CALCULATION OF HIGH ORDER OPERATOR TERMS IN VARIATIONAL MANY-BODY THEORIES

I.E. LAGARIS
Department of Physics University of Ioannina
Ioannina - Greece

Abstract. - A recursion relation method capable of treating high order operator terms, in variational many-body theories with operator correlations, is suggested. As an illustration of the method two simple cases are solved exactly. First is the case of Fermi hypernetted $\sigma$ (or $\tau$) chains; second is the case of direct double $\sigma$ (or $\tau$) chains. The method can be extended to treat other interesting cases as well, since it attacks counting problems common to many calculations.

1. - Introduction.

In variational many-body theories of Nuclear matter (NM), with operator correlations, such as those of ref. 1 (from here on called PW), and of ref. 2, the trial wavefunction is of the form:

$$|\Psi\rangle = \sum_{ijkl} c_{ijkl} |\psi_1\rangle |\psi_2\rangle$$ (1.1)

where $|\psi\rangle$ is the non-interacting ground state wavefunction, $F_{ij}$ is the correlation two-body operator and $S$ is the appropriate symmetrizer since the operators $F_{ij}$ and $F_{ji}$ do not in general commute. In PW theory the correlation operator is taken to be:

$$F_{ij} = \frac{1}{2} \left( \delta_{ij} + \frac{1}{2} \right) \left( f_{ij} + f_{ji} \right)$$ (1.2)

where $f_{ij}(r_{ij})$ are functions of the internucleon distance and $\delta_{ij}$ are the operators:

$$\delta_{ij} = \delta_{ij}^{(0)} \delta_{ij}^{(1)} \delta_{ij}^{(2)} \delta_{ij}^{(3)}$$

$$\delta_{ij}^{(0)} = \frac{1}{2} \left( \sigma_i \sigma_j + \sigma_j \sigma_i \right)$$

$$\delta_{ij}^{(1)} = \frac{1}{2} \left( \sigma_i + \sigma_j \right)$$

$$\delta_{ij}^{(2)} = \frac{1}{2} \left( \sigma_i \sigma_j + \sigma_j \sigma_i \right)$$

$$\delta_{ij}^{(3)} = \frac{1}{2} \left( \sigma_i \sigma_j + \sigma_j \sigma_i \right)$$

The Pauli spin-isospin matrices, $t_{ij}$ is the tensor operator and $b_{ij}$ is the spin-orbit operator.

Then one calculates and minimizes:

$$E = \langle \Psi | H | \Psi \rangle$$

$$\langle \Psi | \Psi \rangle$$

to get an upper bound for the ground state energy of NM. The method employs a diagrammatic cluster expansion, described in PW, which has been proven to be very convenient for calculations of this kind. Since the exact calculation is a impossible task, several approximations are used. For instance only central chains are hypernetted, while operator chains are calculated in the 'single operator' scheme (SOC), while multiple operator chains are neglected. Wiringa\textsuperscript{3} tested the validity of those approximations, by evaluating the supposedly leading terms of higher order corrections. The net contribution to the energy from those terms was found to be quite small indeed, despite the fact that separately some of them were appreciable, due to cancelation of different corrections. Wiringa calculated these terms at densities around the equilibrium density of NM. However at higher densities these terms become more important. For asymmetric NM or for Nucleon-matter calculations, much higher densities are relevant and so one may need a more accurate calculation.

Fantoni et al\textsuperscript{4} developed an approximate treatment of hypernetted operator chains (HOC) which neglects various different operator orders. Their results for the Reid V-6 model are quite different (as quoted in PW) from the results of other calculations, such as Owen's\textsuperscript{5}, PW, BBG\textsuperscript{6}, which are in reasonable agreement. This suggests that one has to take seriously in account the various orders in the operator products.

The present work intends to illustrate the general idea of a method which may lead to an accurate calculation of higher order operator terms of the cluster expansion. We picked two examples to illustrate the technique. First is the hypernetting of the simplest $\sigma$ (or $\tau$) chain (which can be extended to more complicated $\sigma$ (or $\tau$) chains). This is described in section 3. Second is the calculation of double $\sigma$ (or $\tau$) chains (without any exchange links). This is described in section 4. The necessary mathematical notation is presented in section 2.

