# MAGNETIZATION CURLING IN AN ELASTIC SPHERE (AN UPPER BOUND TO THE NUCLEATION FIELD)

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The magnetization reversal by the curling mode in a ferromagnetic sphere is examined in order to account for magnetostriction effects. We constrain the strains to simple extension or compression. The nucleation field obtained, is an upper bound to the correct one.

*Keywords:* Micromagnetics; Curling; Nucleation Field; Magnetostriction; Magnetization Reversal; Elastic Sphere.

## I. INTRODUCTION

Understanding the micromagnetics of magnetostrictive materials is very important for the development of efficient micro electromechanical systems (MEMS) [1]. The usual manner to take into account magnetostriction in magnetization reversal is by assuming some kind of inhomogeneity in the magnetocrytsalline anisotropy, in the form of dislocations [2] and voids [3]. The theory of magnetoelastic interactions [4] has never been fully applied to specific problems due to its complexity. Some crude estimations about the effect of magnetostriction on the nucleation field are given in Refs. [5,6], based on a perturbation approach. It has been proved that magnetostriction does not affect the nucleation field for magnetization curling in an infinite cylinder [7]. It is the aim of this work to calculate a rigorous upper bound to the nucleation field in an elastic sphere for the curling mode. The solution methodology is similar to that of Ref. [8]. Two simple deformation modes (a uniform and a non-uniform one), of "hydrostatic" type are considered. The Brown' s paradox is discussed, within the limitations of the deformation modes considered, and the shift of the critical size for curling vs. coherent rotation is computed.

## II. THE MICROMAGNETIC FORMULATION

### A. General Theory

According to micromagnetism [4], the state of a deformable ferromagnetic material is described by the magnetization vector $\boldsymbol{\mu}(\boldsymbol{x})$ per unit mass ($\mu_i\,\mu_i = \mu_s^2 = const.$) and by the displacement vector $\boldsymbol{u}(\boldsymbol{x})$. Stable equilibrium states correspond to minima of the Gibbs free energy functional:

$$G = \int_V \rho_\circ \left( F - \frac{1}{2}\mu_\circ\, \boldsymbol{\mu}\cdot\boldsymbol{H}' - \mu_\circ\, \boldsymbol{\mu}\cdot\boldsymbol{H}^\circ \right) dV_\circ - \int_{V_\circ} \rho_\circ\, \boldsymbol{f}\cdot\boldsymbol{u}\, dV_\circ - \int_{S_\circ} \boldsymbol{T}\cdot\boldsymbol{u}\, dS_\circ, \tag{1}$$

where: $F(\mu_i,\, \mu_{i,j},\, e_{ij})$ is the local internal energy per unit mass, $e_{ij}$ are the infinitesimal strains ($e_{ij}(\boldsymbol{x}) = (u_{i,j} + u_{j,i})/2$), $\boldsymbol{H}'$ is the field generated by the volume $(\nabla\cdot\boldsymbol{M})$ and surface $(\boldsymbol{n}\cdot\boldsymbol{M})$ magnetic charges, $\boldsymbol{H}^\circ$ is the applied magnetic field, $\rho_\circ$ is the mass density in the undeformed configuration, $\boldsymbol{f}$ is the body force, $\boldsymbol{T}$ is the surface traction, $\mu_\circ$ is the magnetic permeability of vacuum, $\boldsymbol{n}$ is the outward unit vector from the boundary surface, $\boldsymbol{M}$ is the magnetization per unit volume ($=\rho\,\boldsymbol{\mu}$), $\rho = \rho_\circ(1 - \nabla\cdot\boldsymbol{u})$ and $(\,)_{,i} \equiv \partial/\partial x_i$. The field $\boldsymbol{H}'$ is determined from the scalar potential $U$, $\boldsymbol{H}' = -\nabla U$, where $U$ obeys the well known potential problem of magnetostatics [9]. The vanishing of the first variation of the energy functional (1) ($\delta G = 0$) results in Brown' s equilibrium magnetoelastic equations, which are highly nonlinear. For the mechanical problem, the presence of the magnetization results in non-symmetric stress tensor $t_{ij}$ and in additional body force density $\mu_\circ \boldsymbol{M}\cdot\nabla\boldsymbol{H}'$ and surface force density $\frac{1}{2}\mu_\circ M_n^2\boldsymbol{n}$ [4]. Thus the variation of the magnetic self-energy is according to Brown [4]:

