IMPLICIT-EXPLICIT MULTISTEP METHODS FOR QUASILINEAR PARABOLIC EQUATIONS

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ABSTRACT. Efficient combinations of implicit and explicit multistep methods for nonlinear parabolic equations were recently studied in [1]. In this note we present a refined analysis to allow more general nonlinearities. The abstract theory is applied to a quasilinear parabolic equation.

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1. INTRODUCTION

In this paper we extend our study of implicit-explicit multistep finite element schemes for parabolic problems to quasilinear equations. In particular, we establish abstract convergence results for our methods under weaker stability and consistency conditions. Thus the abstract theory can be applied to various nonlinear parabolic problems yielding convergence under mild meshconditions. We consider problems of the form: Given $T > 0$ and $u^0 \in H$, find $u : [0, T] \to D(A)$ such that

$$\begin{aligned} &u'(t) + Au(t) = B(t, u(t)), \quad 0 < t < T, \\ &u(0) = u^0, \end{aligned} \tag{1.1}$$

with A a positive definite, selfadjoint, linear operator on a Hilbert space $(H, (\cdot, \cdot))$ with domain $D(A)$ dense in H, and $B(t, \cdot) : D(A) \to H$, $t \in [0, T]$, a (possibly) nonlinear operator. To motivate the construction of the fully discrete schemes we first consider the semidiscrete problem approximating (1.1): For a given finite dimensional subspace V_h of V, $V = D(A^{1/2})$, we seek a function u_h, $u_h(t) \in V_h$, defined by

$$\begin{aligned} &u_h'(t) + A_h u_h(t) = B_h(t, u_h(t)), \quad 0 < t < T, \\ &u_h(0) = u_h^0; \end{aligned} \tag{1.2}$$

here $u_h^0 \in V_h$ is a given approximation to u^0, and A_h, B_h are appropriate operators on V_h with A_h a positive definite, selfadjoint, linear operator.

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Following [1] and [4], we let (α, β) be a strongly $A(0)-$stable $q-$step scheme and (α, γ) be an explicit $q-$step scheme, characterized by three polynomials α, β and γ,

$$\alpha(\zeta) = \sum_{i=0}^{q} \alpha_i \zeta^i, \quad \beta(\zeta) = \sum_{i=0}^{q} \beta_i \zeta^i, \quad \gamma(\zeta) = \sum_{i=0}^{q-1} \gamma_i \zeta^i.$$

Letting $N \in \mathbb{N}$, $k = \frac{T}{N}$ be the time step, and $t^n = nk, n = 0, \ldots, N$, we combine the (α, β) and (α, γ) schemes to obtain an (α, β, γ) scheme for discretizing (1.2) in time, and define a sequence of approximations U^n, $U^n \in V_h$, to $u^n := u(t^n)$, by

$$\sum_{i=0}^{q} \alpha_i U^{n+i} + k \sum_{i=0}^{q} \beta_i A_h U^{n+i} = k \sum_{i=0}^{q-1} \gamma_i B_h(t^{n+i}, U^{n+i}). \tag{1.3}$$

Given $U^0, \ldots, U^{q-1}$ in $V_h, U^q, \ldots, U^N$ are well defined by the (α, β, γ) scheme, see [1]. The scheme (1.3) is efficient, its implementation to advance in time requires solving a linear system with the same matrix for all time levels.

Stability and consistency assumptions. Let $|\cdot|$ denote the norm of H, and introduce in V the norm $\|\cdot\|$ by $\|v\| := |A^{1/2}v|$. We identify H with its dual, and denote by V' the dual of V, again by $(\cdot, \cdot)$ the duality pairing on V' and V, and by $\|\cdot\|_\star$ the dual norm on V'. Let T_u be a tube around the solution u, $T_u := \{v \in V : \min_t \|u(t) - v\| \le 1\}$, say. For stability purposes, we assume that $B(t, \cdot)$ can be extended to an operator from V into V', and an estimate of the form

$$\|B(t, v) - B(t, w)\|_\star \le \lambda \|v - w\| + \mu |v - w| \qquad \forall v, w \in T_u \tag{1.4}$$

holds, uniformly in t, with two constants λ and μ, $\lambda < 1$. Indeed, depending on the particular (α, β, γ) scheme, we shall need to assume that λ be appropriately small. The smallness of λ is essential for our analysis, while the tube T_u is defined in terms of the norm of V for concreteness. Under these conditions we will show convergence, provided that a mild meshcondition is satisfied, see Theorem 2.1. The proof can be easily modified to yield convergence under conditions analogous to (1.4) for v and w belonging to tubes defined in terms of other norms, not necessarily the same for both arguments; milder or stronger meshconditions, respectively, are required if the tubes are defined in terms of weaker or stronger norms, cf. Remark 2.2 and Section 3.

We will assume in the sequel that (1.1) possesses a solution which is sufficiently regular for our results to hold. Local uniqueness of smooth solutions follows easily in view of (1.4).

For the space discretization we use a family V_h, $0 < h < 1$, of finite dimensional subspaces of V. In the sequel the following discrete operators will play an essential role: Define $P_o : V' \to V_h$, $A_h : V \to V_h$ and $B_h(t, \cdot) : V \to V_h$ by

$$(P_o v, \chi) = (v, \chi) \quad \forall \chi \in V_h$$
$$(A_h \varphi, \chi) = (A\varphi, \chi) \quad \forall \chi \in V_h$$
$$(B_h(t, \varphi), \chi) = (B(t, \varphi), \chi) \quad \forall \chi \in V_h.$$

Let $B(t,\cdot) : V \to V'$ be differentiable, and assume that the linear operator $M(t)$, $M(t) := A - B'(t,u(t)) + \sigma I$, is uniformly positive definite, for an appropriate constant σ. We introduce the 'elliptic' projection $R_h(t) : V \to V_h, t \in [0,T]$, by

$$P_o M(t) R_h(t) v = P_o M(t) v. \tag{1.5}$$

We will show consistency of the (α, β, γ) scheme for $R_h(t)u(t)$; to this end we shall use approximation properties of the elliptic projection operator $R_h(t)$. We assume that $R_h(t)$ satisfies the estimates

$$|u(t) - R_h(t)u(t)| + h^{d/2}\|u(t) - R_h(t)u(t)\| \le Ch^r, \tag{1.6}$$

and

$$|\frac{d}{dt}[u(t) - R_h(t)u(t)]| \le Ch^r, \tag{1.7}$$

with two integers r and d, $2 \le d \le r$. We further assume that

$$\|\frac{d^j}{dt^j}[R_h(t)u(t)]\| \le C, \quad j = 1, \dots, p+1, \tag{1.8}$$

p being the order of both multistep schemes.

