STUDY OF THE DYNAMIC CHARACTERISTICS OF THE SPHEROIDAL HOLLOW CAVITY

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Study of the Dynamic Characteristics of the Spheroidal Hollow Cavity

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Abstract

A method for adjusting the solution of the elasticity equation in the spheroidal geometry space is presented. The first task concerns the construction of a basis, which contains the Navier eigenvectors in the spheroidal geometry and the second the satisfaction of the boundary conditions which leads to the eigenfrequencies determination.

Key words: Navier Eigenvectors; Dynamic Characteristics.

AMS subject classifications: 35Q72, 92C10, 92C50.

1 Introduction

The examination of scattering and vibration problems in the framework of linear elasticity is accompanied with several complexity factors having physical as well as mathematical origin. From the physical point of view, the coexistence of two different waves (propagating or stationary) connected through the boundary conditions and travelling with different velocities renders the investigation of the elastic problem rather difficult. From the mathematical point of view, every approach to the problem incorporates in its treatment the above mentioned peculiarity accompanying elastic waves and usually the price for that is high. As an example, the integral equations of elasticity contain kernel functions reflecting this physical situation and their handling differs not only quantitatively but also qualitatively from other physical phenomena corresponding formulations (acoustics or electromagnetism).

The adopted methodology for the solution of the emerged boundary value elastic problems depends on several parameters of the problem. The main two factors orientating the mathematical framework are the particular physical situation as well as the geometrical characteristics of the system. Indeed the situation changes drastically from treatment of scattering procedures - exterior boundary value problems - to treatment of vibration problems, which refer to bounded domains. In addition, it is obvious that the underlying geometry constitutes a difficulty factor to overcome.

This work aims at the study of stationary elastic waves occuring in structures occupied by elastic materials and fitting geometrically to the spheroidal coordinate system. The motivation to this work lies on the necessity to study the dynamic characteristics of structures simulating the human head system but it is not restricted to this case. Systematic and hierarchical analysis of several models [1, 2] belonging to this biomechanics area, has proved that the dynamic characteristis behave smoothly as the geometrical characteristics of the investigated multilayer structures change slightly. We mention here the application of perturbation method techniques in [1], in order to study how the eigenvalues and the corresponding eigenmodes change when the system is transformed from the spherical to the perturbed spherical case, which is equivalent to spheroidal of slightly different axis. However, this is an asymptotic case and can not provide with the results for the spheroidal cavity of arbitrary axis.

The fundamental ingredients of the method is the expansion of the sought elastic fields in terms of the basis of the Navier eigenvectors. More precisely, the generalized Sturm-Liouville theory assures the existence of a countable

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The spheroidal surfaces S_0, S_1 are described by the equations $\delta = \delta_0$ and $\delta = \delta_1$, correspondingly. The hollow cavity is represented as the region $\delta_0 < \delta < \delta_1$. Under the status of elastic vibrations, the physical motion of the system is described by the time - harmonic displacement field $\mathbf{u}(\mathbf{r}, t) = exp(-i\omega t)\mathbf{u}(\mathbf{r})$, where the time-independent function $\mathbf{u}(\mathbf{r})$ satisfies the reduced equation of linearized elasticity:

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \rho\omega^2\mathbf{u} = \mathbf{0}, \qquad \delta_0 < \delta < \delta_1 \tag{2}$$

where ω stands for the frequency of the harmonic motion.

We suppose that the surfaces of the hollow system are stress free. Mathematically, this is realized through the boundary conditions

$$\mathbf{T}\mathbf{u}(\mathrm{r}) = \mathbf{0}, \qquad \delta = \delta_0 \text{ and } \delta = \delta_1 \tag{3}$$

where $\mathbf{T}$ stands for the surface stress operator, given in general form as

$$\mathbf{T} = 2\mu\hat{\mathbf{n}} \cdot \nabla + \lambda\hat{\mathbf{n}}\nabla \cdot + \mu\hat{\mathbf{n}} \times \nabla\times \tag{4}$$

and $\hat{\mathbf{n}}$ is the unit normal to the operator application surface.

The problem consisted of Eqs. (1) and (2) is a well - posed homogeneous boundary value problem and its solution is the aim of this work. More precisely, we are interested in determining the eigenfrequencies ω as well as the corresponding eigenmodes $\mathbf{u}(\mathbf{r})$.

