Digital Image Processing

Wavelets and Multiresolution Processing
(Multiresolution Analysis)

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Multiresolution Analysis

Image pyramids, subband coding and the Haar transform play an important role in a mathematical framework called *multiresolution analysis* (MRA).

In MRA, a *scaling function* is used to create a series of approximations of a signal each differing a factor of 2 in resolution from its nearest neighbour approximation.

Additional functions, called *wavelets* are then used to encode the difference between adjacent approximations.

Multiresolution Analysis
Series Expansions

A signal or a function $f(x)$ may be analyzed as a linear combination of *expansion functions*:

$$f(x) = \sum_k a_k \phi_k(x)$$

If the expansion is unique then the expansion functions are called basis functions and the *expansion set* $\{\phi_k(x)\}$ is called *a basis*. The functions that may be expressed as a linear combination of $\phi_k(x)$ form a function space called the *closed span*:

$$V = \overline{\operatorname*{Span}_k\{\phi_k(x)\}}$$

Multiresolution Analysis
Series Expansions (cont…)

For any function space V and corresponding expansion set $\{\phi_k(x)\}$ there is a set of dual functions $\{\tilde{\phi}_k(x)\}$ used to compute the expansion coefficients α_k for any $f(x) \in V$ as the inner products:

$$\alpha_k = \left\langle \tilde{\phi}_k(x), f(x) \right\rangle = \int \tilde{\phi}_k^*(x) f(x)\, dx$$

Depending on the orthogonality of the expansion set we have three cases for these coefficients.

Multiresolution Analysis Series Expansions (cont…)

Case 1: The expansion functions form an orthonormal basis for V:

$$\left\langle \phi_k(x), \phi_j(x) \right\rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

Then, the basis functions and their duals are equivalent:

$$\phi_k(x) = \tilde{\phi}_k^*(x)$$

and the expansion coefficients are:

$$\alpha_k = \left\langle \phi_k(x), f(x) \right\rangle$$

Multiresolution Analysis
Series Expansions (cont…)

Case 2: The expansion functions are not orthonormal but they are an orthogonal basis for V then:

$$\left\langle \phi_k(x), \phi_j(x) \right\rangle = 0, \quad j \neq k$$

and the basis and its dual are called *biorthogonal*.
The expansion coefficients are:

$$\alpha_k = \left\langle \tilde{\phi}_k(x), f(x) \right\rangle = \int \tilde{\phi}_k^*(x) f(x)\, dx$$

and the biorthogonal basis and its dual are such that:

$$\left\langle \phi_j(x), \tilde{\phi}_k^*(x) \right\rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

Multiresolution Analysis
Series Expansions (cont…)

Case 3: The expansion set is not a basis for V then there is more than one set of coefficients α_k for any $f(x)\in V$.

The expansion functions and their duals are said to be *overcomplete* or *redundant*.

They form a *frame* in which:

$$A\|f(x)\|^2 \leq \sum_k \left|\left\langle \tilde{\phi}_k(x), f(x)\right\rangle\right|^2 \leq B\|f(x)\|^2,\ A>0,\ B<\infty, \forall f(x)\in V$$

Dividing by the squared norm of the function we see that A and B frame the normalized inner products.

Multiresolution Analysis
Series Expansions (cont…)

Case 3 (continued): Equations similar to cases 1 and 2 may be used to find the expansion coefficients. If $A=B$, then the expansion is called a tight frame and it can be shown that (Daubechies [1992]):

$$f(x) = \frac{1}{A} \sum_k \left\langle \phi_k(x), f(x) \right\rangle \phi_k(x)$$

Except from the normalization term, this is identical to the expression obtained for orthonormal bases.

Multiresolution Analysis
Scaling Functions

Consider the set of expansion functions composed of integer translations and binary scalings of a real, square-integrable function $\phi(x)$:

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k),\ j,k \in \mathbb{Z},\ \phi(x) \in L^2(\mathbb{R}).$$

Parameter k determines the position of $\phi_{j,k}(x)$ along the horizontal axis.
Parameter j determines how broad or narrow it is along the horizontal axis. The term $2^{j/2}$ controls the amplitude. Because of its shape, $\phi(x)$ is called *scaling function*.

Multiresolution Analysis Scaling Functions (cont…)

By choosing the scaling function $\phi(x)$ properly, the set $\{\phi_{j,k}(x)\}$ can be made to span the set of all measurable, square-integrable functions $L^2(\mathbb{R})$.

If we restrict j to a specific value $j=j_0$, the resulting expansion set $\{\phi_{j_0,k}(x)\}$ is a subset of $\{\phi_{j,k}(x)\}$ that spans a subspace of $L^2(\mathbb{R})$:

$$V_{j_0} = \overline{\underset{k}{\mathrm{Span}}\{\phi_{j_0,k}(x)\}}$$

That is, V_{j_0} is the span of $\phi_{j_0,k}(x)$ over k.
If $f(x) \in V_{j_0}$ then we can write: $f(x) = \sum_k a_k \phi_{j_0,k}(x)$

Multiresolution Analysis
Scaling Functions (cont...)

If $f(x) \in V_{j_0}$ then we can write: $f(x) = \sum_k a_k \phi_{j_0,k}(x)$

More generally,

$$V_j = \overline{\underset{k}{\text{Span}}\{\phi_{j,k}(x)\}}$$

Increasing j, increases the size of V_j allowing functions with fine details to be included in the subspace.