For consistency purposes, we assume for the nonlinear part the estimate

$$\|B(t,u(t)) - B(t, R_h(t)u(t)) - B'(t,u(t))(u(t) - R_h(t)u(t))\|_\star \le Ch^r. \tag{1.9}$$

Then, under some mild meshconditions and for appropriately small λ and appropriate starting values $U^0, \dots, U^{q-1}$, we shall derive optimal order error estimates in $|\cdot|$.

Implicit-explicit multistep methods for linear parabolic equations with time dependent coefficients were first introduced and analyzed in [4]. Recently, [1], we analyzed implicit-explicit multistep finite element methods for nonlinear parabolic problems, under stronger conditions on the nonlinearity. More precisely, we took B independent of t, and assumed for stability purposes the global condition

$$|(B'(v)w, \omega)| \le \lambda\|w\|\|\omega\| + \mu(v)|w|\,|\omega| \qquad \forall v, w, \omega \in V \tag{1.4'}$$

with a functional $\mu(v)$ bounded for v bounded in V, and for consistency purposes that

$$\|B(u(t)) - B(R_h u(t))\|_\star \le Ch^r \tag{1.9'}$$

with elliptic projection operator R_h defined, in terms of the linear operator A only, by $(AR_h v, \chi) = (Av, \chi)\ \forall \chi \in V_h$.

It is easily seen that (1.4) follows from (1.4′). Besides the fact that (1.4) is local, in contrast to the global condition (1.4′), the major difference between the two conditions consists in the norm of ω used in their last term: in (1.4′) the H−norm while in (1.4), implicitly, the V−norm is used.

Condition (1.9′) restricts essentially the order of the derivatives contained in B to $d/2$, if A is a differential operator of order d. It was already mentioned in [1] that, for some concrete differential equations, one can get by with a less stringent condition by taking into account in the definition of the elliptic projection operator the terms of B of order higher than $d/2$; an attempt in this direction is the definition of the elliptic projection considered in this note. Condition (1.9) may be satisfied even if A and B are differential operators of the same order.

To emphasize that the new stability and consistency conditions do indeed allow more general nonlinearities than the corresponding conditions used in [1], we mention two simple examples of initial and boundary value problems in one space variable in a bounded interval. It is easily seen that condition (1.4′) is satisfied for the equation

$$u_t - u_{xx} = (f(u))_x,$$

provided that f' is uniformly bounded by a small constant; condition (1.4) on the other hand is satisfied with $\lambda = 0$ for any smooth function f. Next we consider the equation

$$u_t - u_{xx} = (a(x,t,u)u_x)_x.$$

It is easily seen in this case that condition (1.9′) is not satisfied whereas condition (1.9) is satisfied, cf. Section 3. These two examples are particular cases of the quasilinear equation

$$u_t = \operatorname{div}(c(x,t,u)\nabla u + g(x,t,u)) + f(x,t,u)$$

which will be considered in Section 3.

The crucial tool for proving stability of the (α,β,γ) scheme in [4] and [1] is Lemme 3.1 of [4]. Here we modify it, see Lemma 2.1 below, and give a proof which may be used to derive explicit bounds for the stability constant λ of (1.4), which, as already mentioned, depends on the particular (α,β,γ) scheme; cf. Remark 2.4 for an example concerning a second order scheme.

An outline of the paper is as follows: Section 2 is devoted to the abstract analysis of the implicit-explicit multistep schemes. In the last section, we apply our abstract results to a quasilinear parabolic partial differential equation.

2. Multistep schemes

In this section we shall analyze implicit-explicit multistep schemes for the abstract parabolic initial value problem (1.1).

Let (α,β) be an implicit strongly $A(0)$−stable q−step scheme, and (α,γ) be an explicit q−step scheme. We assume that both methods (α,β) and (α,γ) are of order p, i.e.,

$$\sum_{i=0}^{q} i^\ell \alpha_i = \ell \sum_{i=0}^{q} i^{\ell-1}\beta_i = \ell \sum_{i=0}^{q-1} i^{\ell-1}\gamma_i, \quad \ell = 0,1,\ldots,p.$$

For examples of (α, β, γ) schemes satisfying these stability and consistency properties we refer to [1] and the references therein.

Our main concern in this section is to analyze the approximation properties of the sequence $\{U^n\}$. As an intermediate step, we shall show consistency of the scheme (1.3) for the elliptic projection W of the solution u of (1.1), $W(t) = R_h(t)u(t)$.

Consistency. The consistency error E^n of the scheme (1.3) for W is given by

$$kE^n = \sum_{i=0}^{q} \alpha_i W^{n+i} + k\sum_{i=0}^{q} \beta_i A_h W^{n+i} - k\sum_{i=0}^{q-1} \gamma_i B_h(t^{n+i}, W^{n+i}), \tag{2.1}$$

$n = 0, \ldots, N-q$. Using (1.5), the definition of A_h and B_h, and (1.1), and letting $\gamma_q := 0$, we split E^n as $E^n = E_1^n + E_2^n + E_3^n + E_4^n$, with

$$kE_1^n = \sum_{i=0}^{q} \alpha_i [R_h(t^{n+i}) - P_o]u^{n+i}, \tag{2.2i}$$

$$kE_2^n = P_o \sum_{i=0}^{q} [\alpha_i u^{n+i} - k\gamma_i u'(t^{n+i})], \tag{2.2ii}$$

$$E_3^n := \sum_{i=0}^{q} (\beta_i - \gamma_i) A_h W^{n+i}, \tag{2.2iii}$$

and

$$E_4^n := \sum_{i=0}^{q} \gamma_i \{A_h W^{n+i} - P_o A u^{n+i} + P_o B(t^{n+i}, u^{n+i}) - B_h(t^{n+i}, W^{n+i})\}. \tag{2.2iv}$$

