A TWO-STEP BDF GALERKIN CHARACTERISTIC SCHEME FOR A NONLINEAR COUPLED AGE-SPACE STRUCTURED SIR MODEL

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Abstract. Properties of the two-step backward difference formula (BDF2) for a nonlinear coupled age-space structured SIR model are investigated. A numerical scheme is devised via a two-step BDF in time and age integration along the characteristic age-time direction together with a composite trapezoidal quadrature rule for the birth boundary value, while a finite element method is applied for the spatial discretization of the system. Second order convergence in both time and space is established. Some numerical experiments are given to illustrate and confirm the conclusions.

1. Introduction

Since Lotka and McKendrick proposed the age-structured model to describe age-dependent dynamics, in [27] and [28], scholars began to pay attention to age-structured models. The age structure arises in many realistic models such as the phytoplankton model, [10], the HBV model, [17], the vector–host epidemic model, [18], and so on.

The SIR model, in which the individuals are divided into three categories, namely, *susceptible*, *infected*, and *removed*, is one of the famous infectious disease models. Recently, age-space SIR models have become a hot topic since they reflect the impact of various age and spatial factors on the transmission of diseases. The threshold dynamics of diffusive age-structured models have been systematically studied; see [8], [9], [36], and [40].

Important properties of numerical methods for age-structured models have attracted the attention of scholars for decades. In [20], Iannelli et al. constructed an implicit finite difference scheme along the characteristic lines for an age-structured model and proved first order convergence. In [3], Angulo designed an improved numerical method for age-structured models by employing numerical quadrature rules for the integrals resulting from the explicit integration in the model along characteristics. In [22], the authors applied discontinuous Galerkin finite element methods to an age-structured model with unbounded mortality. In [12], [13], and [41], positivity-preserving methods for some age-structured models as well as numerical stability thresholds converging to the exact stability threshold and playing an important role in stability criteria were proposed.

In recent decades, scholars began to pay attention to numerical properties of age-structured models with spatial diffusion. In [16], Deng and Hallam developed a positivity-preserving numerical scheme for a linear age-structured model in a spatially heterogeneous environment via a finite difference approximation in time and age integration combined with a finite element method for space discretization. In [15], Cusulin and Gerardo-Giorda numerically studied a linear age-space structured model and constructed a numerical scheme by separately integrating both in age and time by finite difference methods

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and by a finite element method for space discretization. In [4], Ayati designed a numerical scheme for an age-structured population model by transforming the model to parabolic equations with moving reference frame in age and then employing the implicit Euler method in time together with a Galerkin method in space. In [21], Kim proposed a numerical scheme for an integro-differential equation with age structure and proved optimal order convergence. Positivity-preserving numerical methods for age-space structured models were devised in [24], [26], and [42]. Higher order methods have also been investigated. In [31], a second order numerical scheme for nonlinear age-structured models was developed; it is based on the Crank–Nicolson scheme. Subsequently, Liang et al. extended this scheme to age-structured nonlinear diffusion models in [25].

BDF methods are frequently employed for the discretization of differential equations of parabolic type as they combine good stability properties and high order accuracy with low computational cost. In particular, BDF methods have proved advantageous for some famous models such as the Kuramoto–Sivashinsky equation, [2], the Cahn–Hilliard equation, [11], [39], the Ericksen–Leslie equation, [29], the phase field crystal model, [23], and the Swift–Hohenberg equation, [34]. So far, BDF methods have not been employed to age-space structured models.

Although the Crank–Nicolson scheme is A-stable, it lacks L-stability. Consequently, high-frequency modes induced by non-smooth or discontinuous initial data remain undamped. These modes give rise to persistent spurious numerical oscillations and can further degrade the temporal order of convergence, even though the method is unconditionally stable; see [32, 33]. In contrast, the two-step BDF method has a smoothing property, it is both A-stable and L-stable. It introduces inherent numerical dissipation to rapidly decaying stiff components, thereby effectively suppressing spurious oscillations.

In this article, we employ the two-step BDF scheme to the following nonlinear age-space structured SIR model, proposed by Chekroun and Kuniya in [8],

$$
\begin{cases}
S_t(t,x) = d_1\Delta S(t,x) + b - S(t,x)\displaystyle\int_0^\infty \beta(a)I(t,a,x)\,\mathrm{d}a - \mu S(t,x),\\
I_t(t,a,x) + I_a(t,a,x) = d_2\Delta I(t,a,x) - [\mu+\gamma(a)]\,I(t,a,x),\\
R_t(t,x) = d_3\Delta R(t,x) + \displaystyle\int_0^\infty \gamma(a)I(t,a,x)\,\mathrm{d}a - \mu R(t,x)\\
I(t,0,x) = S(t,x)\displaystyle\int_0^\infty \beta(a)I(t,a,x)\,\mathrm{d}a,
\end{cases}
\tag{1.1}
$$

for $t>0,\ a>0,\ x\in\Omega$, subject to initial conditions,

$$
S(0,x)=s_0(x),\quad I(0,a,x)=i_0(a,x),\quad R(0,x)=r_0(x),\quad x\in\Omega,
$$

and homogeneous Dirichlet boundary conditions,

$$
S(t,x)=I(t,a,x)=R(t,x)=0,\quad x\in\partial\Omega,
$$

with $\Omega\subset\mathbb{R}^d, d\leqslant 3$, a bounded convex domain with boundary $\partial\Omega$. Here, $S(t,x)$ and $R(t,x)$ denote the densities of susceptible and recovered individuals at time t and location x, while $I(t,a,x)$ denotes the density of infected individuals of age a at time t and location x. The constant b is the birth rate, μ is the mortality rate, and $\beta(a)$ and $\gamma(a)$ denote age-dependent transmission and recovered rates, respectively. The authors make also the following assumptions:

Hypothesis 1.1 (see [8])**.**

(1) $b>0,\ \mu>0,\ d_i>0$ for $i=1,2,3$,

(2) $\beta\in L^\infty_+(\mathbb{R}_+)$ with support in a bounded interval $[0,a^\star]$,

(3) $\gamma \in L^\infty_+(\mathbb{R}_+)$,

with $L^\infty_+(\mathbb{R}_+)$ consisting of the non-negative elements of $L^\infty(\mathbb{R}_+)$. From a biological point of view, no individuals will live forever. Hence, it is reasonable to impose the following hypothesis on γ, as illustrated in [1], which admits approximation of the exact solution in a finite age interval and is realistic from a biological perspective.

Hypothesis 1.2. *The function γ has support in the interval $[0, a^\star]$.*

Existence and boundedness of exact solutions to model (1.1) can be obtained by Theorem 3.1 in [8] as follows.

Theorem 1.1 ([8, Theorem 3.1])**.** *Let $\mathbb{X} = C(\bar\Omega, \mathbb{R})$ with norm $\|\phi\|_{\mathbb{X}} = \sup_{x\in\Omega} |\phi(x)|$ and $\mathbb{Y} = BC(\mathbb{R}_+, \mathbb{X})$ be the space consisting of bounded and continuous functions in $\mathbb{X}$ equipped with the norm $\|\psi\|_{\mathbb{Y}} = \sup_{a\geqslant 0} \|\psi(a)\|_{\mathbb{X}} = \sup_{(a,x)\in\mathbb{R}_+\times\Omega} |\psi(a,x)|$. Assume that hypothesis 1.1 holds and that the initial values s_0, r_0, and i_0 belong to the positive cones $\mathbb{X}^0_+$ and $\mathbb{Y}^0_+$, respectively, of $\mathbb{X}^0$ and $\mathbb{Y}^0$, where*

$$\mathbb{X}^0 = \{\varphi \in \mathbb{X} : \varphi = 0 \quad \forall x \in \partial\Omega\},$$
$$\mathbb{Y}^0 = \{\psi \in \mathbb{Y} : \psi(\cdot, x) = 0 \quad \forall x \in \partial\Omega\}.$$

Then, (1.1) has a unique, uniformly bounded, positive solution in $[0,\infty) \times [0, a^\star] \times \bar\Omega$.

In this paper, we consider a fully discrete method for the nonlinear coupled age-structured model (1.1). We employ an implicit time-stepping scheme along the characteristic lines, based on the two-step BDF method, and a finite element discretization in space. To reduce the computational complexity, we adopt an explicit-implicit composite trapezoidal quadrature rule to approximate the integrals at each initial age. In most numerical convergence proofs for age-structured models in existing literature, such as [12], [13], and [41], numerical positivity is essential. In view of the order barrier (unconditional positivity restricts the order of linear multistep methods and Runge–Kutta methods to 1) of [5] and [6], it is not feasible to establish unconditional positivity for our second-order numerical method. Hence, we need to prove convergence without numerical positivity. Another difficulty in the analysis of our numerical method for this model is caused by the coupling of S and I via the boundary condition for $I(t, 0, x)$; this coupling prevents us from employing fully implicit numerical methods.

The article is structured as follows. In Section 2, we construct the numerical scheme by employing a finite element method for the discretization in space combined with a two-step BDF method for time stepping along the directional derivative together with an explicit–implicit composite trapezoidal quadrature rule to approximate the birth boundary value. In Section 3, we estimate the numerical errors and establish convergence; our main result is Theorem 3.1. In Section 4, we provide results of numerical experiments that confirm the convergence and stability of our numerical scheme. In Section 5, we present conclusions and discuss possible extensions.

2. Construction of the numerical scheme

2.1. **A characteristic two-step BDF scheme for a model equation.** To construct the numerical scheme for the model equation

$$u_t(t,a) + u_a(t,a) = f(t,a), \tag{2.1}$$

for a fixed point (t_0, a_0), we consider the functions

$$v(s) = u(t_0 + s, a_0 + s), \quad g(s) = f(t_0 + s, a_0 + s). \tag{2.2}$$

It follows that

$$v'(s) = u_t(t_0+s, a_0+s) + u_a(t_0+s, a_0+s) = g(s). \tag{2.3}$$

The two-step BDF method, with time step τ and nodes $s_n = n\tau$, for equation (2.3) reads

$$\frac{3}{2}v^n - 2v^{n-1} + \frac{1}{2}v^{n-2} = \tau g(s_n).$$

In view of (2.2), we are led to a two-step BDF characteristic scheme for the model equation (2.1),

$$\frac{3}{2}u^{n,m} - 2u^{n-1,m-1} + \frac{1}{2}u^{n-2,m-2} = \tau f^{n,m}, \tag{2.4}$$

where

$$u^{n,m} \approx u(t_0+n\tau, a_0+m\tau), \quad f^{n,m} := f(t_0+n\tau, a_0+m\tau).$$

For a sequence $(v^n)_{n\in\mathbb{N}_0}$, we shall use the notation

$$\bar{D}v^n = \left(\frac{3}{2}v^n - 2v^{n-1} + \frac{1}{2}v^{n-2}\right)\Big/\tau \quad \text{and} \quad \bar{\partial}v^1 = (v^1 - v^0)/\tau \tag{2.5}$$

for the discrete derivatives associated with the two-step BDF and implicit Euler methods.

