ON THE STRUCTURE OF A-FREE GRAPHS: PART II

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On the Structure of A-free Graphs: Part II

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Abstract — A graph G is called an A-free graph if for every edge (x, y) of G, we have $N[x] \subseteq N[y]$ or $N[x] \supseteq N[y]$, where $N[x]$ denotes the closed neighbourhood of x. We first show that an A-free graph G has some interesting properties, for example, $\{x \in V(G) \mid N[x] = V(G)\}$ is not an empty set, and a graph H is an A-free if and only if H has no induced subgraph isomorphic to P_4 or C_4. Then by making use of these properties, we obtain important structural and algorithmic properties. Based on these results, we show the relationships between A-free graphs and several classes of perfect graphs known as chordal graphs, cographs, comparability, cocomparability, interval, permutation, ptolemaic, distance-hereditary and (t, c, s)-perfect.

1. A-free Graphs and their Structures

In this paper we consider an undirected simple graph G with vertex set $V(G)$ and edge set $E(G)$. Then hereafter we can call G a graph instead of a simple graph. The *neighbourhood* of a vertex x is the set $N(x) = N_G(x)$ consisting of all the vertices of G which are adjacent with x. The *closed neighbourhood* of x is defined by $N[x] = N_G[x] := \{x\} \cup N(x)$. The subgraph of a graph G induced by a subset $S \subseteq V(G)$ is denoted by $G[S]$. Let X and Y be two subsets of a certain set. Then $X \subset Y$ means that X is a proper subset of Y, and if $Y \subseteq X$, then let $X - Y$ denote $X \setminus Y$.

Given a graph G, an edge $(x, y) = (y, x)$ of G is classified as follows according to relationship of closed neighbourhoods [6, 10].

$$
\begin{array}{rll}
(x, y) \text{ is } \textit{free} & \text{if} & N[x] = N[y]; \\
(x, y) \text{ is } \textit{semi-free} & \text{if} & N[x] \subset N[y], \text{ (in particular } N[x] \neq N[y]\text{); and} \\
(x, y) \text{ is } \textit{actual} & \text{if} & N[x] \setminus N[y] \neq \emptyset \text{ and } N[y] \setminus N[x] \neq \emptyset.
\end{array}
$$

Obviously $E(G)$ can be partitioned into the three subsets of free edges, semi-free edges and of actual edges, respectively.

A graph G is called an *A-free graph* if every edge of G is either free or semi-free. Thus G is an A-free graph if and only if for every edge (x, y) of G, we have $N[x] \subseteq N[y]$ or $N[x] \supseteq N[y]$. The graph G in Figure 1 is an A-free graph, while the graphs H and I in the same figure are not A-free graphs.

For a vertex subset S of a graph G, we define $G - S$ by $G[V(G) - S]$. The following lemma follows immediately from the fact that for every subset $S \subset V(G)$ and for a vertex $x \in S$, we have $N_{G[S]}[x] = N[x]$ and that $G - S$ is an induced subgraph.

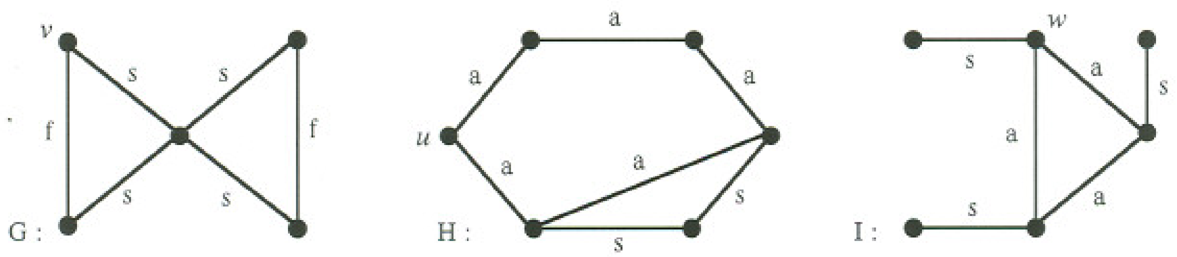

Figure 1. Three undirected graphs. Free, semi-free and actual edges are denoted by f, s and a, respectively.

