WAVE PROPAGATION IN HUMAN LONG BONES

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1. Introduction

Many researchers have worked on the modeling of the wave propagation problem in long bones. Wave propagation and vibration of bone can be used to determine the properties of bone *in vivo* which is of great importance in the examination of growth mechanisms. Vayo and Chista [1] modeled the wave propagation problem in long bones assuming that the bone is composed of two-layered elastic transversely isotropic cylinders corresponding to the cortical and spongy bone. Nowinski and Davis [2] have investigated theoretically the wave propagation in wet bone. They assumed that long bone is a solid poroelastic cylinder composed of linear perfect elastic solid and perfect fluid. Paul and Murali [3] analyzed the propagation of flexural waves in bone, treating bone as a hollow poroelastic cylinder. The problem of free vibrations of a double layered elastic isotropic cylinder was studied by Charalambopoulos et al. [4], by simulating the outer layer with the space of cortical bone and the inner one with the medullary space.

The bone tissue has been studied as a piezoelectric material of crystal class 6. Ambardar and Ferris [5] investigated the wave propagation problem in long bone considering it as a two-layered piezoelectric cylindrical shell of crystal class 6mm. Güzelsu and Saha [6] studied electromechanical wave propagation in dry long bone modeled as a piezoelectric transverse isotropic hollow cylinder. The wave propagation in a piezoelectric bone of arbitrary cross section with a cylindrical cavity of arbitrary shape was studied by Paul and Vankatensan [7], [8]. In Ref. [9] the human long bone was simulated with a hollow piezoelectric cylinder of crystal class 6 and the propagation of harmonic waves was studied.

In the present work an attempt has been made to study the wave propagation problem in a piezoelectric cylinder of crystal class 6, filled with an incompressible viscous fluid. The system under consideration simulates the human long bones where the piezoelectric solid cylinder corresponds to the cortical bone and the viscous fluid to the bone marrow. The mathematical modeling of the solid cylinder is based on the three-dimensional theory of piezoelectricity while the linearized Navier-Stokes equations and the continuity equation are employed to describe the dynamic behavior of the fluid medium. The present study is based on the analysis proposed in Ref. [9] and that of Ref. [10]. The frequency equation is obtained for the case where the exterior lateral surface of the cylinder is stress free and coated with electrodes that are shorted and the conditions of continuity of the fields entering the problem in the interior lateral surface.

2. Mathematical Model

Let us consider an infinite hollow cylinder filled with an incompressible viscous fluid (Fig. 1).

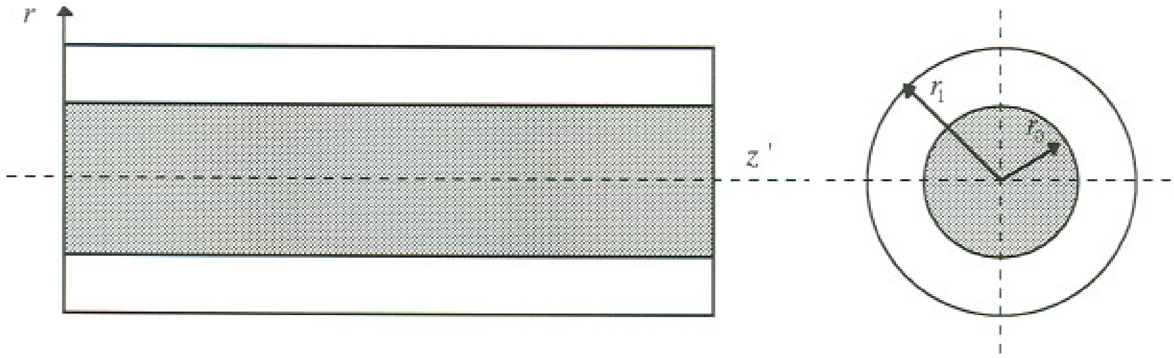

Figure 1: Bone Geometry

Piezoelectric Cylinder

The equations of motion and the Gaussian equation for a piezoelectric cylinder of crystal class 6 are [9]:

$$c_{11}\left(u'_{r,rr}+r^{-1}u'_{r,r}-r^{-2}u'_r\right)+c_{66}r^{-2}u'_{r,\theta\theta}+c_{44}u'_{r,z'z'}+\left(c_{66}+c_{12}\right)r^{-1}u'_{\theta,r\theta}$$

$$-\left(c_{66}+c_{11}\right)r^{-2}u'_{\theta,\theta}+\left(c_{44}+c_{13}\right)u'_{z,rz'}+\left(e_{15}+e_{31}\right)V'_{,rz'}-e_{14}r^{-1}V'_{,\theta z'}=\rho_s\frac{\partial^2 u'_r}{\partial t'^2}, \qquad (1)$$

$$\left(c_{66}+c_{12}\right)r^{-1}u'_{r,r\theta}+\left(c_{66}+c_{11}\right)r^{-2}u'_{r,\theta}+c_{66}\left(u'_{\theta,rr}+r^{-1}u'_{\theta,r}-r^{-2}u'_\theta\right)+c_{11}r^{-2}u'_{\theta,\theta\theta}$$

$$+c_{44}u'_{\theta,z'z'}+(c_{44}+c_{13})r^{-1}u'_{z,\theta z'}+e_{14}V'_{,rz'}+(e_{15}+e_{31})r^{-1}V'_{,\theta z'}=\rho_s\frac{\partial^2 u'_\theta}{\partial t'^2}, \qquad (2)$$

$$(c_{44}+c_{13})\left(u'_{r,rz'}+r^{-1}u'_{r,z'}+r^{-1}u'_{\theta,\theta z'}\right)+c_{44}\left(u'_{z,rr}+r^{-1}u'_{z,r}+r^{-2}u'_{z,\theta\theta}\right)+c_{33}u'_{z,z'z'}$$

$$+e_{15}\left(V'_{,rr}+r^{-1}V'_{,r}+r^{-2}V'_{,\theta\theta}\right)+e_{33}V'_{,z'z'}=\rho_s\frac{\partial^2 u'_z}{\partial t'^2}, \qquad (3)$$

$$\in_{11}\left(V'_{,rr}+r^{-1}V'_{,r}+r^{-2}V'_{,\theta\theta}\right)+\in_{33}V'_{,z'z'}-(e_{15}+e_{31})\left(u'_{r,rz'}+r^{-1}u'_{r,z'}+r^{-1}u'_{\theta,\theta z'}\right)$$

$$+e_{14}\left(r^{-1}u'_{r,\theta z'}-u'_{\theta,rz'}-r^{-1}u'_{\theta,z'}\right)-e_{15}\left(u'_{z,rr}+r^{-1}u'_{z,r}+r^{-1}u'_{z,\theta\theta}\right)-e_{33}u'_{z,z'z'}=0. \quad (4)$$

where u'_r, u'_θ and u'_z are the components of the displacement vector, V' is the electrostatic potential, c_{ij} are the elastic constants with $c_{66}=\frac{1}{2}(c_{11}-c_{12})$, e_{ij} are the piezoelectric constants, $\in_{ij}$ are the dielectric constants, ρ_s is the density of the solid cylinder and $(\ )_{,\xi}\equiv\partial(\ )/\partial\xi$.

We introduce the following dimensionless variables:

$$x=\frac{r}{R},\qquad z=\frac{z'}{R},\qquad u_x=\frac{1}{R}u'_r,\qquad u_\theta=\frac{1}{R}u'_\theta,\qquad u_z=\frac{1}{R}u'_z$$

$$V=\frac{e_{33}}{Rc_{44}}V'\quad \bar{c}_{ij}=\frac{c_{ij}}{c_{44}},\qquad \bar{e}_{ij}=\frac{e_{ij}}{e_{33}},\qquad \varepsilon_{i3}^2=\frac{e_{33}^2}{c_{44}\in_{ii}},\qquad t=\frac{c_s}{R}\ t',\ c_s^2=\frac{c_{44}}{\rho_s}$$

where $R=r_1-r_0$.

We seek a solution of the following form:

$$u_x=\left(G_{,x}+\frac{1}{x}\psi_{,\theta}\right)e^{i(\lambda z-\Omega t)},\quad u_\theta=\left(\frac{1}{x}G_{,\theta}-\psi_{,x}\right)e^{i(\lambda z-\Omega t)},$$

$$u_z=i\,w\,e^{i(\lambda z-\Omega t)},\ V=i\,\phi\ e^{i(\lambda z-\Omega t)}, \qquad (5)$$

where G, ψ, w and ϕ are functions of x and θ, $\lambda=R\gamma$, γ is the wavenumber, $\Omega^2=\frac{(R\omega)^2\rho_s}{c_{44}}$, ω is the angular frequency and $i=\sqrt{-1}$.

