FINITE DIFFERENCE METHODS FOR THE WIDE-ANGLE ‘PARABOLIC’ EQUATION

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Abstract. We consider a model initial and boundary value problem for the wide-angle ‘parabolic’ equation $Lu_r = icu$ of underwater acoustics, where L is a second-order differential operator in the depth variable z with depth- and range-dependent coefficients. We discretize the problem by the Crank–Nicolson finite difference scheme and also by the forward Euler method using nonuniform partitions both in depth and in range. Assuming that the problem admits a smooth solution, and L is invertible for all r under the posed boundary and interface conditions, we show stability of both schemes and derive error estimates.

Dedicated to the memory of Prof. Dr. Günther Hämmerlin

1. Introduction

In this paper we shall analyze finite difference methods for a model initial and boundary value problem with interface for a third-order partial differential equation, the *wide-angle* ‘parabolic’ equation of underwater acoustics. Given $R > 0, \mu \geq 0, \rho > 0, \alpha, \lambda$ and q real constants, $\alpha q \neq 0$, and $z^\star \in (0, 1)$, we seek a complex-valued function u defined on $[0, 1] \times [0, R]$ and satisfying

$$
\begin{cases}
[1 + qb(z, r)]u_r + \alpha q u_{zzr} = -i\dfrac{\lambda}{q}u, & z \in (0, z^\star) \cup (z^\star, 1), r \in [0, R],\\
u(0, \cdot) = 0, & \text{in } [0, R],\\
u(z^\star-, \cdot) = u(z^\star+, \cdot), & \text{in } [0, R],\\
u_z(z^\star-, \cdot) = \rho u_z(z^\star+, \cdot), & \text{in } [0, R],\\
u_z(1, \cdot) + \mu u(1, \cdot) = 0, & \text{in } [0, R],\\
u(\cdot, 0) = u_0 & \text{in } [0, 1];
\end{cases}
\tag{1.1}
$$

here b is a complex-valued function, $b = \beta + i\gamma$ with β and γ real-valued functions on $[0, z^\star) \times [0, R]$ and $(z^\star, 1] \times [0, R]$, which can be smoothly extended to $[0, z^\star] \times [0, R]$ and $[z^\star, 1] \times [0, R]$ but have a possible jump discontinuity across $\{z^\star\} \times [0, R]$, and u_0

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a given complex-valued function on [0,1]. Let $\widetilde{L}$ denote the Lipschitz constant of b with respect to the second variable,

$$\sup_z |b(z,r) - b(z,s)| \le \widetilde{L}|r - s| \qquad \forall r, s \in [0, R]. \tag{1.2}$$

As a matter of fact, the third-order wide-angle equation is

$$(1 + qb)v_r + \alpha q v_{zzr} = i\alpha\lambda v_{zz} + i\lambda b v,$$

but the change of variables $u = v\exp(-i\frac{\lambda}{q}r)$ transforms it into the partial differential equation (PDE) of (1.1), cf. [1].

The existence of solutions of (1.1) for all smooth initial values u_0 is called into question if the second-order operator $L(r), L(r)v := \alpha q v_{zz} + [1 + qb(\cdot, r)]v$, is not invertible under the indicated boundary and interface conditions for all $r \in [0, R]$; we refer the reader to [5] and [1] for relevant commentary. In the sequel we will assume that $L(r)$ is invertible for all $r \in [0, R]$, and that the data are smooth and compatible such that problem (1.1) possesses a solution u which is sufficiently regular for all our results to hold.

We will approximate the solution of (1.1) by a finite difference scheme of Crank–Nicolson type of second order accuracy in the depth and range variables. For the discretization in depth, let $J \in \mathbb{N}$ and $0 = z_0 < z_1 < \cdots < z_J = 1$ be an arbitrary partition of [0,1] such that $z^\star$ is a node, $z_m = z^\star$ say. Let $h_j := z_j - z_{j-1}, j = 1, \dots, J, h_{J+1} := 0, \widehat{h}_j := (h_j + h_{j+1})/2$ for $j \ne m$, and $\widehat{h}_m := (h_m + \rho h_{m+1})/2$. Further, let $H := (h_1, \dots, h_J)$, and $\mathbb{C}_0^{J+1}$ denote the space of complex $J + 1$–vectors $v = (v_0, \dots, v_J)^T$ with $v_0 = 0$. We introduce an operator $\varDelta_H$ in $\mathbb{C}_0^{J+1}$ by $(\varDelta_H v)_j = \varDelta_H v_j$ and

$$\begin{aligned}
\varDelta_H v_0 &:= 0,\\
\varDelta_H v_j &:= \frac{1}{\widehat{h}_j}\Big(\frac{v_{j+1} - v_j}{h_{j+1}} - \frac{v_j - v_{j-1}}{h_j}\Big), \quad 1 \le j \le J-1, \quad j \ne m,\\
\varDelta_H v_m &:= \frac{1}{\widehat{h}_m}\Big(\rho\frac{v_{m+1} - v_m}{h_{m+1}} - \frac{v_m - v_{m-1}}{h_m}\Big),\\
\varDelta_H v_J &:= \frac{2}{h_J^2}(v_{J-1} - v_J).
\end{aligned}$$

Thus, $\varDelta_H v_j$ is the usual centered difference quotient approximation to the second derivative at the interior points $z_j, j \ne m$, and is suitably defined at $j = m$ and $j = J$ in anticipation of the approximation of the interface conditions at $z^\star$ and the bottom mixed boundary condition.