First, we will estimate E_1^n. Using (1.7) and the fact that $\alpha_1 + \cdots + \alpha_q = 0$, it is easily seen that

$$\max_{0 \le n \le N-q} |E_1^n| \le Ch^r. \tag{2.3i}$$

Further, in view of the consistency properties of (α, γ),

$$\Big| \sum_{i=0}^{q} [\alpha_i u^{n+i} - k\gamma_i u'(t_{n+i})] \Big| \le Ck^{p+1},$$

i.e.,

$$\max_{0 \le n \le N-q} |E_2^n| \le Ck^p. \tag{2.3ii}$$

Now, using (1.8) and the consistency properties of (α, β) and (α, γ), we have

(2.3iii) $$\max_{0\le n\le N-q} \|E_3^n\|_\star \le Ck^p\,.$$

Finally, we will estimate E_4^n. First, from (1.5) we deduce that

$$[A_h - B_h'(t, u(t)) + \sigma I]R_h(t)u(t) = P_o[A - B_h'(t, u(t)) + \sigma I]u(t)$$

and rewrite (2.2iv) as

$$\begin{aligned} E_4^n =& P_o \sum_{i=0}^{q} \gamma_i \{B(t^{n+i}, u^{n+i}) - B(t^{n+i}, W^{n+i}) - B'(t^{n+i}, u^{n+i})(u^{n+i} - W^{n+i})\} \\ &+ \sigma P_o \sum_{i=0}^{q} \gamma_i (u^{n+i} - W^{n+i})\,. \end{aligned}$$

Then, in view of (1.9) and (1.6), we obtain

(2.3iv) $$\max_{0\le n\le N-q} \|E_4^n\|_\star \le Ch^r\,.$$

Thus, we have the following estimate for the consistency error E^n,

(2.4) $$\max_{0\le n\le N-q} \|E^n\|_\star \le C(k^p + h^r)\,.$$

Convergence. In the sequel assume that we are given initial approximations U^0, U^1, $\dots$, $U^{q-1} \in V_h$ to $u^0, \dots, u^{q-1}$ such that

(2.5) $$\sum_{j=0}^{q-1} \Big(|W^j - U^j| + k^{1/2} \|W^j - U^j\| \Big) \le c(k^p + h^r),$$

cf. Remark 2.5. Let $U^n \in V_h$, $n = q, \dots, N$, be recursively defined by the (α, β, γ) scheme (1.3). Let $\vartheta^n = W^n - U^n, n = 0, \dots, N$. Then (2.1) and (1.3) yield the error equation for ϑ^n

$$\sum_{i=0}^{q} \alpha_i \vartheta^{n+i} + k \sum_{i=0}^{q} \beta_i A_h \vartheta^{n+i} = k \sum_{i=0}^{q-1} \gamma_i \{B_h(t^{n+i}, W^{n+i}) - B_h(t^{n+i}, U^{n+i})\} + kE^n,$$

i.e.,

(2.6) $$\begin{aligned} \sum_{i=0}^{q} \alpha_i \vartheta^{n+i} + k \sum_{i=0}^{q} \beta_i A_h \vartheta^{n+i} =& k \sum_{i=0}^{q-1} \gamma_i \{B_h(t^{n+i}, W^{n+i}) - B_h(t^{n+i}, U^{n+i})\} \\ &+ kE^n, \quad n = 0, \dots, N - q. \end{aligned}$$

In the sequel we shall use the notation

$$\Theta^n := \begin{pmatrix} \vartheta^{n+q-1} \\ \vdots \\ \vartheta^n \end{pmatrix}, \quad \mathcal{E}^n := \begin{pmatrix} E^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \delta_i(x) := -\frac{\alpha_i + \beta_i x}{\alpha_q + \beta_q x},$$

$$\Delta_i := \delta_i(kA_h), \;\; \Gamma_n^i := \gamma_i \int_0^1 B_h'(t^{n+i}, W^{n+i} - s\vartheta^{n+i})\, ds\,,$$

$$\Lambda := \Lambda(kA_h) = \begin{pmatrix} \Delta_{q-1} & \Delta_{q-2} & \dots & \Delta_0 \\ I & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & I & 0 \end{pmatrix}, \quad \Gamma_n := \begin{pmatrix} \Gamma_n^{q-1} & \dots & \Gamma_n^0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix},$$

and

$$(\alpha_q + k\beta_q A_h)\,\Theta^n = \begin{pmatrix} (\alpha_q + k\beta_q A_h)\vartheta^{n+q-1} \\ \vdots \\ (\alpha_q + k\beta_q A_h)\vartheta^n \end{pmatrix}.$$

Equation (2.6) can then be written in the form

$$(\alpha_q + k\beta_q A_h)\Theta^{n+1} = (\alpha_q + k\beta_q A_h)\;\Lambda\Theta^n + k\;\Gamma_n\Theta^n + k\;\mathcal{E}^n. \tag{2.7}$$

The following result concerning strongly $A(0)-$stable multistep schemes will play a crucial role in our stability analysis, cf. also Crouzeix, [4].

Lemma 2.1. *There exist three constants η, K_1 and K_2, with $0 < \eta < 1$, and a map $\mathcal{H} : \bar{\mathbb{R}}^+ \to \mathbb{R}^{q\times q}$ such that, for all $x \in \bar{\mathbb{R}}^+$, $\|\mathcal{H}(x)\|_2 = 1$, the matrix $\mathcal{H}(x)$ is invertible, $\|\mathcal{H}(x)^{-1}\|_2 \le K_1$, $\|(\mathcal{H}(x)^{-1})^T e_1\|_2 \le K_2$, and the matrix $\mathcal{L}(x)$ defined by*

$$\mathcal{L}(x) = \frac{\alpha_q + \beta_q x}{\alpha_q + \eta\beta_q x}\mathcal{H}(x)^{-1}\Lambda(x)\mathcal{H}(x)$$

satisfies $\|\mathcal{L}(x)\|_2 \le 1, \|\cdot\|_2$ denoting both the Euclidean norm of a vector and the spectral norm of a matrix.

Proof. Let $\rho(x)$ be the spectral radius of $\Lambda(x)$; ρ is clearly a continuous function on $[0,+\infty]$. From the Dahlquist $0-$stability condition, i.e., "$\rho(0) = 1$ and the eigenvalues of modulus 1 of $\Lambda(0)$ are simple", we deduce that the right-derivative of ρ at 0, $\rho'(0_+)$, exists. By definition, the strong $A(0)-$stability of the method means that "$\rho(x) < 1$ for all $x \in (0,\infty]$ and $\rho'(0_+) < 0$". Therefore, we can select $\eta \in (0,1)$ such that

$$\text{for all } x \in (0,+\infty], \quad \frac{\alpha_q + \beta_q x}{\alpha_q + \eta\beta_q x}\rho(x) < 1.$$