Lemma 1 If G is an A-free graph, then for every subset $S \subset V(G)$, both $G[S]$ and $G - S$ are also A-free graphs.

The following results provide important properties for the class of A-free graphs. Let P_4 and C_4 denote the path and the cycle of order four, respectively. For convenience, we here define

$$cent(G) = \{x \in V(G) \mid N[x] = V(G)\}.$$

Theorem 1 Let G be a simple graph. Then the following three statements are equivalent.

(i) G is an A-free graph;

(ii) G has no induced subgraph isomorphic to P_4 or C_4;

(iii) Every connected induced subgraph $G[S]$, $S \subseteq V(G)$, satisfies $cent(G[S]) \neq \emptyset$.

Proof. (i) $\Rightarrow$ (iii) Suppose that there exists a connected induced subgraph $G[S]$ such that $cent(G[S]) = \emptyset$. We now consider the graph $G[S]$, which is an A-free graph by Lemma 1 and whose vertex set is S. It is clear that $|S| \geq 4$. Let x be a vertex of $G[S]$ with maximum degree. Then $N_{G[S]}[x] \neq S$ by $cent(G[S]) = \emptyset$, and so we can find two vertices y and w such that $y \in N_{G[S]}[x]$ and $w \in N_{G[S]}[y] \setminus N_{G[S]}[x]$. Since $w \in N_{G[S]}[y] \setminus N_{G[S]}[x]$ and $G[S]$ is an A-free graph, we have $N_{G[S]}[x] \subsetneq N_{G[S]}[y]$, which implies that the degree of y is greater than that of x. This contradicts the choice of x.

(iii) $\Rightarrow$ (ii) Suppose that there exists an induced subgraph $G[S]$ isomorphic to P_4 or C_4. Then $cent(G[S]) \neq \emptyset$, a contradiction.

(ii) $\Rightarrow$ (i) Suppose that G is not an A-free graph. Then G contains an actual edge, say (x, y). Then there exist two vertices $u \in N[x] \setminus N[y]$ and $v \in N[y] \setminus N[x]$. Hence the induced subgraph $G[\{u, x, y, v\}]$ is isomorphic to P_4 or C_4, a contradiction. □

Lemma 2 The following two statements hold.

(i) A graph G is an A-free if and only if $G - cent(G)$ is an A-free graph.

(ii) Let G be a connected A-free graph. Then $cent(G) \neq \emptyset$. Moreover, if $G - cent(G) \neq \emptyset$, then $G - cent(G)$ contains at least two components.

Proof. (i) By Lemma 1, $G - cent(G)$ is an A-free graph. Conversely, we assume that $G - cent(G)$ is an A-free graph but not G is. By Theorem 1, G has an induced subgraph $G[S]$ isomorphic to P_4 or C_4. Then $S \cap cent(G) = \emptyset$, and thus $S \in V(G - cent(G))$, which implies $(G - cent(G))[S] = G[S]$ is isomorphic to P_4 or C_4. This contradicts (ii) of Theorem 1.

We next prove (ii). It is clear that $cent(G) \neq \varnothing$ by (ii) of Theorem 1 with $S = V(G)$. Next assume that $G - cent(G) \neq \varnothing$ and $G - cent(G)$ is connected. Then since $G - cent(G)$ is an A-free graph, $cent(G - cent(G)) \neq \varnothing$. But it follows that $cent(G - cent(G)) \subset cent(G)$, which is a contradiction. Hence $G - cent(G)$ is not connected. □

Let G be a connected A-free graph. Then $V_1 := cent(G)$ is not an empty set by Lemma 2. Put $G_1 = G$, and $G - V_1 = G_2 \cup G_3 \cup \ldots \cup G_r$, where each G_i is a component of $G - V_1$ and $r \geq 3$. Then since each G_i is an induced subgraph of G, G_i is also an A-free graph, and so let $V_i := cent(G_i) \neq \varnothing$ for $1 \leq i \leq r$. Since each component G_j of G_i-$cent(G_i)$ is also an A-free graph, we can continue this procedure until we get an empty graph. Then we finally obtain the following partition of $V(G)$.

$$V(G) = V_1 + V_2 + \ldots + V_k, \quad \text{where} \quad V_i = cent(G_i).$$

Moreover we can define a partial order $\leq$ on $\{V_1, V_2, \ldots, V_k\}$ as follows:

$$V_i \leq V_j \qquad \text{if} \quad V_i = cent(G_i) \text{ and } V_j \subseteq V(G_i).$$

It is easy to see that this partition possesses the following properties.