Following the method introduced in Ref. 9, the elastic displacements and the electric potential can be written as:

$$u_x = \sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}\delta_j^{p1}\frac{\partial\zeta^{m,l}(k_j x)}{\partial x}+\beta_j^{m,l}\delta_j^{p2}\frac{m}{x}\zeta^{m,l}(k_j x)\right]\cos(m\theta)\right.$$
$$\left.+\left[-a_j^{m,l}\delta_j^{p2}\frac{m}{x}\zeta^{m,l}(k_j x)+\beta_j^{m,l}\delta_j^{p1}\frac{\partial\zeta^{m,l}(k_j x)}{\partial x}\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)},$$

$$u_\theta = \sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[-a_j^{m,l}\delta_j^{p2}\frac{\partial\zeta^{m,l}(k_j x)}{\partial x}+\beta_j^{m,l}\delta_j^{p1}\frac{m}{x}\zeta^{m,l}(k_j x)\right]\cos(m\theta)\right.$$
$$\left.-\left[a_j^{m,l}\delta_j^{p1}\frac{m}{x}\zeta^{m,l}(k_j x)+\beta_j^{m,l}\delta_j^{p2}\frac{\partial\zeta^{m,l}(k_j x)}{\partial x}\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)},$$

$$u_z = i\sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}\delta_j^{p3}\zeta^{m,l}(k_j x)\right]\cos(m\theta)+\left[\beta_j^{m,l}\delta_j^{p3}\zeta^{m,l}(k_j x)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)},$$

$$V = i\sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}\delta_j^{p4}\zeta^{m,l}(k_j x)\right]\cos(m\theta)+\left[\beta_j^{m,l}\delta_j^{p4}\zeta^{m,l}(k_j x)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \qquad (6)$$

where $a_j^{m,l}$ and $\beta_j^{m,l}$ are arbitrary constants,

$$\delta_j^{pq} = -d_1^{pq}k_j^6 + d_2^{pq}k_j^4 - d_3^{pq}k_j^2 + d_4^{pq}, \qquad p,q,j = 1,2,3,4,$$

$$\zeta^{m,l}(k_j x) = \begin{cases} \begin{cases} J^m(k_j x), & l=1, \quad (Bessel\ of\ 1st\ kind) \\ Y^m(k_j x), & l=2, \quad (Bessel\ of\ 2nd\ kind) \end{cases} & if\ k_j^2>0, \\ \begin{cases} I^m(k_j x), & l=1, \quad (mod.\ Bessel\ of\ 1nd\ kind) \\ K^m(k_j x), & l=2, \quad (mod.\ Bessel\ of\ 2nd\ kind) \end{cases} & if\ k_j^2<0, \end{cases}$$

$k_j = \left|k_j^2\right|^{1/2}$ and k_j^2, $j = 1,2,3,4$ are the roots of the equation:

$$ak_j^8 - bk_j^6 + ck_j^4 - dk_j^2 + e = 0,$$

and the coefficients d_s^{pq}, $p,q,s = 1,2,3,4$ and a, b, c, d, e are given in Appendices A and B, respectively.

Using (6) the stresses are expressed as follows:

$$T_{xx} = \sum_{j=1}^{4}\sum_{l=1}^{2}\Big\{\Big[a_j^{m,l} P_{p,j}^{m,l}(k_j x) + \beta_j^{m,l} Q_{p,j}^{m,l}(k_j x)\Big]\cos(m\theta)$$

$$+\Big[-a_j^{m,l} Q_{p,j}^{m,l}(k_j x) + \beta_j^{m,l} P_{p,j}^{m,l}(k_j x)\Big]\sin(m\theta)\Big\} e^{i(\lambda z - \Omega t)},$$

$$T_{x\theta} = \tilde{c}_{66}\sum_{j=1}^{4}\sum_{l=1}^{2}\Big\{\Big[a_j^{m,l} R_{p,j}^{m,l}(k_j x) + \beta_j^{m,l} S_{p,j}^{m,l}(k_j x)\Big]\cos(m\theta)$$

$$+\Big[-a_j^{m,l} S_{p,j}^{m,l}(k_j x) + \beta_j^{m,l} R_{p,j}^{m,l}(k_j x)\Big]\sin(m\theta)\Big\} e^{i(\lambda z - \Omega t)},$$

$$T_{xz} = i\sum_{j=1}^{4}\sum_{l=1}^{2}\Big\{\Big[a_j^{m,l} T_{p,j}^{m,l}(k_j x) + \beta_j^{m,l} U_{p,j}^{m,l}(k_j x)\Big]\cos(m\theta)$$

$$+\Big[-a_j^{m,l} S_{p,j}^{m,l}(k_j x) + \beta_j^{m,l} R_{p,j}^{m,l}(k_j x)\Big]\sin(m\theta)\Big\} e^{i(\lambda z - \Omega t)}, \qquad (7)$$

where $P_{p,j}^{m,l}$, $Q_{p,j}^{m,l}$, $R_{p,j}^{m,l}$, $S_{p,j}^{m,l}$, $T_{p,j}^{m,l}$ and $U_{p,j}^{m,l}$ are given Appendix C.

Incompressible Viscous Fluid

We consider the following dimensionless variables:

$$x = \frac{r}{R}, \qquad z = \frac{z'}{R}, \qquad t = \frac{c_s}{R}t', \qquad c_s^2 = \frac{c_{44}}{\rho_s},$$

$$v_x = \frac{1}{c_s}v'_r, \qquad v_\theta = \frac{1}{c_s}v'_\theta, \qquad v_z = \frac{1}{c_s}v'_z, \qquad p = \frac{1}{\rho_f c_s^2}p', \qquad R_e = \frac{\rho_f c_s R}{\eta},$$

where v_x, v_θ and v_z are the components of the velocity of the fluid, p is the pressure, ρ_f is the density of the fluid, η is the viscosity and R_e is the Reynolds number.

The equations governing the motion of the fluid are the linearized Navier-Stokes equations [11]:

$$\dot{v}_x = -p_{,x} + \frac{1}{R_e}\left(v_{x,xx} + \frac{1}{x}v_{x,x} + \frac{1}{x^2}v_{x,\theta\theta} + v_{x,zz} - \frac{2}{x^2}v_{\theta,\theta} - \frac{1}{x^2}v_x\right), \qquad (8)$$

$$\dot{v}_\theta = -\frac{1}{x}p_{,\theta} + \frac{1}{R_e}\left(v_{\theta,xx} + \frac{1}{x}v_{\theta,x} + \frac{1}{x^2}v_{\theta,\theta\theta} + v_{\theta,zz} + \frac{2}{x^2}v_{x,\theta} - \frac{1}{x^2}v_\theta\right), \qquad (9)$$

$$\dot{v}_z = -p_{,z} + \frac{1}{R_e}\left(v_{z,xx} + \frac{1}{x}v_{z,x} + \frac{1}{x^2}v_{z,\theta\theta} + v_{z,zz}\right), \qquad (10)$$

and the continuity equation:

$$v_{x,x} + \frac{1}{x} v_x + \frac{1}{x} v_{\theta,\theta} + v_{z,z} = 0. \tag{11}$$

We seek a solution of the form:

$$v_x = \left(U_{,x} + \frac{1}{x} \Psi_{,\theta} \right) e^{i(\lambda z - \Omega t)}, \; v_\theta = \left(\frac{1}{x} U_{,\theta} - \Psi_{,x} \right) e^{i(\lambda z - \Omega t)},$$

$$v_z = W \;\; e^{i(\lambda z - \Omega t)}, \; p = P \;\; e^{i(\lambda z - \Omega t)}, \tag{12}$$

where U, Ψ, W and P are functions of x and θ.

The system of equations (8) - (11) is simplified as follows:

$$\left[\nabla^2 + iR_e \Omega - \lambda^2 \right] \Psi = 0, \tag{13}$$

$$\mathbf{C} \begin{bmatrix} U \\ W \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

where

$$\mathbf{C} = \begin{bmatrix} \nabla^2 + iR_e \Omega - \lambda^2 & 0 & -R_e \\ 0 & \nabla^2 + iR_e \Omega - \lambda^2 & -i\lambda R_e \\ \nabla^2 & i\lambda & 0 \end{bmatrix},$$

and $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial \theta^2}$.

Following Ref. 10, we introduce a new function $g = g(x, \theta)$ such that $\{\det \mathbf{C}\} g = 0$ or equivalently,

$$\left(\nabla^2 - \mu_1^2 \right) \left(\nabla^2 - \mu_2^2 \right) g = 0. \tag{15}$$

where $\mu_1^2 = \lambda^2 - i\Omega R_e$ and $\mu_2^2 = \lambda^2$.