Analogously, for a double sequence $(v^{n,i})_{n,i\in\mathbb{N}_0}$, we denote by ${}^c\bar{D}v^{n,i}$ and ${}^c\bar{\partial}v^{n,i}$ the discrete *directional* derivatives associated with the two-step BDF and implicit Euler methods,

$$\begin{cases} {}^c\bar{D}v^{n,i} := \left(\frac{3}{2}v^{n,i} - 2v^{n-1,i-1} + \frac{1}{2}v^{n-2,i-2}\right)\Big/\tau, \quad i = 2,\dots,N_a,\ n = 2,\dots, \\ {}^c\bar{\partial}v^{n,i} = (v^{n,i} - v^{n-1,i-1})/\tau, \quad i = 1 \ \text{ or } \ n = 1. \end{cases} \tag{2.6}$$

We introduced ${}^c\bar{\partial}v^{n,i}$ only for $i = 1$ or $n = 1$ since we shall employ the implicit Euler method only to initialize the two-step BDF method. Notice that, for $v^{n,i} := v(t_n, a_i, \cdot)$, with a smooth function v, ${}^c\bar{D}v^{n,i}$ is an approximation to the directional derivative $Dv(t_n, a_i, \cdot)$ of v along the characteristic line $t = a + c$, where c is an arbitrary constant,

$$Dv(t_n, a_i, \cdot) = \lim_{s\to 0} \frac{v(t_n+s, a_i+s, \cdot) - v(t_n, a_i, \cdot)}{s};$$

see [25], [30], and [31]. With this notation, (2.4) takes the form

$${}^c\bar{D}u^{n,m} = f^{n,m}.$$

2.2. **A two-step BDF scheme for the age-space structured system.** We consider a quasi-uniform family $(\mathscr{T}_h)_{h>0}$ of partitions $\mathscr{T}_h$ of Ω with mesh size h. With $H_0^1(\Omega) = \{v : v \in H^1(\Omega) \text{ and } v = 0 \text{ on } \partial\Omega\}$, let $S_h \subset H_0^1(\Omega)$ consist of continuous and piecewise affine polynomials with respect to $\mathscr{T}_h$, i.e., affine in each element of $\mathscr{T}_h$.

We denote the final time by T and assume that $a^\star/T \in \mathbb{Q}_+$; this can always be achieved by possibly slightly increasing $a^\star$. Let $N, N_a \in \mathbb{N}, N_a > 1$, be such that $T/N = a^\star/N_a$. To discretize the second equation in system (1.1) in the same way as we discretized the model equation (2.1), we shall use the same step size τ both in time and age. Thus, we let $\tau := a^\star/N_a$, and consider uniform partitions $t_n := n\tau, n = 0, 1, \dots, N$, and $a_i := i\tau, i = 0, 1, \dots, N_a$, of the intervals $[0, T]$ and $[0, a^\star]$, rspectively.

Let $S^n, I^{n,i}, R^n \in S_h$ denote approximations to $S(t_n, \cdot), I(t_n, a_i, \cdot)$, and $R(t_n, \cdot)$, respectively. Then, we devise an implicit two-step BDF characteristic scheme for model (1.1) based on a second-order backward difference quotient of the characteristic derivative together with the composite trapezoidal rule for the birth integrals as follows: with the

notation (2.5) and (2.6), seek $S^n, I^{n,i}, R^n \in S_h$ for $n = 2, \dots, N,\ i = 0, 1, \dots, N_a$, such that the following relations hold for every $\chi \in S_h$,

$$
\begin{cases}
(\bar{D}S^n, \chi) + d_1(\nabla S^n, \nabla \chi) = (b - I^{n,0} - \mu S^n, \chi), \\
({}^c\bar{D}I^{n,i}, \chi) + d_2(\nabla I^{n,i}, \nabla \chi) = -(\mu + \gamma(a_i))(I^{n,i}, \chi), \\
(\bar{D}R^n, \chi) + d_3(\nabla R^n, \nabla \chi) = \left(\tau {\sum_{i=0}^{N_a}}'' \gamma(a_i) I^{n,i}, \chi\right) - \mu(R^n, \chi), \\
J^{n,0} = S^n \left(\dfrac{\tau}{2}\beta(a_0) I^{n-1,0} + \tau {\sum_{i=1}^{N_a}}' \beta(a_i) I^{n,i}\right), \\
I^{n,0} = P_h J^{n,0}, \quad n = 2, \dots, N,\ i = 2, 3, \dots, N_a,
\end{cases}
\tag{2.7}
$$

with the convention that the last term in $\sum'$ and the first and the last term in $\sum''$ are halved. Here, P_h denotes the L^2 projection onto S_h.

We initialize this scheme by employing one step of the implicit Euler method,

$$
\begin{cases}
(\bar{\partial}S^1, \chi) + d_1(\nabla S^1, \nabla \chi) = (b - I^{1,0} - \mu S^1, \chi), \\
({}^c\bar{\partial}I^{1,i}, \chi) + d_2(\nabla I^{1,i}, \nabla \chi) = -(\mu + \gamma(a_i))(I^{1,i}, \chi), \qquad i = 1, 2, \dots, N_a, \\
({}^c\bar{\partial}I^{n,1}, \chi) + d_2(\nabla I^{n,1}, \nabla \chi) = -(\mu + \gamma(a_1))(I^{n,1}, \chi), \quad n = 1, 2, \dots, N, \\
(\bar{\partial}R^1, \chi) + d_3(\nabla R^1, \nabla \chi) = \left(\tau {\sum_{i=0}^{N_a}}'' \gamma(a_i) I^{1,i}, \chi\right) - \mu(R^1, \chi), \\
J^{1,0} = S^1 \left(\dfrac{\tau}{2}\beta(a_0) I^{0,0} + \tau {\sum_{i=1}^{N_a}}' \beta(a_i) I^{1,i}\right), \\
I^{1,0} = P_h J^{1,0}.
\end{cases}
\tag{2.8}
$$

We assume here that suitable starting approximations $S^0, I^{0,i}, R^0 \in S_h$ are given.

Remark 2.1. Although (2.7) seems a coupled nonlinear system, we can implement it by solving linear systems as follows.

Step 1: Compute $\{I^{n,i}\}_{i=1}^{N_a}$ from the second equation in (2.7).

Step 2: Substitute $I^{n,0}$ from the last two relations in (2.7) (depending on the unknown S^n) into the first equation in (2.7) to compute S^n. With the approximations $I^{n,1}, \dots, I^{n,N_a}$ available, the first equation in (2.7) reduces to a linear system for S^n; no linearization is required.

Step 3: Compute $I^{n,0}$ from the last two relations and R^n from the third equation in (2.7).

Remark 2.2 (Existence and uniqueness of numerical solutions)**.** As illustrated in Remark 2.1, system (2.7) is essentially linear. We can obviously replace $I^{n,0}$ by $J^{n,0}$ in the first relations of (2.7) and (2.8). Assuming for the time being that the coefficients of S^n in $J^{n,0}$ are uniformly bounded in the L^∞ norm, we can easily see that existence and uniqueness of the numerical solutions, for sufficiently small time step τ, follow from the Lax–Milgram lemma. Under a mild meshcondition, these coefficients are indeed uniformly bounded in the L^∞ norm; see Remark 3.1.

3. Convergence analysis

We shall use the standard notation $\|\cdot\|_{L^\infty} = \|\cdot\|_{L^\infty(\Omega)}$, $\|\cdot\|$, and $\|\cdot\|_2$ for the $L^\infty(\Omega)$- and $L^2(\Omega)$-norms, and for the norm on the Sobolev space $H^2(\Omega)$,

$$\|v\| = \|v\|_{L^2} = \left(\int_\Omega |v|^2 dx\right)^{1/2}, \quad \|v\|_2 = \|v\|_{H^2} = \left(\sum_{|\alpha|\leqslant 2} \|D^\alpha v\|^2\right)^{1/2}.$$

To simplify the notation, following [25] and [30], for sequences $\chi^n = (\chi^{n,i})_{i=0,\dots,N_a} \subset L^2(\Omega)$, we introduce the discrete norms $\|\cdot\|_{\ell^1(L^2)}$ and $\|\cdot\|_{\ell^2(L^2)}$ by

$$\|\chi^n\|_{\ell^1(L^2)} := \tau{\sum_{i=0}^{N_a}}''\|\chi^{n,i}\|, \quad \|\chi^n\|_{\ell^2(L^2)} := \left(\tau{\sum_{i=0}^{N_a}}''\|\chi^{n,i}\|^2\right)^{1/2}.$$

By the elementary inequalities $\|x\|_{\ell^2} \leqslant \|x\|_{\ell^1} \leqslant m^{1/2}\|x\|_{\ell^2}$ for $x \in \mathbb{R}^m$, with $\|\cdot\|_{\ell^p}$ the standard ℓ^p-norm on $\mathbb{R}^m$, there exists a constant C, independent of τ, such that

$$\|\chi^n\|_{\ell^1(L^2)} \leqslant C\|\chi^n\|_{\ell^2(L^2)}. \tag{3.1}$$

To split the errors in suitable ways, we shall use the notation

$$\begin{aligned}
&\xi_S^n = S^n - S(t_n,\cdot), && \xi_R^n = R^n - R(t_n,\cdot), && \xi_I^{n,i} = I^{n,i} - I(t_n,a_i,\cdot),\\
&\theta_S^n = S^n - R_hS(t_n,\cdot), && \theta_R^n = R^n - R_hR(t_n,\cdot), && \theta_I^{n,i} = I^{n,i} - R_hI(t_n,a_i,\cdot),\\
&\rho_S^n = R_hS(t_n,\cdot) - S(t_n,\cdot), && \rho_R^n = R_hR(t_n,\cdot) - R(t_n,\cdot), && \rho_I^{n,i} = R_hI(t_n,a_i,\cdot) - I(t_n,a_i,\cdot),
\end{aligned}$$

with R_h the Ritz projection operator onto S_h, defined by

$$(\nabla R_hu, \nabla v) = (\nabla u, \nabla v) \quad \forall v \in S_h. \tag{3.2}$$

The following lemma provides standard error estimates for the projection R_h (see [37, Lemma 1.1] and [35, (1.3)]); we shall use them in the sequel.

Lemma 3.1 (Error estimate for the Ritz projection)**.** *Assume that S_h satisfies the approximation property*

$$\inf_{\chi\in S_h}\{\|v-\chi\| + h\|\nabla(v-\chi)\|\} \leqslant Ch^2\|v\|_2 \quad \forall v \in H^2(\Omega)\cap H_0^1(\Omega).$$

Then, the projection R_h defined by (3.2) satisfies the estimate

$$\|v - R_hv\| + h\|\nabla(v - R_hv)\| \leqslant Ch^2\|v\|_2 \quad \forall v \in H^2(\Omega)\cap H_0^1(\Omega).$$

Furthermore, in view of our quasi-uniformilty assumption, we have the L^∞ error estimate

$$\|v - R_hv\|_{L^\infty} \leqslant Ch^2\ln\frac{1}{h}\|u\|_{W^{2,\infty}(\Omega)} \quad \forall v \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega). \tag{3.3}$$

For the analysis, it is advantageous to consider also a slight modification of the original scheme (2.7)–(2.8). So, with M_0 such that $\|S(t,\cdot)\|_{L^\infty} \leqslant M_0, 0 \leqslant t \leqslant T$, and $M_1 := M_0+1$, we introduce the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(x) := x \ \text{ for } \ |x| \leqslant M_1, \quad g(x) := -M_1 \ \text{ for } \ x < -M_1, \quad g(x) := M_1 \ \text{ for } \ x > M_1,$$

and notice that g is globally Lipschitz continuous with constant $L = 1$. The modification consists in replacing S^n by $g(S^n)$ on the right-hand sides of the fourth relation in (2.7) and the fifth relation in (2.8). The motivation for this modification is that we have control on the L^∞-norm of $g(S^n)$, while this is not known a priori for S^n. For simplicity, we use the same symbols for the numerical solutions for the modified and the original schemes. Obviously, in case $\|S^n\|_{L^\infty} \leqslant M_1$, a solution of the original scheme is also solution of the modified scheme and vice versa.

We next present the analysis for the *modified* scheme. We will turn to the analysis for the *original* scheme in section 3.5.

We begin by deriving a preliminary estimate for $\theta_I^{n,0}$.