This is a consequence of the fact that, as j increases, the $\left\{\phi_{j,k}(x)\right\}$ that are used to represent the subspace functions become narrower.

Multiresolution Analysis Scaling Functions (cont…)

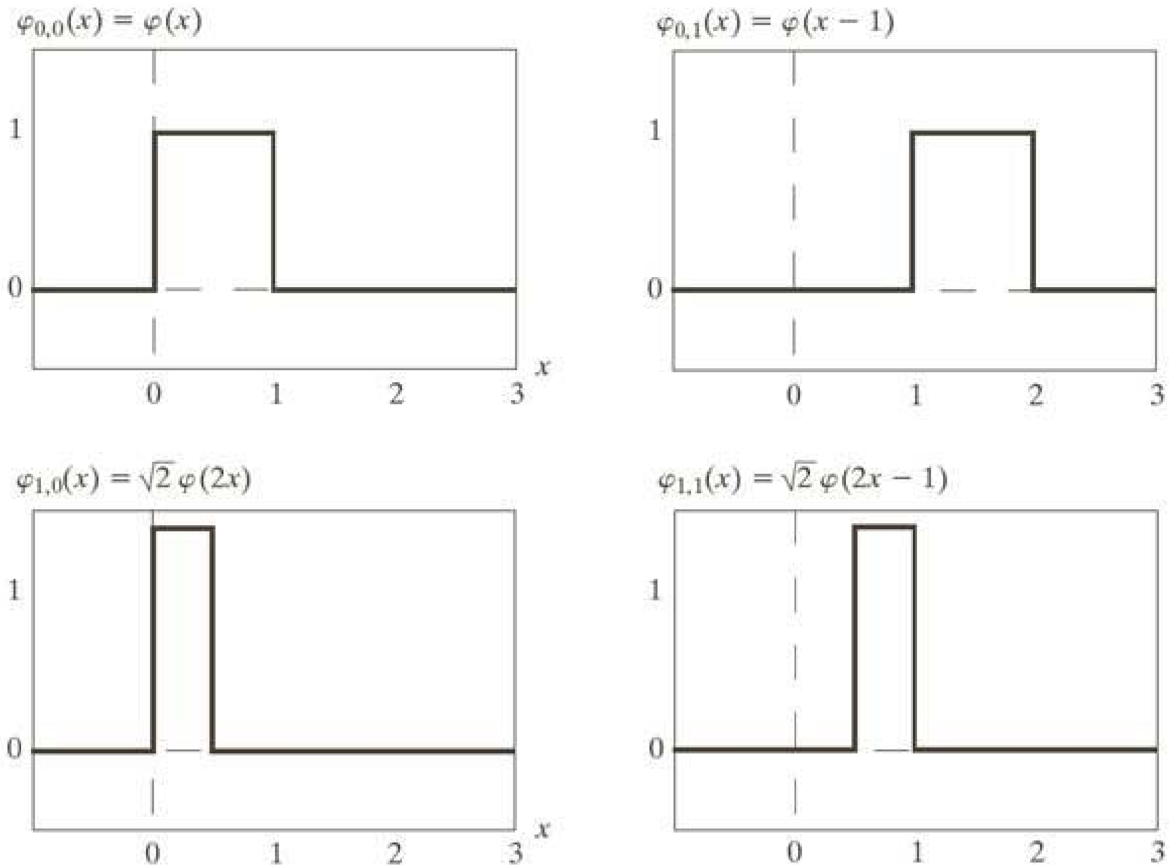
Consider the unit-height, unit-width Haar scaling function:

$$\phi(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and observe some of the expansion functions $\phi_{j,k}(x)$ generated by scaling and translations of the original function.