Let η be as above. We consider the set

$$\mathcal{E}(x) = \left\{ N \in \mathbb{R}^{q\times q} \;:\; N \text{ is invertible and } \frac{\alpha_q + \beta_q x}{\alpha_q + \eta\beta_q x} \|N^{-1}\Lambda(x)N\|_2 \le 1 \right\},$$

and introduce

$$m(x) = \min\{\|N\|_2\|N^{-1}\|_2 \;:\; N \in \mathcal{E}(x)\} \;=\; \min\{\|N^{-1}\|_2 \;:\; N \in \mathcal{E}(x) \text{ and } \|N\|_2 = 1\}.$$

It is clear that, for $x \in [0, \infty]$, the set $\mathcal{E}(x)$ is not empty, and there exists $\mathcal{H}(x) \in \mathcal{E}(x)$ which realizes $m(x)$ and satisfies $\|\mathcal{H}(x)\|_2 = 1$. Then the lemma follows if we prove that

$$K_1 = \sup_{x\in[0,+\infty]} m(x) = \sup_{x\in[0,+\infty]} \|\mathcal{H}(x)^{-1}\|_2 < +\infty.$$

For this, it suffices (in view of the compactness of $[0, +\infty]$) to prove that, for all $x \in [0, +\infty]$, there exists a neighborhood $v(x)$ of x such that $\sup_{y\in v(x)} m(y) < +\infty$. We shall distinguish four cases:

<u>1^{st} case</u> $\quad x \in (0, \infty]$

Then, there exists $N \in \mathbb{R}^{q\times q}$ such that $\frac{\alpha_q+\beta_q x}{\alpha_q+\eta\beta_q x}\|N^{-1}\Lambda(x)N\|_2 < 1$. Thus, there exists a neighborhood $v(x)$ of x such that $N \in \mathcal{E}(y)$ for all $y \in v(x)$; therefore, $\sup_{y\in v(x)} m(y) \le \|N\|_2\|N^{-1}\|_2$.

<u>2^{nd} case</u> $\quad x = 0$ *and the eigenvalues of* $\Lambda(0)$ *are real and distinct.* In this case we can find an interval $v(0) = [0, b]$, with $b > 0$, q analytic functions $\lambda_1, \ldots, \lambda_q$ (eigenvalues) from $v(0)$ into $\mathbb{R}$, and q analytic functions $h_1, \ldots, h_q$ (eigenvectors) from $v(0)$ into $\mathbb{R}^q$, such that

$$\Lambda(y)h_i(y) = \lambda_i(y)h_i(y), \;\; i = 1, \ldots, q, \quad \forall y \in v(0).$$

We consider now the matrix $N(y) = (h_1(y), \ldots, h_q(y))$; then we have

$$N(y)^{-1}\Lambda(y)N(y) = \text{diag }\{\lambda_1(y), \ldots, \lambda_q(y)\}.$$

Therefore, for $y \in v(0)$, we have $\|N(y)^{-1}\Lambda(y)N(y)\|_2 = \rho(y)$, so that $N(y) \in \mathcal{E}(y)$ and thus $\sup_{y\in v(0)} m(y) \le \max_{y\in[0,b]} \|N(y)\|_2\|N(y)^{-1}\|_2 < +\infty$.

<u>3^{rd} case</u> $\quad x = 0$ *and the eigenvalues of* $\Lambda(0)$ *are distinct.* If $\Lambda(0)$ admits some nonreal eigenvalues, the previous construction provides a complex matrix N. But in each occurrence where $\lambda_k = \bar{\lambda}_{k+1} = a_k + ib_k$ is a nonreal eigenvalue, if $h_k = m_k + in_k$ denotes the corresponding eigenvector, we replace in the columns of N the pair (h_k, h_{k+1}) by (m_k, n_k). In this way we obtain a matrix $\widetilde{N} \in \mathbb{R}^{q\times q}$, and in the matrix $\widetilde{N}^{-1}\Lambda\widetilde{N}$ the block $\begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_{k+1} \end{pmatrix}$ of $N^{-1}\Lambda N$ is replaced by $\begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix}$.

We still have $\|\widetilde{N}(y)^{-1}\Lambda(y)\widetilde{N}(y)\|_2 = \rho(y)$ and $\widetilde{N}(y) \in \mathcal{E}(y)$.

<u>4^{th} case</u> $\quad x = 0$ *and* $\Lambda(0)$ *has multiple eigenvalues.* From the Dahlquist 0−stability condition we know that the eigenvalues $\lambda_1(0), \ldots, \lambda_s(0)$ of $\Lambda(0)$ of modulus 1 are simple.

There exists a real $0 < \rho_1 < 1$ such that the remaining eigenvalues satisfy $|\lambda_k(0)| < \rho_1,\ k = s+1, \dots, q$.

Now we denote by C_{ρ_1} the circle (positive oriented) with center 0 and radius ρ_1. We choose a positive real b such that, for all $y \in v(0) = [0, b]$, we have

$$|\lambda_k(y)| > \rho_1 \quad \text{for } k = 1, \dots, s, \quad \text{and} \quad |\lambda_k(y)| < \rho_1 \quad \text{for } k = s+1, \dots, q.$$

For all $y \in v(0)$, the linear operator $P(y)$,

$$P(y) = \frac{1}{2\pi \mathrm{i}} \int_{C_{\rho_1}} (\zeta I - \Lambda(y))^{-1} d\zeta,$$

is well defined and depends analytically on y; $P(y)$ is a projector. The set $Range(I - P(y))$ (respectively $Range(P(y))$) is the invariant subspace of $\Lambda(y)$ associated to the eigenvalues $\lambda_1(y), \dots, \lambda_s(y)$ (resp. $\lambda_{s+1}(y), \dots, \lambda_q(y)$). We have $\mathbb{R}^q = Range(I - P(y)) \oplus Range(P(y))$ and we can choose a matrix $M(y) = \big(m_1(y), m_2(y), \dots, m_q(y)\big)$ which depends analytically on y such that the columns $\{m_1(y), \dots, m_s(y)\}$ constitute a basis of $Range(I - P(y))$ and $\{m_{s+1}(y), \dots, m_q(y)\}$ a basis of $Range(P(y))$. Then, the matrix $M(y)^{-1}\Lambda(y)M(y) = \begin{pmatrix} \Lambda_1(y) & 0 \\ 0 & \Lambda_2(y) \end{pmatrix}$ is block diagonal, the eigenvalues $\lambda_1(y), \dots, \lambda_s(y)$ of $\Lambda_1(y)$ are simple, and the eigenvalues of $\Lambda_2(y)$ satisfy $|\lambda_k(y)| < \rho_1$. Arguing as in the previous case, we can find a matrix $N_1(y) \in \mathbb{R}^{s \times s}$, depending analytically on y, such that $\|N_1(y)^{-1}\Lambda_1(y)N_1(y)\|_2 = \rho(\Lambda_1(y)) = \rho(y)$. Similarly to the first case, and reducing b if necessary, we can find a matrix N_2 such that $\|N_2^{-1}\Lambda_2(y)N_2\|_2 \le \rho_1$ for all $y \in [0, b)$.