Lemma 3.2 (Estimate for $\theta_I^{n,0}$)**.** *Assume Hypothesis 1.1 holds, $\beta \in C^3[0,a^\star]$, $S(t,\cdot), I(t,a,\cdot) \in H^2(\Omega)\cap H_0^1(\Omega)\cap L^\infty(\Omega)$, $S(\cdot,x)\in C^3([0,T])$, and $I(\cdot,\cdot,x)\in C^3([0,T]\times[0,a^\star])$. Then, there exists a constant C, independent of the stepsize τ and the meshsize h, such that the following estimate holds*

$$\|\theta_I^{n,0}\| \leqslant C(\tau^2+h^2) + \beta^\star a^\star M_0\|\theta_S^n\| + \tau\beta^\star M_1\Big(\frac{1}{2}\|\theta_I^{n-1,0}\| + \sum_{i=1}^{N_a}{}'\|\theta_I^{n,i}\| + Ch^2\Big), \tag{3.4}$$

where $\beta^\star = \max\limits_{0\leqslant a\leqslant a^\star}\beta(a)$.

Proof. Since $\theta_I^{n,0} = \xi_I^{n,0} - \rho_I^{n,0}$ and, with $\zeta_I^{n,0} := I(t_n,0,\cdot) - J^{n,0}$,

$$\xi_I^{n,0} = I(t_n,0,\cdot) - P_hI(t_n,0,\cdot) + P_h\zeta_I^{n,0},$$

we have

$$\theta_I^{n,0} = P_h\zeta_I^{n,0} + [I(t_n,0,\cdot) - P_hI(t_n,0,\cdot)] - \rho_I^{n,0}.$$

Now, the second and third terms on the right-hand side are of order $O(h^2)$ in the L^2-norm, and we obtain

$$\|\theta_I^{n,0}\| \leqslant \|\zeta_I^{n,0}\| + Ch^2.$$

Hence, it suffices to estimate $\zeta_I^{n,0}$. To this end, we split $\zeta_I^{n,0}$ in the form

$$\begin{aligned}
\zeta_I^{n,0} &= S(t_n,\cdot)\bigg[\int_0^{a^\star}\beta(a)I(t_n,a,\cdot)\,\mathrm{d}a - \tau\sum_{i=0}^{N_a}{}''\beta(a_i)I(t_n,a_i,\cdot)\bigg]\\
&\quad + \frac{\tau}{2}\beta(a_0)S(t_n,\cdot)\big(I(t_n,a_0,\cdot) - I(t_{n-1},a_0,\cdot)\big)\\
&\quad + \bigg[\frac{\tau}{2}\beta(a_0)I(t_{n-1},a_0,\cdot) + \tau\sum_{i=1}^{N_a}{}'\beta(a_i)I(t_n,a_i,\cdot)\bigg]\big(S(t_n,\cdot) - g(S^n)\big)\\
&\quad + g(S^n)\bigg[\frac{\tau}{2}\beta(a_0)\big(I(t_{n-1},a_0,\cdot) - I^{n-1,0}\big) + \tau\sum_{i=1}^{N_a}{}'\beta(a_i)\big(I(t_n,a_i,\cdot) - I^{n,i}\big)\bigg].
\end{aligned} \tag{3.5}$$

We estimate the L^2-norm of each term on the right-hand side of (3.5) by the L^2-norm of the corresponding difference multiplied by the L^∞-norms of the remaining factors and use the uniform boundedness of both S and I. Now, the L^2-norms of the differences in the first two terms are of order $O(\tau^2)$ and $O(\tau)$, respectively; but notice that the second term contains an additional factor τ. Furthermore, in view of the fact that $g(S(t_n,\cdot)) = S(t_n,\cdot)$ and the Lipschitz continuity of g,

$$\|S(t_n,\cdot) - g(S^n)\| \leqslant \|\theta_S^n\| + \|\rho_S^n\| \leqslant \|\theta_S^n\| + Ch^2,$$

and, analogously,

$$\|I(t_{n-1},a_0,\cdot) - I^{n-1,0}\| \leqslant \|\theta_I^{n-1,0}\| + Ch^2, \quad \|I(t_n,a_i,\cdot) - I^{n,i}\| \leqslant \|\theta_I^{n,i}\| + Ch^2.$$

Combining these estimates, we obtain (3.4). ∎

3.1. **Estimation of $\theta_I^{n,i}$.** Let $\hat{I}^{n,i} = I(t_n, a_i, \cdot)$ and $\omega_I^{n,i}$ be the consistency errors,

$$\omega_I^{n,i} := \begin{cases} ({}^c\bar{D} - D)\hat{I}^{n,i} + (R_h - E){}^c\bar{D}\hat{I}^{n,i} & \text{for } 2 \leqslant i \leqslant N_a \text{ and } 2 \leqslant n \leqslant N, \\ ({}^c\bar{\partial} - D)\hat{I}^{n,i} + (R_h - E){}^c\bar{\partial}\hat{I}^{n,i} & \text{for } i = 1 \text{ or } n = 1, \end{cases} \tag{3.6}$$

where E denotes the identity operator. Then, by (3.2) we have, for $\chi \in S_h$,

$$\begin{aligned} &({}^c\bar{D}\theta_I^{n,i}, \chi) + d_2(\nabla\theta_I^{n,i}, \nabla\chi) \\ &\qquad = ({}^c\bar{D}I^{n,i}, \chi) - ({}^c\bar{D}R_h\hat{I}^{n,i}, \chi) + d_2(\nabla I^{n,i}, \nabla\chi) - d_2(\nabla R_h\hat{I}^{n,i}, \nabla\chi) \\ &\qquad = -(\mu + \gamma(a_i))(I^{n,i}, \chi) - (R_h({}^c\bar{D}\hat{I}^{n,i}), \chi) - d_2(\nabla\hat{I}^{n,i}, \nabla\chi) \\ &\qquad = (\mu + \gamma(a_i))(\hat{I}^{n,i} - I^{n,i}, \chi) - (R_h({}^c\bar{D}\hat{I}^{n,i}), \chi) + (D\hat{I}^{n,i}, \chi) \\ &\qquad = (\mu + \gamma(a_i))(\hat{I}^{n,i} - I^{n,i}, \chi) - (\omega_I^{n,i}, \chi), \quad 2 \leqslant i \leqslant N_a, 2 \leqslant n \leqslant N. \end{aligned}$$

Choosing here the test function $\chi := \theta_I^{n,i}$, for $2 \leqslant i \leqslant N_a, 2 \leqslant n \leqslant N$, we obtain

$$({}^c\bar{D}\theta_I^{n,i}, \theta_I^{n,i}) + d_2\|\nabla\theta_I^{n,i}\|^2 = (\mu + \gamma(a_i))(\hat{I}^{n,i} - I^{n,i}, \theta_I^{n,i}) - (\omega_I^{n,i}, \theta_I^{n,i}). \tag{3.7}$$

In the same way, for $i = 1, \dots, N_a$ and $j = 2, \dots, N$, we have

$$({}^c\bar{\partial}\theta_I^{1,i}, \theta_I^{1,i}) + d_2\|\nabla\theta_I^{1,i}\|^2 = (\mu + \gamma(a_i))(\hat{I}^{1,i} - I^{1,i}, \theta_I^{1,i}) - (\omega_I^{1,i}, \theta_I^{1,i}), \tag{3.8}$$

$$({}^c\bar{\partial}\theta_I^{j,1}, \theta_I^{j,1}) + d_2\|\nabla\theta_I^{j,1}\|^2 = (\mu + \gamma(a_1))(\hat{I}^{j,1} - I^{j,1}, \theta_I^{j,1}) - (\omega_I^{j,1}, \theta_I^{j,1}). \tag{3.9}$$

Replacing n by j in (3.7) and summing over j from $j = 2$ to $j = n$, and adding the result to (3.8) multiplied by τ, we see that

$$\begin{aligned} &\tau({}^c\bar{\partial}\theta_I^{1,i}, \theta_I^{1,i}) + \tau\sum_{j=2}^{n}({}^c\bar{D}\theta_I^{j,i}, \theta_I^{j,i}) + \tau d_2\sum_{j=1}^{n}\|\nabla\theta_I^{j,i}\|^2 \\ &\qquad = \tau(\mu + \gamma(a_i))\sum_{j=1}^{n}(\hat{I}^{j,i} - I^{j,i}, \theta_I^{j,i}) - \tau\sum_{j=1}^{n}(\omega_I^{j,i}, \theta_I^{j,i}). \end{aligned} \tag{3.10}$$

Then, we sum (3.10) over i from $i = 2$ to $i = N_a$ to get

$$\begin{aligned} &\tau{\sum_{i=2}^{N_a}}'({}^c\bar{\partial}\theta_I^{1,i}, \theta_I^{1,i}) + \tau{\sum_{i=2}^{N_a}}'\sum_{j=2}^{n}({}^c\bar{D}\theta_I^{j,i}, \theta_I^{j,i}) + \tau d_2{\sum_{i=2}^{N_a}}'\sum_{j=1}^{n}\|\nabla\theta_I^{j,i}\|^2 \\ &\qquad = \tau{\sum_{i=2}^{N_a}}'\sum_{j=1}^{n}(\mu + \gamma(a_i))(\hat{I}^{j,i} - I^{j,i}, \theta_I^{j,i}) - \tau{\sum_{i=2}^{N_a}}'\sum_{j=1}^{n}(\omega_I^{j,i}, \theta_I^{j,i}). \end{aligned} \tag{3.11}$$

Multiplying (3.9) by τ and summing over j from $j = 2$ to $j = n$, and adding the result to (3.8) with $i = 1$, we obtain

$$\begin{aligned} \tau\sum_{j=1}^{n}({}^c\bar{\partial}\theta_I^{j,1}, \theta_I^{j,1}) + \tau d_2\sum_{j=1}^{n}\|\nabla\theta_I^{j,1}\|^2 &= \tau\sum_{j=1}^{n}(\mu + \gamma(a_1))(\hat{I}^{j,1} - I^{j,1}, \theta_I^{j,1}) \\ &\quad - \tau\sum_{j=1}^{n}(\omega_I^{j,1}, \theta_I^{j,1}). \end{aligned} \tag{3.12}$$

Adding (3.11) and (3.12), we have

$$\begin{aligned}(3.13)\quad &\tau\sum_{i=2}^{N_a}{}'({}^c\bar\partial\theta_I^{1,i},\theta_I^{1,i})+\tau\sum_{j=1}^{n}({}^c\bar\partial\theta_I^{j,1},\theta_I^{j,1})+\tau\sum_{i=2}^{N_a}{}'\sum_{j=2}^{n}({}^c\bar D\theta_I^{j,i},\theta_I^{j,i})+\tau d_2\sum_{i=1}^{N_a}{}'\sum_{j=1}^{n}\|\nabla\theta_I^{j,i}\|^2\\ &=\tau\sum_{i=1}^{N_a}{}'\sum_{j=1}^{n}(\mu+\gamma(a_i))(\hat I^{j,i}-I^{j,i},\theta_I^{j,i})-\tau\sum_{i=1}^{N_a}{}'\sum_{j=1}^{n}(\omega_I^{j,i},\theta_I^{j,i}).\end{aligned}$$

With $G=(g_{ij})_{i,j=1,2}, g_{11}=5/4, g_{12}=g_{21}=-1/2, g_{22}=1/4$, the positive definite symmetric matrix associated with the two-step BDF method, we shall use the notation

$$\|V^m\|_G^2=\sum_{i,j=1}^{2}g_{ij}(v^{m+1-i},v^{m+1-j}),\quad \text{where}\quad V^m:=\begin{pmatrix}v^m\\ v^{m-1}\end{pmatrix}.$$

Then, the G-stability property of the two-step BDF method yields

$$\Big(\frac{3}{2}v^m-2v^{m-1}+\frac{1}{2}v^{m-2},v^m\Big)\geqslant\|V^m\|_G^2-\|V^{m-1}\|_G^2;$$

see [19, Example 6.5].

Analogously, by the G-stability of the method, we can estimate $\tau({}^c\bar D\theta_I^{j,i},\theta_I^{j,i})$ from below, as illustrated in the following Lemma.

