

Digital Image Processing

Wavelets and Multiresolution Processing (Multiresolution Analysis)

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Multiresolution Analysis

Image pyramids, subband coding and the Haar transform play an important role in a mathematical framework called *multiresolution analysis* (MRA).

In MRA, a *scaling function* is used to create a series of approximations of a signal each differing a factor of 2 in resolution from its nearest neighbour approximation.

Additional functions, called *wavelets* are then used to encode the difference between adjacent approximations.

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Multiresolution Analysis Series Expansions

A signal or a function $f(x)$ may be analyzed as a linear combination of *expansion functions*:

$$f(x) = \sum_k a_k \phi_k(x)$$

If the expansion is unique then the expansion functions are called basis functions and the *expansion set* $\{\phi_k(x)\}$ is called a *basis*. The functions that may be expressed as a linear combination of $\phi_k(x)$ form a function space called the *closed span*:

$$V = \overline{\text{Span}\{\phi_k(x)\}}$$

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Multiresolution Analysis Series Expansions (cont...)

For any function space V and corresponding expansion set $\{\phi_k(x)\}$ there is a set of dual functions $\{\tilde{\phi}_k(x)\}$ used to compute the expansion coefficients α_k for any $f(x) \in V$ as the inner products:

$$\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle = \int \tilde{\phi}_k^*(x) f(x) dx$$

Depending on the orthogonality of the expansion set we have three cases for these coefficients.

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Multiresolution Analysis Series Expansions (cont...)

Case 1: The expansion functions form an orthonormal basis for V :

$$\langle \phi_k(x), \phi_j(x) \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

Then, the basis functions and their duals are equivalent:

$$\phi_k(x) = \tilde{\phi}_k^*(x)$$

and the expansion coefficients are:

$$\alpha_k = \langle \phi_k(x), f(x) \rangle$$

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7 Multiresolution Analysis Series Expansions (cont...)

Case 2: The expansion functions are not orthonormal but they are an orthogonal basis for V then:

$$\langle \phi_k(x), \phi_j(x) \rangle = 0, \quad j \neq k$$

and the basis and its dual are called *biorthogonal*.
The expansion coefficients are:

$$\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle = \int \tilde{\phi}_k^*(x) f(x) dx$$

and the biorthogonal basis and its dual are such that:

$$\langle \phi_j(x), \tilde{\phi}_k^*(x) \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

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8 Multiresolution Analysis Series Expansions (cont...)

Case 3: The expansion set is not a basis for V then there is more than one set of coefficients α_k for any $f(x) \in V$.
The expansion functions and their duals are said to be *overcomplete* or *redundant*.

They form a *frame* in which:

$$A \|f(x)\|^2 \leq \sum_k |\langle \tilde{\phi}_k(x), f(x) \rangle|^2 \leq B \|f(x)\|^2, \quad A > 0, B < \infty, \forall f(x) \in V$$

Dividing by the squared norm of the function we see that A and B frame the normalized inner products.

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9 Multiresolution Analysis Series Expansions (cont...)

Case 3 (continued): Equations similar to cases 1 and 2 may be used to find the expansion coefficients. If $A=B$, then the expansion is called a tight frame and it can be shown that (Daubechies [1992]):

$$f(x) = \frac{1}{A} \sum_k \langle \phi_k(x), f(x) \rangle \phi_k(x)$$

Except from the normalization term, this is identical to the expression obtained for orthonormal bases.

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10 Multiresolution Analysis Scaling Functions

Consider the set of expansion functions composed of integer translations and binary scalings of a real, square-integrable function $\phi(x)$:

$$\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k), \quad j, k \in \mathbb{Z}, \phi(x) \in L^2(\mathbb{R}).$$

Parameter k determines the position of $\phi_{j,k}$ along the horizontal axis.
Parameter j determines how broad or narrow it is along the horizontal axis. The term $2^{j/2}$ controls the amplitude. Because of its shape, $\phi(x)$ is called *scaling function*.

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11 Multiresolution Analysis Scaling Functions (cont...)

By choosing the scaling function $\phi(x)$ properly, the set $\{\phi_{j,k}(x)\}$ can be made to span the set of all measurable, square-integrable functions $L^2(\mathbb{R})$.