$$\delta\int_V \mu_\circ \boldsymbol{M}\cdot\boldsymbol{H}'dV = \int_V 2\mu_\circ\left[M_s\,\boldsymbol{H}'\cdot\delta\boldsymbol{\alpha} + (\boldsymbol{M}\cdot\nabla)\,\boldsymbol{H}'\cdot\delta\boldsymbol{u}\right]dV - \int_S \mu_\circ M_n^2\,\boldsymbol{n}\cdot\delta\boldsymbol{u}dS, \tag{2}$$

where $M_n \equiv \boldsymbol{n}\cdot\boldsymbol{M}$ and $\boldsymbol{\alpha}$ are the direction cosines of the magnetization vector. The local internal energy per unit undeformed volume $\rho_\circ F$ can be expanded as:

$$\rho_\circ F = w_{ex}(\mu_{i,j}) + w_{exs}(\mu_{i,j},\, e_{ij}) + w_a(\mu_i) + w_{mel}(\mu_i,\, e_{ij}) + w_{el}(e_{ij}) \tag{3}$$

which includes exchange, exchange-strictive, anisotropy, magnetoelastic and elastic terms, respectively. In what follows we will neglect exchange-strictive phenomena and assume zero body forces ($\boldsymbol{f} = \boldsymbol{0}$).

### B. The Model

We consider an elastic ferromagnetic sphere of radius $R$ in its virgin undeformed state, with cubic crystallographic symmetry. For convenience we assume that in the undeformed state, the Cartesian coordinate axes are identical with the crystallographic axes. The uniform external field is applied along the $z$-axis. The geometry of the problem is shown in Fig. 1

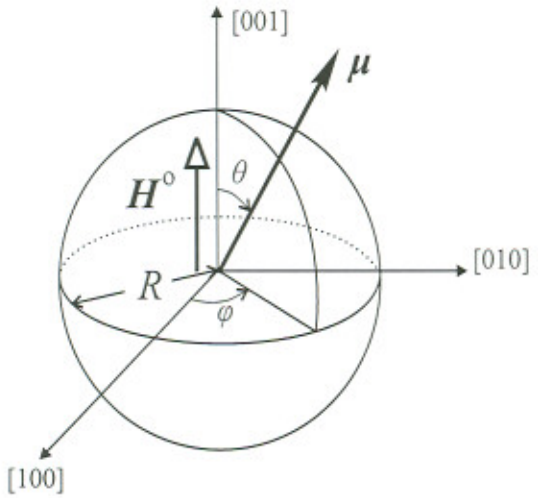


FIG. 1. The geometry of the problem.

The deformations are constrained in two specified "hydrostatic" deformation modes:
*Non-Uniform Strain Mode 1 (M1):* $u_r = P$, $u_\theta = u_\phi = 0$, $P = const.$ with $e_{xx} = P(y^2+z^2)/\left(x^2+y^2+z^2\right)^{3/2}$, $e_{xy} = -Pxy/\left(x^2+y^2+z^2\right)^{3/2}$ (the others are obtained by cyclic permutation).
*Uniform Strain Mode 2 (M2):* $u_r = pr$, $u_\theta = u_\phi = 0$, $p = const.$ with $e_{xx} = e_{yy} = e_{zz} = p$ and $e_{ij} = 0$ for $i \neq j$.
Notice that M1 introduces a singularity at $r = 0$, but the additional accuracy which might be obtained by selecting $u_i$' s that are unique at the origin does not justify the additional amount of work needed to calculate the nucleation field $h_n$. The calculated $h_n$ will be an upper bound to the correct one, since the minimization of (1) is performed under an additional constraint [4]. The magnetization reverses by curling [8]:

$$\begin{aligned} &\alpha_x = -\sqrt{1-\alpha_z^2}\sin\phi,\ \alpha_y = -\sqrt{1-\alpha_z^2}\cos\phi, \\ &\alpha_z = \alpha_z(r,\theta) \simeq g_0(r) + [1-g_0(r)]\cos^2\theta, \\ &U = U(r,\theta) \simeq U_1(r)\cos\theta + U_3(r)\cos^3\theta. \end{aligned} \tag{4}$$

The surface tractions are radial ones ($T_r = const.$, $T_\theta = T_\phi = 0$). From hereafter, the superscript $(i)$ will denote the mode considered.