Lemma 3.3 (G-stability property)**.** *The following estimate holds,*

$$(3.14)\qquad \tau({}^c\bar D\theta_I^{j,i},\theta_I^{j,i})\geqslant\|\Theta_I^{j,i}\|_G^2-\|\Theta_I^{j-1,i-1}\|_G^2\quad \textit{with}\quad \Theta_I^{j,i}:=\begin{pmatrix}\theta_I^{j,i}\\ \theta_I^{j-1,i-1}\end{pmatrix}.$$

Summing in (3.14) over i and j, we obtain an estimate for the third term on the left-hand side of (3.13) from below,

$$(3.15)\qquad \tau\sum_{i=2}^{N_a}{}'\sum_{j=2}^{n}({}^c\bar D\theta_I^{j,i},\theta_I^{j,i})\geqslant\sum_{i=2}^{N_a}{}'\sum_{j=2}^{n}\left(\|\Theta_I^{j,i}\|_G^2-\|\Theta_I^{j-1,i-1}\|_G^2\right).$$

Corollary 3.1. *By (3.15), we get a lower bound for the expression on the right-hand side of relation (3.13)*

$$\begin{aligned}&\tau\sum_{i=1}^{N_a}{}'\sum_{j=1}^{n}(\mu+\gamma(a_i))(\hat I^{j,i}-I^{j,i},\theta_I^{j,i})-\tau\sum_{i=1}^{N_a}{}'\sum_{j=1}^{n}(\omega_I^{j,i},\theta_I^{j,i})\geqslant\tau\sum_{i=2}^{N_a}{}'({}^c\bar\partial\theta_I^{1,i},\theta_I^{1,i})\\ &+\tau\sum_{j=1}^{n}({}^c\bar\partial\theta_I^{j,1},\theta_I^{j,1})+\sum_{i=2}^{N_a}{}'\sum_{j=2}^{n}\left(\|\Theta_I^{j,i}\|_G^2-\|\Theta_I^{j-1,i-1}\|_G^2\right).\end{aligned}$$

3.2. **Consistency.** Here, we estimate the consistency errors $\omega_I^{j,i}$; see also section 3.3 for the consistency error ω_S^j.

Lemma 3.4 (Consistency estimate)**.** *Assume that $I(\cdot,\cdot,x)\in C^3([0,T]\times[0,a^\star])$ and $I(t,a,\cdot)\in H^2(\Omega)\cap H_0^1(\Omega)$ for $(t,a)\in[0,T]\times[0,a^\star]$. Then, there exists a constant $C>0$, independent of the stepsize τ and the meshsize h, such that the following estimate holds*

$$\tau\sum_{j=1}^{n}\|\omega_I^{j,i}\|\leqslant C(\tau^2+h^2),\quad 1\leqslant i\leqslant N_a,\ 1\leqslant n\leqslant N,$$

with $\omega_I^{j,i}$ the consistency error defined in (3.6).

Proof. For $j = 1$ or $i = 1$, we obtain the following estimate by Lemma 3.1 and Taylor expansion,

$$\begin{aligned}\tau\|\omega_I^{j,i}\| &\leqslant \|(\hat I^{j,i} - \hat I^{j-1,i-1} - \tau D\hat I^{j,i})\| + \|\tau(R_h - E)\ {}^c\bar\partial \hat I^{j,i}\| \\ &= \|I(t_j, a_i, \cdot) - I(t_j - \tau, a_i - \tau, \cdot) - \tau(I_t(t_j, a_i, \cdot) + I_a(t_j, a_i, \cdot))\| + C\tau h^2 \\ &= \left\| \int_{-\tau}^0 \left[\frac{\mathrm{d}}{\mathrm{d}s} I(t_j + s, a_i + s, \cdot) - \frac{\mathrm{d}}{\mathrm{d}s} I(t_j + s, a_i + s, \cdot)\big|_{s=0} \right] \mathrm{d}s \right\| + C\tau h^2 \\ &\leqslant \int_{-\tau}^0 \int_0^s \left\| \frac{\mathrm{d}^2}{\mathrm{d}t^2} I(t_j + t, a_i + t, \cdot)(s - t) \right\| \mathrm{d}t\, \mathrm{d}s + C\tau h^2\end{aligned}$$

and thus

$$\|\omega_I^{j,i}\| \leqslant C(\tau^2 + h^2) \quad \text{for } j = 1 \text{ or } i = 1. \tag{3.16}$$

For $i \geqslant 2$ and $j \geqslant 2$, Taylor expansion yields

$$\begin{aligned}&\left\| \frac{3}{2}\hat I^{j,i} - 2\hat I^{j-1,i-1} + \frac{1}{2}\hat I^{j-2,i-2} - \tau D\hat I^{j,i} \right\| \\ &\leqslant \left\| \frac{3}{2}\hat I^{j,i} - 2\left(\hat I^{j,i} - \tau D\hat I^{j,i} + \frac{\tau^2}{2} D^2 \hat I^{j,i} + \frac{1}{2}\int_0^{-\tau} \frac{\mathrm{d}^3 I(t_j + s, a_i + s, \cdot)}{\mathrm{d}s^3} (\tau + s)^2 \mathrm{d}s \right) + \right. \\ &\left. \frac{1}{2}\left(\hat I^{j,i} - 2\tau D\hat I^{j,i} + 2\tau^2 D^2 \hat I^{j,i} + \frac{1}{2}\int_0^{-2\tau} \frac{\mathrm{d}^3 I(t_j + s, a_i + s, \cdot)}{\mathrm{d}s^3} (2\tau + s)^2\, \mathrm{d}s \right) - \tau D\hat I^{j,i} \right\| \\ &\leqslant C\tau^2 \int_0^{-\tau} \left\| \frac{\mathrm{d}^3 I(t_j + s, a_i + s, \cdot)}{\mathrm{d}s^3} \right\| \mathrm{d}s + C\tau^2 \int_0^{-2\tau} \left\| \frac{\mathrm{d}^3 I(t_j + s, a_i + s, \cdot)}{\mathrm{d}s^3} \right\| \mathrm{d}s \leqslant C\tau^3.\end{aligned}$$

Now,

$$\tau\|\omega_I^{j,i}\| \leqslant \left\| \frac{3}{2}\hat I^{j,i} - 2\hat I^{j-1,i-1} + \frac{1}{2}\hat I^{j-2,i-2} - \tau D\hat I^{j,i} \right\| + \left\| \tau(R_h - E)\ {}^c\bar D \hat I^{j,i} \right\|$$

and thus, in view also of Lemma 3.1,

$$\|\omega_I^{j,i}\| \leqslant C(\tau^2 + h^2), \quad i, j \geqslant 2. \tag{3.17}$$

Therefore, we obtain the conclusion by summing (3.16) and (3.17) over j from $j = 1$ to $j = n$. □

For the sake of brevity, we introduce the notation

$$\|\Theta_I^m\|_{G,\ell^2(L^2)} := \begin{cases} \left(\dfrac{\tau}{2}\|\theta_I^{m,0}\|^2 + \tau {\displaystyle\sum_{i=1}^{N_a}}' \|\Theta_I^{m,i}\|_G^2 \right)^{1/2} & \text{if } m \geqslant 1 \\ \|\theta_I^0\|_{\ell^2(L^2)} & \text{if } m = 0. \end{cases}$$

Clearly, there exists a constant C, independent of τ, such that

$$\|\Theta_I^m\|_{\ell^2(L^2)} \leqslant C\|\Theta_I^m\|_{G,\ell^2(L^2)}. \tag{3.18}$$

Corollary 3.2. *Let M and M_j be such that $\|\omega_I^{j,M_j}\| = \max_{1\leqslant i\leqslant N_a}\|\omega_I^{j,i}\|$ and $\|\Theta_I^M\|_{G,\ell^2(L^2)} = \max_{1\leqslant j\leqslant n}\|\Theta_I^j\|_{G,\ell^2(L^2)}$. Then, by (3.1) and (3.18), we obtain the preliminary estimate*

$$\begin{aligned}\tau^2\sum_{j=1}^{n}\sum_{i=1}^{N_a}{}'\left(\omega_I^{j,i},\theta_I^{j,i}\right) &\leqslant \tau^2\sum_{j=1}^{n}\sum_{i=1}^{N_a}{}'\|\omega_I^{j,i}\|\cdot\|\theta_I^{j,i}\| \leqslant \tau\sum_{j=1}^{n}\|\omega_I^{j,M_j}\|\cdot\|\theta_I^j\|_{\ell^1(L^2)}\\ &\leqslant C\tau\sum_{j=1}^{n}\|\omega_I^{j,M_j}\|\cdot\|\theta_I^j\|_{\ell^2(L^2)} \leqslant C\|\Theta_I^M\|_{G,\ell^2(L^2)}\sum_{j=1}^{n}\tau\|\omega_I^{j,M_j}\|\\ &\leqslant \frac{1}{\varepsilon}\Big(C\tau\sum_{j=1}^{n}\|\omega_I^{j,M_j}\|\Big)^2+\varepsilon\|\Theta_I^M\|^2_{G,\ell^2(L^2)} \leqslant C(\tau^4+h^4)+\varepsilon\|\Theta_I^M\|^2_{G,\ell^2(L^2)},\end{aligned}$$

where ε is a positive constant to be determined in the subsequent analysis.

We shall use this preliminary estimate in the proof of Lemma 3.5.

Based on Lemmata 3.2 and 3.4, and on Corollary 3.1, we next derive an estimate for $\|\Theta_I^n\|_{G,\ell^2(L^2)}$ as follows.

Lemma 3.5 (Estimate for $\|\Theta_I^n\|_{G,\ell^2(L^2)}$)**.** *Let $\gamma^\star = \max_{0\leqslant a\leqslant a^*}\gamma(a)$, and assume that $\|\theta_I^{0,0}\| \leqslant Ch^2$ and the hypotheses in Lemma 3.2 hold. Then, there exist constants C and C_1, independent of τ and h, such that the following estimate holds*

$$\begin{aligned}\Big(\frac{1}{2}-C_1\tau\Big)\|\Theta_I^n\|^2_{G,\ell^2(L^2)} &\leqslant C\left(\|\Theta_I^1\|^2_{G,\ell^2(L^2)}+\|\Theta_I^0\|^2_{G,\ell^2(L^2)}\right)+C\tau\sum_{j=0}^{n-1}\|\Theta_I^j\|^2_{G,\ell^2(L^2)}\\ &\quad+C\tau\sum_{j=1}^{n}\|\theta_S^j\|^2+C(\tau^2+h^2)^2.\end{aligned}\tag{3.19}$$

Proof. By Corollary 3.1, we obtain

$$\begin{aligned}&\sum_{i=2}^{N_a}{}'\sum_{j=2}^{n}\left(\|\Theta_I^{j,i}\|_G^2-\|\Theta_I^{j-1,i-1}\|_G^2\right)\\ &\leqslant \tau\sum_{i=1}^{N_a}{}'\sum_{j=1}^{n}(\mu+\gamma(a_i))(\hat{I}^{j,i}-I^{j,i},\theta_I^{j,i})-\tau\sum_{i=1}^{N_a}{}'\sum_{j=1}^{n}(\omega_I^{j,i},\theta_I^{j,i})\\ &\quad-\tau\sum_{i=2}^{N_a}{}'({}^c\bar{\partial}\theta_I^{1,i},\theta_I^{1,i})-\tau\sum_{j=1}^{n}({}^c\bar{\partial}\theta_I^{j,1},\theta_I^{j,1}).\end{aligned}$$

