - addcost/findmin: O(log n) time:
    - values stored in difference form.

Contractions

- ST-Trees [ST83, ST85]:
  - first data structure to handle paths within trees efficiently.
  - It is clearly path-oriented:
    - relevant paths explicitly exposed and dealt with.
- Other approaches are based on contractions:
  - Original tree is progressively contracted until a structure representing only the relevant path (or tree) is left.

Contractions

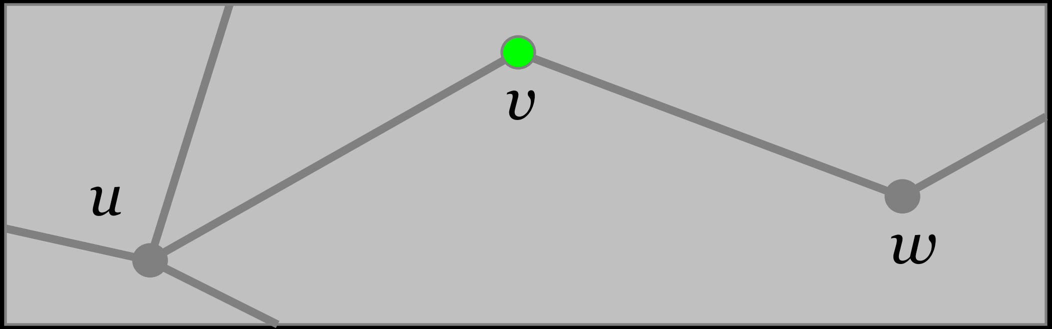
- Assume we are interested in the path from a to b:



  - Using only local information, how can we get closer to the solution?

Contractions

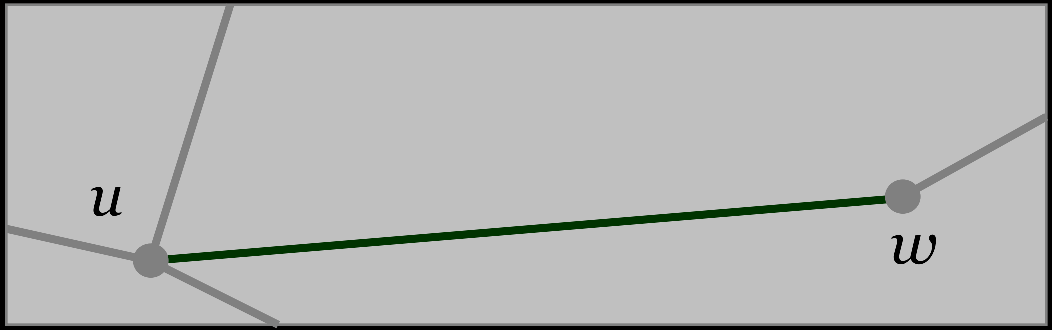
- Consider any vertex v with degree 2 in the tree



- Possibilities if v is neither a nor b:
  - a and b on same "side": v is not in $a \cdots b$.
  - If a and b on different sides: v belongs to path $a \cdots b$.

Contractions

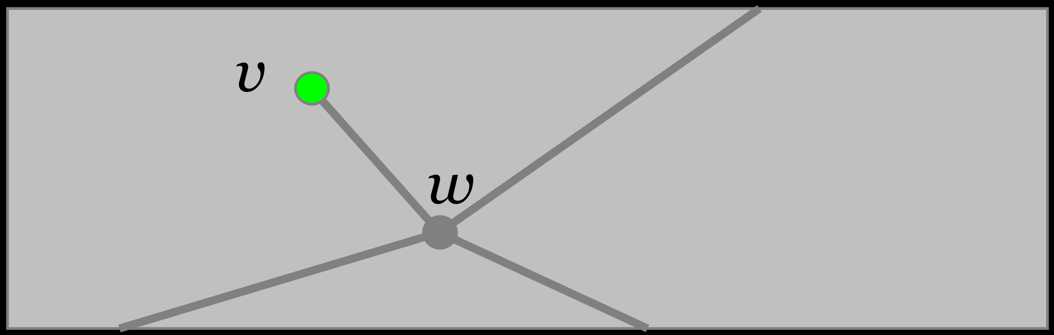
- Consider any vertex v with degree 2 in the tree



- Possibilities if v is neither a nor b:
  - a and b on same "side": v is not in $a \cdot\cdot b$.
  - If a and b on different sides: v belongs to path $a \cdot\cdot b$.
- We can replace (u,v) and (v,w) with a new edge (u,w):
  - This is a compress operation.