Theorem 2 Let G be a connected A-free graph, and let $V(G) = V_1 + V_2 + \ldots + V_k$ be the partition defined above, in particular, $V_1 := cent(G)$. Then this partition and the partially ordered set $(\{V_i\}, \leq)$ have the following properties:

(P1) If $V_i \leq V_j$, then every vertex of V_i and every vertex of V_j are joined by an edge of G.
(P2) For every V_i, $cent(G[\{\cup V_i \mid V_j \geq V_i\}]) = V_i$.
(P3) For every two V_s and V_t such that $V_s \leq V_t$, $G[\{\cup V_i \mid V_s \leq V_i \leq V_t\}]$ is a complete graph. Moreover, for every maximal element V_t of $(\{V_i\}, \leq)$, $G[\{\cup V_i \mid V_1 \leq V_i \leq V_t\}]$ is a maximal complete subgraph of G.
(P4) Every edge with both endpoints in V_i is a free edge.
(P5) Every edge with one endpoint in V_i and the other endpoint in V_j, where $V_i \neq V_j$, is a semi-free edge.

The results of Theorem 2 provide algorithmic and structural properties for the class of A-free graphs. A typical structure of such a graph is shown in Figure 2. We shall refer to the structure which meets the properties of Theorem 2 as *cent-tree* $T_c(G)$. The cent-tree is a rooted tree with root V_1; every node V_i of $T_c(G)$ is either a leaf or has at least two children. Moreover, $V_s \leq V_t$ if and only if V_s is an ancestor of V_t.

If V_i and V_j are disjoint vertex sets of an A-free graph G, we say that V_i and V_j are *clique-adjacent* and denote $V_i \approx V_j$ if $V_i \leq V_j$ or $V_j \leq V_i$.

Let us now examine the effect of property (P3) of Theorem 2 on the structure of an A-free graph. This property ensures that all the edges with both endpoints in a vertex set V_i are free edges, $1 \leq i \leq k$. A consequence of this property is that the vertex set $V_1 \cup V_i$ is not always a maximal clique. We can easily see that $V_1 \cup V_i$ is not a maximal clique if there exists a vertex set V_j such that $V_i \approx V_j$, $2 \leq j \leq k$.

