STABILITY PROPERTIES OF IMPLICIT–EXPLICIT MULTISTEP METHODS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS

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Abstract. We consider the discretization of a special class of nonlinear parabolic equations, including the complex Ginzburg–Landau equation, by implicit–explicit multistep methods and establish stability under a best possible linear stability condition.

1. Introduction

Let $T > 0, u^0 \in H$, and consider the initial value problem of seeking a function $u : [0,T] \to \mathscr{D}(A)$ satisfying

$$\begin{cases} u'(t) + A(t)u(t) = B(t,u(t)), & 0 < t < T, \\ u(0) = u^0, \end{cases} \tag{1.1}$$

with

$$A(t) = A + \mathrm{i}a(t)A, \tag{1.2}$$

$A : \mathscr{D}(A) \to H$ a time-independent, positive definite, self-adjoint linear operator on a Hilbert space $(H, (\cdot,\cdot))$, with domain $\mathscr{D}(A)$ dense in H, i the imaginary unit, $a(t) : [0,T] \to \mathbb{R}$ a continuous real-valued function, and $B(t,\cdot) : \mathscr{D}(A) \to H, t \in [0,T]$, (possibly) nonlinear operators. We assume that (1.1) possesses a smooth solution.

An example of a parabolic equation with linear operator of the form (1.2) is the complex Ginzburg–Landau equation, given here in its simplest form in one space dimension,

$$u_t = (1 + \mathrm{i}\tilde{a})u_{xx} + (1 + \mathrm{i}c)u - (1 + \mathrm{i}d)|u|^2 u, \tag{1.3}$$

with $\tilde{a}, c$ and d real numbers; see, e.g., [8, 9, 13, 18]. The complex Ginzburg–Landau equation is encountered in several diverse branches of physics, for example in superconductivity and superfluidity, non-equilibrium fluid dynamics and chemical systems, nonlinear optics, Bose–Einstein condensates and Rayleigh–Bénard convection. Variants of (1.3), also of the form of the differential equation in (1.1), like the cubic-quintic

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complex Ginzburg–Landau equation, appear in applications as well, for instance, in nonlinear optics; see, e.g., [8, 10] and references therein. Furthermore, the generalized cubic quintic complex Swift–Hohenberg equation, given here for simplicity again in one space dimension,

$$u_t + (1 + \mathrm{i}\tilde{a})u_{xxxx} = \delta u + \beta u_{xx} + \mu|u|^2u + \nu|u|^4u, \tag{1.4}$$

with $\tilde{a}$ and δ real numbers, and β, μ and ν complex numbers, see [16], belongs also to the class of parabolic equations considered in this paper.

Let (α, β) be a strongly $A(0)-$stable $q-$step scheme and (α, γ) be an explicit $q-$step scheme, characterized by three polynomials α, β and γ,

$$\alpha(\zeta) = \sum_{i=0}^{q} \alpha_i \zeta^i, \quad \beta(\zeta) = \sum_{i=0}^{q} \beta_i \zeta^i, \quad \gamma(\zeta) = \sum_{i=0}^{q-1} \gamma_i \zeta^i.$$

Let $N \in \mathbb{N}$, $k := T/N$ be the constant time step, and $t^n := nk, n = 0, \dots, N$, be a uniform partition of the interval $[0, T]$. Since we consider $q-$step schemes, we assume that starting approximations $U^0, \dots, U^{q-1}$ are given. We consider the discretization of the initial value problem (1.1) by the implicit–explicit $(\alpha, \beta, \gamma)-$scheme: More precisely, we use the implicit scheme (α, β) for the discretization of the linear part and the explicit scheme (α, γ) for the discretization of the nonlinear part of the equation; see [3, 4, 1]. We thus recursively define a sequence of approximations U^m to the nodal values $u^m := u(t^m)$ of the solution u of (1.1) by

$$\sum_{i=0}^{q} \big(\alpha_i I + k\beta_i A(t^{n+i})\big) U^{n+i} = k \sum_{i=0}^{q-1} \gamma_i B(t^{n+i}, U^{n+i}), \tag{1.5}$$

$n = q, \dots, N$. The unknown U^{n+q} appears in (1.5) only linearly, since $\gamma_q = 0$; therefore, to advance with (1.5) in time, we need to solve, at each time level, just one linear equation, which reduces to a linear system of equations, if we discretize also in space.

1.1. **Abstract setting.** Let $|\cdot|$ denote the norm of H, and introduce in $V, V := \mathscr{D}(A^{1/2})$, the norm $\|\cdot\|$ by $\|v\| := |A^{1/2}v|$. We identify H with its dual, and denote by V' the dual of V, and by $\|\cdot\|_\star$ the dual norm on V', $\|v\|_\star := |A^{-1/2}v|$. We use the notation $(\cdot, \cdot)$ also for the duality pairing between V' and V; then $\|v\| = (Av, v)^{1/2}$ and $\|v\|_\star = (v, A^{-1}v)^{1/2}$. We assume that $B(t, \cdot)$ can be extended to operators from V into V', and satisfy the local Lipschitz condition

$$\|B(t, v) - B(t, w)\|_\star \le \lambda_2\|v - w\| + \mu_2|v - w| \quad \forall v, w \in T_u, \tag{1.6}$$

in a tube $T_u, T_u := \{v \in V : \min_t \|v - u(t)\| \le 1\}$, around the solution u, uniformly in t, with the *stability constant* λ_2 and a constant μ_2; this is actually the condition needed, but for simplicity we have also assumed that $B(t, \cdot) : \mathscr{D}(A) \to H, t \in [0, T]$.