Alternatively, Eq. (2) can be written as

$$c_s^2\nabla^2\mathbf{u} + (c_p^2 - c_s^2)\nabla(\nabla \cdot \mathbf{u}) + \omega^2\mathbf{u} = \mathbf{0} \tag{5}$$

where $c_p = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}, c_s = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$ stand for the velocities of the longitudinal and transverse waves, respectively [6]. It is well known [3] that every elastic field can be decomposed as the superposition of the longitudinal and transverse elastic components according to the following formula

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}_p(\mathbf{r}) + \mathbf{u}_s(\mathbf{r}) \tag{6}$$

where the components satisfy the vector Helmholtz equation

$$\nabla^2\mathbf{u}_\alpha + k_\alpha^2\mathbf{u}_\alpha = \mathbf{0}, \qquad \alpha = p, s, \tag{7}$$

and $k_\alpha = \frac{\omega}{c_\alpha}, \alpha = p, s$ denote the wave numbers of the p- and s- elastic waves.

As it is shown in [4], the transverse elastic fields $\mathbf{u}_s$ can be represented through the following basis, family of Navier eigenvectors

$$^{\mathbf{a}}\mathbf{M} = \nabla\psi \times \mathbf{a}, \tag{8}$$

$$^{\mathbf{a}}\mathbf{N} = \nabla \times {}^{\mathbf{a}}\mathbf{M} = \nabla \times (\nabla\psi \times \mathbf{a}), \tag{9}$$

where $\mathbf{a} \in \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \mathbf{r}\}$ ($\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are the unit vectors in the x, y, z directions respectively).

These vector functions do satisfy vector Helmholtz equation and constitute eight choices to represent the transverse components of the vector fields, if and only if the scalar function ψ runs over the countable basis set of scalar Helmholtz equation with wavenumber k_s.

In addition, the longitudinal elastic field $\mathbf{u}_p(\mathbf{r})$ can be expanded in terms of the eigenvectors

$$\mathbf{L} = \nabla\phi \tag{10}$$

where ϕ exhausts all the solutions of the scalar Helmholtz equation with wavenumber k_p.

We recognise then the crucial role of Helmholtz equation

$$\nabla^2\psi_\alpha + k_\alpha^2\psi_\alpha = 0, \qquad \alpha = p, s, \tag{11}$$

whose study in spheroidal coordinates is proved necessary. Omitting indices for simplicity we express the Laplace operator in the spheroidal system:

$$\nabla^2\psi = \frac{\frac{4}{\alpha^2}}{\cosh\delta^2 - \cos^2\theta}\left[\frac{1}{\sinh\delta}\frac{\partial}{\partial\delta}\left(\sinh\delta\frac{\partial\psi}{\partial\delta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\sinh\delta^2 + \sin^2\theta}{\sinh\delta^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]. \tag{12}$$

$$
\begin{aligned}
{}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn} &= \frac{2(\xi^2-1)^{\frac{1}{2}}}{\alpha(\xi^2-\eta^2)^{\frac{1}{2}}}\left[-\frac{d}{d\xi}R^{(i)}_{mn}(\xi,c)\frac{d}{d\eta}\left(\frac{\eta(1-\eta^2)S_{mn}(\eta,c)}{\xi^2-\eta^2}\right)\right.\\
&+ \left.\xi R^{(i)}_{mn}(\xi,c)\frac{d}{d\eta}\left(\frac{1-\eta^2}{\xi^2-\eta^2}\frac{d}{d\eta}S_{mn}(\eta,c)\right)-\frac{m^2\xi}{(\xi^2-1)(1-\eta^2)}R^{(i)}_{mn}(\xi,c)S_{mn}(\eta,c)\right]\left\{\begin{array}{c}\cos(m\phi)\\ \sin(m\phi)\end{array}\right\}\hat{\xi}\\
&+ \frac{2(1-\eta^2)^{\frac{1}{2}}}{\alpha(\xi^2-\eta^2)^{\frac{1}{2}}}\left[\frac{d}{d\xi}\left(\frac{\xi(\xi^2-1)R^{(i)}_{mn}(\xi,c)}{\xi^2-\eta^2}\right)\frac{d}{d\eta}S_{mn}(\eta,c)\right.\\
&+ \left.\eta S^{(i)}_{mn}(\xi,c)\frac{d}{d\xi}\left(\frac{\xi^2-1}{\xi^2-\eta^2}\frac{d}{d\xi}R_{mn}(\xi,c)\right)-\frac{m^2\eta}{(\xi^2-1)(1-\eta^2)}R^{(i)}_{mn}(\xi,c)S_{mn}(\eta,c)\right]\left\{\begin{array}{c}\cos(m\phi)\\ \sin(m\phi)\end{array}\right\}\hat{\eta}\\
&- \frac{2m(\xi^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}{\alpha(\xi^2-\eta^2)}\left[-\frac{R^{(i)}_{mn}(\xi,c)}{\xi^2-1}\frac{d}{d\eta}\left(\eta S_{mn}(\eta,c)\right)-\frac{d}{d\xi}\left(\xi R^{(i)}_{mn}(\xi,c)\right)\frac{S_{mn}(\eta,c)}{1-\eta^2}\right]\left\{\begin{array}{c}-\sin(m\phi)\\ \cos(m\phi)\end{array}\right\}\hat{\phi}.
\end{aligned}
\tag{18}
$$