If we restrict j to a specific value $j=j_0$, the resulting expansion set $\{\phi_{j_0,k}(x)\}$ is a subset of $\{\phi_{j,k}(x)\}$ that spans a subspace of $L^2(\mathbb{R})$:

$$V_{j_0} = \overline{\text{Span}\{\phi_{j_0,k}(x)\}}$$

That is, V_{j_0} is the span of $\phi_{j_0,k}(x)$ over k .
If $f(x) \in V_{j_0}$ then we can write: $f(x) = \sum_k a_k \phi_{j_0,k}(x)$

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12 Multiresolution Analysis Scaling Functions (cont...)

If $f(x) \in V_{j_0}$ then we can write: $f(x) = \sum_k a_k \phi_{j_0,k}(x)$

More generally,

$$V_j = \overline{\text{Span}\{\phi_{j,k}(x)\}}$$

Increasing j , increases the size of V_j allowing functions with fine details to be included in the subspace.
This is a consequence of the fact that, as j increases, the $\{\phi_{j,k}(x)\}$ that are used to represent the subspace functions become narrower.

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13 Multiresolution Analysis
Scaling Functions (cont...)

Consider the unit-height, unit-width Haar scaling function:

$$\phi(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and observe some of the expansion functions $\phi_{j,k}(x)$ generated by scaling and translations of the original function.

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14 Multiresolution Analysis
Scaling Functions (cont...)

$$\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k)$$

As j increases, the functions become narrower.

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15 Multiresolution Analysis
Scaling Functions (cont...)

Function $f(x)$ does not belong to V_0 because the V_0 expansion functions are too coarse to represent it. Higher resolution functions are required.

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16 Multiresolution Analysis
Scaling Functions (cont...)

Indeed, $f(x) \in V_1$. $f(x) = 0.5\phi_{1,0}(x) + \phi_{1,1}(x) - 0.25\phi_{1,4}(x)$

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17 Multiresolution Analysis
Scaling Functions (cont...)

Note also that $\phi_{0,0}(x)$ may be decomposed as a sum of V_1 expansion functions.

$$\phi_{0,0}(x) = \frac{1}{\sqrt{2}} \phi_{1,0}(x) + \frac{1}{\sqrt{2}} \phi_{1,1}(x) = \phi(2x) + \phi(2x-1)$$

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18 Multiresolution Analysis
Scaling Functions (cont...)

In a similar manner, any V_0 expansion function may be decomposed as a sum of V_1 expansion functions:

$$\phi_{0,k}(x) = \frac{1}{\sqrt{2}} \phi_{1,2k}(x) + \frac{1}{\sqrt{2}} \phi_{1,2k+1}(x)$$

Therefore, if $f(x) \in V_0$, then $f(x) \in V_1$. This is because all V_0 expansion functions are contained in V_1 . Mathematically, we say that V_0 is a subspace of V_1 :

$$V_0 \subset V_1$$

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19 Multiresolution Analysis
Scaling Functions (cont...)

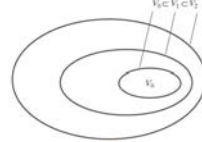
The simple scaling function in the preceding example obeys the four fundamental requirements of multiresolution analysis [Mallat 1989].

MRA Requirement 1: The scaling function is orthogonal to its integer translates.
Easy to see for the Haar function. Hard to satisfy for functions with support different than $[0, 1]$.

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20 Multiresolution Analysis
Scaling Functions (cont...)

MRA Requirement 2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.
This means that:

$$V_{-\infty} \subset \dots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots \subset V_{\infty}$$


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21 Multiresolution Analysis
Scaling Functions (cont...)

Moreover, if $f(x) \in V_j$, then $f(2x) \in V_{j+1}$.

The fact that the Haar scaling function satisfies this requirement is not an indication that any function with support of width 1 satisfies the condition.

For instance, the simple function:

$$\phi(x) = \begin{cases} 1, & 0.25 \leq x \leq 0.75 \\ 0, & \text{otherwise} \end{cases}$$

is not a valid scaling function for MRA.

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22 Multiresolution Analysis
Scaling Functions (cont...)

MRA Requirement 3: The only common function to all subspaces V_j is $f(x)=0$.
In the coarsest possible expansion the only representable function is the function with no information $f(x)=0$. That is:

$$j \rightarrow -\infty, V_{-\infty} = \{0\}$$

MRA Requirement 4: Any function may be represented with arbitrary precision.
This means that in the limit:

$$j \rightarrow \infty, V_{\infty} = \{L^2(\mathbb{R})\}$$

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23 Multiresolution Analysis
Scaling Functions (cont...)