For the discretization in range, let $N \in \mathbb{N}$ and $0 = r^0 < r^1 < \cdots < r^N = R$ be an arbitrary partition of $[0, R]$, and $r^{n+\frac12} := (r^n + r^{n+1})/2$. Let $k_n := r^{n+1} - r^n$, $n = 0, \dots, N-1$. For $v^0, \dots, v^N \in \mathbb{C}_0^{J+1}$ define $\partial v^n := (v^{n+1} - v^n)/k_n$ and $v^{n+\frac12} := (v^{n+1} + v^n)/2$.

We associate with a complex-valued function f on $[0, 1]$, the right- and left-hand-side limits of which exist at $z^\star = z_m$, a vector $\widehat{f} \in \mathbb{C}_0^{J+1}$ given by $\widehat{f}_j := f(z_j)$,

$1 \le j \le J, j \ne m$, and $\widehat{f}_m := \widehat{f}(z_m) := [h_m f(z^\star-) + \rho h_{m+1} f(z^\star+)]/2\widehat{h}_m$; $\widehat{b}(z_m, r)$ and the vectors $\widehat{b}(r) \in \mathbb{C}_0^{J+1}$, $r \in [0, R]$, are defined analogously. Clearly, if f is continuous at $z^\star$, $\widehat{f}_m = \widehat{f}(z_m) = f(z^\star)$.

We define finite difference approximations $U^n \in \mathbb{C}_0^{J+1}$ to u^n, $u^n := (u(z_0, r^n), \dots, u(z_J, r^n))^T$, as follows: For $n = 0$, let $U^0 := u^0$. Then, for $n = 0, \dots, N-1$, we require for $j = 1, \dots, J-1, j \ne m$,

$$[1 + qb(z_j, r^{n+\frac{1}{2}})]\partial U_j^n + \alpha q \partial \varDelta_H U_j^n = -i\frac{\lambda}{q} U_j^{n+\frac{1}{2}}. \tag{1.3}$$

Discretizing the interface condition of (1.1) in the customary way leads, for $n = 0, \cdots, N-1$, to

$$[1 + q\widehat{b}(z_m, r^{n+\frac{1}{2}})]\partial U_m^n + \alpha q \partial \varDelta_H U_m^n = -i\frac{\lambda}{q} U_m^{n+\frac{1}{2}}. \tag{1.4}$$

Finally, discretizing the mixed boundary condition at $z = 1$ by centered differences in the customary way, we complete the definition of the difference approximations letting, for $n = 0, \dots, N-1$,

$$[1 + qb(z_J, r^{n+\frac{1}{2}})]\partial U_J^n + \alpha q \partial \varDelta_H U_J^n - 2\alpha q \frac{\mu}{h_J} \partial U_J^n = -i\frac{\lambda}{q} U_J^{n+\frac{1}{2}}. \tag{1.5}$$

Let $\delta := (0, \dots, 0, 1)^T \in \mathbb{C}_0^{J+1}$, and for $v, w \in \mathbb{C}_0^{J+1}$ set $v \otimes w := (v_0 w_0, \dots, v_J w_J)^T$. With this notation in place, we may rewrite (1.3)–(1.5) in the form

$$\partial U^n + q\widehat{b}(r^{n+\frac{1}{2}}) \otimes \partial U^n - 2\alpha q \frac{\mu}{h_J} \delta \otimes \partial U^n + \alpha q \partial \varDelta_H U^n = -i\frac{\lambda}{q} U^{n+\frac{1}{2}}. \tag{1.6}$$

Let $h := \max_j h_j$ and $k := \max_n k_n$. In this paper we establish second-order estimates in various norms for the error $u^n - U^n$, for sufficiently small k and h, under the natural condition that $L(r)$ be invertible for all $r \in [0, R]$. Similar results are proved in [1] under some conditions on the coefficients of the PDE in (1.1). More precisely, for the estimates in [1], $\lambda\gamma \ge 0$ in $[0, 1] \times [0, R]$ is required; the estimates in the discrete H_0^1 and maximum norms are proved under the additional hypothesis that $\gamma = 0$ or $\alpha q > 0$. Also, the technique in [1] is restricted to uniform partitions in range.

The forward Euler approximations $U^n \in \mathbb{C}_0^{J+1}$ to u^n are defined by $U^0 := u^0$ and, for $n = 0, \dots, N-1$,

$$\partial U^n + q\widehat{b}(r^n) \otimes \partial U^n - 2\alpha q \frac{\mu}{h_J} \delta \otimes \partial U^n + \alpha q \partial \varDelta_H U^n = -i\frac{\lambda}{q} U^n. \tag{1.7}$$

For sufficiently small h, we show that the scheme is stable under no meshconditions and derive error estimates in various norms of second order in h and of first order in k, assuming that $L(r)$ be invertible for all $r \in [0, R]$. Analogous results, under some conditions on the coefficients of the PDE —see (4.9) and (4.10) below— are given in [1].