Finally, we choose $N(y) = M(y)\begin{pmatrix} N_1(y) & 0 \\ 0 & N_2 \end{pmatrix}$ and obtain

$$\forall y \in [0, b), \quad \|N(y)^{-1}\Lambda(y)N(y)\|_2 \le \rho(y).$$

The proof is now complete. □

Now let

$$\mathcal{H} = \mathcal{H}(kA_h), \quad \mathcal{L} = \mathcal{L}(kA_h),$$

and

$$Y^n = \mathcal{H}^{-1}\Theta^n,\ \widetilde{\Gamma}_n = \mathcal{H}^{-1}\Gamma_n,\ \widetilde{\mathcal{E}}^n = \mathcal{H}^{-1}\mathcal{E}^n;$$

then, we can rewrite (2.7) as

$$(\alpha_q + k\beta_q A_h)Y^{n+1} = (\alpha_q + k\eta\beta_q A_h)\ \mathcal{L}Y^n + k\widetilde{\Gamma}_n\Theta^n + k\widetilde{\mathcal{E}}^n. \tag{2.8}$$

In view of the boundedness of the functions $\|\mathcal{H}(x)\|_2, \|\mathcal{H}(x)^{-1}\|_2$, it suffices to estimate Y^n. We assume without loss of generality that both α_q and β_q are positive, and introduce in V an appropriate to the scheme under consideration norm $|\!|\!|\cdot|\!|\!|$ by $|\!|\!|v|\!|\!| :=$

$(\alpha_q|v|^2+\beta_q k\|v\|^2)^{1/2}$, $v\in V$. Further, for $V=(v_1,\dots,v_q)^T$ and $W=(w_1,\dots,w_q)^T$ in H^q or in V^q we shall use the notation

$$(V,W):=\sum_{i=1}^{q}(v_i,w_i),\quad |V|:=\Big(\sum_{i=1}^{q}|v_i|^2\Big)^{1/2},$$
$$\|V\|:=\Big(\sum_{i=1}^{q}\|v_i\|^2\Big)^{1/2},\quad |\!|\!|V|\!|\!|:=\Big(\sum_{i=1}^{q}|\!|\!|v_i|\!|\!|^2\Big)^{1/2},\quad \|V\|_\star:=\Big(\sum_{i=1}^{q}\|v_i\|_\star^2\Big)^{1/2},$$

and, for a linear operator $M:H^q\to H^q$, we set $|M|:=\sup_{V\in H^q,V\neq 0}\frac{|MV|}{|V|}$.

The main result in this paper is given in the following theorem:

Theorem 2.1. *Assume that the constant λ in (1.4) is appropriately small (depending on the particular scheme) and that k and $h^{2r}k^{-1}$ are sufficiently small. Then, we have the local stability estimate*

$$|\vartheta^n|+k^{1/2}\|\vartheta^n\|\le Ce^{c\mu^2T}\Big\{\sum_{j=0}^{q-1}\Big(|\vartheta^j|+k^{1/2}\|\vartheta^j\|\Big)^2+\sum_{j=0}^{n-q}\|E^j\|_\star^2\Big\}^{1/2}, \tag{2.9}$$

$n=q-1,\dots,N$, and the error estimate

$$\max_{0\le n\le N}|u(t^n)-U^n|\le C(k^p+h^r). \tag{2.10}$$

Proof. Let $\rho^n=u^n-W^n, n=0,\dots,N$. Then, according to (1.6),

$$\max_{0\le n\le N}|\rho^n|\le Ch^r \tag{2.11}$$

and, for sufficiently small h,

$$\max_{0\le n\le N}\|\rho^n\|\le 1/2. \tag{2.12}$$

Now, if we assume that (2.9) holds, using (2.5) and (2.4), we obtain

$$\max_{0\le n\le N}|\vartheta^n|\le C(k^p+h^r)\,, \tag{2.13}$$

and (2.10) follows immediately from (2.11) and (2.13). Thus, it remains to prove (2.9). According to (2.5) and (2.4), there exists a constant $C_\star$ such that the right-hand side of (2.9) can be estimated by $C_\star(k^p+h^r)$,

$$Ce^{c\mu^2T}\Big\{\sum_{j=0}^{q-1}\Big(|\vartheta^j|+k^{1/2}\|\vartheta^j\|\Big)^2+k\sum_{j=0}^{N-q}\|E^j\|_\star^2\Big\}^{1/2}\le C_\star(k^p+h^r)\,. \tag{2.14}$$

We will estimate ϑ^n by estimating Y^n. In fact, we shall show that for some positive ε and ζ, $\varepsilon + \zeta < (1-\eta^2)\beta_q$ with η as in Lemma 2.1, and a constant c depending on ε,

$$\|Y^n\| \le e^{c\mu^2 t^n}\{\|Y^0\|^2 + \frac{k}{\zeta}\sum_{j=0}^{n-1} \|\widetilde{\mathcal{E}}^j\|_\star^2\}^{1/2}. \tag{2.15}$$

From Lemma 2.1 we deduce $\|\Theta^n\| \le \|Y^n\|$, $\|Y^0\| \le K_1\|\Theta^0\|$ and $\|\widetilde{\mathcal{E}}^j\|_\star \le K_2\|E^j\|_\star$. Then, (2.9) follows, and the proof will be complete. We shall use induction: The estimate (2.15) is valid for $n = 0$. Assume that it holds for $0,\dots,n$, $0 \le n \le N-q$. Then, according to (2.14) and (2.9), which is then valid for $0,\dots,n+q-1$, we have, for k and $k^{-1}h^{2r}$ small enough,

$$\max_{0\le j\le n+q-1} \|\vartheta^j\| \le C_\star(k^{p-1/2} + k^{-1/2}h^r) \le 1/2,$$

i.e., using also (2.12),

$$W^j, U^j \in T_u, \qquad j = 0,\dots,n-q+1. \tag{2.16}$$

Taking in (2.8) the inner product with Y^{n+1}, we have

$$\|Y^{n+1}\|^2 = ((\alpha_q + k\eta\beta_q A_h)\mathcal{L}Y^n, Y^{n+1}) + k(\widetilde{\Gamma}_n\Theta^n, Y^{n+1}) + k(\widetilde{\mathcal{E}}^n, Y^{n+1}). \tag{2.17}$$