Under these conditions, the expansion functions of subspace V_j may be expressed as a weighted sum of the expansion functions of subspace V_{j+1} :

$$\phi_{j,k}(x) = \sum_n a_n \phi_{j+1,n}(x)$$

Substituting $\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k)$

and changing variable a_n to $h_{\phi}(n)$, we obtain:

$$\phi_{j,k}(x) = \sum_n h_{\phi}(n) 2^{(j+1)/2} \phi(2^{j+1} x - n)$$

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24 Multiresolution Analysis
Scaling Functions (cont...)

Because $\phi_{0,0}(x) = \phi(x)$

we can set $j=k=0$ to obtain a simpler expression:

$$\phi(x) = \sum_n h_{\phi}(n) \sqrt{2} \phi(2x - n)$$

The coefficients $h_{\phi}(n)$, are called *scaling function coefficients*.
This equation is fundamental to MRA and is called the *refinement equation*, the *MRA equation* or the *dilation equation*.

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25 Multiresolution Analysis
Scaling Functions (cont...)

$$\phi(x) = \sum_n h_\phi(n) \sqrt{2} \phi(2x - n)$$

The refinement equation states that the expansion functions of any subspace may be obtained from double-resolution copies of themselves, that is, the expansion functions of the next higher resolution space.

Note that the choice of reference V_0 is arbitrary. We can start at any resolution level.

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26 Multiresolution Analysis
Scaling Functions (cont...)

The scaling function coefficients for the Haar function are the elements of the first row of matrix $\mathbf{H}_2$, that is:

$$h_\phi(0) = h_\phi(1) = \frac{1}{\sqrt{2}}$$

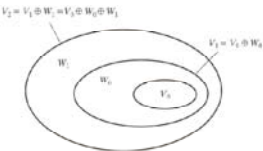
Thus, the refinement equation is:

$$\phi(x) = \frac{1}{\sqrt{2}} [\sqrt{2} \phi(2x)] + \frac{1}{\sqrt{2}} [\sqrt{2} \phi(2x-1)] = \phi(2x) + \phi(2x-1)$$

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27 Multiresolution Analysis
Wavelet Functions

Given a scaling function that meets the MRA criteria we can define a wavelet function $\psi(x)$ that together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces V_j and V_{j+1} .



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28 Multiresolution Analysis
Wavelet Functions (cont...)

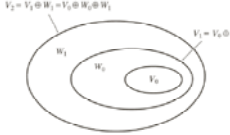
We define the set $\{\psi_{j,k}(x)\}$ of wavelets

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}$$

That span the W_j spaces. As with scaling functions:

$$W_j = \overline{\text{Span}\{\psi_{j,k}(x)\}}$$

If $f(x) \in W_j$, $f(x) = \sum_k a_k \psi_{j,k}(x)$



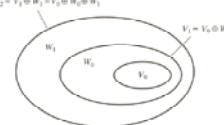
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29 Multiresolution Analysis
Wavelet Functions (cont...)

The scaling and wavelet function subspaces are related by

$$V_{j+1} = V_j \oplus W_j$$

where the symbol $\oplus$ denotes the union of spaces. The orthogonal complement V_j of in V_{j+1} is W_j and all members of V_j are orthogonal to the members of W_j .

$$\langle \phi_{j,k}(x), \psi_{j,l}(x) \rangle = 0, \quad \forall j, k, l.$$


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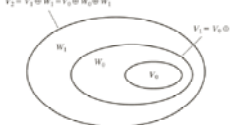
30 Multiresolution Analysis
Wavelet Functions (cont...)

We can now express the space of all measurable, square-integrable function as

$$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \dots = V_1 \oplus W_1 \oplus W_2 \oplus \dots \Leftrightarrow$$

$$L^2(\mathbb{R}) = \dots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \dots$$

which eliminates the scaling function and uses only wavelet functions



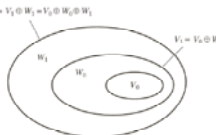
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31 Multiresolution Analysis Wavelet Functions (cont...)

If $f(x) \in V_1$ but $f(x) \notin V_0$ its expansion using

$$L^2(\mathbb{R}) = V_0 \oplus W_0$$

contains an *approximation* using scaling functions V_0 and wavelets from W_0 would encode the *difference* between this approximation and the actual function.