Contractions

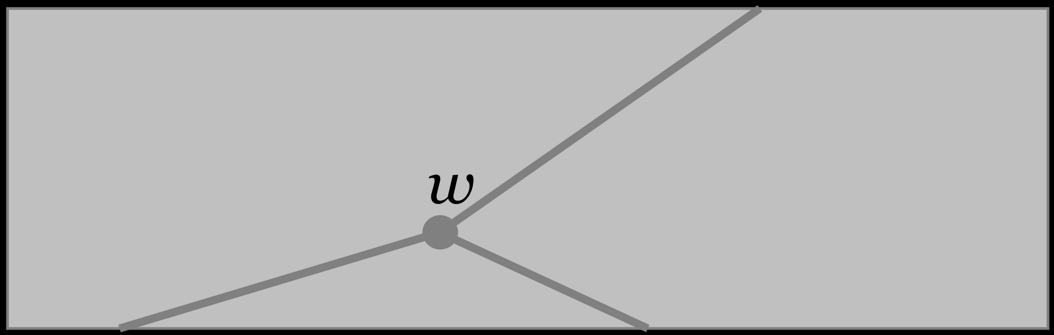
- Consider any vertex v with degree 1 in the tree:



  - If v is neither a nor b, it is clearly not in $a \cdots b$.

Contractions

- Consider any vertex v with degree 1 in the tree:



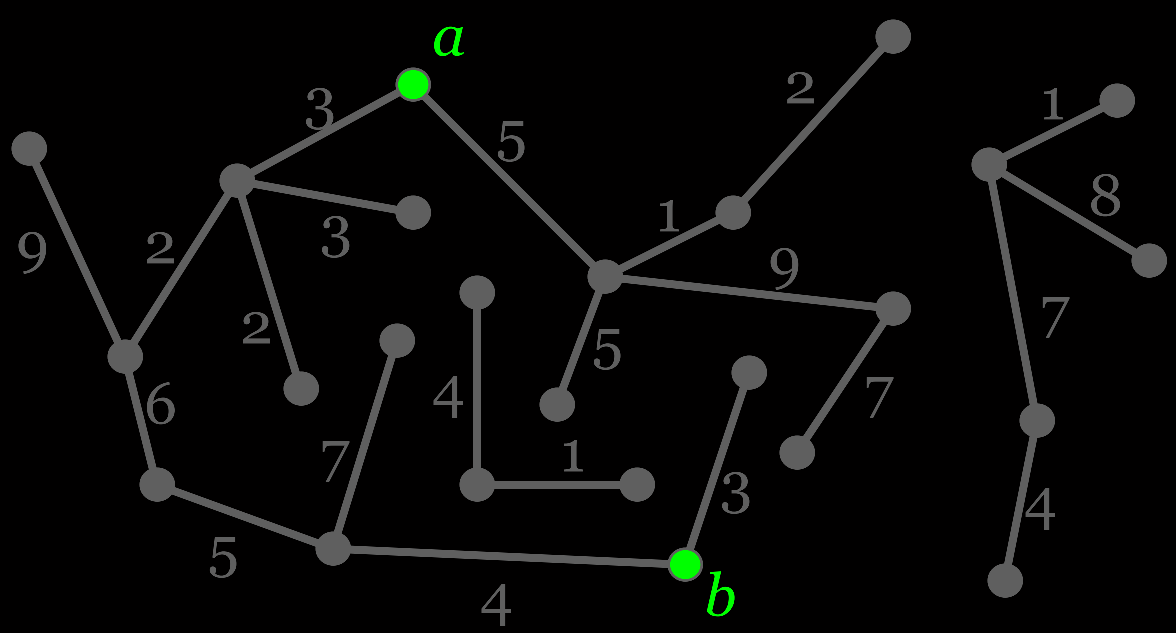
  - If v is neither a nor b, it is clearly not in $a \cdots b$.
  - We can simply eliminate (v,w), reducing the problem size.
    - This is a rake operation.

Contractions

- A contraction-based algorithm:
  - Work in rounds;
  - In each round, perform some rakes and/or compresses:
    - this will create a new, smaller tree;
    - moves within a round are usually "independent".
  - Eventually, we will be down to a single element (vertex/edge) that represents a path (or the tree).

