Filtering in the Frequency Domain

Filter: A device or material for suppressing or minimizing waves or oscillations of certain frequencies.

Frequency: The number of times that a periodic function repeats the same sequence of values during a unit variation of the independent variable.

Webster’s New Collegiate Dictionary

Jean Baptiste Joseph Fourier

Fourier was born in Auxerre, France in 1768.

– Most famous for his work “La Théorie Analytique de la Chaleur” published in 1822.


Nobody paid much attention when the work was first published.

One of the most important mathematical theories in modern engineering.

The Big Idea

Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a Fourier series

1D continuous signals

• It may be considered both as continuous and discrete.

• Useful for the representation of discrete signals through sampling of continuous signals.

1D continuous signals (cont.)

Impulse train function

\[ S_{\alpha}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \]

\[ x[n] = x(t)S_{\alpha}(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - n\Delta T) = \sum_{n=-\infty}^{\infty} x(n\Delta T)\delta(t - n\Delta T) \]
1D continuous signals (cont.)

\[ x[n] = x(t) \delta(t - n \Delta T) = \sum_{n=-\infty}^{\infty} x(n \Delta T) \delta(t - n \Delta T) \]

- The Fourier series expansion of a periodic signal \( f(t) \).

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi \mu n / T} \]

\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi \mu t / T} dt \]

1D continuous signals (cont.)

- The Fourier transform of a continuous signal \( f(t) \).

\[ F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi \mu t} dt \]

\[ f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi \mu t} d\mu \]

- Attention: the variable is the frequency (Hz) and not the radial frequency \( \Omega = 2\pi \mu \) as in the Signals and Systems course.

1D continuous signals (cont.)

- Convolution property of the FT.

\[ f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \]

\[ f(t) * h(t) \leftrightarrow F(\mu)H(\mu) \]

\[ f(t) h(t) \leftrightarrow F(\mu) * H(\mu) \]

1D continuous signals (cont.)

- Intermediate result
  - The Fourier transform of the impulse train.

\[ \sum_{n=-\infty}^{\infty} \delta(t - n \Delta T) \leftrightarrow \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - n / \Delta T) \]

- It is also an impulse train in the frequency domain.

- Impulses are equally spaced every \( 1/\Delta T \).
1D continuous signals (cont.)

Sampling

\[ x[n] = x(t) \delta_n(t) = \sum_{n=-\infty}^{\infty} x(t) \delta(t-n\Delta T) = \sum_{n=-\infty}^{\infty} x(n\Delta T) \delta(t-n\Delta T) \]

Nyquist theorem

\[ \frac{1}{\Delta T} \geq 2\mu_{\text{max}} \]

1D continuous signals (cont.)

Sampling

- The spectrum of the discrete signal consists of repetitions of the spectrum of the continuous signal every \( 1/\Delta T \).
- The Nyquist criterion should be satisfied.

\[ f(t) \leftrightarrow F(\mu) \]

\[ \tilde{f}(n\Delta T) = f[n] \leftrightarrow \tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right) \]

Reconstruction

- Provided a correct sampling, the continuous signal may be perfectly reconstructed by its samples.

\[ f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T) \text{sinc} \left(\frac{t-n\Delta T}{n\Delta T}\right) \]
1D continuous signals (cont.)

• Under aliasing, the reconstruction of the continuous signal not correct.

The Discrete Fourier Transform

• The Fourier transform of a sampled (discrete) signal is a continuous function of the frequency.

\[ \hat{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right) \]

• For a \( N \)-length discrete signal, taking \( N \) samples of its Fourier transform at frequencies:

\[ \mu_k = \frac{k}{N\Delta T}, \quad k = 0, 1, \ldots, N - 1 \]

provides the discrete Fourier transform (DFT) of the signal.

The Discrete Fourier Transform (cont.)

• Property: sum of complex exponentials

\[ \sum_{n=0}^{N-1} w_N^n = \begin{cases} 1, & k = rN, \quad r \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} \]

The proof is left as an exercise.
The Discrete Fourier Transform (cont.)