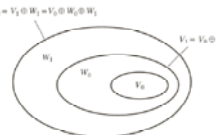
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32 Multiresolution Analysis Wavelet Functions (cont...)

The representation may be generalized to yield

$$L^2(\mathbb{R}) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \dots$$

starting from an arbitrary scale and adding the appropriate wavelet functions that capture the difference between the coarse scale representation and the actual function.



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33 Multiresolution Analysis Wavelet Functions (cont...)

Any wavelet function, like its scaling function counterpart, reside in the space spanned by the next higher resolution level. Therefore, it can be expressed as a weighted sum of shifted, double-resolution scaling functions:

$$\psi(x) = \sum_n h_\psi(n) \sqrt{2} \phi(2x - n)$$

The coefficients $h_\psi(n)$, are called *wavelet function coefficients*. It can also be shown that

$$h_\psi(n) = (-1)^n h_\phi(1 - n)$$

Note the similarity with the analysis-synthesis filters.

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34 Multiresolution Analysis Wavelet Functions (cont...)

The Haar scaling function coefficients were defined as

$$h_\phi(0) = h_\phi(1) = \frac{1}{\sqrt{2}}$$

The corresponding wavelet coefficients are

$$h_\psi(0) = (-1)^0 h_\phi(1 - 0) = \frac{1}{\sqrt{2}}, \quad h_\psi(1) = (-1)^1 h_\phi(1 - 1) = -\frac{1}{\sqrt{2}}$$

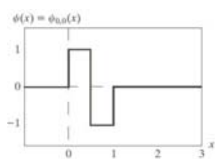
These coefficients are the elements of the second row of the Haar transformation matrix $\mathbf{H}_2$.

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35 Multiresolution Analysis Wavelet Functions (cont...)

Substituting this result into $\psi(x) = \sum_n h_\psi(n) \sqrt{2} \phi(2x - n)$

we get
$$\psi(x) = \phi(2x) - \phi(2x - 1) = \begin{cases} 1 & 0 \leq x \leq 0.5 \\ -1 & 0.5 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



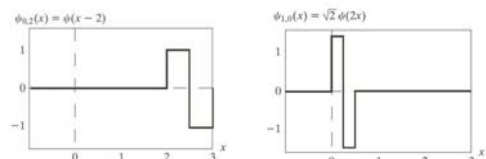
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36 Multiresolution Analysis Wavelet Functions (cont...)

Using

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}$$

we can now generate the universe of translated and scaled Haar wavelets.

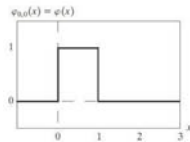


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Multiresolution Analysis Wavelet Functions (cont...)

Any function $f(x) \in V_0$ may be expressed by the scaling function $\phi(x)$:

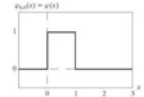


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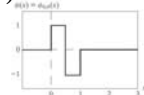
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Multiresolution Analysis Wavelet Functions (cont...)

Any function $f(x) \in V_1$ may be expressed by the scaling function $\phi(x)$ describing the coarse form



and the wavelet function $\psi(x)$ describing the details that cannot be represented in V_0 by $\phi(x)$.

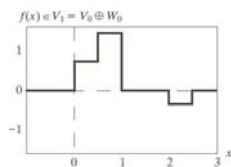


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Multiresolution Analysis Wavelet Functions (cont...)

Remember the function of an earlier example $f(x) \in V_1$ but $f(x) \notin V_0$. This indicates that it could be expanded using V_0 to capture the coarse characteristics of the function and W_0 to encode the details that cannot be represented by V_0 .



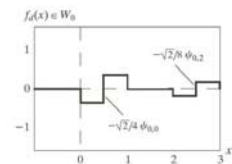
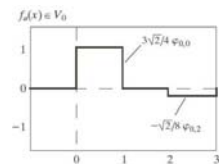
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Multiresolution Analysis Wavelet Functions (cont...)

$$f(x) = f_a(x) + f_d(x)$$

$$f_a(x) = \frac{3\sqrt{2}}{4} \phi_{0,0}(x) - \frac{\sqrt{2}}{8} \phi_{0,2}(x) \quad f_d(x) = \frac{-\sqrt{2}}{4} \psi_{0,0}(x) - \frac{\sqrt{2}}{8} \psi_{0,2}(x)$$



Notice the equivalence to low pass and high pass filtering.

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