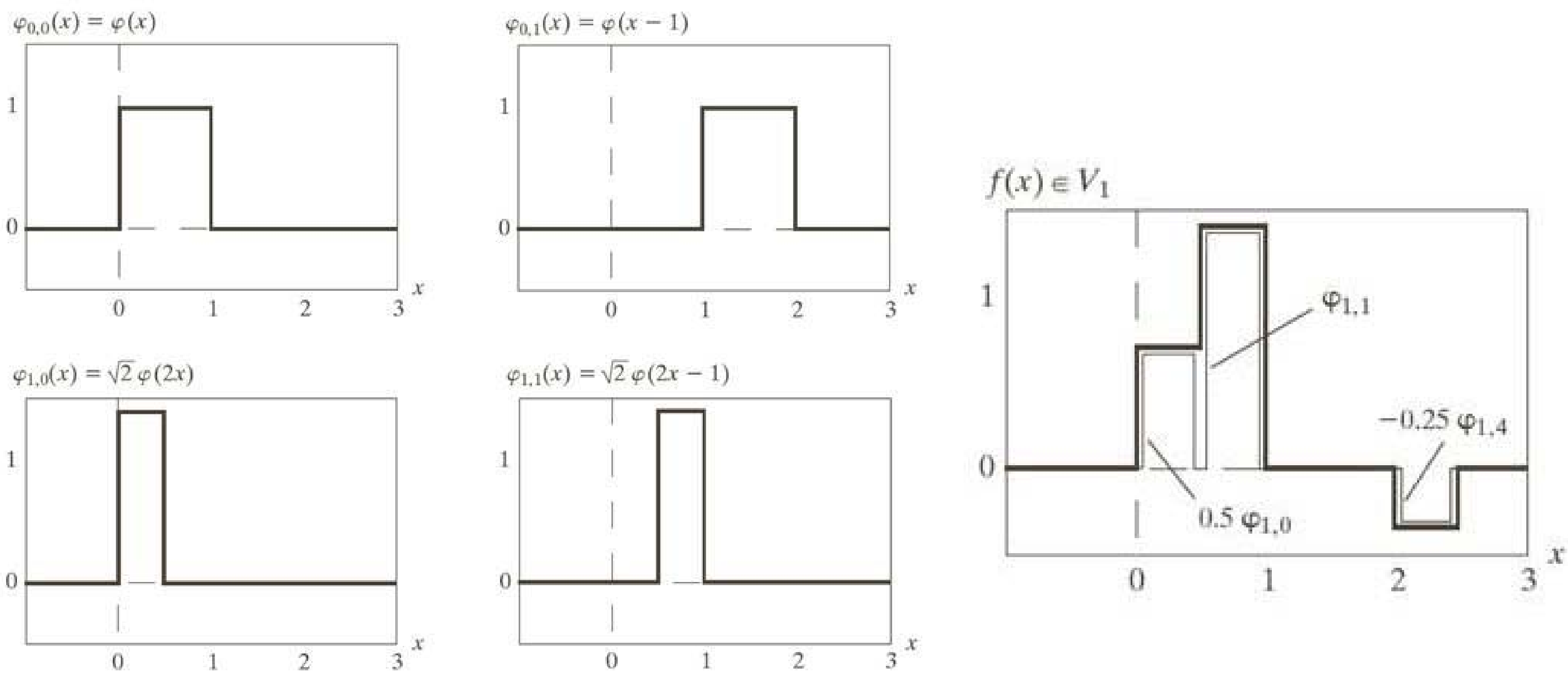
Multiresolution Analysis Scaling Functions (cont…)

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$$



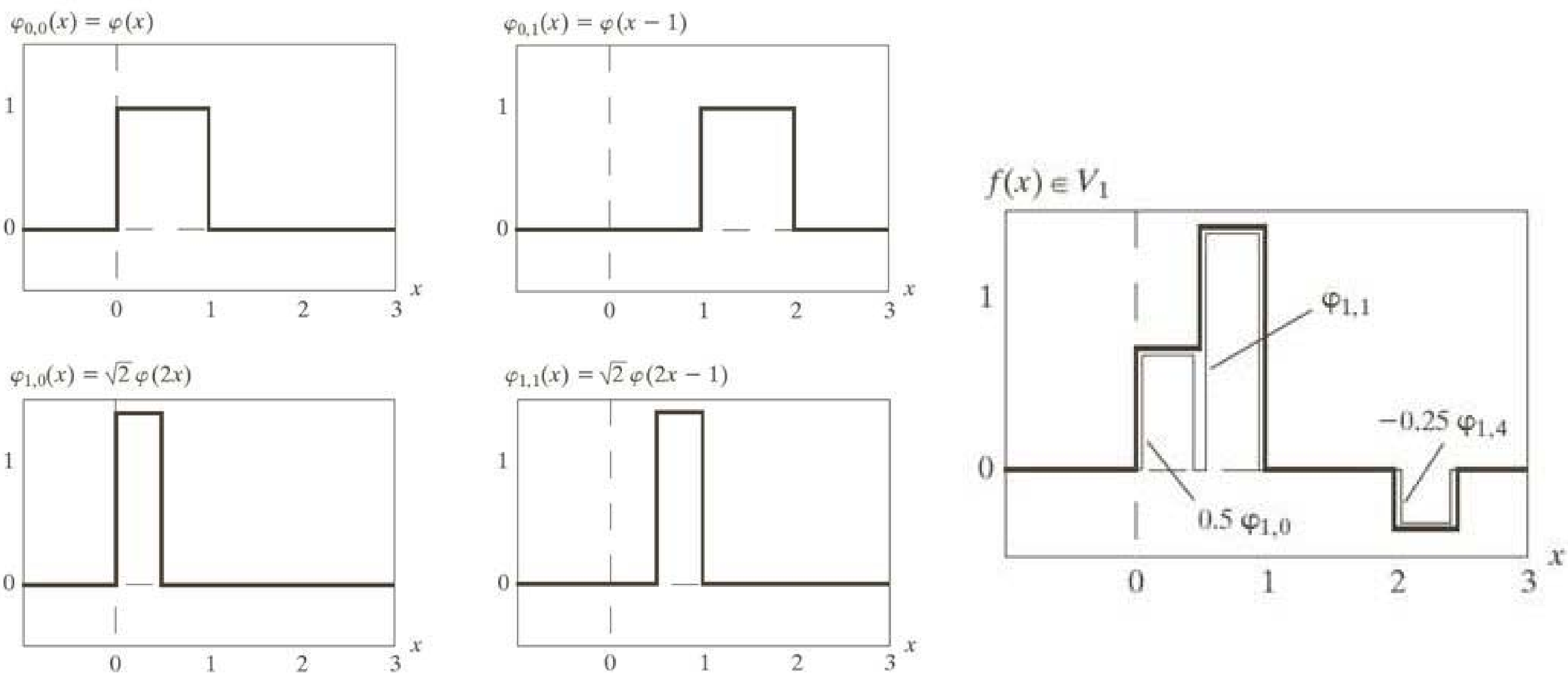
As j increases, the functions become narrower.