Since the implicit scheme (α, β) is $A(0)-$stable, the product $\alpha_q\beta_q$ is positive. It then follows immediately from the Lax–Milgram lemma that, given a $w \in V'$, the equation

$$\alpha_q v + k\beta_q\big[1 + \mathrm{i}a(t)\big]Av = w \tag{1.7}$$

possesses a unique solution $v \in V$, for any fixed $t \in [0,T]$. Therefore, given the starting approximations $U^0,\dots,U^{q-1} \in V$, the approximations $U^q,\dots,U^N \in V$ are well defined by the implicit–explicit scheme (1.5), under the mild condition that $|a(t)|$ is bounded by an arbitrary constant λ_1.

Let us now briefly consider the complex Ginzburg–Landau equation (1.3), posed in a bounded interval $[a,b]$ and subject to homogeneous Dirichlet boundary conditions, with initial value $u(\cdot,0)=u^0$. We consider complex-valued functions on (a,b) and, with standard notation for Sobolev spaces, let $H:=L^2:=L^2(a,b), V:=H_0^1:=H_0^1(a,b)$, and $V'=H^{-1}:=H^{-1}(a,b)$.

To write the corresponding initial and boundary value problem for the complex Ginzburg–Landau equation (1.3) in the form (1.1), we let the time-independent, positive definite, self-adjoint linear operator $A:\mathscr{D}(A)=H^2\cap H_0^1\to L^2$ be given, for instance, by $Av:=-v_{xx}$. Then, the norm in $V=H_0^1=\mathscr{D}(A^{1/2})$ is $\|\cdot\|, \|v\|:=|v_x|$, with $|\cdot|$ the L^2-norm, i.e., the standard H^1-seminorm, which is a norm in H_0^1 equivalent to the standard H^1-norm. It is easily seen that the dual norm $\|\cdot\|_\star$ in V' is given by

$$\|v\|_\star := \sup_{w\in V\setminus\{0\}} \frac{|(v,w)|}{\|w\|} = \min_{c\in\mathbb{C}} |c+\hat v| = |\hat v - \hat v_{\text{ave}}|,$$

with $\hat v$ an antiderivative of v and $\hat v_{\text{ave}}$ its mean in $[a,b]$,

$$\hat v(x) = \int_a^x v(s)\,ds + c, \quad x\in[a,b], \quad \text{and} \quad \hat v_{\text{ave}} := \frac{1}{b-a}\int_a^b \hat v(x)\,dx.$$

In other words, the $H^{-1}-$norm of a function v is equal to the L^2-norm of the antiderivative $\hat v$ of v with vanishing mean in (a,b), $\hat v_{\text{ave}}=0$.

With $B(v):=(1+\mathrm{i}c)v-(1+\mathrm{i}d)|v|^2v$, we write the complex Ginzburg–Landau equation (1.3) as

$$u_t + (1+\mathrm{i}\tilde a)Au = B(u).$$

Now, since $H_0^1(a,b)\subset C[a,b]\subset L^2(a,b)$ and $B: C[a,b]\to C[a,b]$, we obviously have $B:V\to V'$ (as well as $B:\mathscr{D}(A)\to H$). Furthermore, in view of the obvious inequality

$$\big||z_1|^2z_1 - |z_2|^2z_2\big| \le \big(|z_1|+|z_2|\big)^2|z_1-z_2| \quad \forall z_1,z_2\in\mathbb{C}$$

and the fact that the elements of the tube T_u are uniformly bounded in the maximum norm, we easily infer that

$$\|B(t,v)-B(t,w)\|_\star \le \mu_2|v-w| \quad \forall v,w\in T_u, \tag{1.8}$$

with a constant μ_2, i.e., that the local Lipschitz condition (1.6) is satisfied with $\lambda_2=0$.

1.2. **The stability result.** We first introduce two constants that will play an important role in the sequel, namely $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$, by

$$K_{(\alpha,\beta)} := \sup_{x>0}\max_{\zeta\in\mathscr{K}} \Big|\frac{x\beta(\zeta)}{(\alpha+x\beta)(\zeta)}\Big|, \quad K_{(\alpha,\beta,\gamma)} := \sup_{x>0}\max_{\zeta\in\mathscr{K}} \Big|\frac{x\gamma(\zeta)}{(\alpha+x\beta)(\zeta)}\Big|, \tag{1.9}$$

with $\mathscr{K}$ denoting the unit circle in the complex plane, $\mathscr{K} := \{z \in \mathbb{C} : |z| = 1\}$. Under our hypotheses, the constants $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ are finite; cf. [4, 1]. Actually, with ϑ the largest angle for which the scheme (α, β) is $A(\vartheta)-$stable, we have

$$K_{(\alpha,\beta)} = \frac{1}{\sin\vartheta}; \tag{1.10}$$

cf. [1]. Moreover, for some implicit–explicit multistep schemes the constants $K_{(\alpha,\beta,\gamma)}$ are explicitly given in [4] and [5]. The main result of this paper is:

Theorem 1.1 (Stability of the scheme (1.5) for (1.1) with operator $A(t)$ of the form (1.2))**.** *Let* $\lambda_1 := \max_{0\le t\le T} |a(t)|$ *and* λ_2 *be the stability constant of the local Lipschitz condition* (1.6)*. Then, under the linear stability condition*

$$(\cot\vartheta)\lambda_1 + K_{(\alpha,\beta,\gamma)}\lambda_2 < 1 \tag{1.11}$$

on λ_1 *and* λ_2*, the implicit–explicit multistep scheme* (1.5) *is locally stable for* (1.1) *with operator* $A(t)$ *of the form* (1.2) *in the following sense: If* $U^0, \dots, U^N, V^0, \dots, V^N \in T_u$ *satisfy* (1.5) *and*

$$\sum_{i=0}^{q} \big(\alpha_i I + k\beta_i A(t^{n+i})\big)V^{n+i} = k\sum_{i=0}^{q-1} \gamma_i B(t^{n+i}, V^{n+i}),$$