### C. The Differential Equations

We introduce the dimensionless quantities:

$$\begin{aligned} &E = \frac{G}{2\pi\mu_o\left(M_s^o\right)^2 R_o^3},\ \mathcal{E}_{(...)} = \frac{t^2 w_{(...)}}{\mu_o\left(M_s^o\right)^2}, \\ &t = \frac{r}{R},\ p_o = \frac{P}{R},\ \tau = \frac{T_r}{\mu_o\left(M_s^o\right)^2},\ S = \frac{R}{R_o}, \\ &b_1 = \frac{B_1}{\mu_o\left(M_s^o\right)^2},\ b_2 = \frac{B_2}{\mu_o\left(M_s^o\right)^2},\ k = \frac{2K_1}{\mu_o\left(M_s^o\right)^2}, \\ &v_1 = \frac{U_1}{M_s^o R_0},\ v_3 = \frac{U_3}{M_s^o R_0},\ h = \frac{H_z^o}{M_s^o}, \\ &c_{el}^{(1)} = \frac{8c_{11}+12c_{12}+c_{44}}{5\mu_o\left(M_s^o\right)^2},\ c_{el}^{(2)} = \frac{3\left(c_{11}+2c_{12}\right)}{\mu_o\left(M_s^o\right)^2}, \end{aligned} \tag{5}$$

where $(\,)' \equiv d/dt$, $M_s^o = \rho_o\mu_s$, $R_o = \sqrt{C}/\left(\sqrt{2\mu_o}M_s^o\right)$ is the exchange length, with $C$ the exchange constant, $K_1$ is the anisotropy constant, $B_1$, $B_2$ are the magnetoelastic constants, $c_{11}$, $c_{12}$, $c_{44}$ are the elastic constants, $H_z^o$ is the uniform applied magnetic field along the $z-$direction. Following the approximations introduced in Ref. [8] and carrying out the integrations over $\theta$ and $\phi$ the Gibbs free energy (1) reduces to:

$$\begin{aligned} &E = \int_0^1 \left(\mathcal{E}_{loc} + \mathcal{E}_m + \mathcal{E}_H\right) dt - 2S^3\tau p^{(i)}, \\ &\mathcal{E}_{loc} = \mathcal{E}_{ex} + \mathcal{E}_a + \mathcal{E}_{mel} + \mathcal{E}_{el} \end{aligned} \tag{6}$$

where $p^{(1)} = p_o$, $p^{(2)} = p$ and

$$\begin{aligned} &\mathcal{E}_{ex} = St^2\left\{\frac{2(g_0')^2}{(1-g_0)^2}\left[\frac{1+a_0}{\sqrt{a_0}}\tan^{-1}\left(\frac{1}{\sqrt{a_0}}\right)-1\right] + \right. \\ &\quad + \frac{8}{t^2}\left[1-\sqrt{a_0}\tan^{-1}\left(\frac{1}{\sqrt{a_0}}\right)\right] + \\ &\quad \left. + \frac{2(1-g_0)^2}{t^2}\left(a_0+\frac{1}{3}\right)\right\}, \\ &\mathcal{E}_a = kS^3t^2\frac{1}{45}\left[8+4g_0+12g_0^2-8g_0^3-16g_0^4\right], \\ &\mathcal{E}_{mel}^{(1)} = S^3t^2\left(\frac{p_o}{t}\right)\frac{1}{105}\left[b_1\left(148-16g_0+8g_0^2\right) + \right. \\ &\quad \left. + b_2\left(32-8g_0-24g_0^2\right)\right], \\ &\mathcal{E}_{mel}^{(2)} = S^3t^2 2b_1 p, \\ &\mathcal{E}_{el}^{(1)} = S^3t^2\left(\frac{p_o}{t}\right)^2 c_{el}^{(1)}, \\ &\mathcal{E}_{el}^{(2)} = S^3t^2 c_{el}^{(2)} p^2, \\ &\mathcal{E}_m^{(i)} = S^2t^2\left(\frac{M_s^{(i)}}{M_s^o}\right)\left[\frac{2}{15}\frac{v_1}{t}+\frac{8}{15}g_0\frac{v_1}{t}+\frac{6}{35}\frac{v_3}{t}+\right. \\ &\quad \left. + \frac{8}{35}g_0\frac{v_3}{t}+\frac{1}{5}v_1'+\frac{2}{15}g_0v_1'+\frac{1}{7}v_3'+\frac{2}{35}g_0v_3'\right], \\ &\mathcal{E}_H^{(i)} = S^3t^2\frac{2}{3}h\left(\frac{M_s^{(i)}}{M_s^o}\right)(1+2g_0). \end{aligned} \tag{7}$$

