FREQUENCY SPECTRUM OF THE BISPHERICAL HOLLOW SYSTEM: THE CASE OF THE NONUNIFORM THICKNESS HUMAN SKULL

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Abstract

In the present work we discuss the role of the thickness nonuniformities of the human dry skull on its frequency spectrum. The mathematical modelling is based on the three-dimensional theory of elasticity, the approximation of the skull by an isotropic material occupying the region bounded by two non-concentric spheres and the introduction of the bispherical coordinates system. The mathematical analysis is based on the construction of the Navier eigenfunctions of the problem and the representation of the solution in terms of them. The frequency equation is solved numerically and the role of various parameters entering the problem is extensively discussed.

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1. Introduction

The complexity of the human cranial system presents an extremely difficult task to one wishing to perform detailed analysis of the physical processes of the human head by mathematical modelling. For this reason geometrical and material approximations are typically used for analytical investigations [1,2]. The human skull is a complex structure made up of several bones each with its own unique internal and external geometry. In cross section, the skull varies in thickness from about 3/8 to 1/2 in, increasing in this dimension towards the base of the skull. It consists of an inner and outer table of compact bone with a highly vecicular middle layer resembling a honeycomb, called the diploë, and appears to be transversely isotropic. In analytical investigations geometrical approximations and mainly the geometry of spherical and prolate spheroidal shells have been used [3,4,5]. In a recent paper [6] the dynamic characteristics of the human dry skull were presented and the results obtained were favourably compared to the experimental ones [7]. The analysis in Ref. 6 was based on the three-dimensional theory of elasticity and the representation of the displacement field in terms of the Navier eigenvectors. The role of a small deviation of the spherical to spheroidal geometry on the frequency spectrum of the human dry skull was presented very recently in [8]. In Ref. 8 the analysis was based on the three - dimensional theory of elasticity, complex analysis techniques and the constructed for this problem Navier eigenvectors. The conclusion from the results obtained in Ref. 8 is that the deviation of the spherical geometry to spheroidal one with sufficiently small eccentricity does not appreciably influence the frequency spectrum of the human dry skull.

In the present work we deal with the role of the thickness nonuniformities, due to the nature and the existence of the facial bones, of the human skull on its frequency spectrum. The mathematical modelling is based on the three - dimensional theory of elasticity and the approximation of the skull by an isotropic material occupying the region bounded by two non-concentric spheres. The bispherical coordinates system was used to describe the geometry introduced by the skull morphology. We note that the analysis we present is only valid for small deviations of the system from the one of the two concentric spheres. The mathematical analysis is based on the construction of the Navier eigenfunctions for the problem under discussion and the representation of its solution in terms of this complete set of vector functions in the space of solutions of time-independent Navier equation. The frequency equation is constructed by imposing the satisfaction of the boundary conditions and it is solved numerically. From the results obtained we discover the role of the various parameters entering the problem (especially of the thickness nonuniformities) on the frequency spectrum of the human dry skull.

2. The Bispherical System

We consider the region V bounded by two non-concentric spheres. The bispherical coordinate system is the most suitable to describe the geometry introduced by the two non-concentric spheres, S_1 and S_2 (Fig. 1).