Then the solution of the system (14) is given as:

$$U = \mathrm{C}_{i1} g, \qquad W = \mathrm{C}_{i2} g, \qquad P = \mathrm{C}_{i3} g, \qquad i = 1,2,3, \tag{16}$$

where C_{ij} are the algebraic components of the matrix $\mathbf{C}$ satisfying the relations:

$$\mathrm{C}_{ij} = c_1^{ij} \nabla^4 + c_2^{ij} \nabla^2 + c_3^{ij},$$

and the coefficients $c_k^{ij},\ i,j,k=1,2,3$ are given in the Appendix D.

The solution of (14) is equivalent of solving equation(15). This equation admits a solution of the form:

$$g=\sum_{l=1}^{2}\left\{A_l^m\cos(m\theta)+B_l^m\sin(m\theta)\right\}I^m(\mu_l x),\qquad m\in\mathrm{N} \tag{17}$$

where $A_l^m,\ B_l^m$ are arbitrary constants and $I^m(\mu_l x)$ are the modified Bessel functions.

Solving equation (13) we have:

$$\Psi=\left\{A_3^m\sin(m\theta)+B_3^m\cos(m\theta)\right\}I^m(\mu_3 x),\qquad m\in\mathrm{N} \tag{18}$$

where $A_3^m,\ B_3^m$ are arbitrary constants and $\mu_3=\mu_1$.

Using the relations (12), (16) -(18), the solution for the system (8)-(11) is expressed as:

$$v_x=\sum_{l=1}^{2}\left\{\left[A_l^m\sigma_l^{i1}\frac{\partial I^m(\mu_l x)}{\partial x}+A_3^m\frac{m}{x}I^m(\mu_3 x)\right]\cos(m\theta)\right.$$

$$\left.+\left[B_l^m\sigma_l^{i1}\frac{\partial I^m(\mu_l x)}{\partial x}-B_3^m\frac{m}{x}I^m(\mu_3 x)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)},$$

$$v_\theta=\sum_{l=1}^{2}\left\{\left[B_l^m\sigma_l^{i1}\frac{m}{x}I^m(\mu_l x)-B_3^m\frac{\partial I^m(\mu_3 x)}{\partial x}\right]\cos(m\theta)\right.$$

$$\left.-\left[A_l^m\sigma_l^{i1}\frac{m}{x}I^m(\mu_l x)+A_3^m\frac{\partial I^m(\mu_3 x)}{\partial x}\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)},$$

$$v_z=\sum_{l=1}^{2}\left\{\sigma_l^{i2}\left[A_l^m\cos(m\theta)+B_l^m\sin(m\theta)\right]I^m(\mu_l x)\right\}e^{i(\lambda z-\Omega t)},$$

$$p=\sum_{l=1}^{2}\left\{\sigma_l^{i3}\left[A_l^m\cos(m\theta)+B_l^m\sin(m\theta)\right]I^m(\mu_l x)\right\}e^{i(\lambda z-\Omega t)}, \tag{19}$$

where $\sigma_l^{ij}=c_1^{ij}\mu_l^4+c_2^{ij}\mu_l^2+c_3^{ij}$.

For an incompressible viscous fluid the stresses $\tau_{xx},\ \tau_{x\theta},\ \tau_{xz}$ are given by

$$\tau_{xx} = -p + \frac{2}{R_e} v_{x,x}, \qquad \tau_{x\theta} = \frac{1}{R_e}\left(\frac{1}{x} v_{x,\theta} + v_{\theta,x} - \frac{1}{x} v_\theta\right), \qquad \tau_{xz} = \frac{1}{R_e}\left(v_{z,x} + v_{x,z}\right).$$

Replacing into the above expressions the relations (19) we obtain

$$\tau_{xx} = \sum_{j=1}^{3}\left[A_j^m q_j^m(\mu_j x)\cos(m\theta) + B_j^m r_j^m(\mu_j x)\sin(m\theta)\right]e^{i(\lambda z - \Omega t)},$$

$$\tau_{x\theta} = \sum_{j=1}^{3}\left[B_j^m s_j^m(\mu_j x)\cos(m\theta) + A_j^m t_j^m(\mu_j x)\sin(m\theta)\right]e^{i(\lambda z - \Omega t)},$$

$$\tau_{xz} = \sum_{j=1}^{3}\left[A_j^m w_j^m(\mu_j x)\cos(m\theta) + B_j^m z_j^m(\mu_j x)\sin(m\theta)\right]e^{i(\lambda z - \Omega t)}, \tag{20}$$

where $q_j^m(\mu_j x)$, $r_j^m(\mu_j x)$, $s_j^m(\mu_j x)$, $t_j^m(\mu_j x)$, $w_j^m(\mu_j x)$ and $z_j^m(\mu_j x)$ are given in Appendix E.

3. Frequency Equation

For the system under discussion we consider the following boundary conditions:

$$\left.\begin{array}{l} T_{xx} = -\tau_{xx},\ T_{x\theta} = -\tau_{x\theta},\ T_{xz} = -\tau_{xz} \\ \dot{u}_x = v_x,\ \dot{u}_\theta = v_\theta,\ \dot{u}_z = v_z,\ V = 0 \end{array}\right\} \text{for } x = \frac{r_0}{R}, \tag{21}$$

$$T_{xx} = 0,\ T_{x\theta} = 0,\ T_{xz} = 0,\ V = 0 \text{ for } x = \frac{r_1}{R}. \tag{22}$$

The conditions (22) express the fact that the lateral surface of the piezoelectric cylinder is free of mechanical traction and coated with electrodes which are shorted, while (21) indicates that conditions of continuity of the dynamic, kinetic and electrical quantities are met on the internal lateral surface of the cylinder.

Satisfaction of the boundary conditions leads to a 22×22 linear system of the form:

$\mathcal{M}\mathbf{x} = \mathbf{0}$

where

$$\mathbf{x} = \left(\alpha_j^{m,1},\ \alpha_j^{m,2},\ \beta_j^{m,1},\ \beta_j^{m,2},\ A_\alpha^m,\ B_\alpha^m\right), \qquad j = 1,2,3,4, \quad \alpha = 1,2,3.$$

In order for a nontrivial solution to exist, the determinant of the coefficients matrix must vanish:

$$\det(\mathcal{M}_{rs}) = 0, \qquad r,s = 1,2,\dots,22, \tag{23}$$

(the elements of the matrix $\mathcal{M}_{rs}$ are given in Appendix F).

Equation (23) relates the wavenumber λ with the frequency Ω. The frequency is generally complex and can be written as the sum of real and imaginary parts, e.g. $\Omega = \Omega_1 + i\Omega_2$. For various values of λ the angular frequency Ω_1 and the attenuation Ω_2 can be evaluated numerically [12]

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Appendix A

$$d_1^{11} = \tilde{c}_{66}\left(\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\right),$$

$$d_2^{11} = -\left\{\tilde{e}_{14}^2 + \tilde{e}_{15}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{66}\left[2\tilde{e}_{15} + \tilde{c}_{33}\varepsilon_{13}^{-2} + \varepsilon_{33}^{-2}\right]\right\}\lambda^2 + \left\{\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\left[1 + \tilde{c}_{66}\right]\right\}\Omega^2,$$

$$d_3^{11} = \left\{\tilde{c}_{66} + 2\tilde{e}_{15} + \varepsilon_{33}^{-2} + \tilde{c}_{33}\left(\tilde{e}_{14}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{66}\varepsilon_{33}^{-2}\right)\right\}\lambda^4$$
$$-\left\{\left(\tilde{e}_{14}^2 + 2\tilde{e}_{15}\right) + \varepsilon_{13}^{-2}\left(1 + \tilde{c}_{33}\right) + \varepsilon_{33}^{-2}\left(1 + \tilde{c}_{66}\right)\right\}\lambda^2\Omega^2 + \varepsilon_{13}^{-2}\Omega^4,$$

$$d_4^{11} = -\left\{1 + \tilde{c}_{33}\varepsilon_{33}^{-2}\right\}\lambda^6 + \left\{1 + \varepsilon_{33}^{-2}\left(1 + \tilde{c}_{33}\right)\right\}\lambda^4\Omega^2 - \varepsilon_{33}^{-2}\lambda^2\Omega^4,$$

$$d_1^{12} = 0,$$

$$d_2^{12} = -\tilde{e}_{14}\left\{\tilde{e}_{31} - \tilde{c}_{13}\tilde{e}_{15}\right\}\lambda^2,$$