where $M_s^{(i)}/M_s^o = 1 - e_{kk}^{(i)}$, $e_{kk}^{(1)} = 2p_0/t$, $e_{kk}^{(2)} = 3p$, $a_0 = (1+g_0)/(1-g_0)$ and $i = 1, 2$. The vanishing of the first variation of the energy functional (6), with respect to internal variables $g_0$, $v_1$, $v_3$ and $p^{(i)}$ results in the four Euler equations of the problem. However, the variations of $v_1$ and $v_3$ depend on the variations of $g_0$ and $p^{(i)}$ due to Eq. (2):

$$\delta \int_0^1 \mathcal{E}_m(g_0,\, v_1,\, v_3,\, v_1',\, v_3')dt = \int_0^1 \left[2\frac{\partial \mathcal{E}_m}{\partial g_0}\delta g_0 + 2\mathcal{E}_m(g_0,\, v_1',\, v_3',\, v_1'',\, v_3'')\delta p^{(i)}\right] dt - \quad (8)$$
$$-\left(\frac{M_s^{(i)}}{M_s^o}\right)^2 \frac{2S^3}{105}\left[8g_0(1)^2 + 12g_0(1) + 15\right]\delta p^{(i)}.$$

The Euler' s equations are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{E}_{loc}}{\partial g_0'}\right) - \frac{\partial \mathcal{E}_{loc}}{\partial g_0} - 2\frac{\partial \mathcal{E}_m}{\partial g_0} - \frac{\partial \mathcal{E}_H}{\partial g_0} = 0 \quad (9)$$

$$\frac{\partial \mathcal{E}_{loc}}{\partial p^{(i)}} + 2\mathcal{E}_m\left(g_0, v_1', v_3', v_1'', v_3''\right) = 0, \quad (10)$$

with boundary conditions

$$\left.\frac{\partial \mathcal{E}_{loc}}{\partial g_0}\right|_{t=1} = 0, \quad (11)$$

$$-\frac{M_s^{(i)}}{M_s^o}\frac{1}{105}\left[8g_0(1)^2 + 12g_0(1) + 15\right] = \tau. \quad (12)$$

Eqs. (9) and (10) and boundary conditions (11) and (12) account for magnetic and mechanical equilibrium, respectively. We must add to the equilibrium equations (9-12) the equations for the potential problem (Eqs. 4) of Ref. [8].

## III. LINEARIZATION (NUCLEATION FIELD CALCULATION)

For small deviations from saturation, Euler' s equations (9-10) reduce to the equations that determine the nucleation mode. Setting

$$g_0(t) = 1 - \epsilon\, Q(t)^2 \quad (13)$$

with $0 < \epsilon \ll 1$, taking into account (10) and expanding the rhs of (9) up to terms that are linear in $\epsilon$, one finds for $Q(t)$ the equation (the self-magnetostatic potential is taken to be that corresponding to saturation: $v_1 = St/3, v_3 = 0$):

$$\frac{d^2Q}{dt^2} + \frac{2}{t}\frac{dQ}{dt} + \left[\left(\mathcal{B}^{(i)}\right)^2 - \frac{2}{t^2}\right] Q = 0, \quad (14)$$