Multiplying by τ and adding

$$\tau\sum_{j=2}^{n}\left(\|\Theta_I^{j,1}\|_G^2-\frac{1}{2}\|\Theta_I^{j-1,N_a}\|_G^2-\frac{1}{2}\|\Theta_I^{j-1,N_a-1}\|_G^2\right)+\frac{\tau}{2}\sum_{j=2}^{n}\left(\|\theta_I^{j,0}\|^2-\|\theta_I^{j-1,0}\|^2\right)$$

to both sides, we have

$$
\begin{aligned}
&\sum_{j=2}^{n}\Big[\|\Theta_I^j\|_{G,\ell^2(L^2)}^2-\|\Theta_I^{j-1}\|_{G,\ell^2(L^2)}^2\Big]\\
&\leqslant\frac{\tau}{2}(\mu+\gamma^\star)\sum_{j=1}^{n}\big(\|\hat I^j-I^j\|_{\ell^2(L^2)}^2+\|\theta_I^j\|_{\ell^2(L^2)}^2\big)-\tau^2\sum_{j=1}^{n}{\sum_{i=1}^{N_a}}'\big(\omega_I^{j,i},\theta_I^{j,i}\big)-\tau^2{\sum_{i=2}^{N_a}}'({}^c\bar\partial\theta_I^{1,i},\theta_I^{1,i})\\
&\quad-\tau^2\sum_{j=1}^{n}({}^c\bar\partial\theta_I^{j,1},\theta_I^{j,1})+\tau\sum_{j=2}^{n}\Big(\|\Theta_I^{j,1}\|_G^2-\frac12\|\Theta_I^{j-1,N_a}\|_G^2-\frac12\|\Theta_I^{j-1,N_a-1}\|_G^2\Big)\\
&\quad+\frac{\tau}{2}\sum_{j=2}^{n}\big(\|\theta_I^{j,0}\|^2-\|\theta_I^{j-1,0}\|^2\big)\\
&\leqslant\frac{\tau}{2}(\mu+\gamma^\star)\sum_{j=1}^{n}\big(\|\hat I^j-I^j\|_{\ell^2(L^2)}^2+\|\theta_I^j\|_{\ell^2(L^2)}^2\big)-\tau^2\sum_{j=1}^{n}{\sum_{i=1}^{N_a}}'\big(\omega_I^{j,i},\theta_I^{j,i}\big)\\
&\quad+\tau{\sum_{i=2}^{N_a}}'(\theta_I^{0,i-1},\theta_I^{1,i})+\tau\sum_{j=1}^{n}(\theta_I^{j-1,0},\theta_I^{j,1})+\tau\sum_{j=2}^{n}\|\Theta_I^{j,1}\|_G^2+\frac{\tau}{2}\sum_{j=2}^{n}\|\theta_I^{j,0}\|^2.
\end{aligned}
$$

This estimate can be simplified to

$$
\begin{aligned}
\|\Theta_I^n\|_{G,\ell^2(L^2)}^2&\leqslant\|\Theta_I^1\|_{G,\ell^2(L^2)}^2+\frac{\tau}{2}(\mu+\gamma^\star)\sum_{j=1}^{n}\big(\|\hat I^j-I^j\|_{\ell^2(L^2)}^2+\|\theta_I^j\|_{\ell^2(L^2)}^2\big)\\
&\quad-\tau^2\sum_{j=1}^{n}{\sum_{i=1}^{N_a}}'\big(\omega_I^{j,i},\theta_I^{j,i}\big)+\frac{\tau}{2}{\sum_{i=2}^{N_a}}'(\|\theta_I^{0,i-1}\|^2+\|\theta_I^{1,i}\|^2)+\frac{\tau}{2}\sum_{j=1}^{n}(\|\theta_I^{j-1,0}\|^2+\|\theta_I^{j,1}\|^2)\\
&\quad+\tau\sum_{j=2}^{n}\|\Theta_I^{j,1}\|_G^2+\frac{\tau}{2}\sum_{j=2}^{n}\|\theta_I^{j,0}\|^2.
\end{aligned}
\tag{3.20}
$$

Notice that

$$
\|\hat I^j-I^j\|_{\ell^2(L^2)}^2\leqslant 2\|\rho_I^j\|_{\ell^2(L^2)}^2+2\|\theta_I^j\|_{\ell^2(L^2)}^2\leqslant 2\|\theta_I^j\|_{\ell^2(L^2)}^2+Ch^4;
$$

therefore, by (3.18),

$$
\frac{\tau}{2}(\mu+\gamma^\star)\sum_{j=1}^{n}\big(\|\hat I^j-I^j\|_{\ell^2(L^2)}^2+\|\theta_I^j\|_{\ell^2(L^2)}^2\big)\leqslant C\tau\sum_{j=1}^{n}\|\Theta_I^j\|_{G,\ell^2(L^2)}^2+Ch^4.
\tag{3.21}
$$

In view of $\|\Theta_I^{j,1}\|_G^2\leqslant C(\|\theta_I^{j,1}\|^2+\|\theta_I^{j-1,0}\|^2)$ and

$$
\frac{\tau}{2}{\sum_{i=2}^{N_a}}'(\|\theta_I^{0,i-1}\|^2+\|\theta_I^{1,i}\|^2)\leqslant\|\theta_I^0\|_{\ell^2(L^2)}^2+\|\theta_I^1\|_{\ell^2(L^2)}^2\leqslant C(\|\Theta_I^0\|_{G,\ell^2(L^2)}^2+\|\Theta_I^1\|_{G,\ell^2(L^2)}^2),
$$

combining (3.21) with Corollary 3.2, we obtain from (3.20)

$$
\begin{aligned}
\|\Theta_I^n\|_{G,\ell^2(L^2)}^2&\leqslant C(\tau^4+h^4)+C\tau\sum_{j=1}^{n}\|\Theta_I^j\|_{G,\ell^2(L^2)}^2+C(\|\Theta_I^0\|_{G,\ell^2(L^2)}^2+\|\Theta_I^1\|_{G,\ell^2(L^2)}^2)\\
&\quad+C\tau\sum_{j=1}^{n}\big(\|\theta_I^{j,0}\|^2+\|\theta_I^{j,1}\|^2\big)+\varepsilon\|\Theta_I^M\|_{G,\ell^2(L^2)}^2.
\end{aligned}
$$

Then, for sufficient small τ and ε, by Lemma 3.4 , we have

$$\begin{aligned}&\big(1-C\tau-\varepsilon\big)\|\Theta_I^n\|^2_{G,\ell^2(L^2)}\leqslant\big(1-C\tau-\varepsilon\big)\|\Theta_I^M\|^2_{G,\ell^2(L^2)}\leqslant C(\tau^4+h^4)\\&+C\Big(\|\Theta_I^1\|^2_{G,\ell^2(L^2)}+\|\Theta_I^0\|^2_{G,\ell^2(L^2)}\Big)+C\tau\sum_{j=0}^{n-1}\|\Theta_I^j\|^2_{G,\ell^2(L^2)}+C\tau\sum_{j=1}^{n}\big(\|\theta_I^{j,0}\|^2+\|\theta_I^{j,1}\|^2\big).\end{aligned}$$

On the other hand, in view of (3.9), we have

$$\begin{aligned}\|\theta_I^{j,1}\|&\leqslant\|\theta_I^{j-1,0}\|+\tau(\mu+\gamma(a_1))\|\hat I^{j,1}-I^{j,1}\|+\tau\|\omega_I^{j,1}\|\\&\leqslant\|\theta_I^{j-1,0}\|+\tau(\mu+\gamma(a_1))(\|\theta_I^{j,1}\|+\|\rho_I^{j,1}\|)+\tau\|\omega_I^{j,1}\|\\&\leqslant\|\theta_I^{j-1,0}\|+\tau(\mu+\gamma(a_1))\|\theta_I^{j,1}\|+C(\tau^2+h^2).\end{aligned}$$

Therefore, we infer that

$$\begin{aligned}&(1-C\tau-\varepsilon)\|\Theta_I^n\|^2_{G,\ell^2(L^2)}\leqslant C(\tau^2+h^2)^2\\&+C\Big(\|\Theta_I^1\|^2_{G,\ell^2(L^2)}+\|\Theta_I^0\|^2_{G,\ell^2(L^2)}\Big)+C\tau\sum_{j=1}^{n-1}\|\Theta_I^j\|^2_{G,\ell^2(L^2)}+C\tau\sum_{j=0}^{n}\|\theta_I^{j,0}\|^2.\end{aligned}\tag{3.22}$$

By Lemma 3.1, we have the following estimate for $j\geqslant 1$,

$$\|\theta_I^{j,0}\|^2\leqslant C(\tau^2+h^2)^2+C\|\theta_S^j\|^2+C\left(\|\Theta_I^j\|^2_{G,\ell^2(L^2)}+\|\Theta_I^{j-1}\|^2_{G,\ell^2(L^2)}\right).$$

Substituting this estimate into (3.22) and taking $\varepsilon=1/2$ leads to the conclusion. □

3.3. **Estimation of θ_S^n.** Following the same procedure as in the derivation of (3.7), we see that

$$(\bar D\theta_S^j,\theta_S^j)+d_1\|\nabla\theta_S^j\|^2=(\hat I^{j,0}-I^{j,0},\theta_S^j)+\mu(\hat S^j-S^j,\theta_S^j)-(\omega_S^j,\theta_S^j),\quad 2\leqslant j\leqslant N,\tag{3.23}$$

$$(\bar\partial\theta_S^1,\theta_S^1)+d_1\|\nabla\theta_S^1\|^2=(\hat I^{1,0}-I^{1,0},\theta_S^1)+\mu(\hat S^1-S^1,\theta_S^1)-(\omega_S^1,\theta_S^1),\tag{3.24}$$

where $\hat I^{j,0}=I(t_j,0,\cdot),\hat S^j=S(t_j,\cdot)$, and

$$\omega_S^j=\begin{cases}(R_h-E)\bar D\hat S^j+\bar D\hat S^j-S_t(t_j,\cdot), & \text{if } j\geqslant 2,\\(R_h-E)\bar\partial\hat S^j+\bar\partial\hat S^j-S_t(t_j,\cdot), & \text{if } j=1.\end{cases}$$

Multiplying (3.23) by τ and summing over j from $j=2$ to $j=n$, and adding the result to (3.24) multiplied by τ, we obtain

$$\begin{aligned}&\tau(\bar\partial\theta_S^1,\theta_S^1)+\tau\sum_{j=2}^{n}(\bar D\theta_S^j,\theta_S^j)+\tau d_1\sum_{j=1}^{n}\|\nabla\theta_S^j\|^2\\&=\tau\sum_{j=1}^{n}(\hat I^{j,0}-I^{j,0},\theta_S^j)+\mu\tau\sum_{j=1}^{n}(\hat S^j-S^j,\theta_S^j)-\tau\sum_{j=1}^{n}(\omega_S^j,\theta_S^j).\end{aligned}$$

Using a similar technique as in the previous subsection or in the proof of [37, Theorem 1.7], we get

$$\|\theta_S^n\|\leqslant\frac12\|\theta_S^1\|+\frac12\|\theta_S^0\|+2\tau\sum_{j=1}^{n}\|\theta_I^{j,0}\|+2\mu\tau\sum_{j=1}^{n}\|\theta_S^j\|+C(\tau^2+h^2).\tag{3.25}$$

In view of Lemma 3.2, this yields

$$\Big(1-2(\mu+\beta^\star a^\star M_0)\tau\Big)\|\theta_S^n\| \leqslant \frac{1}{2}\|\theta_S^1\|+\frac{1}{2}\|\theta_S^0\| + C\tau\sum_{j=1}^{n}\big(\|\theta_I^j\|_{\ell^1(L^2)}+\|\theta_I^{j-1}\|_{\ell^1(L^2)}\big)+C\tau\sum_{j=1}^{n-1}\|\theta_S^j\|+C(\tau^2+h^2).$$

Therefore, for sufficient small τ, we have

(3.26)
$$(1-C_2\tau)^2\|\theta_S^n\|^2 \leqslant \frac{1}{2}\|\theta_S^1\|^2+\frac{1}{2}\|\theta_S^0\|^2+C\tau\sum_{j=0}^{n}\|\theta_I^j\|^2_{\ell^2(L^2)}+C\tau\sum_{j=1}^{n-1}\|\theta_S^j\|^2+C(\tau^2+h^2)^2$$
$$\leqslant \frac{1}{2}\|\theta_S^1\|^2+\frac{1}{2}\|\theta_S^0\|^2+C\tau\sum_{j=0}^{n-1}\|\theta_S^j\|^2+C\tau\sum_{j=0}^{n}\|\Theta_I^j\|^2_{G,\ell^2(L^2)}+C(\tau^2+h^2)^2,$$

where $C_2=2(\mu+\beta^\star a^\star M_0)$.