- DFT pair of signal \( f[n] \) of length \( N \) may be expressed in matrix-vector form.

\[
F[k] = \sum_{n=0}^{N-1} f[n] w_N^{nk}, \quad 0 \leq k \leq N-1
\]

\[
f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] w_N^{-nk}, \quad 0 \leq n \leq N-1
\]

\[
w_N = e^{-\frac{2\pi}{N}}
\]

Example for \( N=4 \)

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j \\
\end{bmatrix}
\]

The inverse DFT is then expressed by:

\[
f = A^H F
\]

\[
A^{-1} = \frac{1}{N} (A^H)^{-1}
\]

This is derived by the complex exponential sum property.

Linear convolution

\[
f[n] = \{1, 2, 2\}, \ h[n] = \{1, -1\}, \ N_1 = 3, N_2 = 2
\]

\[
g[n] = f[n] \ast h[n] = \sum_{m=-\infty}^{\infty} f[m] h[n-m]
\]

is of length \( N=N_1+N_2-1=4 \)

Linear convolution (cont.)

\[
f[n] = \{1, 2, 2\}, \ h[n] = \{1, -1\}, \ N_1 = 3, N_2 = 2
\]

\[
g[n] = f[n] \ast h[n] = \sum_{m=-\infty}^{\infty} f[m] h[n-m]
\]

<table>
<thead>
<tr>
<th>( f[m] )</th>
<th>( 1 )</th>
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<td>( n=3 )</td>
<td>( h[3] )</td>
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<td>1</td>
<td>( \rightarrow ) -2</td>
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</table>

\[
g[n] = \{1, 1, 0, -2\}
\]
### Circular shift

- Signal $x[n]$ of length $N$.
- A circular shift ensures that the resulting signal will keep its length $N$.
- It is a shift modulo $N$ denoted by $x[(n-m)_N] = x[(n-m) \mod N]$.
- Example: $x[n]$ is of length $N=8$.
  
  $x[(-2)_N] = x[6]$  
  $x[(10)_N] = x[2]$  

### Circular convolution

Let $f[n] = \{1, 2, 2\}$ and $h[n] = \{-1, 1\}$, $N_1 = 3, N_2 = 2$.

$$g[n] = f[n] \bigotimes h[n] = \sum_{m=-\infty}^{\infty} f[m] h[(n-m)_N]$$

The result is of length $N = \max\{N_1, N_2\} = 3$.

### DFT and convolution

- The property holds for the circular convolution.
- In signal processing we are interested in linear convolution.
- Is there a similar property for the linear convolution?

### Zero-padding to length $N=N_1+N_2-1=4$

The result is the same as the linear convolution.

### DFT and convolution (cont.)

- Zero-padded signals

### DFT and convolution (cont.)

- Zero-padded signals
DFT and convolution (cont.)

Verification using DFT

\[
\mathbf{F} = A \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -j & -1 & j & 2 \\ 0 & 1 & -j & -1 \\ 1 & j & -1 & -j \\ \end{bmatrix} = \begin{bmatrix} 5 \\ -1-j2 \\ 1 \\ -1+j2 \\ \end{bmatrix}
\]

\[
\mathbf{H} = A \mathbf{h} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -j & -1 & j & -j \\ 0 & 1 & -j & 0 \\ 1 & j & -1 & -j \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 1+j \\ 2 \\ 1-j \\ \end{bmatrix}
\]

\[
\mathbf{G} = \mathbf{F} \times \mathbf{H} = \begin{bmatrix} 5 \times 0 \\ (-1-j2) \times (1+j) \\ 1 \times 2 \\ (-1+j2) \times (1-j) \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 1-j3 \\ 2 \\ 1+j3 \\ \end{bmatrix}
\]

DFT and convolution (cont.)

Inverse DFT of the result

\[
\mathbf{g} = A^* \mathbf{G} = \frac{1}{4} (A^*)^T \mathbf{G} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ j & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ j & -1 & j & 1+j3 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ \end{bmatrix}
\]

The same result as their linear convolution.

DFT and convolution (cont.)