Multiresolution Analysis Scaling Functions (cont…)



Function $f(x)$ does not belong to V_0 because the V_0 expansion functions are to coarse to represent it. Higher resolution functions are required.

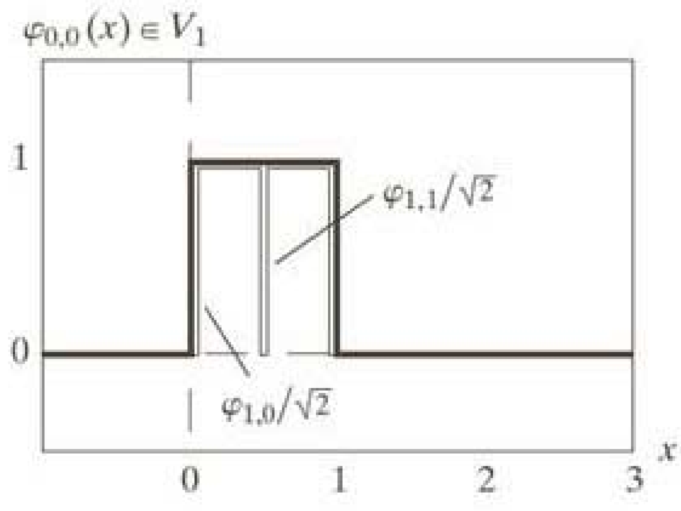
Multiresolution Analysis Scaling Functions (cont…)



Indeed, $f(x) \in V_1$. $f(x) = 0.5\phi_{1,0}(x) + \phi_{1,1}(x) - 0.25\phi_{1,4}(x)$

Multiresolution Analysis Scaling Functions (cont…)

Note also that $\phi_{0,0}(x)$ may be decomposed as a sum of V_1 expansion functions.



$$\phi_{0,0}(x) = \frac{1}{\sqrt{2}}\phi_{1,0}(x) + \frac{1}{\sqrt{2}}\phi_{1,1}(x) = \phi(2x) + \phi(2x-1)$$

Multiresolution Analysis Scaling Functions (cont...)

In a similar manner, any V_0 expansion function may be decomposed as a sum of V_1 expansion functions:

$$\phi_{0,k}(x) = \frac{1}{\sqrt{2}}\phi_{1,2k}(x) + \frac{1}{\sqrt{2}}\phi_{1,2k+1}(x)$$

Therefore, if $f(x) \in V_0$, then $f(x) \in V_1$.
This is because all V_0 expansion functions are contained in V_1. Mathematically, we say that V_0 is a subspace of V_1:

$$V_0 \subset V_1$$

Multiresolution Analysis Scaling Functions (cont…)

The simple scaling function in the preceding example obeys the four fundamental requirements of multiresolution analysis [Mallat 1989].

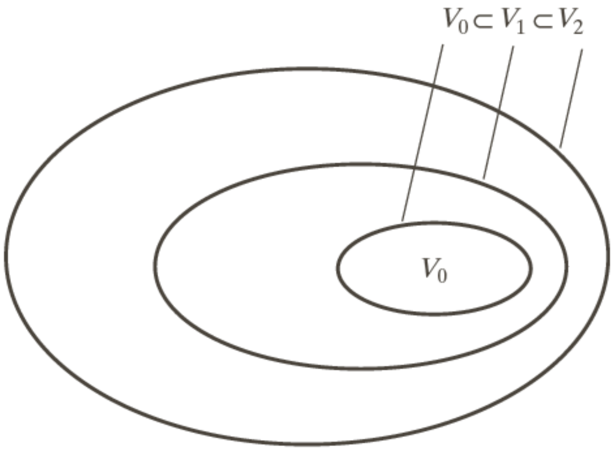
MRA Requirement 1: *The scaling function is orthogonal to its integer translates.*
Easy to see for the Haar function. Hard to satisfy for functions with support different than $[0, 1]$.

Multiresolution Analysis Scaling Functions (cont…)

MRA Requirement 2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.

This means that:

$$V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{\infty}$$



Multiresolution Analysis Scaling Functions (cont…)

Moreover, if $f(x) \in V_j$, then $f(2x) \in V_{j+1}$.

The fact that the Haar scaling function satisfies this requirement is not an indication that any function with support of width 1 satisfies the condition.

For instance, the simple function:

$$\phi(x) = \begin{cases} 1, & 0.25 \leq x \leq 0.75 \\ 0, & \text{otherwise} \end{cases}$$

is not a valid scaling function for MRA.

Multiresolution Analysis Scaling Functions (cont…)

MRA Requirement 3: *The only common function to all subspaces V_j is $f(x)=0$.*

In the coarsest possible expansion the only representable function is the function with no information $f(x)=0$. That is:

$$j \to -\infty, V_{-\infty} = \{0\}$$

MRA Requirement 4: *Any function may be represented with arbitrary precision.*

This means that in the limit:

$$j \to \infty, V_{\infty} = \{L^2(\mathbb{R})\}$$

Multiresolution Analysis
Scaling Functions (cont…)

Under these conditions, the expansion functions of subspace V_j may be expressed as a weighted sum of the expansion functions of subspace V_{j+1}:

$$\phi_{j,k}(x) = \sum_n a_n \phi_{j+1,n}(x)$$

Substituting

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$$

and changing variable a_n to $h_\varphi(n)$, we obtain:

$$\phi_{j,k}(x) = \sum_n h_\phi(n) 2^{(j+1)/2} \phi(2^{j+1} x - n)$$

Multiresolution Analysis Scaling Functions (cont…)

Because $\phi_{0,0}(x) = \phi(x)$

we can set $j=k=0$ to obtain a simpler expression:

$$\phi(x) = \sum_{n} h_{\phi}(n)\sqrt{2}\phi(2x-n)$$

The coefficients $h_{\varphi}(n)$, are called *scaling function coefficients*.