For the physical significance of problem (1.1), and numerical methods for it, we refer the reader to [1], [7], [8], and the references in these papers.

The paper is organized as follows: In Section 2 we investigate a finite difference scheme for an indefinite two-point boundary value problem; the established stability estimate is the heart of the approach of this note. Sections 3 and 4 are devoted to the analysis of the Crank–Nicolson and the forward Euler finite difference schemes for (1.1), respectively.

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2. An indefinite two-point boundary value problem

In this section we study a finite difference scheme for an indefinite two-point boundary value problem; we will use the fact that the finite difference scheme is closely related to a finite element method with numerical integration. The analysis is based on ideas from [3]. The results of this section will play a central role in the analysis of the Crank–Nicolson and the forward Euler method in the next two sections; they may also be of independent interest.

Finite difference methods for indefinite problems with real-valued coefficients are analyzed in [2]. Bramble's approach makes essential use of the fact that the discrete problem reduces to a linear system of equations with normal coefficient matrix; consequently, it can not be easily extended to equations with variable complex-valued coefficients.

Finite element methods for indefinite problems are investigated in [6] and [7]. The fact that the convergence in the L^2-norm is faster than in the H^1-norm plays a crusial role in the analysis of finite element methods for indefinite problems. It is, therefore, not straightforward to directly apply this technique to finite difference methods, since in this case we have second-order convergence both in the discrete L^2- and H^1-norm.

The continuous problem. We consider the following two-point boundary value problem with parameter $r, r \in [0, R]$,

$$(2.1) \qquad \begin{cases} -u_{zz}(z,r) + d(z,r)u(z,r) = f(z), & z \in [0, z^\star) \cup (z^\star, 1], \\ u(0,\cdot) = 0, & \text{in } [0, R], \\ u(z^\star-,\cdot) = u(z^\star+,\cdot), & \text{in } [0, R], \\ u_z(z^\star-,\cdot) = \rho u_z(z^\star+,\cdot), & \text{in } [0, R], \\ u_z(1,\cdot) + \mu u(1,\cdot) = 0, & \text{in } [0, R]; \end{cases}$$

here $\rho, \mu, z^\star$ and R are as in the introduction, and $f : [0, z^\star) \cup (z^\star, 1] \to \mathbb{C}$, $d : ([0, z^\star) \cup (z^\star, 1]) \times [0, R] \to \mathbb{C}$ smooth functions which can be continuously extended to $[0, z^\star], [z^\star, 1]$, and $[0, z^\star] \times [0, R], [z^\star, 1] \times [0, R]$, respectively. Let d be Lipschitz continuous with respect to r uniformly in z,

$$(2.2) \qquad \sup_z |d(z,r) - d(z,s)| \le \widetilde{L}|r - s| \qquad \forall r, s \in [0, R].$$

We assume that, for every $r \in [0, R]$, problem (2.1) possesses a unique solution.

Let us also consider the following auxiliary two-point boundary value problem

$$
\begin{cases}
-v'' = f & \text{in } (0,z^\star)\cup(z^\star,1),\\
v(0)=0,\\
v(z^\star-)=v(z^\star+),\\
v'(z^\star-)=\rho v'(z^\star+),\\
v'(1)+\mu v(1)=0.
\end{cases}
\tag{2.3}
$$

We equip $L^2=L^2(0,1)$ with the natural for problems (2.1) and (2.3) weighted inner product $(\cdot,\cdot)$,

$$(v,w) := \int_0^{z^\star} v(z)\bar{w}(z)dz + \rho\int_{z^\star}^1 v(z)\bar{w}(z)dz,$$

and denote by $\|\cdot\|$ the induced norm. Let H_0^1 consist of the elements of the Sobolev space H^1 which vanish at 0; we will use the norms $\|\cdot\|_1, |\cdot|_1, \|w\|_1 := (\|w\|^2+\|w'\|^2)^{1/2}, |w|_1 := \|w'\|$.

A variational formulation of problem (2.3) is: given $f\in L^2$, seek $v\in H_0^1$ satisfying

$$(v',w')+\rho\mu v(1)\bar{w}(1)=(f,w) \qquad \forall w\in H_0^1. \tag{2.3$'$}$$

Letting T denote the solution operator of (2.3′), $v=Tf$, we rewrite problem (2.1) in the form: Seek $u(\cdot,r)\in H_0^1$ such that

$$u(\cdot,r)+T(d(\cdot,r)u(\cdot,r))=Tf, \quad r\in[0,R]. \tag{2.4}$$

Using the continuity of $T: L^2\to H_0^1$, we easily see that $T(d(r)\cdot): H_0^1\to H_0^1$ is compact, and conclude, in view of our assumption for problem (2.1), that $A(r),\ A(r):=I+T(d(r)\cdot)$, is an isomorphism from H_0^1 to H_0^1. Therefore,

$$\|A(r)^{-1}\|_1\le C(r). \tag{2.5}$$

Using (2.2), we easily see that

$$\|A(r)-A(s)\|_1\le L|r-s| \qquad \forall r,s\in[0,R].$$

Thus, we have

$$\|A(s)^{-1}-A(r)^{-1}\|_1\le L|r-s|\,\|A(r)^{-1}\|_1\|A(s)^{-1}\|_1 \quad \forall r,s\in[0,R], \tag{2.6}$$

i.e., for s sufficiently close to r,

$$\|A(s)^{-1}\|_1\le\frac{\|A(r)^{-1}\|_1}{1-L|r-s|\|A(r)^{-1}\|_1}. \tag{2.7}$$

From (2.6) and (2.7) we obtain

$$|\,\|A(s)^{-1}\|_1-\|A(r)^{-1}\|_1\,|\le L\frac{\|A(r)\|_1^2}{1-L|r-s|\,\|A(r)^{-1}\|_1}|r-s|. \tag{2.8}$$