where

$$\left(\frac{\mathcal{B}^{(1)}}{S}\right)^2 = -\frac{k}{2} + \frac{1}{\left(9c_{el}^{(1)} - 4\right)} \times$$
$$\times \left[\frac{2}{5}b_2(3b_1+1) + \frac{3}{2}\left(4b_1 + 3c_{el}^{(1)}\right)\left(\frac{1}{3} - h\right)\right], \quad (15)$$

$$\left(\frac{\mathcal{B}^{(2)}}{S}\right)^2 = -\frac{k}{2} + \frac{\left[\left(9c_{el}^{(2)} + 27b_1\right)t + 2\right]}{2\left(9c_{el}^{(2)}t - 2\right)}\left(\frac{1}{3} - h\right). \quad (16)$$

Notice that $\mathcal{B}^{(1)} = const.$ while $\mathcal{B}^{(2)} = \mathcal{B}^{(2)}(t)$. Without loosing much from the physics of the problem, we might replace for M2 Eq. (10) with the approximate one $p \simeq -b_1/c_{el}^{(2)}$. Then

$$\left(\mathcal{B}^{(2)}\right)^2 = S^2\left\{-\frac{k}{2} + \frac{(1-3p)}{2}\left(\frac{1}{3} - h\right)\right\}. \quad (17)$$

The general solution of (14) is the spherical Bessel function of first order $Q(t) = j_1(\mathcal{B}t)$. At nucleation $h = h_n$, this function satisfies the boundary condition (11) $j_1'(\mathcal{B}_n) = 0$, the smallest root of which is $\mathcal{B}_n = 2.0816$. Thus the nucleation field is given by:

$$h_n^{(i)} = \frac{1}{3} + \beta_{mel}^{(i)} - \gamma_{mel}^{(i)}\left(k + \frac{2\mathcal{B}_n^2}{S^2}\right), \quad (18)$$

where

$$\beta_{mel}^{(1)} = \frac{4b_2(3b_1+1)}{5\left(9c_{el}^{(1)} + 12b_1\right)}, \qquad \gamma_{mel}^{(1)} = \frac{9c_{el}^{(1)} - 4}{9c_{el}^{(1)} + 12b_1}, \quad (19)$$

$$\beta_{mel}^{(2)} = 0, \qquad \gamma_{mel}^{(2)} = \frac{1}{1-3p}, \qquad p \simeq -\frac{b_1}{c_{el}^{(2)}}. \quad (20)$$

For both deformation modes $h_n$ is affected by magnetostriction (the rigid sphere limit corresponds to $\beta_{mel}^{(i)} = 0$, $\gamma_{mel}^{(i)} = 1$). For small particles the coherent Stoner-Wolfarth (SW) rotation governs the nucleation process. For M1 the SW rotation is not a permissible magnetization reversal mode, while for M2 is, with the well known nucleation field:

$$h_n^{SW} = -\frac{k}{1-3p}. \quad (21)$$

The crossover from SW to curling rotation determines the critical radius, which is affected by magnetostriction:

$$S_c^{el} = \frac{S_c^{rigid}}{\sqrt{1-3p}}, \qquad S_c^{rigid} = \sqrt{6\,\mathcal{B}_n^2} = 5.099. \quad (22)$$

According to the predictions of micromagnetics, the nucleation field $h_n$ serves as a lower bound to the experimental measured coercivity $h_c$ (which is defined as a positive quantity):

$$h_c \geq -h_n. \quad (23)$$

As long as nucleation starts at positive fields ($h_n > 0$) the above inequality is an empty statement. But even for $h_n < 0$ it is not always satisfied (in experiments $h_c \ll -h_n$, sometimes by two or more orders of magnitude). This is known as Brown' s coercivity paradox [10]. It is attributed to the existence of crystalline imperfections such as magnetic impurities and dislocations [3]. Magnetostriction effects might be a step towards resolving that paradox, as long as