$$d_3^{12} = -\tilde{e}_{14}\left\{\left[\left(1 + \tilde{c}_{13}\right) - \tilde{c}_{33}\left(\tilde{e}_{15} + \tilde{e}_{31}\right)\right]\lambda^4 + \left(\tilde{e}_{15} + \tilde{e}_{31}\right)\lambda^2\Omega^2\right\},$$

$$d_4^{12} = 0,$$

$$d_1^{13} = -\tilde{c}_{66}\left\{\tilde{e}_{15}\left(\tilde{e}_{15} + \tilde{e}_{31}\right) + \varepsilon_{13}^{-2}\left(1 + \tilde{c}_{13}\right)\right\}\lambda,$$

$$d_2^{13} = \left\{\left(1 + \tilde{c}_{13}\right)\left(\tilde{e}_{14}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{66}\varepsilon_{33}^{-2}\right) + \left(\tilde{c}_{66} + \tilde{e}_{15}\right)\left(\tilde{e}_{15} + \tilde{e}_{31}\right)\right\}\lambda^3$$
$$-\left\{\tilde{e}_{15}\left(\tilde{e}_{15} + \tilde{e}_{31}\right) + \varepsilon_{13}^{-2}\left(1 + \tilde{c}_{13}\right)\right\}\lambda\Omega^2,$$

$$d_3^{13} = \left\{\varepsilon_{33}^{-2}\left(1 + \tilde{c}_{13}\right) + \left(\tilde{e}_{15} + \tilde{e}_{31}\right)\right\}\left(\Omega^2 - \lambda^2\right)\lambda^3,$$

$$d_4^{13} = 0,$$

$$d_1^{14} = -\tilde{c}_{66}\left(\tilde{c}_{13}\tilde{e}_{15} - \tilde{e}_{31}\right)\lambda,$$

$$d_2^{14} = \left\{\tilde{c}_{66}\left(1+\tilde{c}_{13}\right)+\left(\tilde{c}_{13}\tilde{e}_{15}-\tilde{e}_{31}\right)-\tilde{c}_{66}\tilde{c}_{33}\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\right\}\lambda^3$$
$$-\left\{\left(\tilde{c}_{13}\tilde{e}_{15}-\tilde{e}_{31}\right)-\tilde{c}_{66}\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\right\}\lambda\Omega^2,$$

$$d_3^{14} = -\left\{\left(1+\tilde{c}_{13}\right)-\tilde{c}_{33}\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\right\}\lambda^5$$
$$-\left\{-\left(1+\tilde{c}_{13}\right)+\left(1+\tilde{c}_{33}\right)\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\right\}\lambda^3\Omega^2+\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\lambda\Omega^4,$$

$$d_4^{14} = 0,$$

$$d_1^{21} = 0,$$

$$d_2^{21} = -\tilde{e}_{14}\left(\tilde{e}_{31}-\tilde{c}_{13}\tilde{e}_{15}\right)\lambda^2,$$

$$d_3^{21} = -\tilde{e}_{14}\left\{\left[\left(1+\tilde{c}_{13}\right)-\tilde{c}_{33}\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\right]\lambda^4+\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\lambda^2\Omega^2\right\},$$

$$d_4^{21} = 0,$$

$$d_1^{22} = \tilde{c}_{11}\left(\tilde{e}_{15}^2+\varepsilon_{13}^{-2}\right),$$

$$d_2^{22} = \left\{-\left(\tilde{e}_{15}+\tilde{e}_{31}\right)^2+2\left(1+\tilde{c}_{13}\right)\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\overline{e}_{15}-\tilde{e}_{15}\left(2\tilde{c}_{11}+\tilde{e}_{15}\right)\right.$$
$$\left.+\varepsilon_{13}^{-2}\tilde{c}_{13}\left(2+\tilde{c}_{13}\right)-\tilde{c}_{11}\left(\tilde{c}_{33}\varepsilon_{13}^{-2}+\varepsilon_{33}^{-2}\right)\right\}\lambda^2+\left\{\tilde{e}_{15}^2+\varepsilon_{13}^{-2}\left(1+\tilde{c}_{11}\right)\right\}\Omega^2,$$

$$d_3^{22} = \left\{-2\left(1+\tilde{c}_{13}\right)\left(\tilde{e}_{15}+\tilde{e}_{31}\right)+\tilde{c}_{11}\left(1+\tilde{c}_{33}\varepsilon_{13}^{-2}\right)+\tilde{c}_{33}\left(\tilde{e}_{15}+\tilde{e}_{31}\right)^2+\right.$$
$$\left.+2\tilde{e}_{15}+\tilde{c}_{33}\varepsilon_{13}^{-2}-\tilde{c}_{13}\varepsilon_{33}^{-2}\left(2+\tilde{c}_{13}\right)\right\}\lambda^4$$
$$+\left\{-\left(\tilde{e}_{15}+\tilde{e}_{31}\right)^2-2\tilde{e}_{15}-\varepsilon_{13}^{-2}\left(1+\tilde{c}_{33}\right)-\varepsilon_{33}^{-2}\left(1+\tilde{c}_{11}\right)\right\}\lambda^2\Omega^2+\varepsilon_{13}^{-2}\Omega^4,$$

$$d_4^{22} = -\left\{1+\tilde{c}_{13}\varepsilon_{33}^{-2}\right\}\lambda^6+\left\{1+\left(1+\tilde{c}_{33}\right)\varepsilon_{33}^{-2}\right\}\lambda^4\Omega^2-\varepsilon_{33}^{-2}\lambda^2\Omega^4,$$

$$d_1^{23} = \tilde{c}_{11}\tilde{e}_{14}\tilde{e}_{15}\lambda,$$

$$d_2^{23} = -\tilde{e}_{14}\left\{\left[\tilde{c}_{11}-\tilde{e}_{31}-\tilde{c}_{13}\left(\tilde{e}_{15}+\tilde{e}_{31}\right)\right]\lambda^3-\tilde{e}_{15}\lambda\Omega^2\right\},$$

$$d_3^{23} = -\tilde{e}_{14}\left\{-\lambda^5+\lambda^3\Omega^2\right\},$$

$$d_4^{23} = 0,$$

$$d_1^{24} = -\tilde{e}_{14}\tilde{c}_{11}\lambda,$$

$$d_2^{24} = \tilde{e}_{14}\left\{\left[-\tilde{c}_{13}(2+\tilde{c}_{13})+\tilde{c}_{11}\tilde{c}_{33}\right]\lambda^3-(1+\tilde{c}_{11})\lambda\Omega^2\right\},$$

$$d_3^{24} = \tilde{e}_{14}\left\{-\tilde{c}_{33}\lambda^5+(1+\tilde{c}_{33})\lambda^3\Omega^2-\lambda\Omega^4\right\},$$

$$d_4^{24} = 0,$$

$$d_1^{31} = 0,$$

$$d_2^{31} = \tilde{c}_{66}\left\{\tilde{e}_{15}(\tilde{e}_{15}+\tilde{e}_{31})+\varepsilon_{13}^{-2}(1+\tilde{c}_{13})\right\}\lambda,$$

$$d_3^{31} = \left\{-\tilde{e}_{14}^2(1+\tilde{c}_{13})-(\tilde{e}_{15}+\tilde{e}_{31})(\tilde{c}_{66}+\tilde{e}_{15})-(1+\tilde{c}_{13})\left(\varepsilon_{13}^{-2}+\tilde{c}_{66}\varepsilon_{33}^{-2}\right)\right\}\lambda^3 \\ +\left\{\tilde{e}_{15}(\tilde{e}_{15}+\tilde{e}_{31})+\varepsilon_{13}^{-2}(1+\tilde{c}_{13})\right\}\lambda\Omega^2,$$

$$d_4^{31} = \left\{(\tilde{e}_{15}+\tilde{e}_{31})+\varepsilon_{33}^{-2}(1+\tilde{c}_{13})\right\}\left(\lambda^5-\lambda^3\Omega^2\right),$$

$$d_1^{32} = 0,$$

$$d_2^{32} = -\tilde{e}_{14}\tilde{c}_{11}\tilde{e}_{15}\lambda,$$

$$d_3^{32} = -\tilde{e}_{14}\left[\left\{-\tilde{c}_{11}+\tilde{c}_{13}(\tilde{e}_{15}+\tilde{e}_{31})+\tilde{e}_{31}\right\}\lambda^3+\tilde{e}_{15}\lambda\Omega^2\right],$$