3.4. **Estimation of θ_R^n.** Following the same procedure as in the previous subsection, we conclude that

$$\tau(\bar\partial\theta_R^1,\theta_R^1)+\tau\sum_{j=2}^{n}(\bar D\theta_R^j,\theta_R^j)+\tau d_3\sum_{j=1}^{n}\|\nabla\theta_R^j\|^2$$
$$=\mu\tau\sum_{j=1}^{n}(\hat R^j-R^j,\theta_R^j)-\tau\sum_{j=1}^{n}(\omega_R^j,\theta_R^j)+\tau\sum_{j=1}^{n}\left(\int_0^{a^\star}\gamma(a)I(t_j,a,\cdot)\,\mathrm{d}a-\tilde I_\gamma^j,\theta_R^j\right),$$

where $\hat R^j=R(t_j,\cdot)$, $\tilde I_\gamma^j=\tau\sum\limits_{i=0}^{N_a}{}''\gamma(a_i)I^{j,i}$ and

$$\omega_R^j=\begin{cases}(R_h-E)\bar D\hat R^j+\bar D\hat R^j-R_t(t_j,\cdot), & \text{if } j\geqslant 2,\\ (R_h-E)\bar\partial\hat R^j+\bar\partial\hat R^j-R_t(t_j,\cdot), & \text{if } j=1.\end{cases}$$

Proceeding as in the proof of [37, Theorem 1.7], we obtain

$$\|\theta_R^n\|\leqslant\frac{1}{2}(\|\theta_R^1\|+\|\theta_R^0\|)+2\tau\sum_{j=1}^{n}\left\|\int_0^{a^\star}\gamma(a)I(t_j,a,\cdot)\,\mathrm{d}a-\tilde I_\gamma^j\right\| + 2\mu\tau\sum_{j=1}^{n}\|\theta_R^j\|+C(\tau^2+h^2). \tag{3.27}$$

We now present error estimates for the modified scheme, namely, (2.7)–(2.8) with S^n replaced by $g(S^n)$ on the right-hand sides of the fourth relation in (2.7) and the fifth relation in (2.8).

Proposition 3.1 (Error estimates for the modified scheme)**.** *Assume $\beta,\gamma\in C^3[0,a^\star]$, $S(t,\cdot),R(t,\cdot),I(t,a,\cdot)\in H^2(\Omega)\cap H_0^1(\Omega)$, $S(\cdot,x),R(\cdot,x)\in C^3[0,T]$, $I(\cdot,\cdot,x)\in C^3([0,T]\times[0,a^\star])$, and Hypothesis 1.2. Assume furthermore that the initial functions are approximated with second order accuracy, namely, $\|\xi_S^0\|+\|\xi_I^0\|_{\ell^2(L^2)}+\|\xi_R^0\|\leqslant Ch^2$. Then, there exists a constant C, independent of τ and h, such that, for sufficiently small τ,*

$$\|\xi_I^n\|_{\ell^2(L^2)}+\|\xi_S^n\|+\|\xi_R^n\|\leqslant C(\tau^2+h^2),\quad n=1,\dots,N. \tag{3.28}$$

Proof. By Lemma 3.1, estimate (3.28) is valid with ξ replaced by ρ; hence, it suffices to prove (3.28) with ξ replaced by θ. By (3.26) together with (3.19), we have

$$\Big(\frac{1}{2}-C_3\tau\Big)\big(\|\Theta_I^n\|_{G,\ell^2(L^2)}^2+\|\theta_S^n\|^2\big)\leqslant C\big(\|\Theta_I^1\|_{G,\ell^2(L^2)}^2+\|\theta_S^1\|^2\big)$$
$$+C(\|\Theta_I^0\|_{G,\ell^2(L^2)}^2+\|\theta_S^0\|^2)+C\tau\sum_{j=0}^{n-1}\big(\|\Theta^j\|_{G,\ell^2(L^2)}^2+\|\theta_S^j\|^2\big)+C(\tau^2+h^2)^2.$$

Via a discrete Gronwall inequality, we infer that

$$\begin{aligned}\|\Theta_I^n\|_{G,\ell^2(L^2)}^2+\|\theta_S^n\|^2&\leqslant C\big(\|\Theta_I^1\|_{G,\ell^2(L^2)}^2+\|\theta_S^1\|^2\big)\\&\quad+C(\|\Theta_I^0\|_{G,\ell^2(L^2)}^2+\|\theta_S^0\|^2)+C(\tau^2+h^2)^2.\end{aligned}\tag{3.29}$$

In the case $n=1$ in (3.25), we get

$$\|\theta_S^1\|\leqslant C\|\theta_S^0\|+C\tau\|\theta_I^{1,0}\|+C(\tau^2+h^2).\tag{3.30}$$

On the other hand, we obtain by (3.4) that

$$\|\theta_I^{1,0}\|\leqslant\beta^\star a^\star M_0\|\theta_S^1\|+\beta^\star M_1\|\theta_I^1\|_{\ell^1(L^2)}+C(\tau^2+h^2).$$

Substituting this estimate into (3.30), we see that

$$\begin{aligned}\|\theta_S^1\|^2&\leqslant C\|\theta_S^0\|^2+C\tau\|\theta_I^1\|_{\ell^1(L^2)}^2+C(\tau^2+h^2)^2\\&\leqslant C\|\theta_S^0\|^2+C\tau\|\Theta_I^1\|_{G,\ell^2(L^2)}^2+C(\tau^2+h^2)^2\end{aligned}\tag{3.31}$$

and

$$\|\theta_I^{1,0}\|^2\leqslant C\|\theta_S^1\|^2+C\|\Theta_I^1\|_{G,\ell^2(L^2)}^2+C(\tau^2+h^2)^2.\tag{3.32}$$

Furthermore, by (3.8) and Lemma 3.4, we obtain

$$\|\theta_I^{1,i}\|\leqslant C\|\theta_I^{0,i-1}\|+C(\tau^2+h^2),\quad i=1,\dots,N_a,$$

and infer that

$$\|\Theta_I^{1,i}\|_G^2\leqslant C\|\theta_I^{0,i-1}\|^2+C(\tau^2+h^2)^2.\tag{3.33}$$

It follows by (3.32) and (3.33) that

$$\|\Theta_I^1\|_{G,\ell^2(L^2)}^2\leqslant C\tau\|\theta_S^1\|^2+C\|\theta_I^0\|_{\ell^2(L^2)}^2+C(\tau^2+h^2)^2.\tag{3.34}$$

Using (3.31) and (3.34), we get

$$\begin{aligned}\|\Theta_I^1\|_{G,\ell^2(L^2)}^2+\|\theta_S^1\|^2&\leqslant C(\|\theta_I^0\|_{\ell^2(L^2)}^2+\|\theta_S^0\|^2)+C(\tau^2+h^2)^2\\&=C(\|\Theta_I^0\|_{G,\ell^2(L^2)}^2+\|\theta_S^0\|^2)+C(\tau^2+h^2)^2.\end{aligned}$$

Therefore, we obtain from (3.29) and (3.18)

$$\|\theta_I^n\|_{\ell^2(L^2)}^2+\|\theta_S^n\|^2\leqslant C(\tau^2+h^2)^2,\tag{3.35}$$

which implies second order convergence of S^n and $I^{n,i}$.

To estimate $\|\theta_R^n\|$, we firstly note that

$$\begin{aligned}&\Big\|\int_0^{a^\star}\gamma(a)I(t_j,a,\cdot)\,\mathrm{d}a-\tilde{I}_\gamma^j\Big\|\\&\leqslant\Big\|\int_0^{a^\star}\gamma(a)I(t_j,a,\cdot)\,\mathrm{d}a-\tau{\sum_{i=0}^{N_a}}''\gamma(a_i)I(t_j,a_i,\cdot)\Big\|+\Big\|\tau{\sum_{i=0}^{N_a}}''\gamma(a_i)\big(I(t_j,a_i,\cdot)-I^{j,i}\big)\Big\|\\&\leqslant C\tau^2+C\|\xi_I^j\|_{\ell^1(L^2)}\leqslant C(\tau^2+h^2).\end{aligned}$$

Combining this estimate with (3.27), we obtain by a similar technique as in the proof of [37, Theorem 1.7] that

$$\|\theta_R^n\| \leqslant C(\tau^2 + h^2).$$

Together with (3.35), we conclude that estimate (3.28) is valid with ξ replaced by θ,

$$\|\theta_I^n\|_{\ell^2(L^2)} + \|\theta_S^n\| + \|\theta_R^n\| \leqslant C(\tau^2 + h^2). \tag{3.36}$$ □

3.5. **Error analysis for the original scheme.** The following theorem is our main result; it illustrates our convergence conclusions for the original scheme (2.7)–(2.8).

Theorem 3.1 (Error estimates for the original scheme)**.** *Assume* $\beta, \gamma \in C^3[0, a^\star]$, $S(t,\cdot)$, $R(t,\cdot), I(t,a,\cdot) \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$, $S(\cdot,x), R(\cdot,x) \in C^3([0,T])$, *and* $I(\cdot,\cdot,x) \in C^3([0,T]\times[0,a^\star])$, *and let the assumptions in Hypothesis 1.2 hold. Assume furthermore that the initial functions are approximated with second order accuracy, namely,* $\|\xi_S^0\| + \|\xi_I^0\|_{\ell^2(L^2)} + \|\xi_R^0\| \leqslant Ch^2$. *Then, for sufficiently small* h *and* $\tau h^{-d/4}$, *there exists a constant* C, *independent of* τ *and* h, *such that*

$$\|\xi_I^n\|_{\ell^2(L^2)} + \|\xi_S^n\| + \|\xi_R^n\| \leqslant C(\tau^2 + h^2), \quad n = 1, \dots, N. \tag{3.37}$$

Proof. It suffices to show that the solutions of the modified scheme are such that

$$\|S^n\|_{L^\infty} \leqslant M_1, \quad n = 1, \dots, N; \tag{3.38}$$

then, the solutions of the modified scheme are also solutions of the original scheme and the asserted error estimate (3.37) follows from Proposition 3.1.

In view of our quasi-uniformily assumption, we have the inverse inequality

$$\|\chi\|_{L^\infty} \leqslant ch^{-d/2}\|\chi\| \quad \forall \chi \in S_h;$$

see, for instance, [7, Lemma 4.5.3]. Therefore, from (3.36), we obtain

$$\|\theta_S^n\|_{L^\infty} \leqslant Ch^{-d/2}(\tau^2 + h^2)$$

and, under our assumptions, infer that

$$\|\theta_S^n\|_{L^\infty} \leqslant \frac{1}{2}. \tag{3.39}$$

Furthermore, for sufficiently small h, (3.3) yields

$$\|\rho_S^n\|_{L^\infty} \leqslant \frac{1}{2}. \tag{3.40}$$

Now, (3.38) follows from (3.39) and (3.40), and the proof is complete. □

Remark 3.1 (Boundedness of the coefficients of S^n in $J^{n,0}$)**.** Under the assumptions of Theorem 3.1, proceeding as in its proof, we can see that the coefficients

$$\frac{\tau}{2}\beta(a_0)I^{n-1,0} + \tau{\sum_{i=1}^{N_a}}'\beta(a_i)I^{n,i}$$

of S^n in $J^{n,0}$ are uniformly bounded in the L^∞ norm. This fact ensures existence and uniqueness of the numerical solutions for the original scheme (2.7)–(2.8), for sufficiently small time step τ; see Remark 2.2.