Actually, this is the straightforward construction of the eigenvector ${}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn}$ and previous relation is the preferable form, under some possible slight modifications used in a few works treating spheroidal problems [7, 8, 9]. However the expression (18) is a very complicated expression and becomes much more complex if someone tries to apply boundary differential operators on it. As a matter of fact, it contains a lot of differentiations of second order and this creates the question whether some terms can be simplified after combining this expression with the differential equation itself. In other words there exists the feeling that expression (18) contain some ficticious terms, which should be rearranged suitably to lightening the burden of the equation. Actually these terms stem from the fact that operator curl does not "know" the differential equation satisfied by the functions on which it acts. Nevertheless, instead of rearranging terms, it would be preferable to follow a simple different procedure to obtain an alternative "minimal" expression of ${}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn}$.

Indeed, following some simple arguments based on differential equation properties, we begin with the definition equation

$$
{}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn} = \nabla \times^{\mathbf{r}} \mathbf{M}^{(i)}_{{}^{e}_{o}mn}, \tag{19}
$$

and we obtain

$$
\begin{aligned}
{}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn} &= \nabla\times\left(\nabla\times^{\mathbf{r}}\mathbf{M}^{(i)}_{{}^{e}_{o}mn}\right) = \nabla\times\left[\nabla\times\left(\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}\right)\right]\\
&= \nabla\left[\nabla\cdot\left(\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}\right)\right]-\nabla^2\left(\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}\right) = \nabla\left[\nabla\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}\cdot\mathbf{r}\right]+3\nabla\psi^{(i)}_{{}^{e}_{o}mn}-\nabla^2\left(\psi^{(i)}_{{}^{e}_{o}mn}\right)\mathbf{r}-2\nabla\psi^{(i)}_{{}^{e}_{o}mn}\\
&= \nabla\psi^{(i)}_{{}^{e}_{o}mn}+k^2\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}+\nabla\left[\left(\mathbf{r}\cdot\nabla\right)\psi^{(i)}_{{}^{e}_{o}mn}\right] = 2\nabla\psi^{(i)}_{{}^{e}_{o}mn}+k^2\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}+\left(\mathbf{r}\cdot\nabla\right)\nabla\psi^{(i)}_{{}^{e}_{o}mn}.
\end{aligned}
\tag{20}
$$

The first term of the last part of the representation (20) is equal to $\mathbf{L}^{(i)}_{{}^{e}_{o}mn}$, while the third term is acquired after applying the differential operator $\mathbf{r}\cdot\nabla$ on the same function. As far as the second term is concerned, it has a very simple expression. In other words, we have

$$
{}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn} = 2\mathbf{L}^{(i)}_{{}^{e}_{o}mn}+\left(\mathbf{r}\cdot\nabla\right)\mathbf{L}^{(i)}_{{}^{e}_{o}mn}+k^2\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}. \tag{21}
$$

Another useful representation of ${}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn}$ is the intermediate step of (20) fournishing the formula

$$
{}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn} = \mathbf{L}^{(i)}_{{}^{e}_{o}mn}+\nabla\left(\mathbf{r}\cdot\mathbf{L}^{(i)}_{{}^{e}_{o}mn}\right)+k^2\psi^{(i)}_{{}^{e}_{o}mn}\mathbf{r}. \tag{22}
$$

It is true that expressions (20) - (22) dispose as well second order differentiations (all appearing in the term $\left(\mathbf{r}\cdot\nabla\right)\mathbf{L}^{(i)}_{{}^{e}_{o}mn}$), but now no retractable differentiation is appearing. Special mention must be assigned to the fact that every function involving in transverse field expresions, incorporates the wavenumber k_s in its argument, while every longitudinal component disposes k_p respectively.

3 Solution of the Problem

The completeness of the Navier eigenvectors permits the representation

$$
\mathbf{u}(\mathbf{r}) = \sum\left\{\alpha^{(i)}_{{}^{e}_{o}mn}\mathbf{L}^{(i)}_{{}^{e}_{o}mn}+\beta^{(i)}_{{}^{e}_{o}mn}{}^{\mathbf{r}}\mathbf{M}^{(i)}_{{}^{e}_{o}mn}+\gamma^{(i)}_{{}^{e}_{o}mn}{}^{\mathbf{r}}\mathbf{N}^{(i)}_{{}^{e}_{o}mn}\right\}, \tag{23}
$$

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