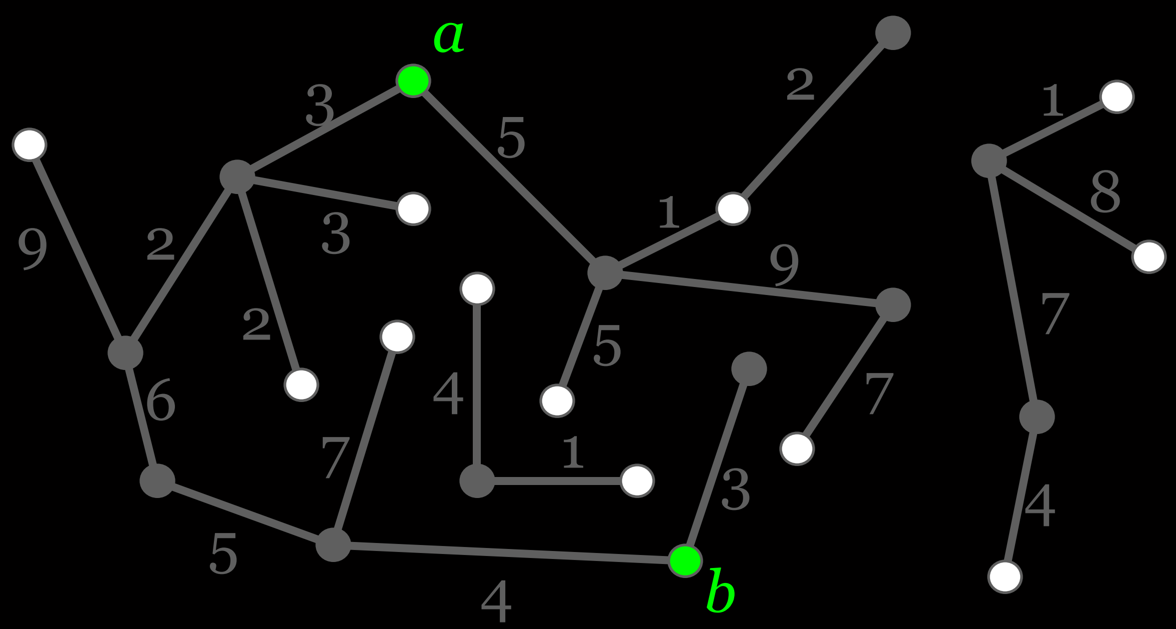
Path Queries

- Computing the minimum cost from a to b:



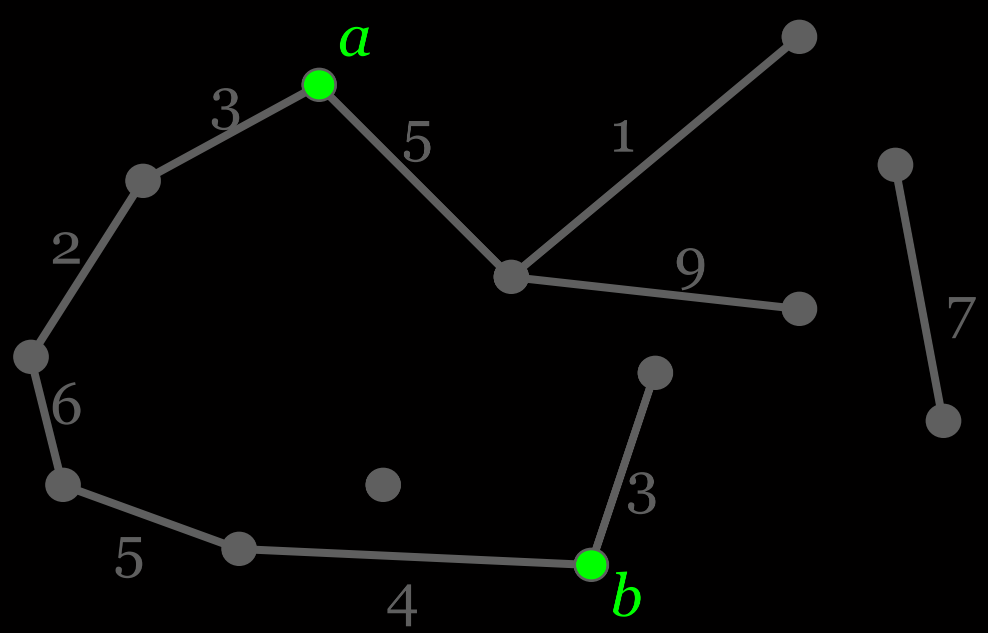
Path Queries

- Computing the minimum cost from *a* to *b*:



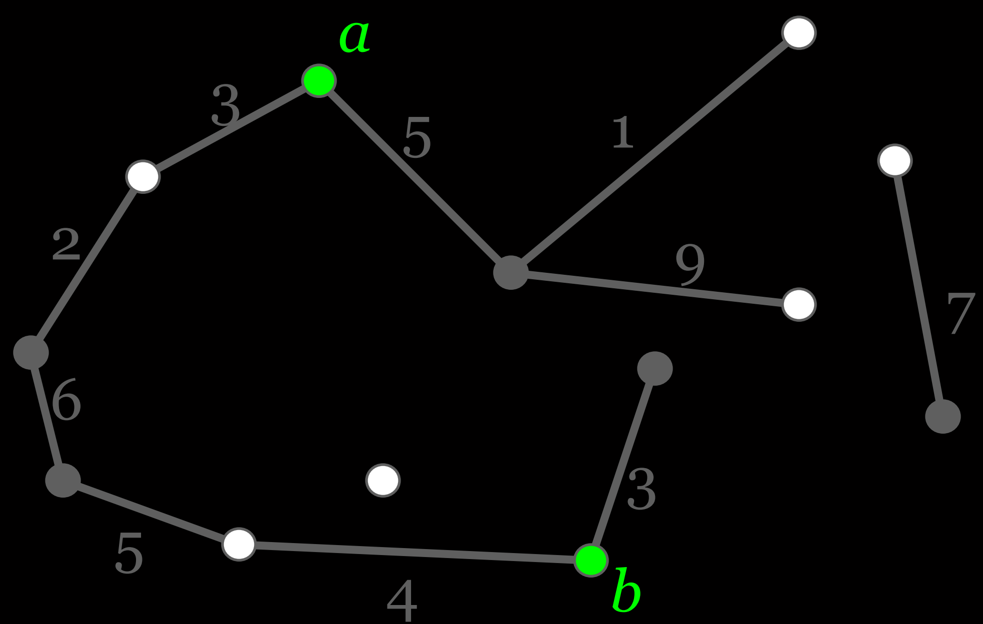
Path Queries

- Computing the minimum cost from a to b:



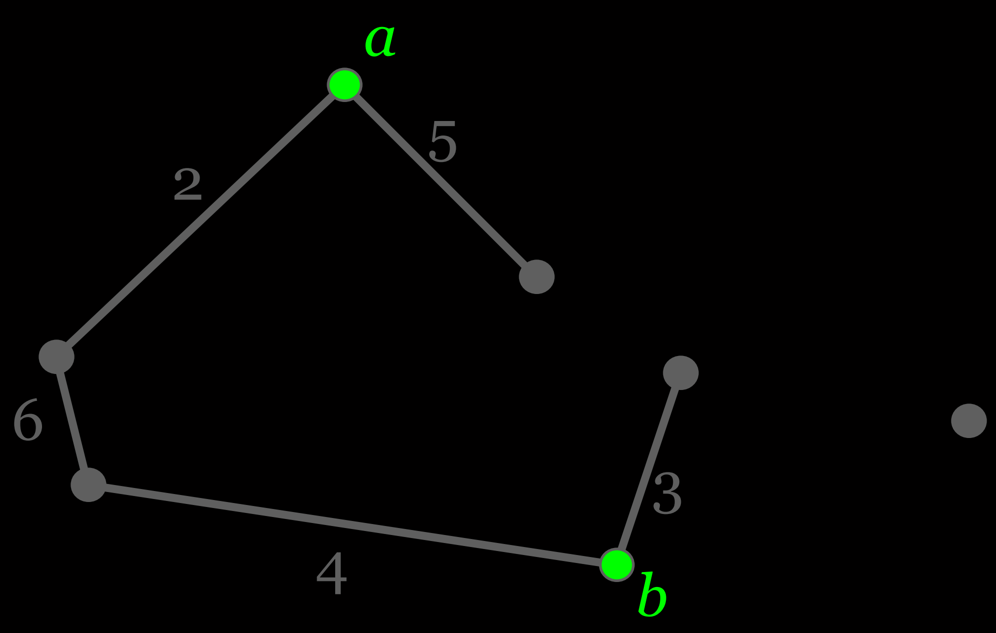
Path Queries

- Computing the minimum cost from *a* to *b*:



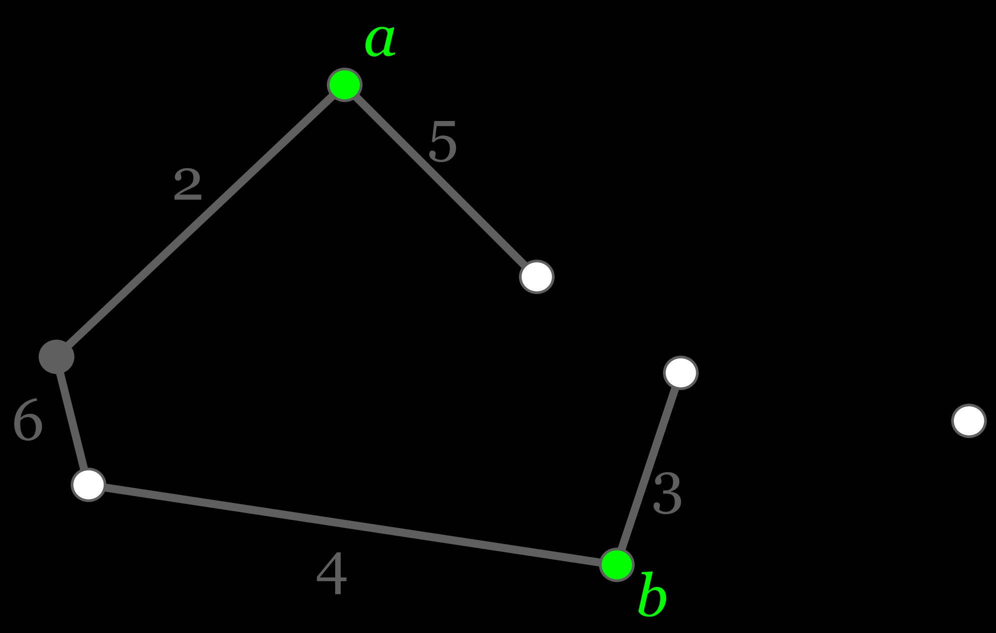
Path Queries

- Computing the minimum cost from a to b:



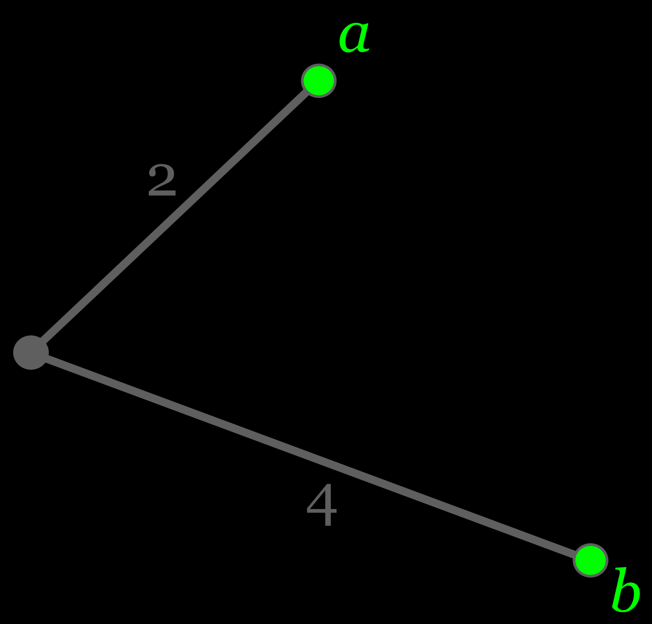
Path Queries

- Computing the minimum cost from a to b:



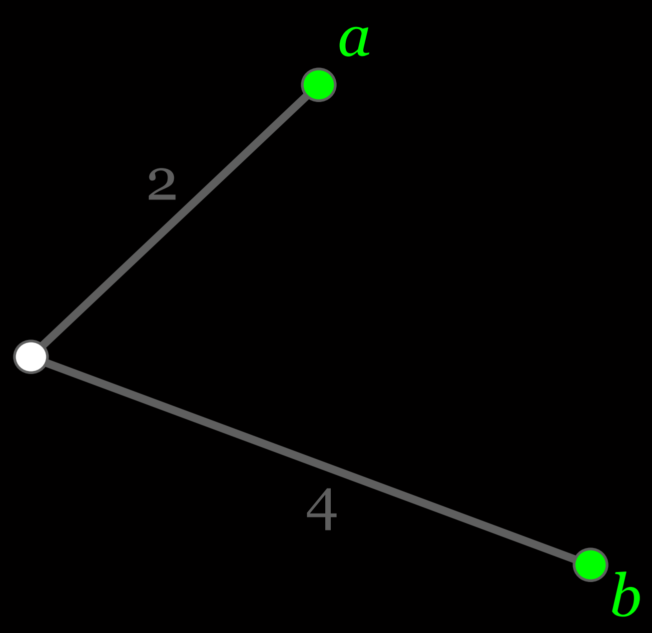
Path Queries

- Computing the minimum cost from *a* to *b*:



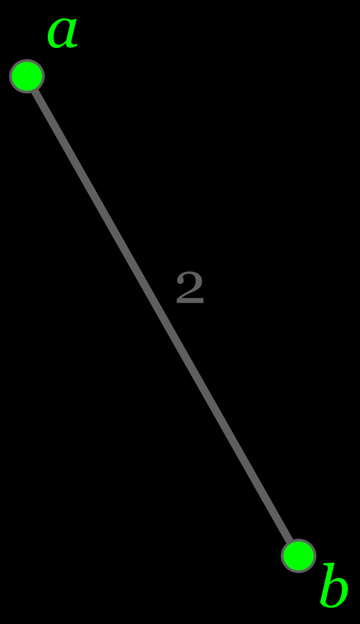
Path Queries

- Computing the minimum cost from *a* to *b*:



Path Queries

- Computing the minimum cost from a to b:



Contractions

- Suppose a definition of independence guarantees that a fraction $1/k$ of all possible rakes and compresses will be executed in a round.
  - All degree-1 vertices are rake candidates.
  - All degree-2 vertices are compress candidates.
  - *Fact*: at least half the vertices in any tree have degree 1 or 2.
  - *Result*: a fraction $1/2k$ of all vertices will be removed.
  - Total number of rounds is $\lceil \log_{(2k)/(2k-1)} n \rceil = O(\log n)$.