First, we shall estimate the second term on the right-hand side of (2.17). Setting $Z^{n+1} := (\mathcal{H}^{-1})^T Y^{n+1}$, we have

$$(\widetilde{\Gamma}_n\Theta^n, Y^{n+1}) = (\Gamma_n\Theta^n, Z^{n+1}) = \sum_{i=0}^{q-1}\gamma_i(B_h(t^{n+i}, W^{n+i}) - B_h(t^{n+i}, U^{n+i}), Z_1^{n+1}).$$

Using here the induction hypothesis, which ensures (2.16), and the assumption (1.4), we see that

$$|(\widetilde{\Gamma}_n\Theta^n, Y^{n+1})| \le \sum_{j=0}^{q-1}|\gamma_i|(\lambda\|\vartheta^{n+i}\| + \mu|\vartheta^{n+i}|)\|Z_1^{n+1}\|,$$

and thus, with $M_1 = \|\gamma\|_2 K_2$, in view of $\|(\mathcal{H}(x)^{-1})^T e_1\|_2 \le K_2$, cf. Lemma 2.1,

$$|(\widetilde{\Gamma}_n\Theta^n, Y^{n+1})| \le M_1(\lambda\|Y^n\| + \mu|Y^n|)\|Y^{n+1}\|. \tag{2.18}$$

Further, Lemma 2.1 implies $|\mathcal{L}| \le 1$, and we have the following estimate for the first term on the right-hand side of (2.17)

$$|((\alpha_q + k\eta\beta_q A_h)\mathcal{L}Y^n, Y^{n+1})| \le \alpha_q|Y^n|\,|Y^{n+1}| + \eta\beta_q k\|Y^n\|\,\|Y^{n+1}\|, \tag{2.19}$$

cf. [1]. From (2.17), (2.18) and (2.19), we obtain

$$
\begin{aligned}
\|Y^{n+1}\|^2 \le & \alpha_q |Y^n|\,|Y^{n+1}| + (\lambda M_1 + \eta\beta_q)k\|Y^n\|\,\|Y^{n+1}\| \\
& + M_1\mu k |Y^n|\,\|Y^{n+1}\| + k\|\widetilde{\mathcal{E}}^n\|_*\,\|Y^{n+1}\|.
\end{aligned} \tag{2.20}
$$

Therefore, with $\lambda = \frac{1}{M_1}\left\{[\beta_q(\beta_q - \varepsilon - \zeta)]^{1/2} - \eta\beta_q\right\}$ for some positive ε and ζ, $\varepsilon + \zeta < (1-\eta^2)\beta_q$,

$$
\begin{aligned}
\|Y^{n+1}\|^2 \le & \frac{\alpha_q}{2}|Y^n|^2 + \frac{\alpha_q}{2}|Y^{n+1}|^2 + [\beta_q(\beta_q - \varepsilon - \zeta)]^{1/2} k\|Y^n\|\,\|Y^{n+1}\| \\
& + \frac{(M_1\mu)^2}{2\varepsilon}k|Y^n|^2 + \frac{\varepsilon}{2}k\|Y^{n+1}\|^2 + \frac{1}{2\zeta}k\|\widetilde{\mathcal{E}}^n\|_\star^2 + \frac{\zeta}{2}k\|Y^{n+1}\|^2.
\end{aligned}
$$

Thus, with $c = \frac{(M_1)^2}{2\alpha_q\varepsilon}$,

$$
\begin{aligned}
\|Y^{n+1}\|^2 \le & \frac{1}{2}(1 + 2c\mu^2 k)\alpha_q|Y^n|^2 + \frac{1}{2}\alpha_q|Y^{n+1}|^2 + \frac{1}{2}\beta_q k\|Y^n\|^2 \\
& + \frac{1}{2}\beta_q k\|Y^{n+1}\|^2 + \frac{1}{2\zeta}k\|\widetilde{\mathcal{E}}^n\|_\star^2,
\end{aligned}
$$

i.e.,

$$
\|Y^{n+1}\|^2 \le (1 + 2c\mu^2 k)\|Y^n\|^2 + \frac{k}{\zeta}\|\widetilde{\mathcal{E}}^n\|_\star^2. \tag{2.21}
$$

From (2.21) and the induction hypothesis, it easily follows that (2.15) holds for $n+1$ as well, and the proof is complete. □

Remark 2.1. Let $\tau \in \mathbb{R}$ be such that $A + \tau I$ is positive semidefinite. It is then easily seen that the results of Theorem 2.1 hold also for the scheme

$$
\sum_{i=0}^{q} \alpha_i U^{n+i} + k\sum_{i=0}^{q} \beta_i (A_h U^{n+i} + \tau U^{n+i}) = k\sum_{i=0}^{q-1} \gamma_i [B_h(t^{n+i}, U^{n+i}) + \tau U^{n+i}].
$$

Remark 2.2. The weak meshcondition "$k^{-1}h^{2r}$ small" is used in the proof of Theorem 2.1 only to show that $\|\vartheta^n\| \le 1/2$ which implies (2.16). If the estimate (1.4) holds in tubes around u defined in terms of weaker norms, not necessarily the same for both arguments v and w, one may get by with an even weaker meshcondition. Assume, for instance, that (1.4) holds for $v, w \in T_u^\star := \{\omega \in V : \min_t \|u(t) - \omega\|^\star \le 1\}$ —or for $v \in T_u$, cf. (2.12), and $w \in T_u^\star$— and the norm $\|\cdot\|^\star$ satisfies an inequality of the form

$$
\|v\|^\star \le |v| + |v|^{1-a}\|v\|^a, \quad v \in V,
$$

for some constant $a, 0 \leq a < 1$. Then, a condition of the form "k and $k^{-a}h^{2r}$ sufficiently small" suffices for (2.9) and (2.10) to hold.

Similarly, when the relation (1.4) is satisfied in tubes around u defined in terms of stronger norms, not necessarily the same for both arguments, the convergence result of Theorem 2.1 may still be valid but under *stronger* meshconditions, cf. [1]; this fact will be used in the next section.

Remark 2.3. If we combine the implicit and explicit Euler methods, then it is easily seen that the result of Theorem 2.1 holds for any $\lambda < 1$, cf. [1].