Thus, the function $\varphi,\ \varphi(r) := \|A(r)^{-1}\|_1$, is continuous; in particular,

$$\sup_r \|(I + T(d(r)\cdot))^{-1}\|_1 \le C. \tag{2.9}$$

Discretization. Let S_H denote the space of continuous functions in $[0,1]$ which vanish at 0 and reduce to polynomials of degree less or equal one on each subinterval (z_j, z_{j+1}). Using the notation $\widetilde{h}_j := \widehat{h}_j, j = 1,\dots,m, \widetilde{h}_j := \rho\widehat{h}_j, j = m+1,\dots,J$, we introduce in $\mathbb{C}_0^{J+1}$ a discrete weighted L^2 inner product $(\cdot,\cdot)_H$ by

$$(v,w)_H := \sum_{j=1}^{J} \widetilde{h}_j v_j \bar{w}_j,$$

and denote by $\|\cdot\|_H$ the induced norm. We shall also use the discrete weighted H_0^1-norm $|\cdot|_{1,H}$ and the discrete $H^{-1}-$norm $\|\cdot\|_{-1,H}$, defined for $w \in \mathbb{C}_0^{J+1}$ by

$$|w|_{1,H} := \Big\{\sum_{j=1}^{m} h_j \Big|\frac{w_j - w_{j-1}}{h_j}\Big|^2 + \rho \sum_{j=m+1}^{J} h_j \Big|\frac{w_j - w_{j-1}}{h_j}\Big|^2\Big\}^{1/2},$$

$$\|w\|_{-1,H} := \Big\{\sum_{j=1}^{J} h_j \Big|\sum_{\ell=1}^{j-1} \widetilde{h}_\ell w_\ell\Big|^2 + \Big|\sum_{\ell=1}^{J} \widetilde{h}_\ell w_\ell\Big|^2\Big\}^{1/2}.$$

Rewriting $(v,w)_H$ in the form

$$(v,w)_H = -\sum_{j=2}^{J} h_j \Big(\sum_{\ell=1}^{j-1} \widetilde{h}_\ell v_\ell\Big) \frac{\bar{w}_j - \bar{w}_{j-1}}{h_j} + \sum_{\ell=1}^{J} \widetilde{h}_\ell v_\ell \bar{w}_J,$$

using the Cauchy–Schwarz inequality and the fact that $|\cdot|_{1,H}$ dominates (modulo a constant factor) the discrete maximum norm, we have

$$|(v,w)_H| \le C\|v\|_{-1,H} |w|_{1,H} \quad \forall v, w \in \mathbb{C}_0^{J+1}, \tag{2.10}$$

with a constant C depending on ρ.

Using the notation $\widehat{w}$ for a vector in $\mathbb{C}_0^{J+1}$ associated with a function $w : [0,1] \to \mathbb{C}$, see section 1, we approximate the solution v of (2.3′) by $v_H \in S_H$ defined by

$$(v_H', \chi') + \rho\mu v_H(1)\bar{\chi}(1) = (\widehat{f}, \widehat{\chi})_H \qquad \forall \chi \in S_H. \tag{2.11}$$

Clearly, (2.11) is uniquely solvable. Let T_H denote its solution operator, $v_H = T_H f$. Taking $\chi := v_H$ in (2.11), and using (2.10) and the easily established relation

$$|\widehat{\chi}|_{1,H} = |\chi|_1 \qquad \forall \chi \in S_H, \tag{2.12}$$

we obtain

$$|T_H f|_1 \le C\|\widehat{f}\|_{-1,H}. \tag{2.13}$$

In the sequel, we will also use the weighted $L^1(0,1)$ norm $\|\cdot\|_{L^1}$,

$$\|f\|_{L^1} := \int_0^{z^\star} |f(s)|ds + \rho \int_{z^\star}^{1} |f(s)|ds.$$

The following result is similar to Lemma 2.4 in [3].