The weight is put on the development of the method and our choice of what to calculate at a first stage was made on these grounds. So no claim is made about the importance of the contribution to the energy of the above particular terms.

\textsuperscript{H} Current address: Dipartimento di Fisica, Università di Pisa, Pisa, Italy
2. Notation and formalism.

Since we deal with spin operators
\[ \sigma_1, \sigma_2, \ldots, \sigma_n \]
that correspond to particles 1, 2, ..., n, it is proper to introduce a convenient notation. Let \( \sigma_1 \) be denoted simply by \( 1 \), \( \sigma_2 \) by \( \bar{2} \) and so on. Then one can write the scalar product of two spin operators as:
\[ \sigma_1 \sigma_2 = \bar{1} \bar{2} = 1 \bar{1} \]
(2.1)
where the summation convention over repeated indices is implied. Also the Pauli Identity is written as:
\[ 1 \bar{i} = \delta_{i}^{j} + i \varepsilon_{ijk} k \]
(2.2)
From here on we will denote the product
\[ 1 \bar{i} \bar{j} \bar{k} \ldots \]
by \( 1_{ijk} \ldots \).

The C-part of the \( i \)-th component of the operator \( 1 \), i.e. \( 1_{i} \) is defined as:
\[ C(1_{i}) \equiv <1_{i}| = \frac{1}{2} (|+1_{i}| + |-1_{i}|) \]
(2.3)

Obviously:
\[ <1_{i}| = 0 \]
and
\[ <1_{ij}| = \delta_{ij} + i \varepsilon_{ijk} k = \delta_{ij} + i j k \]
(2.4)

As example note that:
\[ <\sigma_1 \sigma_2 \sigma_1 \sigma_2> = <1 \bar{1} \bar{j} \bar{j} | 1 \bar{j} \bar{j} | 1 \bar{j} \bar{j} | 1 \bar{j} \bar{j} | = \delta_{ij} \delta_{ij} \]
\[ <\sigma_1 \sigma_2 \sigma_1 \sigma_2> = <1 \bar{1} \bar{j} \bar{j} | 1 \bar{j} \bar{j} | 1 \bar{j} \bar{j} | 1 \bar{j} \bar{j} | = \delta_{ij} \delta_{ij} \]
(2.5)

In the above example the C-part operation was extended for two and three particles correspondingly. The subscripted C-part:
\[ <\sigma_1 \sigma_2 \bar{1} \bar{2} \bar{1} \bar{2}> \]
stands for:
\[ <\sigma_1 \sigma_2 \bar{1} \bar{2} \bar{1} \bar{2}> = 1 \bar{1} \bar{j} \bar{j} = 1 \bar{j} \bar{j} \]
(2.5)
i.e. C-part operation over the states of particles indicated by the subscripts is inhibited. So one obtains:
\[ <\sigma_1 \sigma_2 \bar{1} \bar{2} \bar{1} \bar{2}> = 1 \bar{1} \bar{j} \bar{j} = 1 \bar{j} \bar{j} \]
(2.6)

Generalizing, the C-part of an operator product is:
\[ C(i \bar{j}) \equiv <1 \bar{j} \bar{j}> \]
(2.7)

where the brackets denote expectation over all particles appearing in the operator product. Note that since:
\[ <1 \bar{i} \bar{j} \bar{k} \ldots | = \frac{1}{2} \sum_{m} <\bar{m}| <1 \bar{i} | <1 \bar{j} | <1 \bar{k} | \ldots m > = \]
\[ = \frac{1}{2} \sum_{m} <\bar{m}| <1 \bar{i} | <1 \bar{j} | <1 \bar{k} | \ldots m > = \]
(2.8)

One realizes that under the C-part operation cyclic permutations are allowed. In what follows some useful quantities are defined.
\[ D_{ij} \bar{j} \bar{i} \equiv <1 \bar{i} \bar{j} \bar{j} \bar{i} | = <1 \bar{i} \bar{i} \bar{j} | \]
(2.9)

\[ D_{ij} \bar{j} \bar{i} \equiv D_{ij} \bar{j} \bar{i} = <1 \bar{j} \bar{i} \bar{i} | \]
(2.10)

\[ I \bar{i} \bar{i} \bar{j} \equiv \delta_{ii} \delta_{jj} \]
(2.11)