$n = q, \dots, N$*, respectively, then*

$$|U^n - V^n|^2 + k\sum_{\ell=0}^{n} \|U^\ell - V^\ell\|^2 \le C\sum_{j=0}^{q-1} \big(|U^j - V^j|^2 + k\|U^j - V^j\|^2\big), \tag{1.12}$$

$n = q, \dots, N$*, with a constant* C *independent of the time step* k *and the approximations* U^n, V^n*.*

Actually, (1.11) is the best possible *linear* sufficient stability condition on the constants λ_1 and λ_2 in the sense that none of the coefficients $\cot\vartheta$ and $K_{(\alpha,\beta,\gamma)}$ can be replaced by a smaller coefficient, if we want the scheme (1.5) to be stable for all equations (1.1) with linear operators $A(t)$ of the form (1.2), and $\lambda_1 := \max_{0\le t\le T} |a(t)|$, and operators $B(t, \cdot)$ satisfying the local Lipschitz condition (1.6).

Combining the stability result of Theorem 1.1 with the easily established consistency of the scheme (1.5), we are led to optimal order a priori error estimates; see, e.g., [1]. These results extend also to fully discrete schemes, if we discretize in space, for instance, by the finite element method; cf., e.g., [4].

We note that the local Lipschitz condition (1.6) is typically satisfied in the applications in tubes T_u around the solution u defined in terms of $L^\infty-$based Sobolev norms, often different for each argument, rather than in terms of L^2-based Sobolev norms. In such cases, our error analysis does not directly apply if we only consider the discretization in time, since it cannot ensure that the approximations are sufficiently close to the exact solution in the required norm; it does, however, apply, usually under mild mesh-conditions, in the fully discrete case, i.e., if we combine the time stepping schemes with discretization in space; cf., e.g., [4].

Implicit–explicit multistep methods, for linear parabolic equations, were introduced and analyzed in [11]; the analysis was extended to nonlinear parabolic equations in [3, 4, 1]. Implicit multistep schemes are studied in [14] for nonlinear stiff differential equations and in [19] for linear parabolic equations with time-dependent operators. The analysis in [14, 19, 4, 1] is based on spectral and Fourier techniques. In contrast, in [11, 3] the energy method is applied; the drawback of the specific analysis is that it does not lead to quantified sufficient stability conditions on the stability constants λ_1 or λ_2. Energy methods for high-order multistep schemes that do lead to quantified sufficient stability conditions were only recently employed for backward difference formula (BDF) schemes of order up to five, first in [15] for the implicit BDF schemes for linear parabolic equations on evolving surfaces, and subsequently in [7] and [2, 6] for the implicit–explicit methods for quasi-linear and nonlinear parabolic equations, respectively.

An outline of the paper is as follows: In Section 2, which is of preparatory nature, we recall the main stability result from [1]. In Section 3, combining the result of Section 2 with a suitable decomposition of the operators $A(t)$ in (1.2), we prove Theorem 1.1, comment on the sufficient stability condition (1.11), give a necessary stability condition, and, for the case of implicit–explicit BDF schemes of order up to 5, compare the new stability result with the one established in [7, 2, 6] by the energy method. Actually, for the three-, four-, and five-step implicit–explicit BDF schemes, the new sufficient stability condition (1.11) is milder than the best stability condition used in the energy technique approach.

2. A known stability result

We present here a stability result from [1] that will be used in the sequel to prove Theorem 1.1.

In this section we allow the operators $A(t)$ in (1.1) to be of more general form, namely $A(t) = A + A_1(t)$ with $A : \mathscr{D}(A) \to H$ a time-independent, positive definite self-adjoint linear operator as before, and $A_1(t) : \mathscr{D}(A) \to H$ linear operators. We assume that the linear operators $A_1(t) : V \to V'$ are uniformly bounded,

$$\|A_1(t)v\|_\star \le \lambda_1 \|v\| + \mu_1 |v| \quad \forall v \in V,\ \forall t \in [0, T], \tag{2.1}$$

with the *stability constant* λ_1 and a constant μ_1.

It is shown in [1] that the implicit–explicit $(\alpha, \beta, \gamma)-$scheme (1.5) is locally stable in the tube T_u for (1.1), with operators $A(t)$ as described in this section, provided the stability constants λ_1 and λ_2 in the boundedness condition (2.1) and in the local Lipschitz condition (1.6) are small enough such that

$$K_{(\alpha,\beta)}\lambda_1 + K_{(\alpha,\beta,\gamma)}\lambda_2 < 1. \tag{2.2}$$

Furthermore, (2.2) is the best possible *linear* sufficient stability condition on the constants λ_1 and λ_2 in the sense that none of the coefficients $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ can be replaced by a smaller coefficient, if we want the scheme (1.5) to be stable for all equations (1.1) satisfying (2.1) and (1.6); see [1].

In [1] a necessary stability condition on the constants λ_1 and λ_2 for the scheme (1.5) is also given, namely

$$\sup_{x>0} \max_{\zeta\in\mathscr{K}} \frac{\lambda_1|\beta(\zeta)| + \lambda_2|\gamma(\zeta)|}{|x^{-1}\alpha(\zeta) + \beta(\zeta)|} \le 1. \tag{2.3}$$

As before, (2.3) is necessary if we want the implicit–explicit $(\alpha,\beta,\gamma)-$scheme (1.5) to be locally stable for all equations satisfying assumptions (2.1) and (1.6) with the given stability constants λ_1 and λ_2. The difference in the left-hand sides of the sufficient and necessary, respectively, stability conditions (2.2) and (2.3) is that the sum of the suprema in the former is replaced by the supremum of the sum in the latter. In particular, in the case of an $A-$stable implicit scheme (α,β), the left-hand sides of the sufficient and necessary stability conditions (2.2) and (2.3) coincide; in other words, in this case the sufficient stability condition (2.2) is best possible, even compared to not necessarily linear conditions on λ_1 and λ_2.