4. Numerical experiments

4.1. **Convergence.**

Example 4.1. To illustrate the numerical experiments, we take $\Omega = (0,1)$ and the parameters $b = \mu = 1, d_1 = d_2 = 0.01$, $d_3 = 0.02$, and the initial values

$$i_0(a,x) = \left(a + \frac{(a^\star)^3}{6 - 3(a^\star)^2}\right) x,\ r_0(x) = 0.1x + 0.2, \quad x \in [0,1],$$
$$\beta(a) = a^\star - a,\ \gamma(a) = 0.2 - 0.04a.$$

It is easy to check that the compatibility condition

$$i_0(0,x) = s_0(x) \int_0^{a^\star} \beta(a) i_0(a,x)\,\mathrm{d}a \tag{4.1}$$

is satisfied in this case if we take $s_0(x) = 1$. We take $a^\star = 1$ and calculate the numerical solutions at $t = 5$ and $t = 10$, and estimate the time convergence order by

$$p = \log \frac{E(\tau,h)}{E(\frac{\tau}{2},h)} \Big/ \log 2,$$

where $E(\tau,h)$ denotes the L^2 norm of the error for time and age stepsize τ and spatial mesh size h. Note that β and γ have different supports. We apply a standard composite trapezoidal rule to approximate $\int_0^\infty \gamma(a) I(t,a,x)\,\mathrm{d}a$ here. We display the numerical errors at $t = 5$ and $t = 10$ in Table 4.1; we see that the numerical solutions converge with second order in time.

We emphasize that the convergence order may be reduced for $t < a^\star$ if the compatibility

Table 4.1. Numerical errors and convergence orders for Example 4.1 with $h = 1/64$ and $s_0 = 1$ (compatible initial values).

τ	$t=5$			$t=10$		
	e_S	e_I	e_R	e_S	e_I	e_R
1/20	8.4783e-03	2.7065e-02	2.9523e-03	1.0936e-02	1.0081e-02	2.0597e-03
1/40	1.9910e-03	6.7807e-03	7.4404e-04	2.6527e-03	2.4360e-03	5.1606e-04
1/80	4.8868e-04	1.7018e-03	1.8714e-04	6.5750e-04	6.0255e-04	1.2934e-04
1/160	1.2150e-04	4.2678e-04	4.6958e-05	1.6398e-04	1.5013e-04	3.2394e-05
1/320	3.0322e-05	1.0690e-04	1.1763e-05	4.0964e-05	3.7488e-05	8.1070e-06
Order	2.0025	1.9972	1.9971	2.0011	2.0017	1.9985

condition (4.1) is not satisfied. To illustrate this conclusion, we take $s_0 = 2$ and display the numerical errors and convergence orders in Table 4.2. It is clear that the numerical approximations to $I(t,a,x)$ retain second order for $t \geqslant a^\star$; this is because the compatibility condition does not affect the solution for $t \geqslant a^\star$, which originates from the boundary value $I(0,a,x)$ rather than the birth boundary value $I(t,0,x)$. In sharp contrast, the non-validity of the compatibility condition results in reduced order of convergence of the method for $t < a^\star$, where the solution does depend on the birth boundary value.

Example 4.2. Following [8], we take $\Omega = (0,1)$ and the parameters

$$b = 1,\ \mu = 1,\ \gamma(a) = 1,\ d_1 = d_2 = 0.05, d_3 = 0.02,\ \beta(a) = \begin{cases} \beta & \text{if } a \leqslant 1, \\ 0 & \text{if } a > 1, \end{cases}$$

TABLE 4.2. Numerical errors and convergence orders for Example 4.1 with $h = 1/64$ and $s_0 = 2$ (incompatible initial values).

τ	$t = 0.5$			$t = 4$		
	e_S	e_I	e_R	e_S	e_I	e_R
1/20	6.2739e-03	1.1381e-04	8.4838e-04	8.4834e-03	1.5404e-02	2.4953e-04
1/40	2.8305e-03	2.8086e-05	3.7753e-04	3.8280e-03	6.6135e-03	7.9379e-05
1/80	1.3350e-03	6.9861e-06	1.7704e-04	1.8048e-03	3.0309e-03	3.0510e-05
1/160	6.4700e-04	1.7426e-06	8.5589e-05	8.7492e-04	1.4474e-03	1.3124e-05
1/320	3.1832e-04	4.3519e-07	4.2062e-05	4.3062e-04	7.0693e-04	6.0480e-06
Order	1.0233	2.0015	1.0249	1.0228	1.0338	1.1177

and the initial value

$$i_0(a,x) = \begin{cases} 5\mathrm{e}^{-(\mu+\gamma)a}(x-0.3)(0.7-x) & \text{if } x \in (0.3,0.7), \\ 0 & \text{otherwise.} \end{cases}$$

Here, we take $r_0(x) = 0.1x + 0.2$ and

$$s_0(x) = \begin{cases} \dfrac{2}{\beta(1-\mathrm{e}^{-2})} & \text{if } a \leqslant 1 \text{ and } x \in (0.3,0.7), \\ 0 & \text{otherwise,} \end{cases}$$

to ensure the validity of the compatibility condition (4.1). We calculated the numerical solutions at $t = 0.5$ and $t = 1$, and displayed the numerical errors in Table 4.3. It is clear that the convergence is of second order in time.

TABLE 4.3. Numerical errors and convergence orders for Example 4.2 subject to compatible initial values with $h = 1/8$.

τ	$t = 0.5$			$t = 1$		
	e_S	e_I	e_R	e_S	e_I	e_R
1/20	4.8319e-04	4.2573e-05	3.0219e-04	3.6940e-04	1.6276e-05	2.9621e-04
1/40	1.2488e-04	9.6501e-06	7.9634e-05	9.8384e-05	3.5512e-06	7.9732e-05
1/80	3.1893e-05	2.3077e-06	2.0598e-05	2.5617e-05	8.3155e-07	2.0861e-05
1/160	8.0730e-06	5.6518e-07	5.2505e-06	6.5536e-06	2.0117e-07	5.3499e-06
1/320	2.0319e-06	1.3990e-07	1.3263e-06	1.6587e-06	4.9470e-08	1.3557e-06
Order	1.9903	2.0143	1.9851	1.9822	2.0238	1.9805

Example 4.3. To test the convergence in two space dimensions, we again follow [8], and take $\Omega = (0,1) \times (0,1)$ and the parameters $b = \mu = \gamma(a) = 1$, $d_1 = d_2 = 0.01, d_3 = 0.02$, $a^\star = 1, \beta(a) = \beta$ for $a \in [0, a^\star]$ and $\beta(a) = 0$ for $a > a^\star$, and the initial values

$$s_0(x,y) = \begin{cases} \dfrac{2}{\beta(1-\mathrm{e}^{-2})} & \text{if } a \leqslant 1 \text{ and } x,y \in (0.3,0.7), \\ 0 & \text{otherwise,} \end{cases}$$

$$i_0(a,x,y) = \begin{cases} 100\mathrm{e}^{-(\mu+\gamma)a}(x-0.3)(0.7-x)(y-0.3)(0.7-y) & \text{if } x,y \in (0.3,0.7), \\ 0 & \text{otherwise,} \end{cases}$$

$$r_0(x,y) = 0.1x + 0.2y.$$

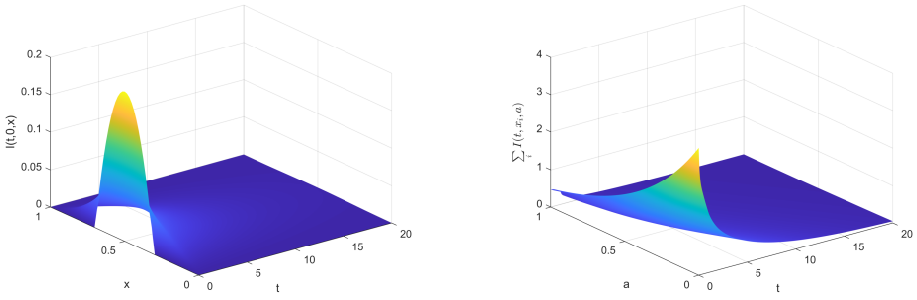


FIGURE 4.1. Numerical approximations to $I(t,0,x)$ and $\sum_i I(t,a,x_i)$ in Example 4.2 with $\beta = 3.6$ for $t \in [0,20]$, $x \in [0,1]$.

It is clear that the compatibility condition $i_0(0,x,y) = s_0(x,y)\int_0^{a^\star} \beta(a) i_0(a,x,y)\mathrm{d}a$ holds for arbitrary constant β. We take $\beta = 4$ and calculate the numerical solutions at $t = 2$ and $t = 4$. We display the numerical errors in Table 4.4; we clearly see second convergence order of the numerical solution.

TABLE 4.4. Numerical errors and convergence orders for Example 4.3 with $h = 1/8$ in a 2d case.

τ	$t=1$			$t=2$		
	e_S	e_I	e_R	e_S	e_I	e_R
1/20	4.3699e-05	1.2204e-05	4.5672e-05	1.1666e-05	3.1497e-06	4.5863e-06
1/40	1.0947e-05	2.7793e-06	1.0386e-05	2.8449e-06	8.5006e-07	1.1513e-06
1/80	2.7568e-06	6.6291e-07	2.4897e-06	7.0766e-07	2.2206e-07	2.9259e-07
1/160	6.9306e-07	1.6185e-07	6.1055e-07	1.7685e-07	5.6845e-08	7.4052e-08
1/320	1.7384e-07	3.9985e-08	1.5125e-07	4.4232e-08	1.4387e-08	1.8647e-08
Order	1.9952	2.0171	2.0132	1.9994	1.9823	1.9896

4.2. **Stability.** To illustrate the numerical stability in a one-dimensional spatial domain, we take the same parameters as in Example 4.2 in the previous section. Firstly, we take $\beta = 3.6$, the case in which the exact solution will tend to the disease-free equilibrium $(S^\star, 0, 0)$ as illustrated in [8, Theorem 5.1]. We plot the numerical solutions in Figures 4.1 and 4.2 to observe the dynamics. It is clear that $S(t,x)$ tends to a steady state while $I(t,a,x)$ and $R(t,x)$ tend to zero in this case. Then, we increased β to 4 and plot the numerical solutions and the steady state at $t = 20$ in Figures 4.3, 4.4, and 4.5. It is obvious that the numerical solutions tend to nonzero equilibria in this case, which is consistent with the conclusions in [8].

To illustrate the numerical stability in a two-dimensional spatial domain, we take the parameters as in Example 4.3 and plot the numerical solutions in $[0,20]$. Firstly, we take $\beta = 0.1$. It is easy to verify, cf. [8, Theorem 5.1], that the equilibrium $(S^\star, 0, 0)$ is globally attractive in this case. We calculated the solutions and plot them in Figures 4.6–4.8. It is obvious that the solutions behave as expected. Then, we increased β to 10. In this case, we expect the exact solutions to tend to a nonzero equilibrium; see [8, Theorems 6.1 and

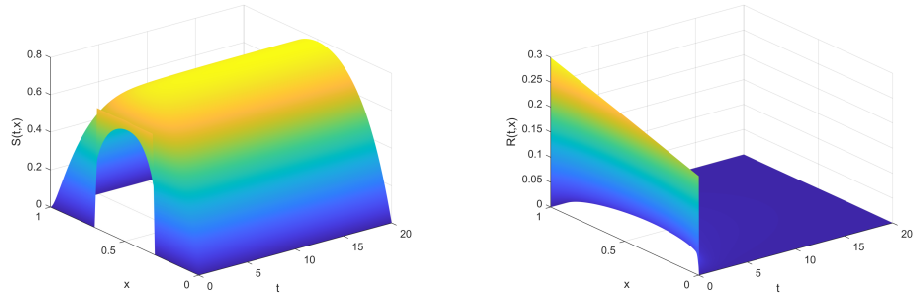


FIGURE 4.2. Numerical approximations to $S(t,x)$ and $V(t,x)$ in Example 4.2 with $\beta = 3.6$ for $t \in [0,20]$, $x \in [0,1]$.