Explicitly one obtains:
\[ D_{ij} \bar{i} \bar{j} \bar{i} = \delta_{ij} \delta_{ij} - \delta_{ii} \delta_{jj} + \delta_{ij} \delta_{ij} \]
(2.12)

Some properties of the above quantities are given below:
\[ D_{ij} \bar{j} \bar{k} \bar{j} \bar{k} = 2 D_{ik} \bar{k} \bar{i} \bar{k} \bar{i} + \delta_{ii} \delta_{kk} \]
(2.13)

or in shorthand: \( D_2 = 2D + I \)
\[ D_{ij} \bar{j} \bar{k} \bar{k} \bar{j} \bar{k} = - \delta_{ii} \delta_{kk} \]
(2.14)

or in shorthand: \( DI = -I \)

Similarly one can show that:
\[ D\bar{D} = \bar{D}D = -I \]
(2.15)
\[ \bar{D}_2 = 2\bar{D} + I \]
\[ \bar{D}I = 3I \]
\[ II = 3I \]

By repeated application of the above "multiplication" properties one obtains:
\[ D^n = 2^{n-1} D + \frac{1}{3} (2^{n-1} + (-1)^n) I \]
(2.16)

\[ D^n = 2^{n-1} D + (3^{n-1} - 2^{n-1}) I \]

and \( D^n = (-1)^k S^{n-1} \)

Also note that:
\[ D_{ij} \bar{j} \bar{i} \bar{j} \bar{i} \bar{j} \bar{i} = 9 \]
(2.17)
\[ D_{ij} \bar{j} \bar{i} \bar{j} \bar{i} \bar{j} \bar{i} = -3 \]

\[ D_{ij} \bar{i} \bar{j} \bar{i} \bar{j} \bar{i} = 9 - 4 \sigma_{12} \]
3. Fermi hypernetted $\sigma$-chains.

3.1 Preliminary analysis

In this and in the following sections we adopt the diagramatic technique and terminology of PW. Some of the diagrams that we are interested in, are displayed in fig. 1.

![Diagram](image)

**Figure 1**

Diagrammatic representation on terms in the cluster expansion - Diagram 1(a) is contained in the SOC calculation of PW - Diagrams 1(b) and 1(c) are examples of terms associated with the hypernetted $\sigma$ (or $\pi$) chains.

Note that each wiggly line corresponds to:

$$ F_{ij}^\sigma \sigma_{ij} \text{ with } F_{ij}^\sigma = 2 f_{ij}^C f_{ij}^C $$

and the thick line, between the interacting particles 1 and 2, corresponds to $f_{12}^C H_{12} f_{12}^C$.

It is instructive to examine in detail a simple diagram. Look for example at diagram 1(b). The relevant operators are:

$\sigma_{12}^C, \sigma_{14}^C, \sigma_{23}^C, \sigma_{24}^C$ and they can come from either side of the Hamiltonian in the expression $\langle \Psi | H | \Psi \rangle$. The possible operator configurations are listed in table 1.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Number of terms</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{12} \sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>41</td>
<td>$1/41$</td>
</tr>
<tr>
<td>$\sigma_{12}^C \sigma_{14}^C \sigma_{23}^C \sigma_{24}^C$</td>
<td>31</td>
<td>$1/31$</td>
</tr>
<tr>
<td>$\sigma_{12}^C \sigma_{14}^C \sigma_{12}^C \sigma_{14}^C$</td>
<td>31</td>
<td>$1/31$</td>
</tr>
<tr>
<td>$\sigma_{12}^C \sigma_{23}^C \sigma_{14}^C \sigma_{24}^C$</td>
<td>31</td>
<td>$1/31$</td>
</tr>
<tr>
<td>$\sigma_{12}^C \sigma_{24}^C \sigma_{14}^C \sigma_{23}^C$</td>
<td>31</td>
<td>$1/31$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\sigma (\psi_{12} \psi_{14} \psi_{23} \psi_{24})$</td>
<td>2x2</td>
<td>$1/4$</td>
</tr>
</tbody>
</table>

The various different cases can be grouped as follows:

1) All four operators are on the right side of $H$
2) Three operators are on the right side of $H$, one on the left
3) Two operators are on the right side of $H$ and two on the left
4) One operator is on the right side of $H$ and three on the left
5) All four operators are on the left side of $H$

In table 1, $H_{12}^C$ stands symbolically instead of $f_{12}^C H_{12} f_{12}^C$. Also the spatial product:

$$ f_{13}^C f_{23}^C f_{24}^C f_{14}^C $$

has been omitted.