$$d_4^{32} = -\tilde{e}_{14}\left\{\lambda^5-\lambda^3\Omega^2\right\},$$

$$d_1^{33} = \tilde{c}_{11}\tilde{c}_{66}\varepsilon_{13}^{-2},$$

$$d_2^{33} = -\left\{\tilde{c}_{11}\tilde{e}_{14}^2+\tilde{c}_{66}(\tilde{e}_{15}+\tilde{e}_{31})^2+\varepsilon_{13}^{-2}(\tilde{c}_{11}+\tilde{c}_{66})+\tilde{c}_{11}\tilde{c}_{66}\varepsilon_{33}^{-2}\right\}\lambda^2 \\ -\varepsilon_{13}^{-2}(\tilde{c}_{11}+\tilde{c}_{66})\Omega^2,$$

$$d_3^{33} = \left\{\tilde{e}_{14}^2+(\tilde{e}_{15}+\tilde{e}_{31})^2+\varepsilon_{13}^{-2}+\varepsilon_{33}^{-2}(\tilde{c}_{11}+\tilde{c}_{66})\right\}\lambda^4 \\ -\left\{\tilde{e}_{14}^2+(\tilde{e}_{15}+\tilde{e}_{31})^2+2\varepsilon_{13}^{-2}+\varepsilon_{33}^{-2}(\tilde{c}_{11}+\tilde{c}_{66})\right\}\lambda^2\Omega^2+\varepsilon_{13}^{-2}\Omega^4,$$

$$d_4^{33} = -\varepsilon_{33}^{-2}\left\{\lambda^6-2\lambda^4\Omega^2+\lambda^2\Omega^4\right\},$$

$$d_1^{34} = \tilde{c}_{11}\tilde{c}_{66}\tilde{e}_{15},$$

$$d_2^{34} = -\left\{\tilde{c}_{11}\tilde{c}_{66}+\tilde{c}_{11}\tilde{e}_{15}-\tilde{c}_{66}\tilde{e}_{31}-\tilde{c}_{13}\tilde{c}_{66}(\tilde{e}_{15}+\tilde{e}_{31})\right\}\lambda^2+\tilde{e}_{15}(\tilde{c}_{11}+\tilde{c}_{66})\Omega^2,$$

$$d_3^{34} = -\left\{-\tilde{c}_{11} - \tilde{c}_{66} + \tilde{c}_{13}\tilde{e}_{15} + \tilde{e}_{31}(1+\tilde{c}_{13})\right\}\lambda^4$$
$$-\left\{\tilde{c}_{11} + \tilde{c}_{66} + \tilde{e}_{15}(1-\tilde{c}_{13}) - \tilde{e}_{31}(1+\tilde{c}_{13})\right\}\lambda^2\Omega^2 + \tilde{e}_{15}\Omega^4,$$

$$d_4^{34} = -\left\{\lambda^6 - 2\lambda^4\Omega^2 + \lambda^2\Omega^4\right\},$$

$$d_1^{41} = 0,$$

$$d_2^{41} = -\tilde{c}_{66}\{\tilde{c}_{13}\tilde{e}_{15} - \tilde{e}_{31}\}\lambda,$$

$$d_3^{41} = -\left\{-\tilde{c}_{66}(1+\tilde{c}_{13}) - \tilde{e}_{15}(\tilde{c}_{13} - \tilde{c}_{33}\tilde{c}_{66}) + \tilde{e}_{31}(1+\tilde{c}_{33}\tilde{c}_{66})\right\}\lambda^3$$
$$-\left\{\tilde{e}_{15}(\tilde{c}_{13} - \tilde{c}_{66}) - \tilde{e}_{31}(1+\tilde{c}_{66})\right\}\lambda\Omega^2,$$

$$d_4^{41} = -\left\{(1+\tilde{c}_{13}) - \tilde{c}_{33}(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda^5 + \left\{(1+\tilde{c}_{13}) - (\tilde{e}_{15} + \tilde{e}_{31})(1+\tilde{c}_{33})\right\}\lambda^3\Omega^2 + (\tilde{e}_{15} + \tilde{e}_{31})\lambda\Omega^4,$$

$$d_1^{42} = 0,$$

$$d_2^{42} = -\tilde{e}_{14}\tilde{c}_{11}\lambda,$$

$$d_3^{42} = -\tilde{e}_{14}\left\{\left[\tilde{c}_{13}(2+\tilde{c}_{13}) - \tilde{c}_{11}\tilde{c}_{33}\right]\lambda^3 + (1+\tilde{c}_{11})\lambda\Omega^2\right\},$$

$$d_4^{42} = \tilde{e}_{14}\left\{-\tilde{c}_{33}\lambda^5 + (1+\tilde{c}_{33})\lambda^3\Omega^2 - \lambda\Omega^4\right\},$$

$$d_1^{43} = -\tilde{c}_{11}\tilde{c}_{66}\tilde{e}_{15},$$

$$d_2^{43} = \left\{\tilde{c}_{66}\left[\tilde{e}_{15} + \tilde{c}_{11} - (1+\tilde{c}_{13})(\tilde{e}_{15} + \tilde{e}_{31})\right] + \tilde{c}_{11}\tilde{e}_{15}\right\}\lambda^2 - \tilde{e}_{15}(\tilde{c}_{11} + \tilde{c}_{66})\Omega^2,$$

$$d_3^{43} = -\left\{\tilde{c}_{66} + \tilde{e}_{15} + \tilde{c}_{11} - (1+\tilde{c}_{13})(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda^4$$
$$+\left\{\tilde{c}_{66} + 2\tilde{e}_{15} + \tilde{c}_{11} - (1+\tilde{c}_{13})(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda^2\Omega^2 - \tilde{e}_{15}\Omega^4,$$

$$d_4^{43} = \lambda^6 - 2\lambda^4\Omega^2 + \Omega^4\lambda^2,$$

$$d_1^{44} = \tilde{c}_{11}\tilde{c}_{66},$$

$$d_2^{44} = \left\{-\tilde{c}_{11}(1+\tilde{c}_{33}\tilde{c}_{66}) + \tilde{c}_{13}\tilde{c}_{66}(2+\tilde{c}_{13})\right\}\lambda^2 + \left\{\tilde{c}_{11} + \tilde{c}_{66} + \tilde{c}_{11}\tilde{c}_{66}\right\}\Omega^2,$$

$$d_3^{44} = \left\{-\tilde{c}_{13}(2+\tilde{c}_{13}) + \tilde{c}_{33}(\tilde{c}_{66} + \tilde{c}_{11})\right\}\lambda^4 +$$
$$\left\{-1 + \tilde{c}_{13}(2+\tilde{c}_{13}) - (1+\tilde{c}_{33})(\tilde{c}_{66} + \tilde{c}_{11})\right\}\Omega^2\lambda^2 + (1+\tilde{c}_{66} + \tilde{c}_{11})\Omega^4,$$

$$d_4^{44} = -\tilde{c}_{33}\lambda^3 + (1 + 2\tilde{c}_{33})\Omega^2\lambda^4 - (2 + \tilde{c}_{33})\Omega^4\lambda^2 + \Omega^6.$$

Appendix B

$$a = \tilde{c}_{11}\tilde{c}_{66}\left(\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\right),$$

$$\begin{aligned} b = -\Big\{&\tilde{c}_{66}\left[\tilde{e}_{31}^2 - \tilde{e}_{15}^2 - 2\tilde{c}_{13}\tilde{e}_{15}\left(\tilde{e}_{15} + \tilde{e}_{31}\right) + \varepsilon_{13}^{-2}(1 + \tilde{c}_{13})^2\right] \\ &+ \tilde{c}_{11}\tilde{e}_{14}^2 + \tilde{c}_{11}\tilde{c}_{66}\left[2\tilde{e}_{15} + \tilde{c}_{33}\varepsilon_{13}^{-2} + \varepsilon_{33}^{-2}\right] - (\tilde{c}_{11} + \tilde{c}_{66})\left(\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\right)\Big\}\lambda^2 \\ &+ \left\{(\tilde{c}_{11} + \tilde{c}_{66})\left(\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\right) + \tilde{c}_{11}\tilde{c}_{66}\varepsilon_{13}^{-2}\right\}\Omega^2, \end{aligned}$$