Lemma 2.1. *Let T and T_H be the solution operators of problems $(2.3')$ and (2.11), respectively. Then, there exists a constant C, independent of H, such that*

$$\|Tf - T_H f\|_1 \le Ch(\|f\| + \|f'\|_{L^1}). \tag{2.14}$$

Proof. Let $P_e v \in S_H$ be given by

$$((P_e v)', \chi') + \rho\mu(P_e v)(1)\bar{\chi}(1) = (f, \chi) \qquad \forall \chi \in S_H. \tag{2.15}$$

Then,

$$((v - P_e v)', \chi') + \rho\mu(v - P_e v)(1)\bar{\chi}(1) = 0 \qquad \forall \chi \in S_H,$$

and, consequently, for $\chi \in S_H$,

$$\begin{aligned}|v - P_e v|_1^2 + \rho\mu|(v - P_e v)(1)|^2 = \\ &= ((v - P_e v)', (v-\chi)') + \rho\mu(v - P_e v)(1)(\bar{v} - \bar{\chi})(1) \\ &\le |v - P_e v|_1 |v-\chi|_1 + \rho\mu|(v - P_e v)(1)|\,|(v-\chi)(1)|.\end{aligned}$$

Choosing here $\chi \in S_H$ to be the interpolant of v, we obtain

$$|v - P_e v|_1^2 + \rho\mu|(v - P_e v)(1)|^2 \le |v - P_e v|_1 ch(|v|_{H^2(0,z^\star)} + |v|_{H^2(z^\star,1)}),$$

i.e.,

$$|v - P_e v|_1 \le Ch\|f\|. \tag{2.16}$$

Further, subtracting (2.11) from (2.15), we get

$$((P_e v - v_H)', \chi') + \rho\mu(P_e v - v_H)(1)\bar{\chi}(1) = (f, \chi) - (\widehat{f}, \widehat{\chi})_H \qquad \forall \chi \in S_H. \tag{2.17}$$

Now

$$(f, \chi) - (\widehat{f}, \widehat{\chi})_H = \sum_{j=0}^{m-1} E_j(f\bar{\chi}) + \rho \sum_{j=m}^{J-1} E_j(f\bar{\chi}),$$

where

$$E_j(\varphi) := \int_{z_j}^{z_{j+1}} \varphi(s)ds - \frac{h_{j+1}}{2}[\varphi(z_j) + \varphi(z_{j+1})];$$

here $\varphi(z_m)$ stands for $\varphi(z^*-)$ in $E_{m-1}(\varphi)$, and for $\varphi(z^*+)$ in $E_m(\varphi)$. Using the fact that the trapezoid rule integrates the elements of S_H exactly, we have $E_j(f\bar{\chi}) = E_j([f - f(z_j)]\bar{\chi})$, i.e.,

$$\begin{aligned}|E_j(f\bar{\chi})| &\le \frac{3}{2}h_{j+1} \max_{z_j \le s \le z_{j+1}} |f(s) - f(z_j)| \max_{z_j \le s \le z_{j+1}} |\chi(s)| \\ &\le \frac{3}{2}h_{j+1} \int_{z_j}^{z_{j+1}} |f'(s)|ds\, \|\chi\|_{L^\infty(0,1)}.\end{aligned}$$

Using also the easily established fact

$$\|\chi\|_{L^\infty(0,1)} \le \max(1, \frac{1}{\sqrt{\rho}})|\chi|_1 \qquad \forall \chi \in S_H, \tag{2.18}$$

we obtain

$$|(f,\chi) - (\widehat{f}, \widehat{\chi})_H| \le \frac{3}{2} h \max(1, \frac{1}{\sqrt{\rho}})\|f'\|_{L^1}\, |\chi|_1. \tag{2.19}$$

Choosing in (2.17) $\chi := P_e v - v_H$, and using (2.19), we obtain

$$|P_e v - v_H|_1 \le Ch\|f'\|_{L^1}. \tag{2.20}$$

From (2.16) and (2.20), we get

$$|v - v_H|_1 \le Ch(\|f\| + \|f'\|_{L^1}), \tag{2.21}$$

and (2.14) follows. □

We approximate the solution u of problem (2.1) by $u_H(\cdot, r) \in S_H, r \in [0,R]$, given by

$$(u_{Hz}(\cdot,r), \chi') + \rho\mu u_H(1,r)\bar{\chi}(1) + (\widehat{du_H}(r), \widehat{\chi})_H = (\widehat{f}, \widehat{\chi})_H \quad \forall \chi \in S_H. \tag{2.22}$$

Problem (2.22) can be equivalently written in the form

$$u_H(\cdot,r) + T_H(d(r)u_H(\cdot,r)) = T_H f. \tag{2.22$'$}$$

Now, $[I + T(d(r)\cdot)] - [I + T_H(d(r)\cdot)] = (T - T_H)(d(r)\cdot)$, and therefore, in view of (2.14),

$$\sup_r \|[I + T(d(r)\cdot)] - [I + T_H(d(r)\cdot)]\|_1 \le Ch. \tag{2.23}$$

From (2.9) and (2.23) we conclude that there exists $h_0 > 0$ such that, for $h \le h_0$, $I + T_H(d(r)\cdot)$ is invertible and

$$\sup_r \|[I + T_H(d(r)\cdot)]^{-1}\|_1 \le C. \tag{2.24}$$

Therefore, in particular, $u_H(\cdot, r)$ is well defined for sufficiently small h. Now, $(2.22')$ can be written in the form

$$u_H(\cdot,r) = [I + T_H(d(r)\cdot)]^{-1} T_H f,$$

and, in view of (2.24), we have

$$\sup_r \|u_H(\cdot,r)\|_1 \le C\|T_H f\|_1, \tag{2.25}$$

i.e.,

$$\sup_r |u_H(\cdot,r)|_1 \le C|T_H f|_1. \tag{2.25$'$}$$

From (2.25′) and (2.13) we obtain the stability estimate

$$\sup_r |u_H(\cdot,r)|_1 \le C||\widehat{f}||_{-1,H}. \tag{2.26}$$

Let $U(r) := \widehat{u}_H(\cdot, r)$. Choosing $\chi = \varphi_j, j = 1, \ldots, J, \varphi_j \in S_H, \varphi_j(z_\ell) = \delta_{j\ell}$, in (2.22) we easily see that

$$\widehat{d}(r) \otimes U(r) + \frac{2\mu}{h_J}\delta \otimes U(r) - \Delta_H U(r) = \widehat{f}, \qquad r \in [0, R]. \tag{2.27}$$