Since we are interested in calculating the C-parts of the operator configurations of table 1, we are allowed to perform cyclic permutations of the operators. For instance, the C-part of:

$$ S(\sigma_{13} \, \sigma_{23}) \, H_{12} \, S(\sigma_{14} \, \sigma_{24}) $$

equals to the C-part of:

$$ H_{12} \, S(\sigma_{14} \, \sigma_{24}) \, S(\sigma_{13} \, \sigma_{23}) $$

If one does that to all configurations shown in table 1, one realizes that all groups have on the right side of the Hamiltonian the same 24 terms, each one being a product of the operators $\sigma_{13}^C, \sigma_{14}^C, \sigma_{23}^C, \sigma_{24}^C$ in all 24 possible orders. Each term has a weight $W$, with:

$$ W = \frac{1}{4!} + \frac{1}{3!} + \frac{1}{4} + \frac{1}{3!} + \frac{1}{4} = \frac{16}{24} $$

i.e. there is no preferred operator order since all orders have the same weight. Each order is multiplied by the weight $W$, the $f_{12}^C f_{12}^C$ factor as well as by $\frac{1}{16} F_{12} F_{23} F_{14} F_{24}$.

$$ \rho_{12}^C $$

This results to:

$$ 16 \times (\frac{1}{16} F_{12} F_{12} F_{14} F_{24}) \times (\rho_{12}^C H_{12} \rho_{12}^C) \times (\rho_{14}^C H_{12} \rho_{12}^C) $$

x (All orders)

$$ C(H_{12} S(\sigma_{13} \, \sigma_{23} \, \sigma_{14} \, \sigma_{24})) = C(H_{12} \rho_{12}^C) (3.1) $$

The quantity of interest is:

$$ C(H_{12} S(\sigma_{13} \, \sigma_{23} \, \sigma_{14} \, \sigma_{24})) = C(H_{12} \rho_{12}^C) (3.1) $$

Table 1
with

$$0_{12} = \hat{S}(q_{13} q_{23} q_{14} q_{24}) > 12$$ (3.2)

One can easily generalize this result for any diagram of the structure shown in fig. 2. Hence one has always to calculate the key-quantity:

$$0_{12} = S(1 14 \cdots 23 24 \cdots) 12$$ (3.3)

![Diagram](image)

**Figure 2**

Diagrammatic representation of a general term considered in the hypernetted $\sigma(t)$ chain calculation.

The general diagram of fig. 2, consists of two factors. One is the correlation spatial part involving the products of $F^{ij}$'s integrated over $F^3, F^4', \ldots$, which is trivial to calculate, (see PW); the other involves the C-part of $(f_{12}^E f_{12}^E) 0_{12}$ the calculation of which is the main task of the following paragraph.

3.b Calculation of $0_{12}$

Starting from the definition for $0_{12}$ (equation (3.3)), and noting that:

$$\langle q_{14} q_{21} \rangle_{12} = 0 \quad i \neq 1, 2$$ (3.4)

one realizes that $0_{12}$ remains unaffected upon commutation of the $q_{1i}$ and $q_{2i}$ operators. That means that one can symmetrize as follows:

$$0_{12} = \hat{S}(q_{13} q_{14} \cdots) S(q_{23} q_{24} \cdots) > 12$$ (3.5)

Using the notation of section 2 one can write:

$$q_{13} q_{14} \cdots q_{23} q_{24} \cdots = s_1 s_2 \cdots r_1 r_2 \cdots s_1 s_2 \cdots$$

and since:

$$s_1 r_1 = s_1 r_1, \quad s_2 r_2 = s_2 r_2 \quad \text{etc.}$$

one obtains:

$$0_{12} = S(1 s_1 \cdots) S(2 \cdots s_1 s_2)$$ (3.6)