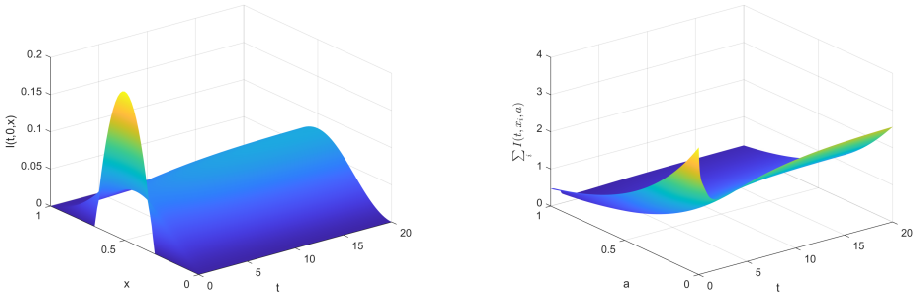


FIGURE 4.3. Numerical approximations to $I(t,0,x)$ and $\sum_i I(t,a,x_i)$ in Example 4.2 with $\beta = 4$ for $t \in [0,20]$, $x \in [0,1]$.

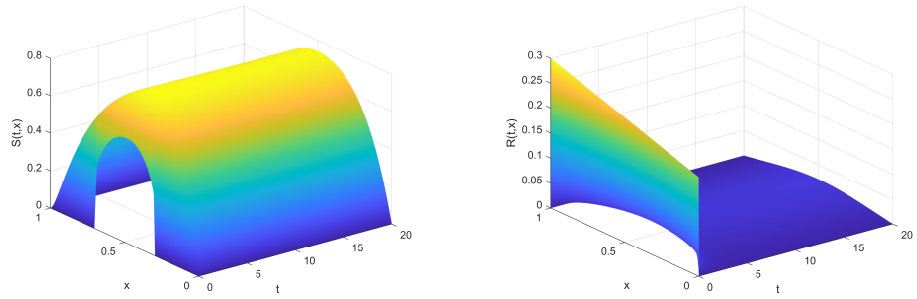


FIGURE 4.4. Numerical approximations to $S(t,x)$ and $R(t,x)$ in Example 4.2 with $\beta = 4$ for $t \in [0,20]$, $x \in [0,1]$.

7.2]. We plot the numerical solutions in Figures 4.9–4.11. It is obvious that our numerical results are consistent with the theory.

5. Conclusion and possible extensions

In this paper, we constructed a numerical scheme for a nonlinear coupled diffusive age-structured SIR model subject to homogeneous Dirichlet boundary conditions. We approximate the directional derivative along the characteristic lines $t = a + c$ by the two-step

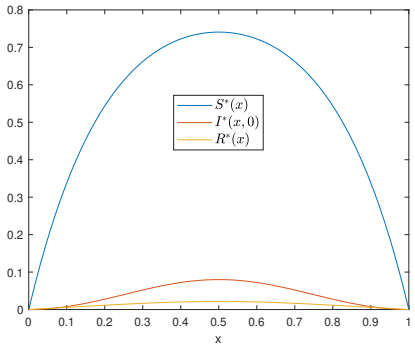


FIGURE 4.5. The numerical steady state of the system in Example 4.2 with $\beta = 4$ at $t = 20$.

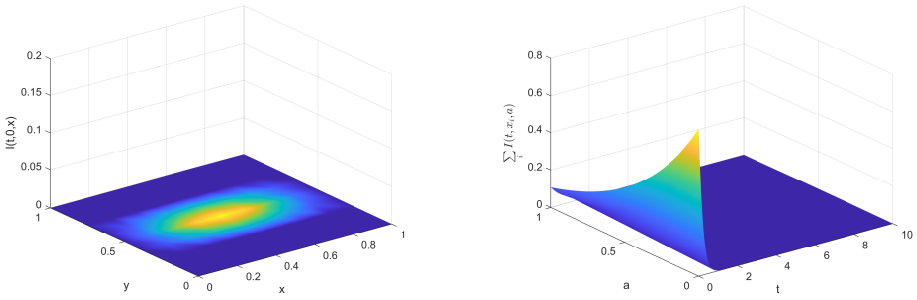


FIGURE 4.6. Numerical approximations to $I(t, 0, x, y)$ in Example 4.3 with $\beta = 0.1$ at $t = 10, (x, y) \in [0, 1] \times [0, 1]$ (left) and $\sum_{i,j} I(t, a, x_i, y_j)$ for $(t, a) \in [0, 10] \times [0, 1]$ (right).

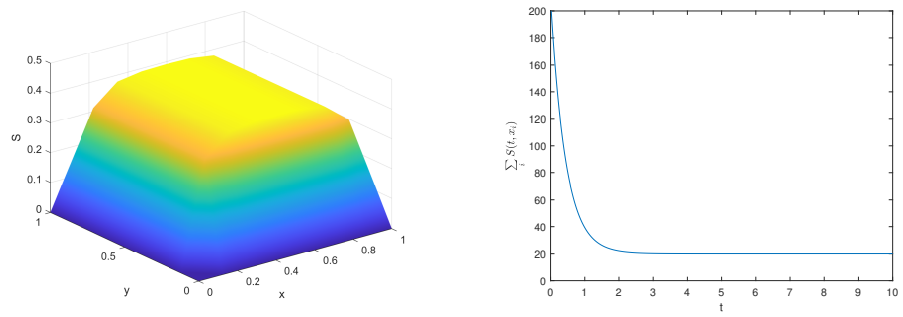


FIGURE 4.7. The numerical approximations of $S(t, x, y)$ in Example 4.3 with $\beta = 0.1$ at $t = 10$, $(x, y) \in [0, 1] \times [0, 1]$ (left) and $\sum_{i,j} S(t, x_i, y_j)$ for $t \in [0, 10]$ (right).

BDF method and combine it with a finite element method for the discretization in space. The convergence of the resulting fully discrete method is systematically analyzed. Some numerical experiments are provided to confirm the conclusions.

This paper does not consider the positivity of numerical solutions, which is essential for numerical approximation of realistic models. In future work, we will consider restrictions on the stepsizes or introduce the cut-off technique as in [14], [38] to ensure numerical positivity. Furthermore, we will investigate the threshold stability of numerical solutions to establish numerical threshold as in [13], [26].

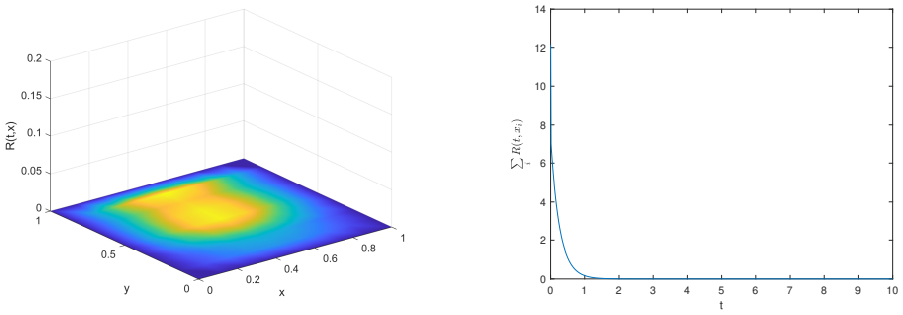


FIGURE 4.8. The numerical approximations of $R(t,x,y)$ in Example 4.3 with $\beta = 0.1$ at $t = 10$, $(x,y) \in [0,1] \times [0,1]$ (left) and $\sum_{i,j} R(t,x_i,y_j)$ for $t \in [0,10]$ (right).

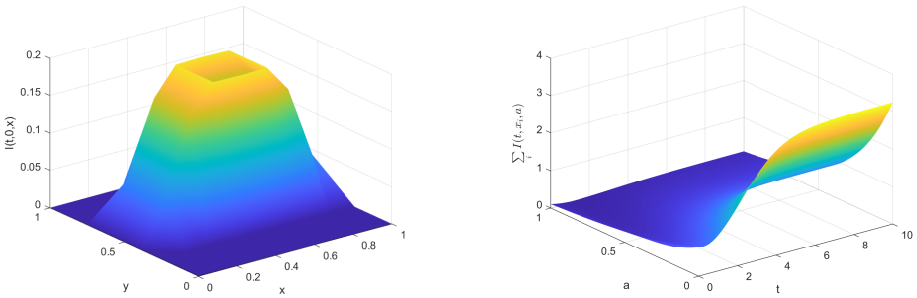


FIGURE 4.9. Numerical approximations to $I(t,0,x,y)$ in Example 4.3 with $\beta = 10$ at $t = 10$, $(x,y) \in [0,1] \times [0,1]$ (left) and $\sum_{i,j} I(t,a,x_i,y_j)$ for $(t,a) \in [0,10] \times [0,1]$ (right).

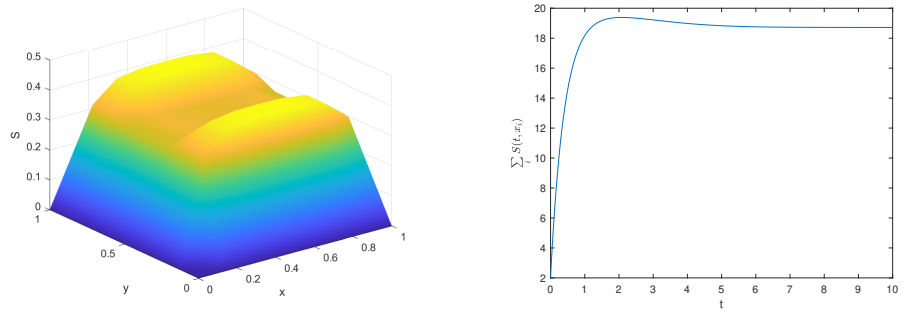


FIGURE 4.10. Numerical approximations to $S(t,x,y)$ in Example 4.3 with $\beta = 10$ at $t = 10$, $(x,y) \in [0,1] \times [0,1]$ (left) and $\sum_{i} S(t,x_i)$ for $t \in [0,10]$ (right).

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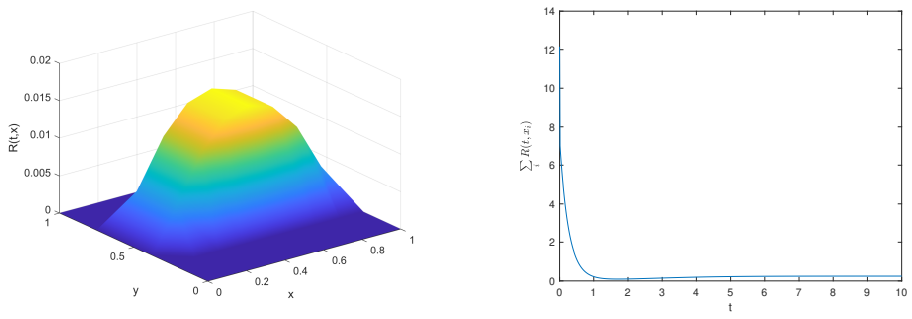


FIGURE 4.11. Numerical approximations to $R(t,x,y)$ in Example 4.3 with $\beta = 10$ at $t = 10$, $(x,y) \in [0,1] \times [0,1]$ (left) and $\sum_{i,j} R(t,x_i,y_j)$ for $t \in [0,10]$ (right).

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