From (2.12) and (2.26) we obtain the stability estimate

$$\sup_r |U(r)|_{1,H} \leq C\|\widehat{f}\|_{-1,H}. \tag{2.28}$$

In the next two sections, we will apply these results with $d(z, r) := -[1 + qb(z, r)]/\alpha q$. For this d, the finite difference scheme (2.27) and the stability estimate (2.28) take the form

$$U(r) + q\widehat{b}(r) \otimes U(r) - 2\alpha q \frac{\mu}{h_J}\delta \otimes U(r) + \alpha q \Delta_H U(r) = -\alpha q \widehat{f}, \quad r \in [0, R], \tag{2.29a}$$

$$\sup_r |U(r)|_{1,H} \leq C\|\widehat{f}\|_{-1,H}. \tag{2.29b}$$

Remark 2.1. Second-order error estimates for the finite difference scheme (2.27) for the indefinite problem (2.1) can be easily established using the stability estimate (2.28). Let $E(r) \in \mathbb{C}_0^{J+1}$ be the consistency error of the finite difference scheme (2.27) for the solution $u(\cdot, r)$ of problem (2.1),

$$E(r) := \widehat{d}(r) \otimes \widehat{u}(r) + \frac{2\mu}{h_J}\delta \otimes \widehat{u}(r) - \Delta_H \widehat{u}(r) - \widehat{f}, \qquad r \in [0, R]. \tag{2.30}$$

By straightforward Taylor expansions we see that $E(r) = E_1(r) + E_2(r)$ with

$$E_{1j}(r) = \begin{cases} -\dfrac{1}{3}(h_{j+1} - h_j)u_{zzz}(z_j, r) & \text{if } 1 \leq j \leq J-1, \ \ j \neq m, \\ -\dfrac{1}{6\widehat{h}_m}[\rho h_{m+1}^2 u_{zzz}(z^\star+, r) - h_m^2 u_{zzz}(z^\star-, r)] & \text{if } j = m, \\ \dfrac{h_J}{3}u_{zzz}(1, r) & \text{if } j = J, \end{cases}$$

and

$$E_{2j}(r) = \begin{cases} -\dfrac{1}{6\widehat{h}_j}[\dfrac{1}{h_{j+1}}\mathcal{R}_j(u) - \dfrac{1}{h_j}\mathcal{L}_j(u)] & \text{if } 1 \leq j \leq J-1, \ \ j \neq m, \\ -\dfrac{1}{6\widehat{h}_m}[\dfrac{\rho}{h_{m+1}}\mathcal{R}_m(u) - \dfrac{1}{h_m}\mathcal{L}_m(u)] & \text{if } j = m, \\ \dfrac{1}{3h_J^2}\mathcal{L}_J(u) & \text{if } j = J, \end{cases}$$

with

$$\mathcal{R}_j(u) := \int_{z_j}^{z_{j+1}} (z_{j+1} - z)^3 u_{zzzz}(z, r)dz, \quad \mathcal{L}_j(u) := \int_{z_{j-1}}^{z_j} (z_{j-1} - z)^3 u_{zzzz}(z, r)dz.$$

It is straightforward to prove that

$$\sup_r(\|E_1(r)\|_{-1,H} + \|E_2(r)\|_H) \le Ch^2. \tag{2.31}$$

Let $e(r) := \widehat{u}(r) - U(r)$. Subtracting (2.27) from (2.30) we obtain

$$\widehat{d}(r) \otimes e(r) + \frac{2\mu}{h_J}\delta \otimes e(r) - \Delta_H e(r) = E(r), \qquad r \in [0, R],$$

i.e., in view of (2.28),

$$\sup_r |e(r)|_{1,H} \le C \sup_r \|E(r)\|_{-1,H}.$$

Thus, using (2.31) we obtain

$$\sup_r |e(r)|_{1,H} \le Ch^2. \tag{2.32}$$

This estimate implies also second-order error estimates in the discrete L^2 and maximum norms. □

Remark 2.2. In the case of definite second-order two-point boundary value problems, it is well known that the standard three-point finite difference formula on nonuniform meshes leads to second-order convergent schemes. For a finite difference formula for the discretization of u_{zzz} on nonuniform meshes leading to second-order convergent finite difference schemes, when the number of grid points is odd, we refer to [4]. □

3. The Crank–Nicolson method

In this section we examine the Crank–Nicolson scheme (1.6) for problem (1.1). We show consistency and stability, and establish second-order error estimates.