Let $n$ be the number of indices $s_1, s_2, \ldots, s_n$ in each of the symmetrized products of eq. (3.6). Clearly $0_{12}$ depends on $n$. It will be shown that:

$$0_{12} = \frac{1}{3} (n+2) c_{12}, \quad \text{if } n \text{ is odd}$$ (3.7)

where the superscript $n$ is added to explicitly display the dependence of $0_{12}$ on the number of indices. Consider first the case where $n$ is even. Then:

$$S(1 s_1 s_2 \cdots s_4 s_3 \cdots) = S(1 s_1 s_2 s_3 s_4)$$

$$+ \frac{1}{3} (n+2) c_{12}$$

This is so because the $c_{12} s_1 s_2 s_3 s_4 \cdots$ terms drop out since they are antisymmetric in the $(s_1, s_2), (s_3, s_4), \ldots$ indices. The number of terms in $S(s_1 s_2 s_3 s_4)$ is $(n-1)!$.

Consider the quantity $S(n)$ defined by:

$$S(n) = S(1 2 \cdots n)$$ (3.9)

Where $s_1, s_2, \ldots, s_n$ stand for $s_1 s_2 s_3 s_4 \cdots$

$$0_{12} = \frac{1}{3} (n+2) \frac{S(n)}{(n+1)!}$$ (3.10)

The quantity $S(n)$ can be constructed as:

$$S(n+2) = S(n+1, n, n+2) + S(n+2, n, n+1)$$

$$+ S(n+1, n, n, n+2) + \ldots$$

$$+ S(n, n+1, n, n, n+2)$$ (3.11)

$$|S(n)|^2$$ (3.12)

(contraction over all repeated indices is implied)

In the first term of eq. (3.12) the factor of 3 comes from contracting two identical Kroncker deltas. There are $(n+1)$ such term in $|S(n+2)|^2$ and thus the factor $n+1$. The second term contains all the "cross" terms of
the form:
\[ S_{12\ldots n,1,n+1,2,n+2}^{(n)} = |S_{12\ldots n}^{(n)}|^2 \delta_{1,n+1} \delta_{2,n+2} \]

There are \( n(n+1) \) terms of this kind and hence the factor of \( n(n+1) \).

\[ 0_{12}^{(n+2)} = \frac{|S_{12\ldots n}^{(n)}|^2}{|S_{12}\ldots n|} = \frac{n(n+1)^2}{(n+1)!} \frac{|S_{12}\ldots n|}{|S_{12}\ldots n|} \]

i.e.:
\[ 0_{12}^{(n+2)} = \frac{n+3}{n+1} 0_{12}^{(n)} \]

Since \( 0_{12}^{(2)} = \frac{|S_{12}\ldots n|^2}{(11)!} = 3 \),

the above recursion relation, (3.13), for \( 0_{12}^{(n)} \), is easily solved to give:

\[ 0_{12}^{(n)} = n(n+1) \]

(for even \( n \)). Now for odd number of indices, \( n \), one should calculate

\[ 0_{12}^{(n+1)} = \frac{n+1}{(n+1)!} \left| S_{12\ldots n}^{(n)} \right|^2 \]

(3.14)

with

\[ S_{12\ldots n,1,n+1}^{(n+1)} = S_{12\ldots n}^{(n)} + S_{12\ldots n,1,n+1}^{(n)} + \ldots + S_{12\ldots n,1,n+1}^{(n)} \]

(3.15)

and

\[ S_{12\ldots n,1,n+1}^{(n+1)} = S_{12\ldots n}^{(n)} + S_{12\ldots n,1,n+1}^{(n)} + \ldots + S_{12\ldots n,1,n+1}^{(n)} \]

(3.16)

\[ S_{12\ldots n}^{(n+1)} = S_{12\ldots n}^{(n)} 0_{12}^{(n+1)} + \left| S_{12\ldots n}^{(n)} \right|^2 \]

(3.17)

The first piece comes from including terms in which the operators 1 and 2 carry the same index. For instance:

\[ S_{12\ldots n,1,n+1,2,n+2}^{(n)} = \left| S_{12\ldots n}^{(n)} \right|^2 \delta_{1,n+1} \delta_{2,n+2} \]

There are \( n(n+1) \) such terms and hence the factor \( n(n+1) \). The second piece comes from "cross" terms.