Contractions

- rake and compress proposed by Miller and Reif [1985].
  - Original context: parallel algorithms.
  - Perform several operations on trees in O(log n) time.

The Update Problem

- Coming up with a definition of independence that results in a contraction with O(log n) levels.
  - But that is not the problem we need to solve.
- Essentially, we want to repair an existing contraction after a tree operation (link/cut).
- So we are interested in the update problem:
  - Given a contraction C of a forest F, find another contraction C' of a forest F' that differs from F in one single edge (inserted or deleted).
  - Fast: O(log n) time.

Our Problem

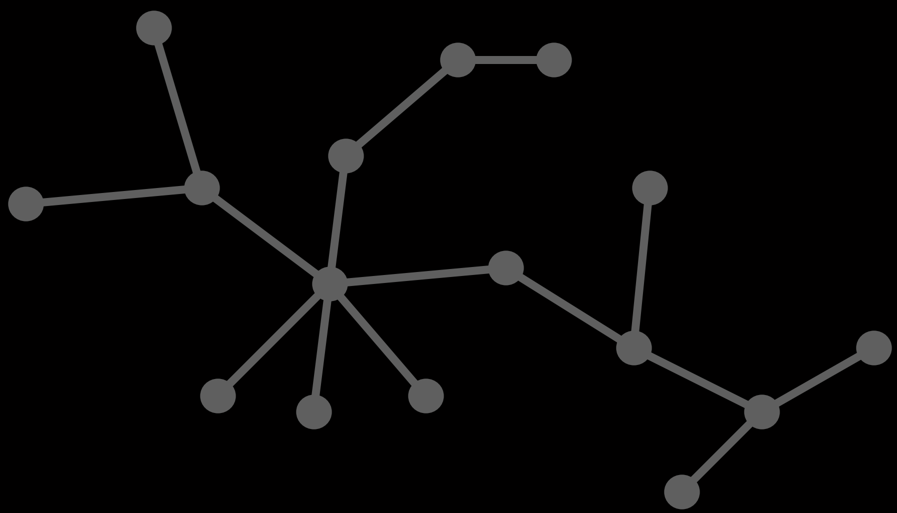
- Several data structures deal with this problem.
  - [Frederickson, 85 and 97]: Topology Trees;
  - [Alstrup et al., 97 and 03]: Top Trees;
  - [Acar et al. 03]: RC-Trees.

Top Trees

- Proposed by Alstrup et al. [1997,2003]
- Handle unrooted (free) trees with arbitrary degrees.
- Key ideas:
  - Associate information with the edges directly.
  - Pair edges up:
    - compress: combines two edges linked by a degree-two vertex;
    - rake: combines leaf with an edge with which it shares an endpoint.
    - All pairs (clusters) must be are disjoint.
  - expose: determines which two vertices are relevant to the query (they will not be raked or compressed).

Top Trees

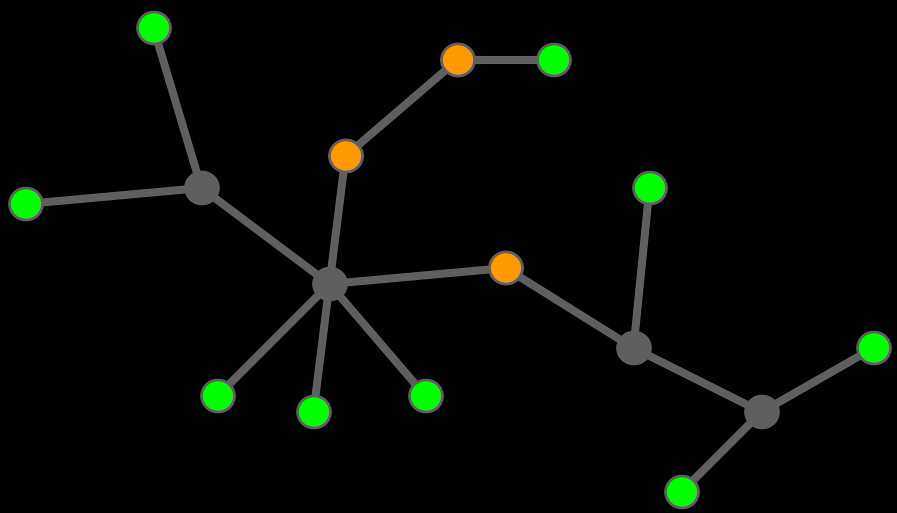
- Consider some free tree.