Let us note that both constants $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta,\gamma)}$ are larger than or equal to 1. This is obvious for $K_{(\alpha,\beta)}$; see (1.10). Furthermore, if the schemes (α,β) and (α,γ) are consistent, i.e., if their orders are at least 1, then $\gamma(1)=\beta(1)$, whence

$$K_{(\alpha,\beta,\gamma)} \ge \lim_{x\to\infty} \Big|\frac{x\gamma(1)}{(\alpha+x\beta)(1)}\Big| = \lim_{x\to\infty} \Big|\frac{x\beta(1)}{(\alpha+x\beta)(1)}\Big| = 1.$$

3. Stability for our special class of parabolic equations

In contrast to Section 2, here we restrict our attention to the initial value problem (1.1) with linear operators $A(t)$ of the form (1.2). In this case the sufficient stability condition (2.2) can be relaxed to (1.11), i.e., the first term on the left-hand side of (2.2) can be multiplied by $\cos\vartheta$. Notice that in the former case there is no restriction on the "direction" of the perturbation $A_1(t)$ and the coefficient of λ_1 in (2.2) is $1/\sin\vartheta$, with $\sin\vartheta$ the ratio of the distance of a positive number a from the boundary of the stability sector $S_\vartheta, S_\vartheta := \{z\in\mathbb{C} : z = \rho e^{i\varphi}, \rho\ge 0, |\varphi|\le\vartheta\}$, over a; analogously, in the latter case, the perturbation $A_1(t) = ia(t)A$ is in the "direction" of the imaginary axis, and the coefficient of λ_1 in (1.11) is $1/\tan\vartheta$, with $\tan\vartheta$ the ratio of the distance in the direction of the imaginary axis of a positive number a from the boundary of the sector S_ϑ over a. These coefficients are best possible; the product of smaller constants with λ_1 may be less than 1 while some of the eigenvalues of the linear operator $A + A_1(t)$ may lie in the exterior of the stability sector S_ϑ, in which case the method is unstable according to the von Neumann criterion.

We will see that Theorem 1.1 follows from the results of Section 2 by using a more favourable decomposition of the operators $A(t)$. We shall also comment on the sufficient stability condition (1.11), give a necessary stability condition, and, for the case of implicit–explicit BDF schemes of order up to 5, compare the new stability result with the one established in [7, 2, 6] by the energy method.

The key point in the proof of Theorem 1.1 is the following choice of a decomposition of the operators $A(t)$,

$$A(t) = \widehat{A} + \widehat{A}_1(t), \text{ with } \widehat{A} := (1+\eta)A \text{ and } \widehat{A}_1(t) := \big(\mathrm{i}a(t) - \eta\big)A, \tag{3.1}$$

with η a nonnegative quantity that may depend on λ_1 and λ_2. We will see that a suitable choice of η is $\eta := (\tan\vartheta)\lambda_1$, with $\lambda_1 = \max_{0\le t\le T}|a(t)|$; see (2.1).

First, obviously, with $A_1(t) = \mathrm{i}a(t)A$ we have $|A^{-1/2}A_1(t)v| = |a(t)|\,|A^{1/2}v|$, whence

$$|A^{-1/2}A_1(t)v| \le \lambda_1|A^{1/2}v| \quad \forall v\in V,\ \forall t\in[0,T], \tag{3.2}$$

i.e., (2.1) is satisfied with the constant λ_1 mentioned above and $\mu_1 = 0$; furthermore, assumption (1.6) may be equivalently written in the form

$$|A^{-1/2}\big(B(t,v) - B(t,w)\big)| \le \lambda_2|A^{1/2}(v-w)| + \mu_2|v-w| \quad \forall v,w\in T_u. \tag{3.3}$$

Now, with the notation of the decomposition (3.1),

$$|\widehat{A}^{-1/2}\widehat{A}_1(t)v| = \frac{|\mathrm{i}a(t)-\eta|}{1+\eta}|\widehat{A}^{1/2}v|,$$

and it is easily seen that the operators $\widehat{A}, \widehat{A}_1(t)$ and $B(t,\cdot)$ satisfy the estimates

$$|\widehat{A}^{-1/2}\widehat{A}_1(t)v| \le \frac{|\mathrm{i}\lambda_1-\eta|}{1+\eta}|\widehat{A}^{1/2}v| \quad \forall v\in V,\ \forall t\in[0,T], \tag{3.4}$$

and

$$|\widehat{A}^{-1/2}\big(B(t,v) - B(t,w)\big)| \le \hat\lambda_2|\widehat{A}^{1/2}(v-w)| + \hat\mu_2|v-w| \quad \forall v,w\in T_u, \tag{3.5}$$

with

$$\hat\lambda_2 := \frac{\lambda_2}{1+\eta}, \quad \hat\mu_2 := \frac{\mu_2}{\sqrt{1+\eta}}. \tag{3.6}$$

Compare (3.4) with (3.2), and (3.5) with (3.3), respectively.

We infer from (2.2) and (3.4), (3.5), (3.6) that the scheme (1.5) is locally stable for (1.1) with operator $A(t)$ of the form (1.2), if λ_1 and λ_2 are such that

$$\frac{1}{\sin\vartheta}\frac{|\mathrm{i}\lambda_1-\eta|}{1+\eta} + K_{(\alpha,\beta,\gamma)}\frac{\lambda_2}{1+\eta} < 1, \tag{3.7}$$

for some nonnegative η. We will now see that the new sufficient stability condition (1.11) follows immediately from (3.7) by a suitable choice of η.