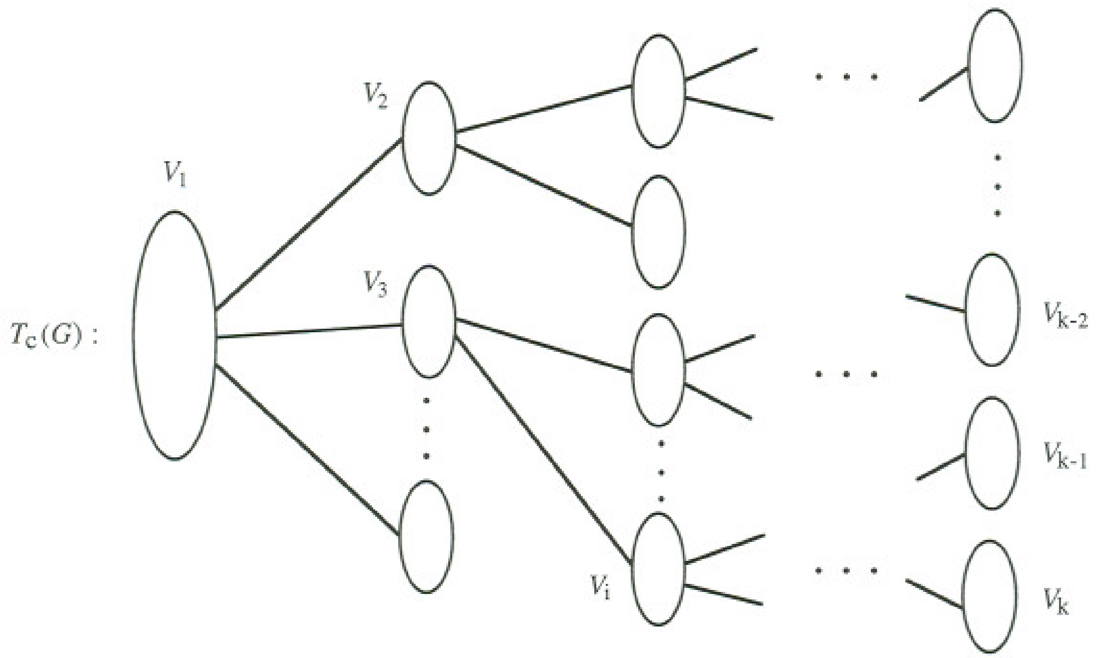

Figure 2. The typical structure of an A-free graph; that is, the *cent-tree* $T_c(G)$. A line between cells V_i and V_j indicates that $V_i \approx V_j$. All edges in V_i are free edges; All edges between cells are semi-free edges.

Let $V = V_1 + V_2 + ... + V_k$ be a partition of V such that $V_1 = cent(G)$. Let $S = \{v_t, v_{t+1}, ..., v_s\}$ be a stable set such that $v_t \in V_t$ and V_t is a maximal element of $(\{V_i\}, \leq)$ or, equivalently, V_t is a leaf node of $T_c(G)$. It is easy to see that S has the maximum cardinality $\alpha(G)$ among all the stable sets of G. On the other hand the sets $\{\cup V_i \mid V_1 \leq V_i \leq V_t\}$, for every maximal element V_t of $(\{V_i\}, \leq)$, provide a clique-cover of size $\varkappa(G)$ which has the property to be a smallest possible clique cover of G; that is $\alpha(G) = \varkappa(G)$. Based on the Theorem 2 or, equivalently, on the cent-tree of G, it is easy to show that the clique number $\omega(G)$ equals the chromatic number $\chi(G)$ of G; that is, $\omega(G) = \chi(G)$. Thus, the following results are obtained.

Theorem 3 Let G be an A-free graph. Let p be the number of maximal elements of $(\{V_i\}, \leq)$ and let q be the number of vertices in a complete subgraph of G with maximum order. Then

(i) $p = \alpha(G) = \varkappa(G)$, and
(ii) $q = \omega(G) = \chi(G)$.

If G is an A-free graph, then for every subset $S \subseteq V(G)$, $G[S]$ is an A-free graph, and each component H of $G[S]$ satisfies Theorem 3, which implies that $\alpha(H) = \varkappa(H)$ = the number of maximal element V_i of the partially ordered set obtained from H and that $\chi(H) = \omega(H)$ = the number of vertices in a complete subgraph with maximum order in H. Thus, we obtain the following result.

Lemma 3 Every induced subgraph H of an A-free graph G is also an A-free graph having $\alpha(H) = \varkappa(H)$ and $\omega(H) = \chi(H)$.

A graph G is said to be *perfect* if it satisfies the following two properties: the χ-Perfect property: $\chi(G[A]) = \omega(G[A])$ for all $A \subseteq V(G)$, and the α-Perfect property: $\alpha(G[A]) = \varkappa(G[A])$ for all $A \subseteq V(G)$, where $\chi(G[A])$, $\omega(G[A])$, $\alpha(G[A])$ and $\varkappa(G[A])$ are the chromatic, clique, stability and clique-cover number of $G[A]$, respectively [5]. Thus an A-free graph is a perfect graph.

2. Relationship between A-free and Perfect Graphs

In this section we show some relationship between A-free graphs and many other perfect graphs.

A graph G is called a *diagonal graph* if for every path in G with edges (x, y), (y, z), (z, w), the graph G also contains at least one of edges (x, z) and (y, w). It was mentioned in [12] that a graph is diagonal if and only if G has no induced subgraph isomorphic to C_4 or P_4. Thus the diagonal graphs are precisely the A-free graphs.