Remark 2.4. If the two constants η and K_2 of Lemma 2.1 are known, then a bound of the constant λ in (1.4) which guarantees that the results of Theorem 2.1 are valid is given by $\frac{1}{M_1}(1-\eta)\beta_q$ with $M_1 = \|\gamma\|_2 K_2$, cf. the proof of Theorem 2.1. As an example, we consider the second order scheme characterized by the polynomials

$$\alpha(x) = \frac{3}{2}x^2 - 2x + \frac{1}{2}, \quad \beta(x) = x^2, \quad \gamma(x) = 2x - 1.$$

In this case we have verified that we can choose

$$\eta = 0.3, \quad K_1 = 2 + \sqrt{5}, \quad K_2 = 3.65. \tag{2.22}$$

Therefore, if (1.4) is valid with

$$\lambda < \frac{1}{3.65\sqrt{5}}\, 0.7 = 0.0857669, \tag{2.23}$$

then the results of Theorem 2.1 hold.

Remark 2.5. *Initial approximations.* Assume that the data of the problem are smooth enough such that one can compute the time derivatives $u^{(j)}(0)$, $j = 1, \dots, p$, of the exact solution at $t = 0$. Then, it is easily seen that $U^0 = W^0$ and $U^m = R_h T_m^p u(0)$, $m = 1, \dots, q-1$, with

$$T_m^p u(0) = u^0 + mku^{(1)}(0) + \cdots + \frac{(mk)^p}{p!}u^{(p)}(0), \quad m = 1, \dots, q-1,$$

satisfy (2.5).

3. Application to a quasilinear equation

In this section we shall apply our results to a class of quasilinear equations: Let $\Omega \subset \mathbb{R}^\nu$, $\nu \leq 3$, be a bounded domain with smooth boundary $\partial\Omega$. For $T > 0$ we seek a real-valued function u, defined on $\bar{\Omega} \times [0, T]$, satisfying

$$\begin{aligned} &u_t - \operatorname{div}(a(x)\nabla u) = \operatorname{div}(b(x,t,u)\nabla u + g(x,t,u)) + f(x,t,u) && \text{in } \Omega \times [0,T], \\ &u = 0 && \text{on } \partial\Omega \times [0,T], \\ &u(\cdot, 0) = u^0 && \text{in } \Omega, \end{aligned} \tag{3.1}$$

with $a : \bar{\Omega} \to (0, \infty)$, $b, f : \bar{\Omega} \times [0, T] \times \mathbb{R} \to \mathbb{R}$, $g : \bar{\Omega} \times [0, T] \times \mathbb{R} \to \mathbb{R}^\nu$, and $u^0 : \bar{\Omega} \to \mathbb{R}$ given smooth functions. We are interested in approximating smooth solutions of this problem, and assume therefore that the data are smooth and compatible such that (3.1) gives rise to a sufficiently regular solution. We assume that $-\text{div}([a(x) + b(x, t, u)]\nabla\cdot)$ is an elliptic operator.

Let $H^s = H^s(\Omega)$ be the usual Sobolev space of order s, and $\|\cdot\|_{H^s}$ be the norm of H^s. The inner product in $H := L^2(\Omega)$ is denoted by $(\cdot, \cdot)$, and the induced norm by $|\cdot|$; the norm of $L^s(\Omega)$, $1 \le s \le \infty$, is denoted by $\|\cdot\|_{L^s}$. Let $Av := -\text{div}(a\nabla v)$ and $B(t, v) := \text{div}(b(\cdot, t, v)\nabla v) + \text{div}g(\cdot, t, v) + f(\cdot, t, v)$. Obviously, $V = H_0^1 = H_0^1(\Omega)$ and the norm $\|\cdot\|$ in V, $\|v\| = |\sqrt{a}\nabla v|$, is equivalent to the H^1-norm.

Let

$$\widetilde{T}_u := \{v \in V \cap L^\infty : \min_t \|u(t) - v\|_{L^\infty} \le 1\},$$

$$\widehat{T}_u := \{v \in V \cap W_\infty^1 : \min_t \|u(t) - v\|_{W_\infty^1} \le 1\},$$

and

$$\lambda := \sup\{|b(x, t, y)|/a(x) : x \in \Omega, t \in [0, T], y \in \mathcal{U}\}$$

with $\mathcal{U} := [-1 + \min_{x,t} u, 1 + \max_{x,t} u]$.

Now, for $v, w, \varphi \in V$,

$$\begin{aligned}(B(t, v) - B(t, w), \varphi) = &- (b(\cdot, t, w)\nabla(v - w), \nabla\varphi) - ([b(\cdot, t, v) - b(\cdot, t, w)]\nabla v, \nabla\varphi) \\ &- (g(\cdot, t, v) - g(\cdot, t, w), \nabla\varphi) + ([f(\cdot, t, v) - f(\cdot, t, w)], \varphi)\,,\end{aligned}$$

and we easily see that

$$\|B(t, v) - B(t, w)\|_\star \le \lambda\|v - w\| + \mu|v - w| \qquad v \in \widehat{T}_u, w \in \widetilde{T}_u. \tag{3.2}$$

Thus, a stability condition of the form (1.4) is satisfied for $v \in \widehat{T}_u$ and $w \in \widetilde{T}_u$.

Further,

$$\begin{aligned}B'(t, v)w =&\text{div}(b(\cdot, t, v)\nabla w) + \text{div}(\partial_3 b(\cdot, t, v)w\nabla v) \\ &+ \text{div}(\partial_3 g(\cdot, t, v)w) + \partial_3 f(\cdot, t, v)w.\end{aligned}$$

and, therefore, $A - B'(t, u(t)) + \sigma I$ is, for an appropriate constant σ, uniformly positive definite in H_0^1.