We first rewrite (3.7) in the equivalent form

$$\Big(\frac{1}{\sin\vartheta}\sqrt{\lambda_1^2+\eta^2} - \eta\Big) + K_{(\alpha,\beta,\gamma)}\lambda_2 < 1, \tag{3.8}$$

and notice that the term in parentheses attains its minimum if and only if $\eta = (\tan\vartheta)\lambda_1$. For this choice of η, we have

$$\frac{1}{\sin\vartheta}\sqrt{\lambda_1^2+\eta^2} - \eta = (\cot\vartheta)\lambda_1$$

and condition (3.8) reduces to the desired sufficient stability condition (1.11).

Let now $\varphi \in [0, \frac{\pi}{2})$ be such that $\tan\varphi = \lambda_1$. Then, the first term on the left-hand side of (1.11) is $\tan\varphi/\tan\vartheta$, and we infer that (1.11) can be satisfied for some positive λ_2 if and only if $\varphi < \vartheta$. This is a sharp condition on λ_1. Indeed, since the eigenvalues of the operator $(1 + \mathrm{i}\lambda_1)A$ are of the form $\rho \mathrm{e}^{\mathrm{i}\varphi}$, with ρ positive, in the case $\varphi > \vartheta$ the scheme (α, β) is unstable for the equation $u' + Au + \mathrm{i}\lambda_1 Au = 0$, according to the von Neumann criterion.

Remark 3.1 (Nonlinear sufficient stability conditions)**.** Conditions (3.7) and (3.8) are obviously equivalent. The left-hand sides of (3.8) and (3.7) attain their minima at $\eta = \eta(\lambda_1) := (\tan\vartheta)\lambda_1$ and at $\eta = \eta(\lambda_1, \lambda_2)$,

$$\eta(\lambda_1, \lambda_2) := \frac{K^2_{(\alpha,\beta)}\lambda_1^2 + K_{(\alpha,\beta,\gamma)}\lambda_1\lambda_2\sqrt{K^2_{(\alpha,\beta)}(1+\lambda_1^2) - K^2_{(\alpha,\beta,\gamma)}\lambda_2^2}}{K^2_{(\alpha,\beta)} - K^2_{(\alpha,\beta,\gamma)}\lambda_2^2}, \tag{3.9}$$

respectively. Substituting the latter value of the parameter η in (3.7), we obtain a nonlinear sufficient stability condition, namely

$$K_{(\alpha,\beta)}\frac{|\mathrm{i}\lambda_1 - \eta(\lambda_1, \lambda_2)|}{1 + \eta(\lambda_1, \lambda_2)} + K_{(\alpha,\beta,\gamma)}\frac{\lambda_2}{1 + \eta(\lambda_1, \lambda_2)} < 1, \tag{3.10}$$

which is obviously equivalent to the linear sufficient stability condition (1.11). In particular, choosing $\lambda_2 = 0$ in (3.9), we see that the first term on the left-hand side of (3.7) attains its minimum at $\eta(\lambda_1, 0) = \lambda_1^2$; see Figure 3.1 for the geometric interpretation. This choice of η leads to the nonlinear sufficient stability condition

$$\frac{1}{\sin\vartheta}\frac{\lambda_1}{\sqrt{1+\lambda_1^2}} + K_{(\alpha,\beta,\gamma)}\frac{\lambda_2}{1+\lambda_1^2} < 1. \tag{3.11}$$

Let $\varphi \in [0, \frac{\pi}{2})$ be such that $\tan\varphi = \lambda_1$. Then, obviously,

$$\frac{\lambda_1}{\sqrt{1+\lambda_1^2}} = \sin\varphi \quad \text{and} \quad 1 + \lambda_1^2 = \frac{1}{\cos^2\varphi},$$

whence (3.11) can be equivalently written in the form

$$\frac{1}{\sin\vartheta}\sin\varphi + K_{(\alpha,\beta,\gamma)}(\cos^2\varphi)\lambda_2 < 1.$$

This condition can be satisfied for some positive λ_2 if and only if $\varphi < \vartheta$. As already mentioned, this is a sharp condition on λ_1.

For positive λ_2, the nonlinear sufficient stability condition (3.11) is less favourable than the equivalent sufficient stability conditions (1.11) and (3.10). □

3.1. **A necessary stability condition.** In the case of an $A-$stable implicit method (α, β) the sufficient stability condition (1.11) takes the form $K_{(\alpha,\beta,\gamma)}\lambda_2 < 1$, which is sharp, even for $\lambda_1 = 0$; cf. [4]. Furthermore, in the case $\lambda_2 = 0$, the sufficient stability condition (1.11) is also sharp, as we already mentioned.

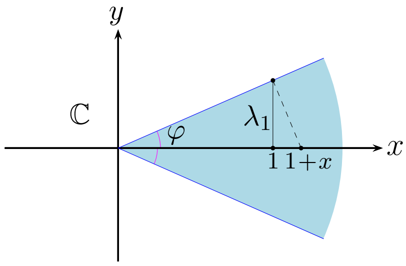

Figure 3.1. Geometric interpretation of the choice $\eta(\lambda_1, 0) = \lambda_1^2$: Let φ be such that $\tan\varphi = \lambda_1$. Obviously $\frac{|x-\mathrm{i}\lambda_1|}{1+x} = \frac{|(1+x)-(1+\mathrm{i}\lambda_1)|}{1+x} \geq \sin\varphi$ for all $x > -1$ and equality is attained if and only if the segment joining $1+x$ and $1+\mathrm{i}\lambda_1$ in the complex plane is orthogonal to the half-line passing through the origin and the point $1+\mathrm{i}\lambda_1$. Then, $1+x = |1+\mathrm{i}\lambda_1|/\cos\varphi = 1/\cos^2\varphi$, whence $x = \tan^2\varphi = \lambda_1^2$.