(level zero: original tree)

Top Trees

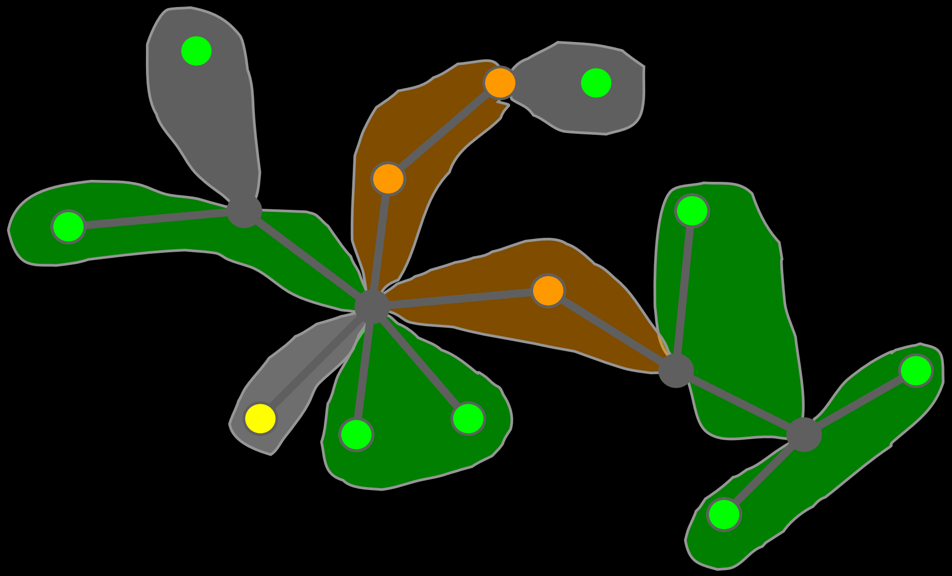
- All degree-1 and degree-2 vertices are candidates for a move (rake or compress).



(level zero: original tree)

Top Trees

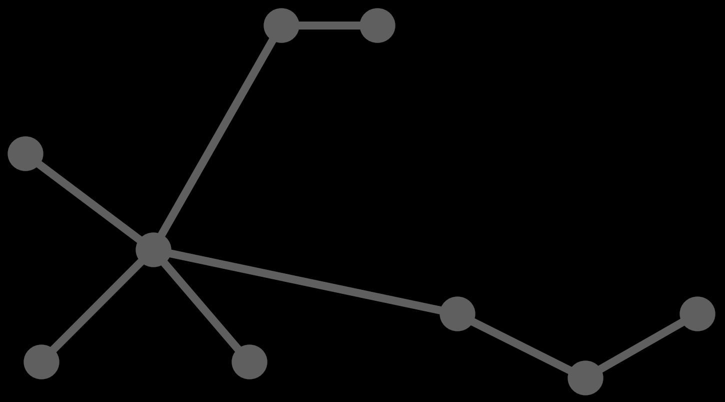
- When two edges are matched, they create new clusters, which are edge-disjoint.



(level zero: original tree)

Top Trees

- Clusters are new edges in the level above:
  - New rakes and compresses can be performed as before.



(level one)

Top Trees

- The top tree itself represents the hierarchy of clusters:
  - original edge: leaf of the top tree (level zero).
  - two edges/clusters are grouped by rake or compress:
    - Resulting cluster is their parent in the level above.
  - edge/cluster unmatched: parent will have only one child.
- What about values?

Top Trees

- Alstrup et al. see top tree as an API.
- The top tree engine handles structural operations:
  - User has limited access to it.
- Engine calls user-defined functions to handle values properly:
  - join(A,B,C): called when A and B are paired (by rake or compress) to create cluster C.
  - split(A,B,C): called when a rake or compress is undone (and C is split into A and B).
  - create(C, e): called when base cluster C is created to represent edge e.
  - destroy(C): called when base cluster C is deleted.

Top Trees

- Example (path operations: findmin/addcost)
  - Associate two values with each cluster:
    - mincost(C): minimum cost in the path represented by C.
    - extra(C): cost that must be added to all subpaths of C.
  - create(C, e): (*called when base cluster C is created*)
    - mincost(C) = cost of edge e.
    - extra(C) = 0
  - destroy(C): (*called when base cluster C is deleted*).
    - Do nothing.