This equation is fundamental to MRA and is called the *refinement equation, the MRA equation* or *the dilation equation*.

Multiresolution Analysis
Scaling Functions (cont…)

$$\phi(x) = \sum_n h_\phi(n)\sqrt{2}\phi(2x-n)$$

The refinement equation states that the expansion functions of any subspace may be obtained from double-resolution copies of themselves, that is, the expansion functions of the next higher resolution space.

Note that the choice of reference V_0 is arbitrary. We can start at any resolution level.

Multiresolution Analysis
Scaling Functions (cont…)

The scaling function coefficients for the Haar function are the elements of the first row of matrix $\mathbf{H}_{2,}$ that is:

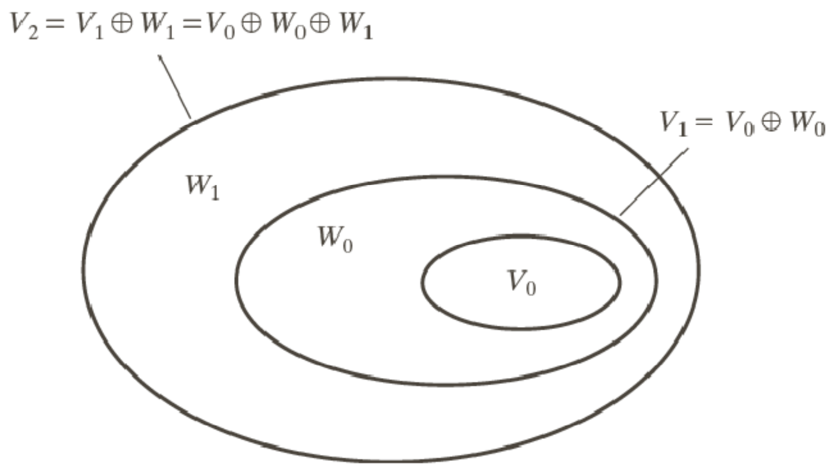
$$h_\phi(0) = h_\phi(1) = \frac{1}{\sqrt{2}}$$

Thus, the refinement equation is:

$$\phi(x) = \frac{1}{\sqrt{2}}\left[\sqrt{2}\phi(2x)\right] + \frac{1}{\sqrt{2}}\left[\sqrt{2}\phi(2x-1)\right] = \phi(2x) + \phi(2x-1)$$

Multiresolution Analysis
Wavelet Functions

Given a scaling function that meets the MRA criteria we can define a wavelet function $\psi(x)$ that together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces V_j and V_{j+1}.



Multiresolution Analysis Wavelet Functions (cont…)

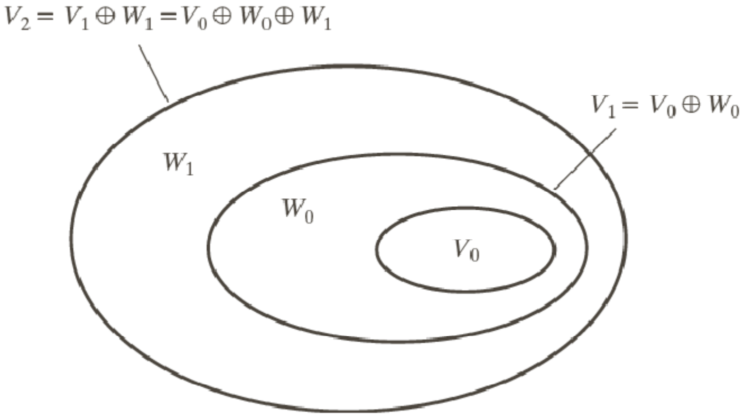
We define the set $\{\psi_{j,k}(x)\}$ of wavelets

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k),\ j,k \in \mathbb{Z}$$

That span the W_j spaces. As with scaling functions:

$$W_j = \overline{\underset{k}{\mathrm{Span}}\{\psi_{j,k}(x)\}}$$

If $f(x) \in W_j$, $f(x) = \sum_k a_k \psi_{j,k}(x)$



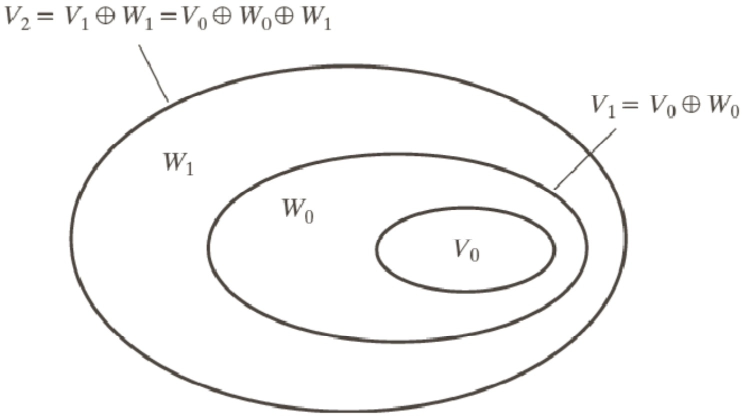
Multiresolution Analysis Wavelet Functions (cont…)