$$\Delta h_n \equiv h_n^{rigid} - h_n^{elastic} < 0, \tag{24}$$

with $h_n^{rigid}$ and $h_n^{elastic}$ both negative. Depending on the values and signs of the magnetoelastic factors $\beta_{mel}^{(i)}$ and $\gamma_{mel}^{(i)}$ Eq. (18) might satisfy Eq. (24), but it cannot help on resolving the paradox, alone, because it computes an upper bound to the correct nucleation field. Only after calculating a lower bound to $h_n$ that also satisfies Eq. (24) and is close to the upper bound (18) this result is of practical importance. For materials for which the upper bound (18) does not satisfy (24) there is no need to compute a lower bound, since in this case the paradox is even more outstanding. The nucleation field is plotted in Fig. 1 as a function of the reduce radius $S$, for a fictitious material with negative magnetostriction ($b_1$, $b_2 > 0$) and positive magnetocrystalline anisotropy constant ($k > 0$).

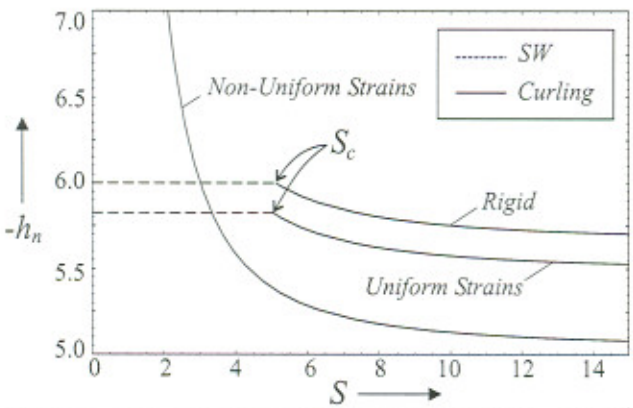


FIG. 2. $-h_n$ vs $S$ for a fictitious material ($b_1 = 10^4$, $b_2 = 7 \times 10^3$, $k = 6$, $c_{el}^{(1)} = 3 \times 10^7$, $c_{el}^{(2)} = 10^6$).

For this particular set of material constants, both deformation modes satisfy Eq. (24). Notice the shift of the critical radius is $\Delta S_c = S_c^{rigid} - S_c^{elastic} = 7.48 \times 10^{-2}$ and that non-uniform strains are better candidates for resolving the paradox than uniform ones. Assuming that $M_s^o = 10^2\, kA/m$ and that the material is "elastically isotropic" ($c_{44} = (c_{11} - c_{12})/2$) [11], Fig. 1 corresponds to $K_1 = 3.77 \times 10^4 J/m^3$, $c_{44} = 5.06 \times 10^{11} J/m^3$ and magnetostriction constants $\lambda_{100} = -8.27 \times 10^{-5}$, $\lambda_{111} = -1.16 \times 10^{-4}$. For Fe [12] $\Delta S_c = -1.92 \times 10^{-6}$, $\Delta h_n^{(1)} = O(10^{-6}) > 0$ and $\Delta h_n^{(2)} = O(-10^{-6}) < 0$ for $S > 1$. Similarly, for Ni [12] $\Delta S_c = 2.39 \times 10^{-5}$, $\Delta h_n^{(1)} = O(10^{-5}) > 0$ and $\Delta h_n^{(2)} = O(-10^{-3}) < 0$ for $S > 1$.

## IV. CONCLUSIONS

Even though it has been stated by Brown [4] that magnetostriction affects the nucleation by curling in an ellipsoidal ferromagnetic particle, no estimates of this effect has ever been given. In our work we computed an upper bound to the nucleation field for an elastic sphere. This value is of practical importance only in combination with a close enough lower bound. Methods already used for obtaining lower bounds to the rigid problem [13], can be applied for elastic ferromagnets, but with the additional cost of solving the "mechanical" problem. We proved that for materials for which the upper bound (18) does not satisfy (24) the paradox is even more outstanding. The shift of the critical radius for curling vs. SW rotation for Fe and Ni was not a significant one. Both deformation modes considered, failed to satisfy the "mechanical" boundary conditions. A more rigorous approach requires consideretion of a mechanical fixed boundary surface [4]. Work is under way in this direction. The magnetization reversal beyond nucleation is also under investigation.

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