Let V_h be the subspace of V defined on a finite element partition $\mathcal{T}_h$ of Ω, and consisting of piecewise polynomial functions of degree at most $r - 1$, $r \ge 2$. Let h_K denote the diameter of an element $K \in \mathcal{T}_h$, and $h := \max_{K \in \mathcal{T}_h} h_K$. We define the elliptic projection operator $R_h(t)$, $R_h(t) : V \to V_h$, $t \in [0, T]$, by

$$\begin{aligned}&([a(\cdot) + b(\cdot, t, u(\cdot, t))]\nabla(v - R_h(t)v), \nabla\chi) \\ &\quad + ([\partial_3 b(\cdot, t, u(\cdot, t))]\nabla u(\cdot, t) + \partial_3 g(\cdot, t, u(\cdot, t))](v - R_h(t)v), \nabla\chi) \\ &\quad - ([\partial_3 f(\cdot, t, u(\cdot, t)) - \sigma](v - R_h(t)v), \chi) = 0 \quad \forall\chi \in V_h.\end{aligned}$$

It is well known from the error analysis for elliptic problems that

$$|v - R_h(t)v| + h\|v - R_h(t)v\| \le Ch^r\|v\|_{H^r}, \quad v \in H^r \cap H_0^1, \tag{3.3}$$

i.e., the estimate (1.6) is satisfied with $d = 2$. Further,

$$\left|\frac{d}{dt}[u(\cdot,t) - R_h(t)u(\cdot,t)]\right| \le Ch^r, \tag{3.4}$$

and

$$\left|\frac{d^j}{dt^j}R_h(t)v\right| + h\left\|\frac{d^j}{dt^j}R_h(t)v\right\| \le Ch^r\|v\|_{H^r}, \quad v \in H^r \cap H_0^1, \quad j = 1,\dots,p+1, \tag{3.5}$$

cf., e.g., [3]; thus (1.7) and (1.8) are valid. We further assume, cf. [7], [10], that

$$\sup_t \|u(\cdot,t) - R_h(t)u(\cdot,t)\|_{W^1_\infty} \le \frac{1}{2}. \tag{3.6}$$

Next, we will verify (1.9). We have

$$\begin{aligned} &B(t,u(t)) - B(t,R_h(t)u(t)) - B'(t,u(t))(R_h(t)u(t) - u(t)) = \\ &\qquad = -\int_0^1 \tau B''(t, R_h(t)u(t) - \tau[R_h(t)u(t) - u(t)]d\tau[R_h(t)u(t) - u(t)]^2 \end{aligned} \tag{3.7i}$$

and

$$\begin{aligned} B''(t,v)w^2 =&\operatorname{div}(\partial_3^2 b(\cdot,t,v)w^2\nabla v) + 2\operatorname{div}(\partial_3 b(\cdot,t,v)w\nabla w) \\ &+ \operatorname{div}(\partial_3^2 g(\cdot,t,v)w^2) + \partial_3^2 f(\cdot,t,v)w^2. \end{aligned} \tag{3.7ii}$$

It easily follows from (3.7) and (3.3), in view of (3.6), that

$$\|B(t,u(t)) - B(t,R_h(t)u(t)) - B'(t,u(t))(u(t) - R_h(t)u(t))\|_{H^{-1}} \le Ch^r, \tag{3.8}$$

i.e., (1.9) is satisfied.

Now, let $W(t) := R_h(t)u(t)$, and assume that we are given approximations $U^0,\dots,U^{q-1} \in V_h$ to $u^0,\dots,u^{q-1}$ such that

$$\sum_{j=0}^{q-1}\Big(|W^j - U^j| + k^{1/2}\|W^j - U^j\|\Big) \le c(k^p + h^r). \tag{3.9}$$

Then, we define $U^n \in V_h$, $n = q,\dots,N$, recursively by the (α,β,γ) scheme

$$\begin{aligned} &\sum_{i=0}^{q}\alpha_i(U^{n+i},\chi) + k\sum_{i=0}^{q}\beta_i(a(\cdot)\nabla U^{n+i},\nabla\chi) = \\ &\qquad = k\sum_{i=0}^{q-1}\gamma_i\{-(b(\cdot,t^{n+i},U^{n+i})\nabla U^{n+i} + g(\cdot,t^{n+i},U^{n+i}),\nabla\chi) \\ &\qquad\qquad + (f(\cdot,t^{n+i},U^{n+i}),\chi)\}, \quad \forall\chi \in V_h, \quad n = 0,\dots,N-q, \end{aligned} \tag{3.10}$$

with (α, β) and (α, γ) multistep schemes of order p, and (α, β) strongly $A(0)$−stable. Then, Theorem 2.1 yields, in view of (3.6), for sufficiently small k and provided that the approximate solutions U^n are in $\widetilde{T}_u$, the error estimate

$$\max_n |u^n - U^n| \leq c(k^p + h^r). \tag{3.11}$$

To ensure that $U^n \in \widetilde{T}_u, n = 0, \ldots, N$, we define $\underline{h} := \min_{K \in \mathcal{T}_h} h_K$ and will distinguish three cases: $\nu = 1$, $\nu = 2$ and $\nu = 3$.

i. $\nu = 1$. First, since the H^1−norm dominates the L^∞−norm in one space dimension, we have

$$\max_{0 \leq j \leq n+q-1} \|\vartheta^j\|_{L^\infty} \leq C \max_{0 \leq j \leq n+q-1} \|\vartheta^j\|,$$

and thus, according to (2.14),

$$\max_{0 \leq j \leq n+q-1} \|\vartheta^j\|_{L^\infty} \leq C(k^{p-1/2} + k^{-1/2} h^r).$$

Therefore, for k and $k^{-1}h^{2r}$ sufficiently small, in view of (3.6), $U^j \in \widetilde{T}_u, j = 0, \ldots, n + q - 1$. We easily conclude that the convergence result holds.

ii. $\nu = 2$. First, we note that

$$\|\chi\|_{L^\infty} \leq C |\log(\underline{h})|^{1/2} \|\chi\|_{H^1} \qquad \forall \chi \in V_h,$$

cf. [8; p. 67]. It is then easily seen that the convergence result holds, if k and h are chosen such that $|\log(\underline{h})| k^{2p-1}$ and $|\log(\underline{h})| k^{-1} h^{2r}$ are sufficiently small.

iii. $\nu = 3$. In this case,

$$\|\chi\|_{L^\infty} \leq C \underline{h}^{-1/2} \|\chi\|_{H^1} \qquad \forall \chi \in V_h,$$

and the result (3.11) holds, provided that $\underline{h}^{-1} k^{2p-1}$ and $k^{-1} \underline{h}^{-1} h^{2r}$ are sufficiently small.

Remark 3.1. Let the quasilinear equation be given in the form

$$u_t = \operatorname{div}(c(x, t, u) \nabla u + g(x, t, u)) + f(x, t, u).$$

It can then be written in the form used in (3.1) by letting, say, $a(x) := c(x, 0, u^0)$ and $b(x, t, u) := c(x, t, u) - a(x)$.

Different splittings might be used on a finite number of subintervals of $[0, T]$. Assume, for instance, that an approximation U to $u(\cdot, t_a)$ has been computed. Then, the splitting $a(x) := c(x, t_a, U)$ and $b(x, t, u) := c(x, t, u) - a(x)$ may be used on a time interval $[t_a, t_b]$.

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