The scaling and wavelet function subspaces are related by

$$V_{j+1} = V_j \oplus W_j$$

where the symbol $\oplus$ denotes the union of spaces. The orthogonal complement V_j of in V_{j+1} is W_j and all members of V_j are orthogonal to the members of W_j.

$$\langle \phi_{j,k}(x), \psi_{j,l}(x) \rangle = 0, \forall j, k, l.$$



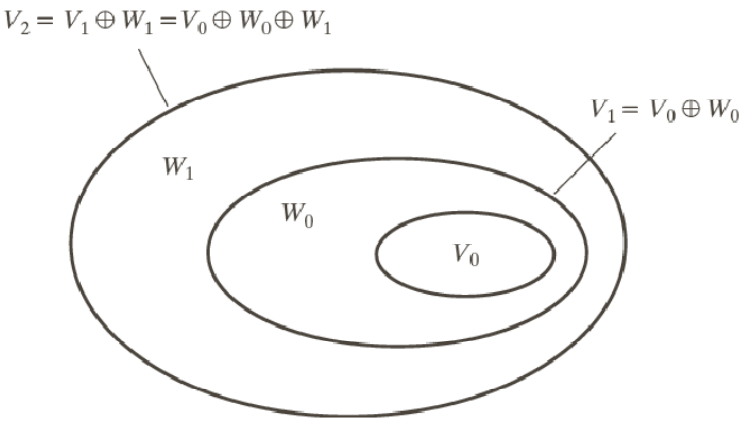
Multiresolution Analysis Wavelet Functions (cont…)

We can now express the space of all measurable, square-integrable function as

$$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots = V_1 \oplus W_1 \oplus W_2 \oplus \ldots \Leftrightarrow$$

$$L^2(\mathbb{R}) = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

which eliminates the scaling function and uses only wavelet functions

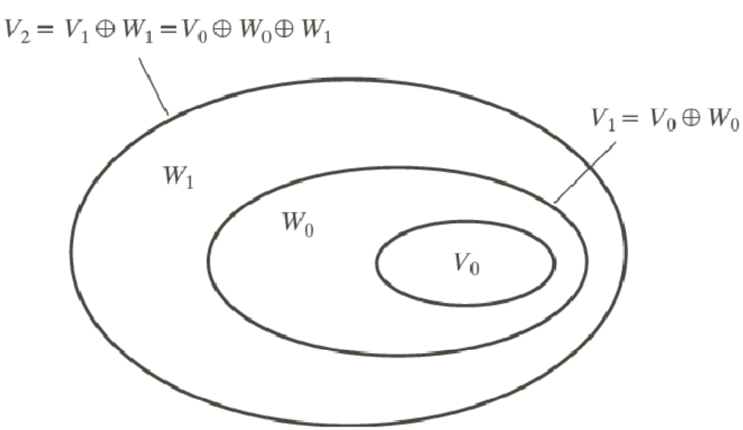


Multiresolution Analysis Wavelet Functions (cont…)

If $f(x) \in V_1$ but $f(x) \notin V_0$ its expansion using

$$L^2(\mathbb{R}) = V_0 \oplus W_0$$

contains an *approximation* using scaling functions V_0 and wavelets from W_0 would encode the *difference* between this approximation and the actual function.

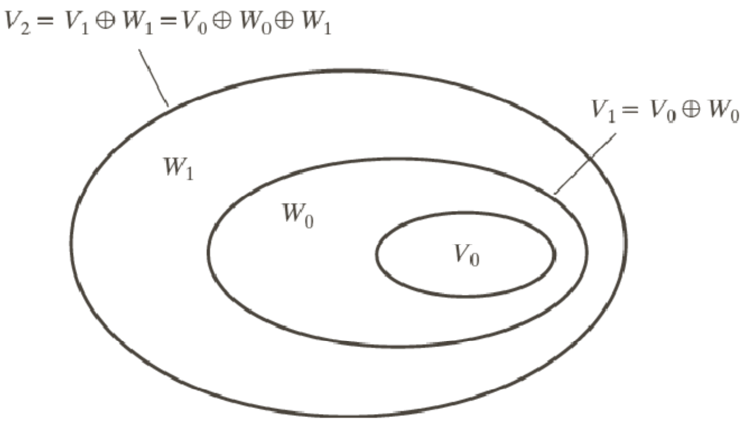


Multiresolution Analysis Wavelet Functions (cont…)

The representation may be generalized to yield

$$L^2(\mathbb{R}) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \ldots$$

starting from an arbitrary scale and adding the appropriate wavelet functions that capture the difference between the coarse scale representation and the actual function.



Multiresolution Analysis Wavelet Functions (cont...)

Any wavelet function, like its scaling function counterpart, reside in the space spanned by the next higher resolution level. Therefore, it can be expressed as a weighted sum of shifted, double-resolution scaling functions:

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\phi(2x-n)$$

The coefficients $h_\psi(n)$, are called *wavelet function coefficients*. It can also be sown that

$$h_\psi(n) = (-1)^n h_\phi(1-n)$$

Note the similarity with the analysis-synthesis filters.

Multiresolution Analysis Wavelet Functions (cont...)