3.1. **Consistency.** The consistency error $E^n \in \mathbb{C}_0^{J+1}, n = 0, \dots, N-1$, of the Crank–Nicolson scheme (1.6) for the solution u of (1.1) is given by

$$E^n := \partial u^n + q\widehat{b}(r^{n+\frac{1}{2}}) \otimes \partial u^n - 2\alpha q\frac{\mu}{h_J}\delta \otimes \partial u^n + \alpha q \partial \Delta_H u^n + i\frac{\lambda}{q}u^{n+\frac{1}{2}}. \tag{3.1}$$

We rewrite the consistency error in the form

$$E^n = \frac{1}{k}\int_{r^n}^{r^{n+1}} [\widehat{u}_r(r) + q\widehat{b}(r) \otimes \widehat{u}_r(r) - 2\alpha q\frac{\mu}{h_J}\delta \otimes \widehat{u}_r(r) + \alpha q\Delta_H \widehat{u}_r(r) + i\frac{\lambda}{q}\widehat{u}(r)]dr$$
$$- i\frac{\lambda}{q}\frac{1}{k}\int_{r^n}^{r^{n+1}} [\widehat{u}(r) - \widehat{u}(r^{n+\frac{1}{2}})]dr - q\frac{1}{k}\int_{r^n}^{r^{n+1}} [\widehat{b}(r) - \widehat{b}(r^{n+\frac{1}{2}})] \otimes \widehat{u}_r(r)dr.$$

It is easily seen that the first term on the right-hand side can be estimated as in Remark 2.1, and the last two terms are of order $O(k^2)$ in the discrete maximum norm. Consequently,

$$\max_{0 \le n \le N-1} \|E^n\|_{-1,H} \le C(k^2 + h^2). \tag{3.2}$$

3.2. **Stability.** Using (2.29), we immediately obtain from (1.6), for sufficiently small h,

$$|\partial U^n|_{1,H} \le C(\|U^n\|_{-1,H} + \|U^{n+1}\|_{-1,H}), \qquad n = 0,\dots,N-1. \tag{3.3}$$

Using now the fact that the discrete H_0^1 norm dominates the discrete L^2 norm which in turn dominates the discrete H^{-1} norm, we obtain

$$\|\partial U^n\|_H \le C(\|U^n\|_H + \|U^{n+1}\|_H), \qquad n = 0,\dots,N-1, \tag{3.4}$$

as well as

$$|\partial U^n|_{1,H} \le C(|U^n|_{1,H} + |U^{n+1}|_{1,H}), \qquad n = 0,\dots,N-1. \tag{3.5}$$

Now, from (3.4) we obtain

$$(1 - Ck_n)\|U^{n+1}\|_H \le (1 + Ck_n)\|U^n\|_H,$$

i.e., for sufficiently small k,

$$\|U^{n+1}\|_H \le (1 + ck_n)\|U^n\|_H, \quad n = 0,\dots,N-1;$$

consequently, stability in the discrete weighted L^2 norm follows,

$$\max_{0\le n\le N} \|U^n\|_H \le C\|U^0\|_H. \tag{3.6}$$

Analogously, for sufficiently small k, from (3.5) we obtain stability in the discrete weighted H_0^1 norm

$$\max_{0\le n\le N} |U^n|_{1,H} \le C|U^0|_{1,H}. \tag{3.7}$$

Stability in the discrete maximum norm follows also easily from (3.3): Estimating the left-hand side from below and the right-hand side from above by the maximum norm, we get, for sufficiently small k,

$$\max_j |U_j^{n+1}| \le (1 + ck_n) \max_j |U_j^n|, \qquad n = 0,\dots,N-1,$$

i.e.,

$$\max_{j,n} |U_j^n| \le C \max_j |U_j^0|. \tag{3.8}$$

From the above stability estimates, it follows, in particular, that, for sufficiently small k and h, the Crank–Nicolson approximations $U^1,\dots,U^N$ are well defined by (1.6).

3.3. **Convergence.** Combining stability and consistency, we next prove optimal order rate of convergence of the Crank–Nicolson approximations.

Theorem 3.1. *Assume that the solution u of (1.1) is sufficiently smooth in $[0, z^\star] \times [0, R]$ and in $[z^\star, 1] \times [0, R]$, and that k and h are sufficiently small. Let $U^0 := u^0$, and $U^1, \dots, U^N$ be the Crank–Nicolson approximations given by the finite difference scheme (1.6). Then, there exists a constant C, independent of $h_1, \dots, h_J$ and $k_1, \dots, k_N$, such that*

$$\max_{0\le n\le N} \|u^n - U^n\|_H \le C(k^2 + h^2), \tag{3.9}$$

$$\max_{0\le n\le N} |u^n - U^n|_{1,H} \le C(k^2 + h^2) \tag{3.10}$$

and

$$\max_{0\le n\le N} \max_{0\le j\le J} |u_j^n - U_j^n| \le C(k^2 + h^2). \tag{3.11}$$

Proof. Let $e^n := u^n - U^n, n = 0, \dots, N$. Subtracting (3.1) from (1.6), we obtain the error equation

$$\partial e^n + q\widehat{b}(r^{n+\frac{1}{2}}) \otimes \partial e^n - 2\alpha q \frac{\mu}{h_J} \delta \otimes \partial e^n + \alpha q \partial \Delta_H e^n = -i\frac{\lambda}{q} e^{n+\frac{1}{2}} + E^n. \tag{3.12}$$

Using (2.29), we immediately get from (3.12)

$$|\partial e^n|_{1,H} \le C(\|e^{n+\frac{1}{2}}\|_{-1,H} + \|E^n\|_{-1,H}),$$

i.e., in view of (3.2),

$$|\partial e^n|_{1,H} \le C|e^{n+\frac{1}{2}}|_H + C(k^2 + h^2), \qquad n = 0, \dots, N-1. \tag{3.13}$$

Using now the fact that the discrete H_0^1 norm dominates the discrete L^2 norm, we obtain

$$|e^{n+1}|_{1,H} \le (1 + ck_n)|e^n|_{1,H} + ck_n(k^2 + h^2),$$

and conclude easily that (3.10) holds.