For instance:

\[ S_{12\ldots n,1}^{(n)} = S_{12\ldots n}^{(n)} S_{12\ldots n,1}^{(n)} \]

Since \( S_{12\ldots n,1}^{(n)} \) is symmetric with respect to the indices 1 and \( n+1 \) it must be proportional to \( \delta_{1,n+1} \). i.e. one has:

\[ S_{12\ldots n,1}^{(n)} = S_{12\ldots n}^{(n)} S_{12\ldots n,1}^{(n)} = \delta_{1,n+1} \]

Then:

\[ S_{12\ldots n}^{(n)} S_{12\ldots n}^{(n)} = 3 \delta_{1,1} \]

or

\[ \beta = \frac{1}{3} S_{12\ldots n}^{(n)} \]

and

\[ S_{12\ldots n}^{(n)} S_{12\ldots n,1}^{(n)} = \frac{1}{3} \left| S_{12\ldots n}^{(n)} \right|^2 \delta_{1,n+1} \]

There are \( n(n+1) \) such terms and hence the factor \( n(n+1) \). From eq. (3.14) and (3.17) one obtains:

\[ 0_{12}^{(n+1)} = \frac{n+3}{n+1} 0_{12}^{(n)} \]

or

\[ 0_{12}^{(n+1)} = \frac{1}{3} (n+3) 0_{12}^{(n+1)} \]

and if \( m = n+1 \) (\( m \) is odd)

\[ 0_{12}^{(m)} = \frac{1}{3} (m+2) 0_{12}^{(m)} \]

If by \( X_{12} \) we denote the spatial part of the diagram of figure 3, the quantity of interest is given by:

\[ h_{12}^{(n)} = \sum_{12}^{n=2,4} \left( \sum_{12}^{n=3,5,\ldots,n} \frac{x^n}{n!} \right) \sum_{12}^{n=3,5,\ldots,n} \frac{x^n}{n!} \]

(3.18)

The basic \( \sigma(\tau) \) chain-diagram that is hypernated in the present work.

Which is trivial to calculate. One obtains:

\[ h_{12}^{(n)} = \cosh(X_{12}) + X_{12} \sinh(X_{12}) - 1 + \frac{1}{3} (2 \sinh(X_{12}) + X_{12} \cosh(X_{12}) - 3X_{12}) \]

4. Direct, double \( \sigma(\tau) \) chains.

4. a Preliminary analysis

The general diagram whose C-part we wish to calculate is shown in figure 4. The quantity of interest is:

A general double \( \sigma(\tau) \) diagram whose C-part is considered in the present work.
contains \( n! \) terms, so that \( 0_{1,n+1} \) contains a total of \( (n!)^2 \) terms. Each of these terms consists of two parts, one comes from the left and the other comes from the right symmetrizer. If two operators that share a common particle index (for instance \( \sigma_{i-1,i} \) and \( \sigma_{i,i+1} \)) are arranged with the same, relative to each other, order in both the left and the right part of same term, a \( \delta \)-factor will be associated with that index, otherwise a \( \delta \)-factor will result. The relative orders of operators that do not share a common index are irrelevant, let \( F(n) \) be a set with elements the \( (n!)^2 \) operator products associated with \( 0_{1,n+1} \). Any element of \( F(n) \) whose left part has its different relevant orders from its right part, and hence it will contribute according to (4.4), is said to be of order \( s \). Let \( F(n,s) \) be a subset of \( F(n) \) that consists of all elements of \( F(n) \) of order \( s \). The number of elements of \( F(n,s) \) is the quantity \( M_{n,s} \), we wish to calculate.

Let \( \lambda_{i,j}^n \) be the number of elements of \( F(n,s) \) that have the \( \sigma_{n,n+1} \) operator placed at the \( i \)th position in their left part and at the \( j \)th position in their right part. Clearly then:

\[
\lambda_{i,j}^n = \frac{\binom{n-1}{i} \binom{n-1}{j}}{n} \quad i,j \leq n \tag{4.7}
\]

Then \( \lambda_{i,j}^{n+1} \), i.e., the number of elements of \( F(n,s) \) that have the operator \( \sigma_{n,n+2} \) at the \( i \)th and \( j \)th position in their left and right parts respectively is given by:

\[
\lambda_{i,j}^{n+1} = \binom{n}{i} \binom{n}{j} \lambda_{i,j}^n + \frac{\binom{n}{i-1} \binom{n-1}{j}}{n} \lambda_{i,j}^{n+1} + \frac{\binom{n}{i} \binom{n-1}{j-1}}{n} \lambda_{i,j}^n \tag{4.8}
\]