$$\begin{aligned} c = \Big\{&(\tilde{e}_{15} + \tilde{e}_{31})^2(1 + \tilde{c}_{33}\tilde{c}_{66}) - (1 + \tilde{c}_{13})^2\left(\tilde{e}_{14}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{66}\varepsilon_{33}^{-2}\right) \\ &- 2(\tilde{e}_{15} + \tilde{e}_{31})(1 + \tilde{c}_{13})(\tilde{c}_{66} + \tilde{e}_{15}) + \tilde{e}_{14}^2(1 + \tilde{c}_{11}\tilde{c}_{33}) \\ &+ (\tilde{c}_{11} + \tilde{c}_{66})\left(\tilde{c}_{33}\varepsilon_{13}^{-2} + \varepsilon_{33}^{-2} + 2\tilde{e}_{15}\right) + \tilde{e}_{15}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{11}\tilde{c}_{66}\left(1 + \tilde{c}_{33}\varepsilon_{33}^{-2}\right)\Big\}\lambda^4 \\ &- \Big\{(\tilde{e}_{15} + \tilde{e}_{31})^2(1 + \tilde{c}_{66}) + \tilde{e}_{14}^2(1 + \tilde{c}_{11}) + (\tilde{c}_{11} + \tilde{c}_{66})\left(\tilde{c}_{33}\varepsilon_{13}^{-2} + \varepsilon_{13}^{-2} + \varepsilon_{33}^{-2} + 2\tilde{e}_{15}\right) \\ &- (1 + \tilde{c}_{13})\left[2(\tilde{e}_{15} + \tilde{e}_{31})\tilde{e}_{15} + (1 + \tilde{c}_{13})\varepsilon_{13}^{-2}\right] + 2\tilde{e}_{15}^2 + 2\varepsilon_{13}^{-2} + \tilde{c}_{11}\tilde{c}_{66}\varepsilon_{33}^{-2}\Big\}\lambda^2\Omega^2 \\ &+ \left\{\tilde{e}_{15}^2 + \varepsilon_{13}^{-2} + (1 + \tilde{c}_{11} + \tilde{c}_{66})\right\}\Omega^4, \end{aligned}$$

$$\begin{aligned} d = \left(\Omega^2 - \lambda^2\right)\Big\{&\Big[-(\tilde{e}_{15} + \tilde{e}_{31})\left[2(1 + \tilde{c}_{13}) - (\tilde{e}_{15} + \tilde{e}_{31})\tilde{c}_{33}\right] + (\tilde{c}_{11} + \tilde{c}_{66})\left(1 + \tilde{c}_{33}\varepsilon_{33}^{-2}\right) \\ &+ \tilde{c}_{33}\left(\tilde{e}_{14}^2 + \varepsilon_{13}^{-2}\right) + 2\tilde{e}_{15} - \tilde{c}_{13}\varepsilon_{33}^{-2}(2 + \tilde{c}_{13})\Big]\lambda^4 \\ &- \left[(\tilde{e}_{15} + \tilde{e}_{31})^2 + \tilde{e}_{14}^2 + 2\tilde{e}_{15} + \varepsilon_{13}^{-2}(1 + \tilde{c}_{33}) + \varepsilon_{33}^{-2}(1 + \tilde{c}_{11}) + \tilde{c}_{66}\varepsilon_{33}^{-2}\right]\lambda^2\Omega^2 + \varepsilon_{13}^{-2}\Omega^4\Big\}, \end{aligned}$$

$$e = \left\{1 + \tilde{c}_{33}\varepsilon_{33}^{-2}\right\}\lambda^8 - \left\{2 + \varepsilon_{33}^{-2}(1 + 2\tilde{c}_{33})\right\}\lambda^6\Omega^2 + \left\{1 + \varepsilon_{33}^{-2}(2 + \tilde{c}_{33})\right\}\lambda^4\Omega^4 - \varepsilon_{33}^{-2}\lambda^2\Omega^6.$$

Appendix C

$$\begin{aligned} P_{p,j}^{m,l}(k_j x) = &\left\{\delta_j^{p1}\left[2\tilde{c}_{66}\frac{m(m-1)}{x^2} + (2\tilde{c}_{66} - \tilde{c}_{12})k_j^2\right] - \lambda\left[\tilde{c}_{13}\delta_j^{p3} + \tilde{e}_{31}\delta_j^{p4}\right]\right\}\zeta^{m,l}(k_j x) \\ &+ 2\tilde{c}_{66}\delta_j^{p1}\frac{k_j}{x}\zeta^{m+,l}(k_j x), \end{aligned}$$

$$Q_{p,j}^{m,l}(k_j x) = 2\tilde{c}_{66}\delta_j^{p2} m\left[\frac{m-1}{x^2}\zeta^{m,l}(k_j x) - \frac{k_j}{x}\zeta^{m+1,l}(k_j x)\right],$$

$$R_{p,j}^{m,l}\left(k_j x\right) = -\delta_j^{p2}\left[2\frac{k_j}{x}\zeta^{m+1,l}\left(k_j x\right) + \left(2\frac{m(m-1)}{x^2} - k_j^2\right)\zeta^{m,l}\left(k_j x\right)\right],$$

$$S_{p,j}^{m,l}(k_j x) = 2\delta_j^{p1} m\left[\frac{m-1}{x^2}\zeta^{m,l}(k_j x) - \frac{k_j}{x}\zeta^{m+1,l}(k_j x)\right],$$

$$T_{p,j}^{m,l}(k_j x) = \left(\lambda\delta_j^{p1} + \delta_j^{p3} + \tilde{e}_{15}\delta_j^{p4}\right)\left[\frac{m}{x}\zeta^{m,l}(k_j x) - k_j\zeta^{m+1,l}(k_j x)\right],$$

$$U_{p,j}^{m,l}(k_j x) = \frac{m}{x}\left(\lambda\delta_j^{p2} - \tilde{e}_{14}\delta_j^{p4}\right)\zeta^{m,l}(k_j x).$$

Appendix D

$$c_1^{11} = c_2^{11} = 0, \qquad c_3^{11} = -\lambda^2 R_e,$$

$$c_1^{12} = 0, \quad c_2^{12} = -i\lambda R_e, \quad c_3^{12} = 0,$$

$$c_1^{13} = -1, \quad c_2^{12} = -\left(i\Omega R_e - \lambda^2\right), \quad c_3^{12} = 0,$$

$$c_1^{21} = c_2^{21} = 0, \qquad c_3^{21} = -i\lambda R_e,$$

$$c_1^{22} = 0, \quad c_2^{22} = R_e, \quad c_3^{22} = 0,$$

$$c_1^{23} = 0, \quad c_2^{23} = -i\lambda, \quad c_3^{23} = -i\lambda\left(i\Omega R_e - \lambda^2\right),$$

$$c_1^{31} = 0, \quad c_2^{31} = R_e, \quad c_3^{31} = R_e\left(i\Omega R_e - \lambda^2\right),$$

$$c_1^{32} = 0, \quad c_2^{32} = i\lambda R_e, \quad c_3^{32} = i\lambda R_e\left(i\Omega R_e - \lambda^2\right),$$

$$c_1^{33} = 1, \quad c_2^{33} = 2\left(i\Omega R_e - \lambda^2\right), \quad c_3^{33} = \left(i\Omega R_e - \lambda^2\right)^2.$$

Appendix E

$$q_l^m(\mu_l x) = \frac{2}{R_e}\sigma_l^{i1}\frac{\partial^2 I^m(\mu_l x)}{\partial x^2} - \sigma_l^{i3} I^m(\mu_l x), \quad l = 1,2$$

$$q_3^m(\mu_3 x) = \frac{2}{R_e}\frac{m}{x}\left(\frac{\partial I^m(\mu_3 x)}{\partial x} - \frac{1}{x}I^m(\mu_3 x)\right),$$

$$r_l^m(\mu_l x) = q_l^m(\mu_l x), \qquad l = 1,2$$

$$r_3^m(\mu_3 x) = -q_3^m(\mu_3 x),$$

$$s_l^m(\mu_l x) = \frac{2}{R_e}\frac{m}{x}\sigma_l^{i1}\left(\frac{\partial I^m(\mu_l x)}{\partial x} - \frac{1}{x}I^m(\mu_l x)\right), \qquad l = 1,2$$

$$s_3^m(\mu_3 x) = -\frac{1}{R_e}\left(\frac{\partial^2 I^m(\mu_3 x)}{\partial x^2} - \frac{1}{x}\frac{\partial I^m(\mu_3 x)}{\partial x} + \frac{m^2}{x^2}I^m(\mu_3 x)\right),$$

$$t_l^m(\mu_l x) = -s_l^m(\mu_l x), \qquad l = 1,2$$

$$t_3^m(\mu_3 x) = s_3^m(\mu_3 x),$$

$$w_l^m(\mu_l x) = \frac{1}{R_e}\left(i\lambda\sigma_l^{i1} + \sigma_l^{i2}\right)\frac{\partial I^m(\mu_l x)}{\partial x}, \qquad l = 1,2$$