Top Trees

- Example (path operations: findmin/addvalue)
  - join(A,B,C): (*called when A and B are combined into C*)
    - compress: mincost(C) = min{mincost(A), mincost(B)}
    - rake: mincost(C) = mincost(B) (assume A is the leaf)
    - Both cases: *extra*(C) = 0
  - split(A,B,C): (*called when C is split into A and B*)
    - compress: for each child $X \in \{A,B\}$:
      - mincost(X) = mincost(X) + extra(C)
      - extra(X) = extra(X) + extra(C)
    - rake: same as above, but only for the edge/cluster that was not raked.

Top Trees

- Example (path operations: findmin/addvalue)
  - To find the minimum cost in path $a \cdots b$:
    - R = expose(a, b);
    - return mincost(R).
  - To add a cost x to all edges in path $a \cdots b$:
    - R = expose(a, b);
    - mincost(R) = mincost(R) + x;
    - extra(R) = extra(R) + x.

Top Trees

- Can handle operations such as:
  - tree costs (just a different way of handling rakes);
  - path lengths;
  - tree diameters.
- Can handle non-local information using the select operation:
  - allows user to perform binary search on top tree.
  - an example: tree center.
- Top trees are implemented on top of topology trees, which they generalize.

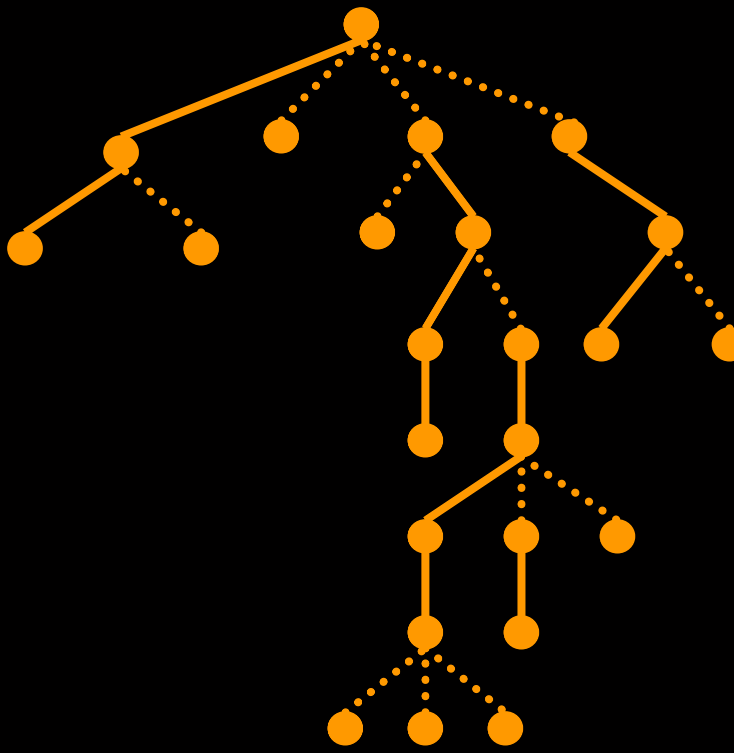
Topology Trees

- Proposed by Frederickson [1985, 1997].
- Work on rooted trees of bounded degree.
  - Assume each vertex has at most two children.
    - Values (and clusters) are associated with vertices.
  - Perform a maximal set of independent moves in each round.
  - Handle updates in O(log n) worst-case time.

RC-Trees

- Proposed by Acar et al. [2003].
- Can be seen as a variant of topology trees.
  - Information stored on vertices.
  - Trees of bounded degree.
- Main differences:
  - Not necessarily rooted.
  - Alternate rake and compress rounds.
  - Not maximal in compress rounds (randomization).
  - Updates in $O(\log n)$ expected time.

Contractions



- Topology, Top, and Trace trees:
  - contraction-based.
- ST-Trees: path-based.
  - But there is a (rough) mapping:
    - dashed $\leftrightarrow$ rake
      - “this is a path that goes nowhere”
    - solid $\leftrightarrow$ compress
      - “both part of a single path”
  - ST-Trees can be used to implement topology trees [AHdLT03].

Chronology

- ST-Trees:
  - Sleator and Tarjan (1983): with balanced and biased search trees;
  - Sleator and Tarjan (1985): splay trees.
- Topology Trees:
  - Frederickson (1985, 1987).
- ET-trees:
  - Hensinger and King (1995);
  - Tarjan (1997).
- Top Trees:
  - Alstrup, de Lichtenberg, and Thorup (1997);
  - Alstrup, Holm, de Lichtenberg, and Thorup (2003).
- RC-Trees:
  - Acar, Blelloch, Harper, and Woo (2003).