The Haar scaling function coefficients were defined as

$$h_\phi(0) = h_\phi(1) = \frac{1}{\sqrt{2}}$$

The corresponding wavelet coefficients are

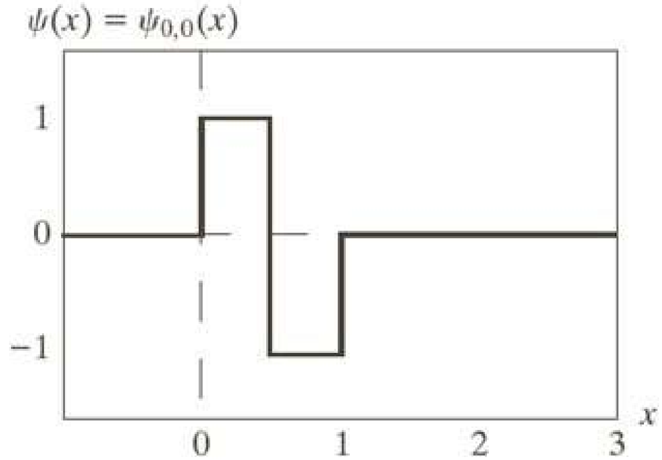
$$h_\psi(0) = (-1)^0 h_\phi(1-0) = \frac{1}{\sqrt{2}},\ h_\psi(1) = (-1)^1 h_\phi(1-1) = -\frac{1}{\sqrt{2}}$$

These coefficients are the elements of the second row of the Haar transformation matrix $\mathbf{H}_2$.

Multiresolution Analysis Wavelet Functions (cont…)

Substituting this result into $\psi(x) = \sum_n h_\psi(n)\sqrt{2}\phi(2x-n)$

we get $\psi(x) = \phi(2x) - \phi(2x-1) = \begin{cases} 1 & 0 \le x \le 0.5 \\ -1 & 0.5 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

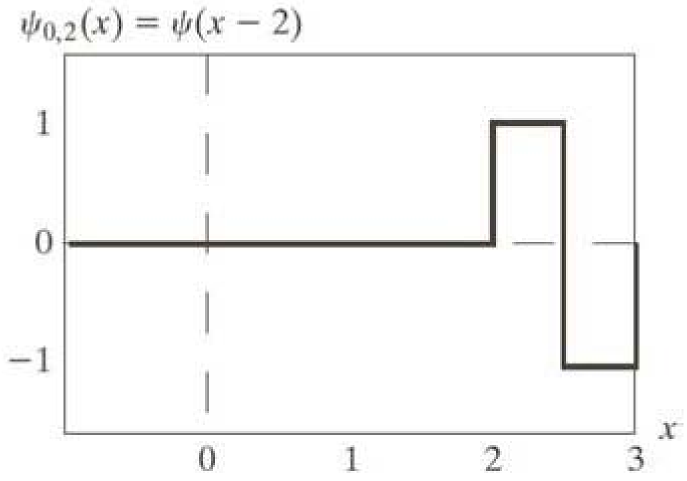


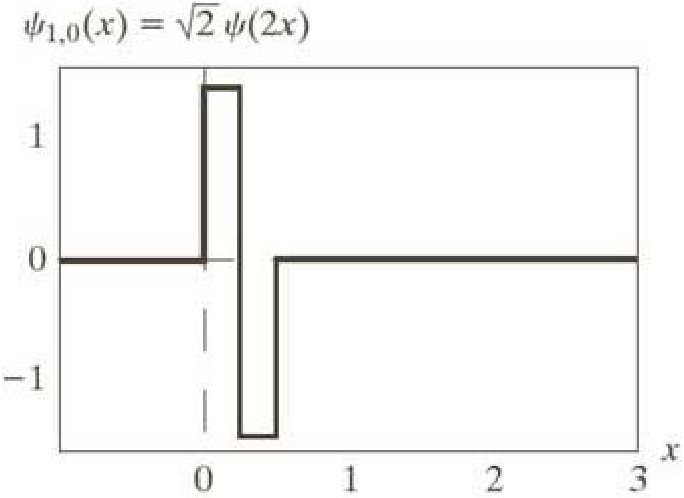
Multiresolution Analysis Wavelet Functions (cont…)

Using

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k),\ j,k \in \mathbb{Z}$$

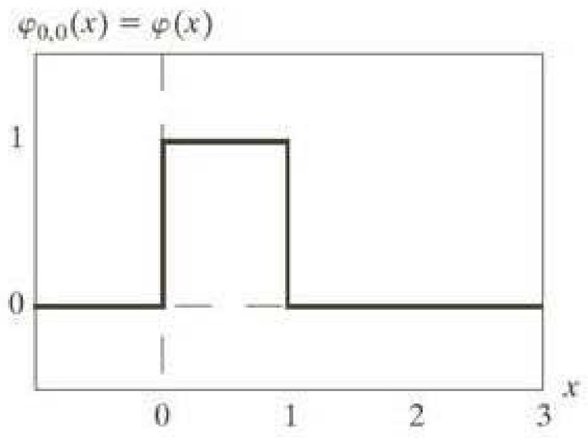
we can now generate the universe of translated and scaled Haar wavelets.