$$w_3^m(\mu_3 x) = \frac{1}{R_e}\frac{m}{x}i\lambda\, I^m(\mu_3 x),$$

$$z_l^m(\mu_l x) = w_l^m(\mu_l x), \qquad l = 1,2$$

$$z_3^m(\mu_3 x) = -w_3^m(\mu_3 x),$$

Appendix F

$$\mathcal{M}_{1,j} = P_{p,j}^{m,1}\left(k_j\bar{r}_0\right), \quad \mathcal{M}_{1,j+4} = P_{p,j}^{m,2}\left(k_j\bar{r}_0\right), \quad \mathcal{M}_{1,j+8} = Q_{p,j}^{m,1}\left(k_j\bar{r}_0\right), \quad \mathcal{M}_{1,j+12} = Q_{p,j}^{m,2}\left(k_j\bar{r}_0\right),$$

$$\mathcal{M}_{1,17} = q_1^m(\mu_1\bar{r}_0), \quad \mathcal{M}_{1,18} = q_2^m(\mu_2\bar{r}_0), \quad \mathcal{M}_{1,19} = q_3^m(\mu_3\bar{r}_0), \; \mathcal{M}_{1,20} = \mathcal{M}_{1,21} = \mathcal{M}_{1,22} = 0,$$

$$\mathcal{M}_{2,j} = -\mathcal{M}_{1,j+8}, \quad \mathcal{M}_{2,j+4} = -\mathcal{M}_{1,j+12}, \quad \mathcal{M}_{2,j+8} = \mathcal{M}_{1,j}, \quad \mathcal{M}_{2,j+12} = \mathcal{M}_{1,j+4},$$

$$\mathcal{M}_{2,17} = \mathcal{M}_{2,18} = \mathcal{M}_{2,19} = 0, \; \mathcal{M}_{2,20} = r_1^m(\mu_1\bar{r}_0), \quad \mathcal{M}_{2,21} = r_2^m(\mu_2\bar{r}_0), \quad \mathcal{M}_{2,22} = r_3^m(\mu_3\bar{r}_0),$$

$$\mathcal{M}_{3,j} = \tilde{c}_{66} R_{p,j}^{m,1}\left(k_j \bar{r}_0\right), \qquad \mathcal{M}_{3,j+4} = \tilde{c}_{66} R_{p,j}^{m,2}\left(k_j \bar{r}_0\right), \qquad \mathcal{M}_{3,j+8} = \tilde{c}_{66} S_{p,j}^{m,1}\left(k_j \bar{r}_0\right),$$

$$\mathcal{M}_{3,j+12} = \tilde{c}_{66} S_{p,j}^{m,2}\left(k_j \bar{r}_0\right), \qquad \mathcal{M}_{3,17} = \mathcal{M}_{3,18} = \mathcal{M}_{3,19} = 0, \quad \mathcal{M}_{3,20} = s_1^m\left(\mu_1 \bar{r}_0\right),$$

$$\mathcal{M}_{3,21} = s_2^m\left(\mu_2 \bar{r}_0\right), \quad \mathcal{M}_{3,22} = s_3^m\left(\mu_3 \bar{r}_0\right),$$

$$\mathcal{M}_{4,j} = -\mathcal{M}_{3,j+8}, \qquad \mathcal{M}_{4,j+4} = -\mathcal{M}_{3,j+12}, \qquad \mathcal{M}_{4,j+8} = \mathcal{M}_{3,j}, \qquad \mathcal{M}_{4,j+12} = \mathcal{M}_{3,j+4},$$

$$\mathcal{M}_{4,17} = t_1^m\left(\mu_1 \bar{r}_0\right), \qquad \mathcal{M}_{4,18} = t_2^m\left(\mu_2 \bar{r}_0\right), \qquad \mathcal{M}_{4,19} = t_3^m\left(\mu_3 \bar{r}_0\right), \qquad \mathcal{M}_{4,20} = \mathcal{M}_{4,21} = \mathcal{M}_{4,22} = 0$$

$$\mathcal{M}_{5,j} = iT_{p,j}^{m,1}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{5,j+4} = iT_{p,j}^{m,2}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{5,j+8} = iU_{p,j}^{m,1}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{5,j+12} = iU_{p,j}^{m,2}\left(k_j \bar{r}_0\right),$$

$$\mathcal{M}_{5,17} = w_1^m\left(\mu_1 \bar{r}_0\right), \qquad \mathcal{M}_{5,18} = w_2^m\left(\mu_2 \bar{r}_0\right), \qquad \mathcal{M}_{5,19} = w_3^m\left(\mu_3 \bar{r}_0\right), \quad \mathcal{M}_{5,20} = \mathcal{M}_{5,21} = \mathcal{M}_{5,22} = 0,$$

$$\mathcal{M}_{6,j} = -\mathcal{M}_{5,j+8}, \qquad \mathcal{M}_{6,j+4} = -\mathcal{M}_{5,j+12}, \qquad \mathcal{M}_{6,j+8} = \mathcal{M}_{5,j}, \qquad \mathcal{M}_{6,j+12} = \mathcal{M}_{5,j+4},$$

$$\mathcal{M}_{6,17} = \mathcal{M}_{6,18} = \mathcal{M}_{6,19} = 0, \quad \mathcal{M}_{6,20} = z_1^m\left(\mu_1 \bar{r}_0\right), \quad \mathcal{M}_{6,21} = z_2^m\left(\mu_2 \bar{r}_0\right), \quad \mathcal{M}_{6,22} = z_3^m\left(\mu_3 \bar{r}_0\right),$$

$$\mathcal{M}_{7,j} = i\delta_j^{p1} F_{p,j}^{m,1}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{7,j+4} = i\delta_j^{p1} F_{p,j}^{m,2}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{7,j+8} = i\delta_j^{p2} G_{p,j}^{m,1}\left(k_j \bar{r}_0\right),$$

$$\mathcal{M}_{7,j+12} = i\delta_j^{p2} G_{p,j}^{m,2}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{7,17} = f_1^m\left(\mu_1 \bar{r}_0\right), \qquad \mathcal{M}_{7,18} = f_2^m\left(\mu_2 \bar{r}_0\right),$$

$$\mathcal{M}_{7,19} = f_3^m\left(\mu_3 \bar{r}_0\right), \qquad \mathcal{M}_{7,20} = \mathcal{M}_{7,21} = \mathcal{M}_{7,22} = 0, \qquad \mathcal{M}_{8,j} = -\mathcal{M}_{7,j+8},$$

$$\mathcal{M}_{8,j+4} = -\mathcal{M}_{7,j+12}, \qquad \mathcal{M}_{8,j+8} = \mathcal{M}_{7,j}, \qquad \mathcal{M}_{8,j+12} = \mathcal{M}_{7,j+4}, \quad \mathcal{M}_{8,17} = 0,$$

$$\mathcal{M}_{8,18} = \mathcal{M}_{8,19} = 0, \quad \mathcal{M}_{8,20} = h_1^m\left(\mu_1 \bar{r}_0\right), \quad \mathcal{M}_{8,21} = h_2^m\left(\mu_2 \bar{r}_0\right), \quad \mathcal{M}_{8,22} = h_3^m\left(\mu_3 \bar{r}_0\right),$$

$$\mathcal{M}_{9,j} = -i\delta_j^{p2} F_{p,j}^{m,1}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{9,j+4} = -i\delta_j^{p2} F_{p,j}^{m,2}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{9,j+8} = i\delta_j^{p1} G_{p,j}^{m,1}\left(k_j \bar{r}_0\right),$$

$$\mathcal{M}_{9,j+12} = i\delta_j^{p1} G_{p,j}^{m,2}\left(k_j \bar{r}_0\right), \quad \mathcal{M}_{9,17} = \mathcal{M}_{9,18} = \mathcal{M}_{9,19} = 0, \quad \mathcal{M}_{9,20} = g_1^m\left(\mu_1 \bar{r}_0\right),$$

$$\mathcal{M}_{9,21} = g_2^m\left(\mu_2 \bar{r}_0\right), \qquad \mathcal{M}_{9,22} = g_3^m\left(\mu_3 \bar{r}_0\right),$$

$$\mathcal{M}_{10,j} = \mathcal{M}_{9,j+8}, \qquad \mathcal{M}_{10,j+4} = \mathcal{M}_{9,j+12}, \qquad \mathcal{M}_{10,j+8} = -\mathcal{M}_{9,j},$$

$$\mathcal{M}_{10,j+12} = -\mathcal{M}_{9,j+4}, \quad \mathcal{M}_{10,17} = l_1^m\left(\mu_1 \bar{r}_0\right), \quad \mathcal{M}_{10,18} = l_2^m\left(\mu_2 \bar{r}_0\right),$$

$$\mathcal{M}_{10,19} = l_3^m\left(\mu_3 \bar{r}_0\right), \qquad \mathcal{M}_{10,20} = \mathcal{M}_{10,21} = \mathcal{M}_{10,22} = 0,$$