A graph is called a *cograph* it contains no induced subgraph isomorphic to P_4 [4]. Then an A-free graph is a cograph, which implies that an A-free graph is a *distance-hereditary* graph and a *parity* graph because a cograph is distance-hereditary [3, 7, 8] and a distance-hereditary graph is a parity graph [1, 4, 9]. An important class of perfect graphs, known as *ptolemaic* graphs [8], forms a subclass of the distance-hereditary graphs. Actually, a graph G is a ptolemaic graph if and only if it is chordal and distance-hereditary graph. Thus, if G is an A-free graph then G is a ptolemaic graph.

A graph is called a *chordal* graph if every cycle of length greater than 3 has a chord [5]. Then the A-free graphs are exactly the chordal cographs.

A *sun* of order p, or *p-sun* ($p \geq 3$) is a chordal graph on vertex set $\{x_1, x_2, ..., x_p, y_1, y_2, ..., y_p\}$, where $\{y_1, y_2, ..., y_p\}$ is an independent set, $(x_1, x_2, ..., x_p)$ is a cycle, and each vertex y_i has exactly two neighbours, x_{i-1} and x_i. By definition, $cent(p\text{-}sun) = \varnothing$ and so *p-sun* is not an A-free graph ($p \geq 3$).

A graph G is called *strongly chordal* if G is chordal and G contains no sun, G is called *balanced chordal* if G is chordal and G contains no sun of odd order, and G is called *compact* if G contains no sun of order 3. We have showed that an A-free graph is a chordal graph and it contains no induced subgraph isomorphic to a p-sun, $p \geq 3$. Thus, we can prove that if G is an A-free graph then G is a strongly chordal graph, a balanced chordal graph and a compact graph.

Let $\gamma(G)$ and $\iota(G)$ be the domination number and independent domination number of a graph G, respectively. A graph G is called a *domination perfect* graph if $\gamma(H) = \iota(H)$, for every induced subgraph H of G. The domination number $\gamma(G)$ is the minimum cardinality taken over all dominating sets of G, and the independent domination number $\iota(G)$ is the minimum cardinality taken over all maximal independent sets of vertices of G. By Lemma 3, $\gamma(H) = \iota(H) = 1$ because $cent(H) \neq \varnothing$, for every induced subgraph H of an A-free graph. Thus, an A-free graph is a domination perfect graph.

Let G be a graph. We define $\mathbf{C}(G)$ to be the set of all maximal cliques of G and similarly, we define $\mathbf{S}(G)$ to be the set of all independent sets of G. Let $F = (V_i)_{i \in I}$ be a family of subsets of the set V. Following the definition in [2], we call a *transversal* of F a subset T of V such that T intersects the sets V_i for all $i \in I$; if all these intersections consist of exactly one vertex, we call T a *perfect transversal*. A perfect transversal of $\mathbf{C}(G)$ ($\mathbf{S}(G)$, respectively) will be called a *stable* (*complete*, respectively) *transversal* of G, since a transversal of $\mathbf{C}(G)$ ($\mathbf{S}(G)$, respectively) is perfect if and only if it is a maximal stable set (maximal clique, respectively) of G.

A graph is called *c-perfect* (*s-perfect*, respectively) if all its induced subgraphs have a stable (complete, respectively) transversal. Let $V = V_1 + V_2 + ... + V_k$ be a partition of V such that $V_1 = cent(G)$. Thus, there exists a stable set $S = \{v_t, v_{t+1}, ..., v_s\}$ such that $v_t \in V_t$ and V_t is a maximal element of $(\{V_i\}, \leq)$. Since the number of maximal elements of $(\{V_i\}, \leq)$ equals $\alpha(G)$, we have that S is a maximal stable set (Theorem 3). Moreover, the set $C = \{\cup V_i \mid V_1 \leq V_i \leq V_t\}$ is a maximal clique, for every maximal element V_t of $(\{V_i\}, \leq)$ (Theorem 2). We can easily conclude that S is a stable transversal and C is a complete transversal of G. Thus, every A-free graph is c-perfect and s-perfect graph. Moreover, a graph is called *t-perfect* if for every induced subgraph H of G, $\alpha(H)$ equals the number of maximal cliques contained in H. By Lemma 3, every induced subgraph H of an A-free graph G is also an A-free graph having $\alpha(H) = \varkappa(H)$. Thus, t-perfect graphs are precisely the A-free graphs.

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