Next, we assume that the implicit (α, β) method is $A(\vartheta)-$stable with $\vartheta < \pi/2$, λ_2 is positive and $\lambda_1 < \tan\vartheta$. We will see that a necessary stability condition on the constants λ_1 and λ_2 for the scheme (1.5), with operators $A(t)$ of the form (1.2), is then

$$\lambda_2 \sup_{x>0} \max_{\zeta\in\mathscr{K}} \max_{-\lambda_1\leq y\leq\lambda_1} \frac{|\gamma(\zeta)|}{|x\alpha(\zeta) + (1+y\mathrm{i})\beta(\zeta)|} \leq 1. \tag{3.12}$$

To this end, we consider the function k,

$$k(x, y, \zeta) := \frac{\lambda_2 x\gamma(\zeta)}{\alpha(\zeta) + x(1+y\mathrm{i})\beta(\zeta)}, \quad x > 0,\ -\lambda_1 \leq y \leq \lambda_1,\ |\zeta| \geq 1, \tag{3.13}$$

which is holomorphic for $|\zeta| \geq 1$, and $x > 0, -\lambda_1 \leq y \leq \lambda_1$, and notice that, if (3.12) is not valid, then we have

$$\exists z \in \mathscr{K},\ x > 0,\ -\lambda_1 \leq y \leq \lambda_1 \quad |k(x, y, z)| > 1. \tag{3.14}$$

Since

$$\lim_{|\zeta|\to\infty} k(x, y, \zeta) = 0,$$

we infer that there exists a $\zeta^\star \in \mathbb{C}$ with $|\zeta^\star| > 1$ such that $|k(x, y, \zeta^\star)| = 1$, i.e.,

$$\frac{\lambda_2 x\gamma(\zeta^\star)}{\alpha(\zeta^\star) + x(1+y\mathrm{i})\beta(\zeta^\star)} = \mathrm{e}^{-\mathrm{i}\varphi},$$

for a suitable $\varphi \in [0, 2\pi)$. Therefore,

$$\alpha(\zeta^\star) + x(1+y\mathrm{i})\beta(\zeta^\star) - \lambda_2 x\mathrm{e}^{\mathrm{i}\varphi}\gamma(\zeta^\star) = 0. \tag{3.15}$$

Then, choosing the linear operator $B(t, \cdot) := \lambda_2 \mathrm{e}^{\mathrm{i}\varphi} A$, we easily see that the Lipschitz condition (1.6) is satisfied. According to the von Neumann criterion, a necessary stability condition is that, if ν is an eigenvalue of A, the solutions of

$$\sum_{i=0}^{q} \left[\alpha_i + k\nu\big((1+y\mathrm{i})\beta_i - \lambda_2\mathrm{e}^{\mathrm{i}\varphi}\gamma_i\big)\right] v^{n+i} = 0 \tag{3.16}$$

are bounded; for $k\nu = x$ this is not the case, since in view of (3.15) the root condition is not satisfied. Therefore, the scheme is not unconditionally stable for the equation $u'(t) + (1+y\mathrm{i})Au(t) = \lambda_2 \mathrm{e}^{\mathrm{i}\varphi} Au(t)$. (In (3.16) we used the notation $\gamma_q = 0$.)

We will next slightly simplify the necessary stability condition (3.12). Let $d(\zeta) := \alpha(\zeta)/\beta(\zeta) = -\rho \mathrm{e}^{\mathrm{i}\varphi(\zeta)}$, for ζ in the unit circle $\mathscr{K}$ such that $\beta(\zeta) \neq 0$, represent the points of the *root locus curve* of the scheme (α, β), with $\rho \geq 0$ and $-\pi \leq \varphi(\zeta) < \pi$. Since the coefficients of α and β are real, we have $d(\bar{\zeta}) = \overline{d(\zeta)}$, i.e., the root locus curve is symmetric with respect to the real axis. For all points $d(\zeta)$ of the root locus curve of an $A(\vartheta)-$stable scheme, with ϑ as large as possible, there holds

$$|\operatorname{Im} d(\zeta)| + \tan\vartheta \operatorname{Re} d(\zeta) \geq 0, \tag{3.17}$$

i.e., the root locus curve is located outside the sector $-S_\vartheta$. Furthermore,

$$\sin\vartheta = \inf\Big\{\frac{|\operatorname{Im} d(\zeta)|}{|d(\zeta)|}: \ \zeta \in \mathscr{K}, \ \operatorname{Re} d(\zeta) < 0\Big\}; \tag{3.18}$$

see, e.g., [1].