The first (second) term in (4.8), takes in account all elements of \( F(n,s) \) that have the \( \sigma_{n,n+1} \) operator in the left part before (after) the \( i \)th position and in the right part after (before) the \( j \)th position. Thus when inserting at the \( i \)th and \( j \)th positions, in the left and right parts respectively, the \( \sigma_{n,n+2} \) operator we obtain elements of the set \( F(n+1,s) \), since the relevant relative orders of \( \sigma_{n,n+1} \) and \( \sigma_{n,n+2} \) are the same in both parts of every element. The third (fourth) term, takes in account all terms of \( F(n,s-1) \) that have the \( \sigma_{n,n+1} \) operator in the left part before (after) the \( i \)th position and in the right part after (before) the \( j \)th position, thus when inserting at the \( i \)th and \( j \)th positions, in the left and right parts respectively, the \( \sigma_{n+1,n+2} \) operator, we obtain elements of the \( F(n+1,s) \) set since the relevant relative orders of \( \sigma_{n+1,n+2} \) are different in the two parts of every element.
There is no other scheme by which one can generate elements of \( P(n+1,s) \), and so \((4.8)\) is complete. Using the recursion relation \((4.8)\) along with \((4.7)\) one can easily compute \( M_{n,s} \).

Initial values for \( n=1 \) and \( n=2 \) are easily calculated. We find:

\[
\begin{align*}
\lambda_{1,1}^1 &= 1, & \lambda_{1,1}^2 &= \lambda_{2,2}^1 = \lambda_{2,1}^2 = 1 \\
\lambda_{1,2}^1 &= 2, & \lambda_{2,1}^2 &= \lambda_{2,1}^2 = 0
\end{align*}
\]

In Table II we give the first few \( M_{n,s} \) numbers calculated using the method described above.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>2</td>
<td>10</td>
<td>88</td>
<td>1216</td>
<td>24176</td>
<td>654424</td>
<td>23136128</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
<td>200</td>
<td>3526</td>
<td>58582</td>
<td>2363728</td>
<td>111842432</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>200</td>
<td>4956</td>
<td>149152</td>
<td>5714472</td>
<td>270769536</td>
<td>33</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>88</td>
<td>3526</td>
<td>149152</td>
<td>7756332</td>
<td>407103104</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1216</td>
<td>58582</td>
<td>149152</td>
<td>7756332</td>
<td>407103104</td>
<td>33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>24176</td>
<td>2363728</td>
<td>270769536</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>654424</td>
<td>111842432</td>
<td>33</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>23136128</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table II** Tabulation of the \( M_{n,s} \) coefficients for \( n=2,3,...,8 \) and for the corresponding relevant \( s \)-values.

5. - Discussion.

The hypernation of \( \sigma \) (or \( \tau \)) chains presented in section 3, can be rather easily extended to treat \( \sigma \tau \) chains as well. Inclusion of exchanges, or of more complicated chains, as well as provision for ring diagrams can be carried out following the same general line with a few modifications in the details of the calculation. Hypernation of chains including tensor operators is certainly more involved.

It is straightforward to extend the method presented in section 4 to treat double \( \sigma \tau \) chains or \( \sigma \sigma \), \( \tau \tau \) or \( \sigma \tau \) double operator chains. Inclusion of exchanges is possible but not as easy. For double tensor chains the counting part is the same as for double \( -\sigma \tau \) chains but the spatial part needs special merit. We want to remark that since it is not always easy to derive a recursion relation that will solve the enumeration problem, one can explicitly generate the various permutations imposed by the symmetrizers and count the required orders one by one using a fast digital computer. A method for generating all permutations of \( n \) different objects is given by Johnson\(^7\) and Trotter\(^8\).

Although these methods as they are described above, are suitable for symmetric Nuclear Matter calculations, they can be extended to facilitate spin and/or isospin polarized matter calculations; also they can play the role of a guide in approximating other more complicated terms of the cluster expansion.

The author would like to acknowledge illuminating discussions with S. Fantoni, A. Fabrocini, S. Rosati, M. Viviani, V.A.Voutsas-Dakis and INFN for partial support.

**References**

7. S.M. Johnson Math. of Comp. 17 (1963) 282