$$\mathcal{M}_{11,j} = K_{p,j}^{m,1}\left(k_j\bar{r}_0\right), \quad \mathcal{M}_{11,j+4} = K_{p,j}^{m,2}\left(k_j\bar{r}_0\right), \quad \mathcal{M}_{11,j+8} = \mathcal{M}_{11,j+12} = 0,$$

$$\mathcal{M}_{11,17} = n_1^m(\mu_1\bar{r}_0), \quad \mathcal{M}_{11,18} = n_2^m(\mu_2\bar{r}_0), \quad \mathcal{M}_{11,19} = n_3^m(\mu_3\bar{r}_0),$$

$$\mathcal{M}_{11,20} = \mathcal{M}_{11,21} = \mathcal{M}_{11,22} = 0,$$

$$\mathcal{M}_{12,j} = 0, \quad \mathcal{M}_{12,j+4} = 0, \quad \mathcal{M}_{12,j+8} = \mathcal{M}_{11,j}, \quad \mathcal{M}_{12,j+12} = \mathcal{M}_{11,j+4},$$

$$\mathcal{M}_{12,17} = \mathcal{M}_{12,18} = \mathcal{M}_{12,19} = 0, \quad \mathcal{M}_{12,20} = n_1^m(\mu_1\bar{r}_0), \quad \mathcal{M}_{12,21} = n_2^m(\mu_2\bar{r}_0),$$

$$\mathcal{M}_{12,22} = n_3^m(\mu_3\bar{r}_0),$$

$$\mathcal{M}_{13,j} = W_{p,j}^{m,1}\left(k_j\bar{r}_0\right), \quad \mathcal{M}_{13,j+4} = W_{p,j}^{m,2}\left(k_j\bar{r}_0\right), \quad \mathcal{M}_{13,j+8} = \mathcal{M}_{13,j+12} = 0,$$

$$\mathcal{M}_{13,17} = \mathcal{M}_{13,18} = \mathcal{M}_{13,19} = \mathcal{M}_{13,20} = \mathcal{M}_{13,21} = \mathcal{M}_{13,22} = 0,$$

$$\mathcal{M}_{14,j} = \mathcal{M}_{14,j+4} = 0, \quad \mathcal{M}_{14,j+8} = \mathcal{M}_{13,j}, \quad \mathcal{M}_{14,j+12} = \mathcal{M}_{13,j+4},$$

$$\mathcal{M}_{14,17} = \mathcal{M}_{14,18} = \mathcal{M}_{14,19} = \mathcal{M}_{14,20} = \mathcal{M}_{14,21} = \mathcal{M}_{14,22} = 0,$$

$$\mathcal{M}_{15,j} = P_{p,j}^{m,1}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{15,j+4} = P_{p,j}^{m,2}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{15,j+8} = Q_{p,j}^{m,1}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{15,j+12} = Q_{p,j}^{m,2}\left(k_j\bar{r}_1\right),$$

$$\mathcal{M}_{15,17} = \mathcal{M}_{15,18} = \mathcal{M}_{15,19} = \mathcal{M}_{15,20} = \mathcal{M}_{15,21} = \mathcal{M}_{15,22} = 0,$$

$$\mathcal{M}_{16,j} = -\mathcal{M}_{15,j+8}, \quad \mathcal{M}_{16,j+4} = -\mathcal{M}_{15,j+12}, \quad \mathcal{M}_{16,j+8} = \mathcal{M}_{15,j}, \quad \mathcal{M}_{16,j+12} = \mathcal{M}_{15,j+4},$$

$$\mathcal{M}_{16,17} = \mathcal{M}_{16,18} = \mathcal{M}_{16,19} = \mathcal{M}_{16,20} = \mathcal{M}_{16,21} = \mathcal{M}_{16,22} = 0,$$

$$\mathcal{M}_{17,j} = R_{p,j}^{m,1}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{17,j+4} = R_{p,j}^{m,2}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{17,j+8} = S_{p,j}^{m,1}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{17,j+12} = S_{p,j}^{m,2}\left(k_j\bar{r}_1\right),$$

$$\mathcal{M}_{17,17} = \mathcal{M}_{17,18} = \mathcal{M}_{17,19} = \mathcal{M}_{17,20} = \mathcal{M}_{17,21} = \mathcal{M}_{17,22} = 0,$$

$$\mathcal{M}_{18,j} = -\mathcal{M}_{17,j+8}, \quad \mathcal{M}_{18,j+4} = -\mathcal{M}_{17,j+12}, \quad \mathcal{M}_{18,j+8} = \mathcal{M}_{17,j}, \quad \mathcal{M}_{18,j+12} = \mathcal{M}_{17,j+4},$$

$$\mathcal{M}_{18,17} = \mathcal{M}_{18,18} = \mathcal{M}_{18,19} = \mathcal{M}_{18,20} = \mathcal{M}_{18,21} = \mathcal{M}_{18,22} = 0,$$

$$\mathcal{M}_{19,j} = T_{p,j}^{m,1}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{19,j+4} = T_{p,j}^{m,2}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{19,j+8} = U_{p,j}^{m,1}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{19,j+12} = U_{p,j}^{m,2}\left(k_j\bar{r}_1\right),$$

$$\mathcal{M}_{19,17} = \mathcal{M}_{19,18} = \mathcal{M}_{19,19} = \mathcal{M}_{19,20} = \mathcal{M}_{19,21} = \mathcal{M}_{19,22} = 0,$$

$$\mathcal{M}_{20,j} = -\mathcal{M}_{19,j+8}, \quad \mathcal{M}_{20,j+4} = -\mathcal{M}_{19,j+12}, \quad \mathcal{M}_{20,j+8} = \mathcal{M}_{19,j}, \quad \mathcal{M}_{20,j+12} = \mathcal{M}_{19,j+4},$$

$$\mathcal{M}_{20,17} = \mathcal{M}_{20,18} = \mathcal{M}_{20,19} = \mathcal{M}_{20,20} = \mathcal{M}_{20,21} = \mathcal{M}_{20,22} = 0,$$

$$\mathcal{M}_{21,j} = W_{p,j}^{m,1}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{21,j+4} = W_{p,j}^{m,2}\left(k_j\bar{r}_1\right), \quad \mathcal{M}_{21,j+8} = \mathcal{M}_{21,j+12} = 0,$$

$$\mathcal{M}_{21,17} = \mathcal{M}_{21,18} = \mathcal{M}_{21,19} = \mathcal{M}_{21,20} = \mathcal{M}_{21,21} = \mathcal{M}_{21,22} = 0,$$

$$\mathcal{M}_{22,j} = \mathcal{M}_{22,j+4} = 0, \quad \mathcal{M}_{22,j+8} = \mathcal{M}_{21,j}, \quad \mathcal{M}_{22,j+12} = \mathcal{M}_{21,j+4},$$

$$\mathcal{M}_{22,17} = \mathcal{M}_{22,18} = \mathcal{M}_{22,19} = \mathcal{M}_{22,20} = \mathcal{M}_{22,21} = \mathcal{M}_{22,22} = 0,$$

for j=1,2,3,4, where

$$G_{p,j}^{m,l}\left(k_jx\right) = \Omega\frac{m}{x}\zeta^{m,l}\left(k_jx\right), \quad F_{p,j}^{m,l}\left(k_jx\right) = \Omega\frac{\partial\zeta^{m,l}\left(k_jx\right)}{\partial x}, \quad K_{p,j}^{m,l}\left(k_jx\right) = \Omega\delta_j^{p3}\zeta^{m,l}\left(k_jx\right),$$

$$f_l^m(\mu_l x) = \sigma_l^{i1}\frac{\partial I^m(\mu_l x)}{\partial x}, \qquad l = 1,2, \qquad f_3^m(\mu_3 x) = \frac{m}{x}I^m(\mu_3 x),$$

$$h_l^m(\mu_l x) = f_l^m(\mu_l x), \qquad l = 1,2, \qquad h_3^m(\mu_3 x) = -f_3^m(\mu_3 x),$$

$$g_l^m(\mu_l x) = \sigma_l^{i1}\frac{m}{x}I^m(\mu_l x), \qquad l = 1,2, \qquad g_3^m(\mu_3 x) = -\frac{\partial I^m(\mu_3 x)}{\partial x},$$

$$l_l^m(\mu_l x) = g_l^m(\mu_l x), \qquad l = 1,2, \qquad l_3^m(\mu_3 x) = -g_3^m(\mu_3 x),$$

$$n_l^m(\mu_l x) = -\sigma_l^{i2}I^m(\mu_l x), \qquad l = 1,2, \qquad n_3^m(\mu_3 x) = 0.$$

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