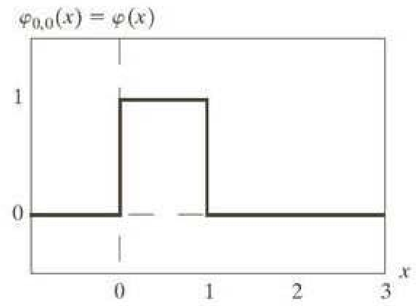
Multiresolution Analysis Wavelet Functions (cont…)

Any function $f(x) \in V_0$ may be expressed by the scaling function $\varphi(x)$:

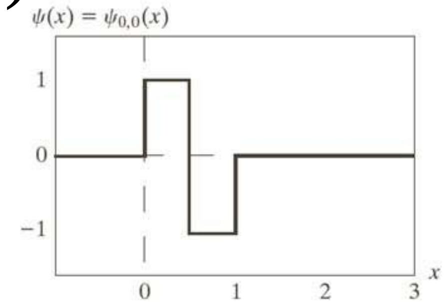


Multiresolution Analysis
Wavelet Functions (cont…)

Any function $f(x) \in V_1$ may be expressed by the scaling function $\varphi(x)$ describing the coarse form

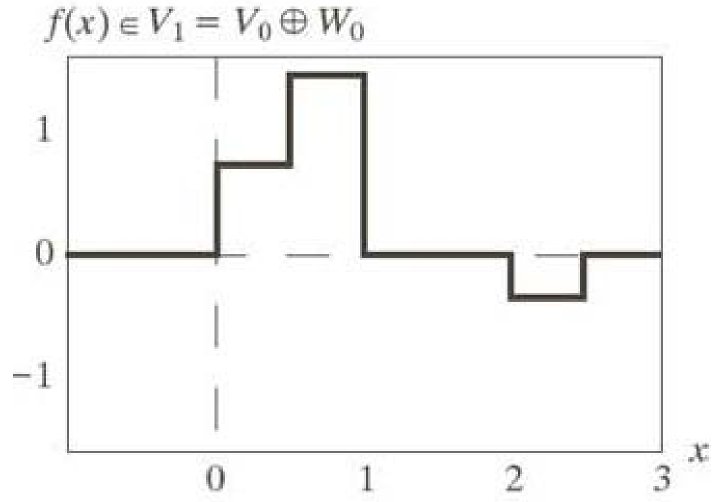


and the wavelet function $\psi(x)$ describing the details that cannot be represented in V_0 by $\varphi(x)$.



Multiresolution Analysis
Wavelet Functions (cont…)

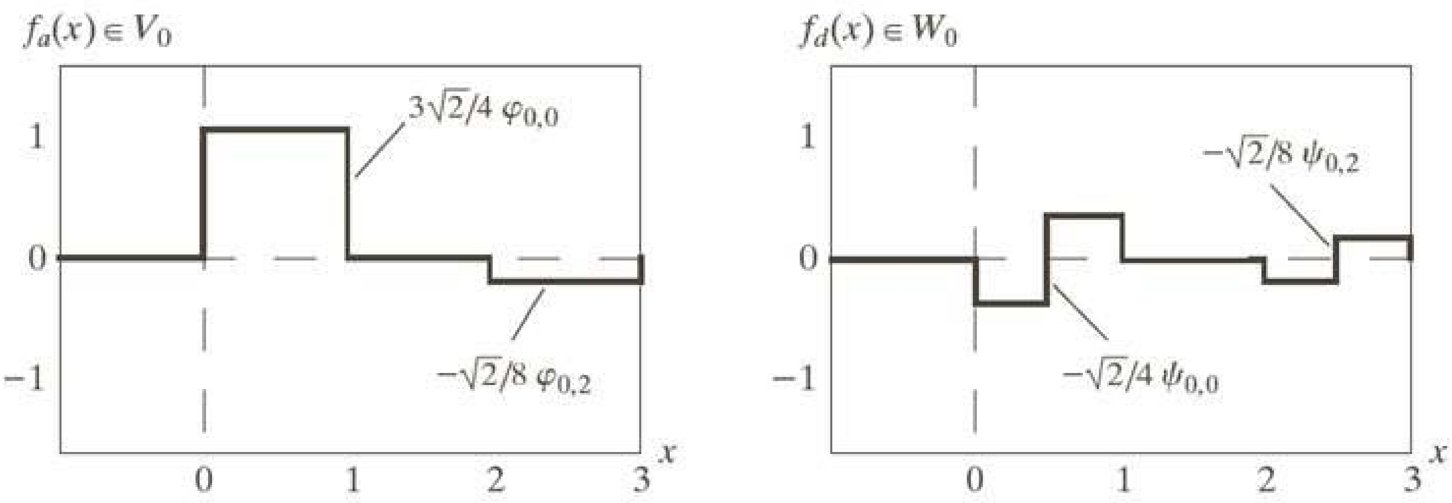
Remember the function of an earlier example $f(x) \in V_1$ but $f(x) \notin V_0$. This indicates that it could be expanded using V_0 to capture the coarse characteristics of the function and W_0 to encode the details that cannot be represented by V_0.



Multiresolution Analysis Wavelet Functions (cont…)

$$f(x) = f_a(x) + f_d(x)$$

$$f_a(x) = \frac{3\sqrt{2}}{4}\phi_{0,0}(x) - \frac{\sqrt{2}}{8}\phi_{0,2}(x) \qquad f_d(x) = \frac{-\sqrt{2}}{4}\psi_{0,0}(x) - \frac{\sqrt{2}}{8}\psi_{0,2}(x)$$



Notice the equivalence to low pass and high pass filtering.