The estimates (3.9) and (3.11) can be established analogously; they also follow from (3.10), since the discrete H_0^1 norm dominates the discrete L^2 norm as well as the discrete maximum norm. □

4. The forward Euler method

In this section we study the forward Euler finite difference scheme (1.7) for problem (1.1).

4.1. **Consistency.** The consistency error $E^n \in \mathbb{C}_0^{J+1}$, $n = 0, \dots, N-1$, of the forward Euler scheme (1.7) for the solution u of (1.1) is given by

$$E^n := \partial u^n + q\widehat{b}(r^n) \otimes \partial u^n - 2\alpha q \frac{\mu}{h_J} \delta \otimes \partial u^n + \alpha q \partial \Delta_H u^n + i\frac{\lambda}{q} u^n. \tag{4.1}$$

As in the case of the Crank–Nicolson scheme, it is easily seen that

$$\max_{0 \le n \le N-1} \|E^n\|_{-1,H} \le C(k + h^2). \tag{4.2}$$

4.2. **Stability.** Using (2.29), we immediately obtain from (1.7), for sufficiently small h,

$$|\partial U^n|_{1,H} \le C \|U^n\|_{-1,H}, \quad n = 0, \dots, N-1. \tag{4.3}$$

Using the fact that the discrete H_0^1 norm dominates the discrete L^2 norm, and the discrete L^2 norm dominates the discrete H^{-1} norm, we obtain

$$\|\partial U^n\|_H \le C \|U^n\|_H, \quad n = 0, \dots, N-1, \tag{4.4}$$

as well as

$$|\partial U^n|_{1,H} \le C |U^n|_{1,H}, \qquad n = 0, \dots, N-1. \tag{4.5}$$

Now, from (4.4) we immediately obtain

$$\|U^{n+1}\|_H \le (1 + ck_n) \|U^n\|_H,$$

and conclude easily that

$$\max_{0 \le n \le N} \|U^n\|_H \le C \|U^0\|_H. \tag{4.6}$$

Analogously, from (4.5) we obtain stability in the discrete weighted H_0^1 norm,

$$\max_{0 \le n \le N} |U^n|_{1,H} \le C |U^0|_{1,H}. \tag{4.7}$$

Moreover, using the fact that the discrete maximum norm dominates the discrete H^{-1} norm and is dominated by the discrete H_0^1 norm, we get from (4.3)

$$\max_j |U_j^{n+1}| \le (1 + ck_n) \max_j |U_j^n|, \qquad n = 0, \dots, N-1,$$

and, consequently,

$$\max_{j,n} |U_j^n| \le C \max_j |U_j^0|. \tag{4.8}$$

The above stability estimates imply, in particular, that, for sufficiently small h, the forward Euler approximations $U^1, \dots, U^N$ are well defined by (1.7).

The stability estimate (4.6) was first derived in [1] for $\alpha > 0$ under the condition that either

$$\gamma \text{ is bounded away from zero} \tag{4.9}$$

or

$$\begin{aligned} &\frac{\alpha q}{C_\rho} > 1 + q \max_{z,r} \beta(z,r), \ \text{ if } q > 0 \\ &\frac{\alpha q}{C_\rho} > -1 + |q| \max_{z,r} \beta(z,r), \ \text{ if } q < 0, \end{aligned} \tag{4.10}$$

where C_ρ is such that $\|\varphi\|^2 \le C_\rho \|\varphi'\|^2$ for all smooth functions vanishing at 0.

Combining stability and consistency, we obtain optimal order rate of convergence of the Euler approximations. The proof goes along the same lines as the proof of Theorem 3.1, and is omitted.

Theorem 4.1. *Assume that the solution u of (1.1) is sufficiently smooth in $[0, z^\star] \times [0, R]$ and in $[z^\star, 1] \times [0, R]$, and that h is sufficiently small. Let $U^0 := u^0$, and $U^1, \dots, U^N$ be the forward Euler approximations given by the finite difference scheme (1.7). Then, there exists a constant C, independent of $h_1, \dots, h_J$ and $k_1, \dots, k_N$, such that*

$$\max_{0 \le n \le N} \|u^n - U^n\|_H \le C(k + h^2) \tag{4.11}$$

$$\max_{0 \le n \le N} |u^n - U^n|_{1,H} \le C(k + h^2) \tag{4.12}$$

and

$$\max_{0 \le n \le N} \max_{0 \le j \le J} |u^n_j - U^n_j| \le C(k + h^2). \quad \square \tag{4.13}$$

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