2D continuous signals

The 2D impulse train is also separable:

\[
S_{\Delta X\Delta Y}(x,y) = S_{\Delta X}(x) S_{\Delta Y}(y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x-n\Delta X, y-n\Delta Y)
\]

2D continuous signals (cont.)

The Fourier transform of a continuous 2D signal \( f(x,y) \).

\[
F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(\mu x + \nu y)} \, dy \, dx
\]

\[
f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu x + \nu y)} \, dv \, d\mu
\]
2D continuous signals (cont.)

• Example: FT of \( f(x,y) = \delta(x) \)

\[
F(\mu,\nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)e^{-j2\pi(\mu x + \nu y)} \, dx \, dy
\]

\[
= \int_{-\infty}^{\infty} \delta(x)e^{-j2\pi\mu x} \, dx \int_{-\infty}^{\infty} e^{-j2\pi\nu y} \, dy
\]

\[
= \int_{-\infty}^{\infty} e^{-j2\pi\nu y} \, dy = \delta(\nu)
\]

2D continuous signals (cont.)

• 2D continuous convolution

\[
f(x,y) * h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-, y)h(\alpha, \beta) \, d\alpha \, d\beta
\]

• We will examine the discrete convolution in more detail.

• Convolution property

\[
f(x,y) * h(x,y) \leftrightarrow F(\mu,\nu)H(\mu,\nu)
\]

2D continuous signals (cont.)

• 2D sampling is accomplished by

\[
S_{\Delta X, \Delta Y}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - n\Delta X, y - n\Delta Y)
\]

• The FT of the sampled 2D signal consists of repetitions of the spectrum of the 1D continuous signal.

\[
\hat{F}(\mu,\nu) = \frac{1}{\Delta X \Delta Y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(\mu - m\Delta X, \nu - n\Delta Y)
\]

2D continuous signals (cont.)

• The Nyquist theorem involves both the horizontal and vertical frequencies.

\[
1 \leq \frac{2\mu_{\text{max}}}{\Delta X}, \frac{2\nu_{\text{max}}}{\Delta Y}
\]
Aliasing

- Effect of sampling a scene with periodic or nearly periodic components (e.g. overlapping grids, TV raster lines and striped materials).
- In image processing the problem arises when scanning media prints (e.g. magazines, newspapers).
- The problem is more general than sampling artifacts.

Aliasing - Moiré Patterns (cont.)

- Superimposed grid drawings (not digitized) produce the effect of new frequencies not existing in the original components.

Aliasing - Moiré Patterns (cont.)

- In printing industry the problem comes when scanning photographs from the superposition of:
  - The sampling lattice (usually horizontal and vertical).
  - Dot patterns on the newspaper image.

Aliasing - Moiré Patterns (cont.)

- The printing industry uses halftoning to cope with the problem.
  - The dot size is inversely proportional to image intensity.
2D discrete convolution

\[ f[m,n] \to h[m,n] \to g[m,n] = f[m,n] * h[m,n] = \sum_{k,l} f[k,l] h[m-k, n-l] \]

• Take the symmetric of one of the signals with respect to the origin.
• Shift it and compute the sum at every position \([m,n]\).


C. Nikou – Digital Image Processing (E12)

The 2D DFT

\[ F[k,l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi \frac{mk}{M} \frac{ln}{N}} \]

\[ f[m,n] = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi \frac{mk}{M} \frac{ln}{N}} \]

\[ \begin{cases} 0 \leq k \leq M-1 & 0 \leq m \leq M-1 \\ 0 \leq l \leq N-1 & 0 \leq n \leq N-1 \end{cases} \]

C. Nikou – Digital Image Processing (E12)


All of the properties of 1D DFT hold.

Particularly:

- Let \( f[m,n] \) be of size \( M_1 \times N_1 \) and \( h[m,n] \) of size \( M_2 \times N_2 \).
- If the signals are zero-padded to size \((M_1 + M_2 - 1) \times (N_1 + N_2 - 1)\) then their circular convolution will be the same as their linear convolution and:

\[ g[m,n] = \tilde{f}[m,n] * \tilde{h}[m,n] \iff \tilde{G}[k,l] = \tilde{F}[k,l] \tilde{H}[k,l] \]