Now,

$$x\alpha(\zeta) + (1+y\mathrm{i})\beta(\zeta) = \beta(\zeta)\big[1 + x\operatorname{Re} d(\zeta) + \mathrm{i}[y + x\operatorname{Im} d(\zeta)]\big],$$

whence

$$|x\alpha(\zeta) + (1+y\mathrm{i})\beta(\zeta)| = |\beta(\zeta)|\big[1 + x^2|d(\zeta)|^2 + 2x\big(\operatorname{Re} d(\zeta) + y\operatorname{Im} d(\zeta)\big)\big]^{1/2}.$$

Thus,

$$\begin{aligned} &\max_{-\lambda_1\leq y\leq\lambda_1} \frac{1}{|x\alpha(\zeta) + (1+y\mathrm{i})\beta(\zeta)|} \\ &\qquad = \frac{1}{|\beta(\zeta)|\big[1 + x^2|d(\zeta)|^2 + 2x\big(\operatorname{Re} d(\zeta) - \lambda_1|\operatorname{Im} d(\zeta)|\big)\big]^{1/2}}. \end{aligned} \tag{3.19}$$

Now, we distinguish two cases, $\zeta \in \mathscr{K}_{\lambda_1}^+$ and $\zeta \in \mathscr{K}_{\lambda_1}^-$, with

$$\begin{aligned} \mathscr{K}_{\lambda_1}^+ &:= \{\zeta \in \mathscr{K} : \operatorname{Re} d(\zeta) - \lambda_1|\operatorname{Im} d(\zeta)| \geq 0\}, \\ \mathscr{K}_{\lambda_1}^- &:= \{\zeta \in \mathscr{K} : \operatorname{Re} d(\zeta) - \lambda_1|\operatorname{Im} d(\zeta)| < 0\}. \end{aligned}$$

For $\zeta \in \mathscr{K}_{\lambda_1}^+$, (3.19) obviously yields

$$\sup_{x>0} \max_{-\lambda_1\leq y\leq\lambda_1} \frac{1}{|x\alpha(\zeta) + (1+y\mathrm{i})\beta(\zeta)|} = \frac{1}{|\beta(\zeta)|}. \tag{3.20}$$

Furthermore, for $\zeta \in \mathscr{K}_{\lambda_1}^-$, it is easily seen that the supremum over all positive x is attained at $x^\star$,

$$x^\star := -\frac{\operatorname{Re} d(\zeta) - \lambda_1|\operatorname{Im} d(\zeta)|}{|d(\zeta)|^2};$$

thus,

$$\sup_{x>0} \max_{-\lambda_1\leq y\leq\lambda_1} \frac{1}{|x\alpha(\zeta) + (1+y\mathrm{i})\beta(\zeta)|} = \frac{1}{|\beta(\zeta)|\Big(1 - \frac{\big(\operatorname{Re} d(\zeta) - \lambda_1|\operatorname{Im} d(\zeta)|\big)^2}{|d(\zeta)|^2}\Big)^{1/2}},$$

whence

$$\begin{aligned}\sup_{x>0}\max_{-\lambda_1\le y\le\lambda_1}&\frac{1}{|x\alpha(\zeta)+(1+y\mathrm{i})\beta(\zeta)|}\\&=\frac{|d(\zeta)|}{|\beta(\zeta)|\,|\operatorname{Im}d(\zeta)|}\frac{1}{\Big(1+\lambda_1^2-2\lambda_1\frac{\operatorname{Re}d(\zeta)}{|\operatorname{Im}d(\zeta)|}\Big)^{1/2}}.\end{aligned}\tag{3.21}$$

Let, now, $\varphi(\zeta)\in(0,\pi)$, denote the angle between the negative real half-axis and the half-line passing through the origin and the point $d(\zeta)$ of the root locus curve. Then, obviously,

$$\frac{|d(\zeta)|}{|\operatorname{Im}d(\zeta)|}=\frac{1}{|\sin\varphi(\zeta)|}\quad\text{and}\quad\frac{\operatorname{Re}d(\zeta)}{|\operatorname{Im}d(\zeta)|}=-\cot\varphi(\zeta).$$

Therefore, (3.21) can be equivalently written as

$$\begin{aligned}\sup_{x>0}\max_{-\lambda_1\le y\le\lambda_1}&\frac{1}{|x\alpha(\zeta)+(1+y\mathrm{i})\beta(\zeta)|}\\&=\frac{1}{|\beta(\zeta)|\,|\sin\varphi(\zeta)|}\frac{1}{\big(1+\lambda_1^2+2\lambda_1\cot\varphi(\zeta)\big)^{1/2}},\end{aligned}\qquad\forall\zeta\in\mathscr{K}^-_{\lambda_1}.\tag{3.22}$$

In view of (3.20) and (3.22), we can write the necessary stability condition (3.12) in the form

$$\lambda_2\max\Big\{\max_{\zeta\in\mathscr{K}^-_{\lambda_1}}\frac{|\gamma(\zeta)|}{|\beta(\zeta)|},\ \sup_{\zeta\in\mathscr{K}^+_{\lambda_1}}\frac{|\gamma(\zeta)|}{|\sin\varphi(\zeta)|\,|\beta(\zeta)|}\frac{1}{\big(1+\lambda_1^2+2\lambda_1\cot\varphi(\zeta)\big)^{1/2}}\Big\}\le 1.\tag{3.23}$$

In the case $\lambda_1=0$, condition (3.23) reduces to $K_{(\alpha,\beta,\gamma)}\lambda_2\le 1$; see [1, (2.10)].

3.2. **The implicit–explicit BDF methods.** A particularly interesting example of multistep schemes satisfying our assumptions are the BDF methods, described by the polynomials

$$\alpha(\zeta)=\sum_{j=1}^{q}\frac{1}{j}\zeta^{q-j}(\zeta-1)^j,\quad\beta(\zeta)=\zeta^q,\quad\gamma(\zeta)=\zeta^q-(\zeta-1)^q.\tag{3.24}$$

The corresponding implicit $(\alpha,\beta)-$schemes are the well-known BDF methods, which are strongly $A(\vartheta_q)-$stable for $q=1,\dots,6$, with $\vartheta_1=\vartheta_2=90°$, $\vartheta_3=86.03°$, $\vartheta_4=73.35°$, $\vartheta_5=51.84°$ and $\vartheta_6=17.84°$; see [12, Section V.2]. Their order is $p=q$. For a given α, the scheme (α,γ) is the unique explicit $q-$step scheme of order $p=q$. The one-step scheme is the implicit–explicit Euler method. For these methods, the constants $K_{(\alpha,\beta,\gamma)}$ in (1.9) are explicitly known, namely

$$K_{(\alpha,\beta,\gamma)}=|\gamma(-1)|=2^q-1;\tag{3.25}$$

see [4].

According to (1.11), the implicit–explicit Euler and the implicit–explicit two-step BDF methods are stable for (1.1), with linear operator $A(t)$ of the form (1.2), for all λ_1, if

$$\lambda_2 < 1 \quad \text{and} \quad \lambda_2 < \frac{1}{3}, \tag{3.26}$$

respectively. For $q = 3, 4, 5, 6,$ we write the sufficient stability condition (1.11) in the form

$$\lambda_2 < \frac{1}{2^q - 1}\big[1 - (\cot\vartheta_q)\lambda_1\big] =: f_q(\lambda_1). \tag{3.27}$$

For $q \in \{1, \dots, 5\}$, stability results for the implicit–explicit BDF methods for (1.1) have also been established via energy techniques that were based on suitable multipliers for the $q-$step BDF schemes; see [7, 2, 6]. More precisely, in [7, 2] the Nevanlinna–Odeh multipliers from [17] were used, while in [6] more favourable multipliers were determined and used for the three- and five-step methods. In these stability results a constant $\hat\eta_q$ plays a crucial role: the values of $\hat\eta_q$ are

$$\hat\eta_1 = \hat\eta_2 = 0, \ \ \hat\eta_3 = 1/13 = 0.07692, \ \ \hat\eta_4 = 0.2878, \ \ \hat\eta_5 = 0.80973; \tag{3.28}$$

see [6].

For the initial value problem (1.1) with operators $A(t)$ of the form (1.2) the best stability results by the energy technique are obtained using in V the norm $\|\cdot\|$ introduced in Section 1; cf. [2, 6]. Since, obviously,

$$\forall v \in V \quad \|\big(A + \mathrm{i}a(t)A\big)v\|_\star = |1 + \mathrm{i}a(t)|\, \|Av\|_\star = |1 + \mathrm{i}a(t)|\, \|v\|,$$

in the notation of [2, 6], we have $\nu(t) = |1 + \mathrm{i}a(t)| \le \sqrt{1 + \lambda_1^2}$. Furthermore,

$$\forall v \in V \quad \mathrm{Re}((A + \mathrm{i}a(t)A)v, v) = (Av, v) = \|v\|^2,$$

whence, again in the notation of [2, 6], we have $\kappa(t) = 1$. According to [6, Theorem 5.1], we infer that the implicit–explicit $q-$step BDF method is stable for the initial value problem (1.1) with operators $A(t)$ of the form (1.2), provided

$$\lambda_2 < \frac{1}{2^q - 1}\frac{1}{1 + \hat\eta_q}\Big(1 - \hat\eta_q\sqrt{1 + \lambda_1^2}\Big) =: g_q(\lambda_1). \tag{3.29}$$

Analogous stability results, with slightly larger constants η_q for the three- and five-step methods, are given in [2, Theorem 2.1] and [7, Theorem 3].

Now, since $\hat\eta_1 = \hat\eta_2 = 0$, for the implicit–explicit Euler method and the implicit–explicit two-step BDF method, the stability condition (3.29) is satisfied for all λ_1 and for $\lambda_2 < 1$ and $\lambda_2 < 1/3$, respectively. Thus, for these two schemes, both the present technique (see (3.26)) and the energy technique lead to best possible stability conditions.

Furthermore, since $\hat\eta_q \ge \cos\vartheta_q$, for $\lambda_1 \le \tan\vartheta_q$, we have

$$(\cot\vartheta_q)\lambda_1 \le (\cos\vartheta_q)\sqrt{1 + \lambda_1^2} \le \hat\eta_q\sqrt{1 + \lambda_1^2}.$$

Therefore, for the implicit–explicit three-, four-, and five-step BDF methods the stability condition (3.27) is more favourable than (3.29); see also the graphs of f_q and $g_q, q = 3, 4, 5,$ in Figure 3.2.

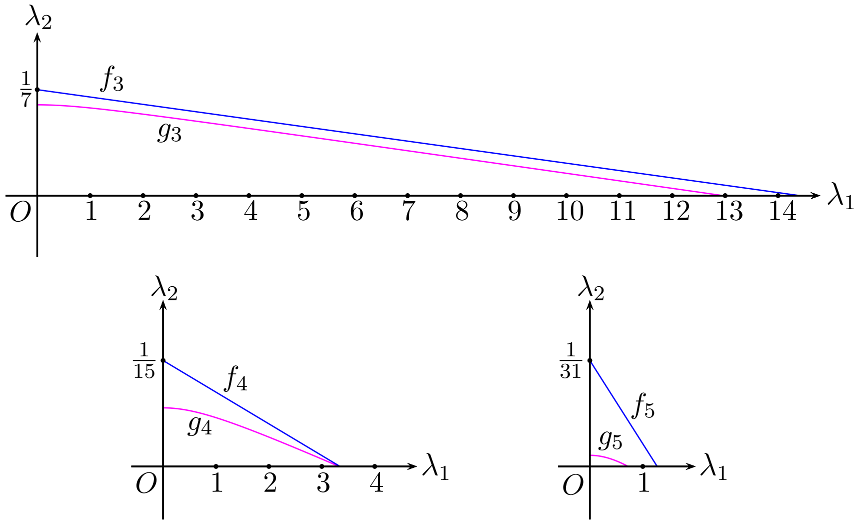

Figure 3.2. The functions f_q and g_q of the stability conditions (3.27) and (3.29).

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