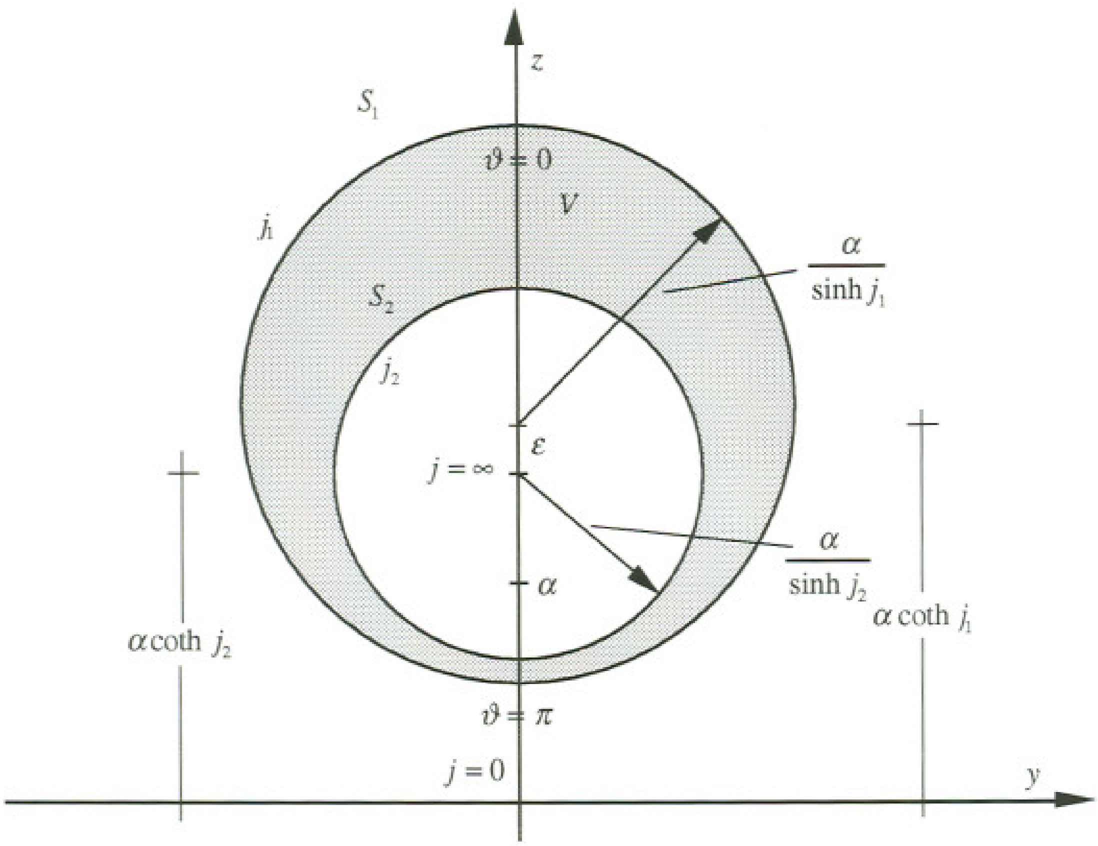

Figure 1: Two Non-concentric Spheres System.

Every point in $\mathbf{R}^3$ is determined by specific values of the bispherical coordinates (ϑ, j, φ) [9] which are related to the cartesian ones by the the following relations (assuming that the z - axis is the rotation axis of the system):

$$x = \frac{\alpha \sin\vartheta \cos\varphi}{\cosh j - \cos\vartheta} \tag{1}$$

$$y = \frac{\alpha \sin\vartheta \sin\varphi}{\cosh j - \cos\vartheta} \tag{2}$$

$$z = \frac{\alpha \sinh j}{\cosh j - \cos\vartheta}. \tag{3}$$

It is well - known for the bispherical system of coordinates that the surfaces of constant j are spheres and the surfaces of constant ϑ are either apple-shaped surfaces with dimples on the $x-$ axis or spindles. Surfaces of constant φ are half planes as in toroidal coordinates.

The variable j has the range from $-\infty$ to ∞. Positive values of j are associated with spheres whose centers lay above the xy - plane and negative values with spheres with centers below the xy - plane. In both cases, the center of the sphere is $a \coth j$ and the radius is $\alpha/|\sinh j|$. Note that $j=0$ corresponds to the xy - plane (sphere of infinite radius) while the values $j=-\infty,+\infty$ correspond to the points $(0,0,-\alpha)$ and $(0,0,+\alpha)$, respectively. That means, as j increases in absolute value, the spheres schrink, changing their centers, tending to coincide with points $(0,0,-\alpha)$ or $(0,0,+\alpha)$ depending on the sign of j.

The variable ϑ ranges from 0 to π. The surfaces $\vartheta = const.$ are apple-shaped if $\vartheta < \pi/2$ and spindle - shaped if $\vartheta > \pi/2$. The case $\vartheta = \pi/2$ corresponds to a sphere of radius α; for the case $\vartheta = \pi$ we have a line segment of length 2α connecting the foci $(0,0,\alpha)$, $(0,0,\text{-}\alpha)$. The case $\vartheta = 0$ corresponds to infinite pieces of z - axis, which are complements of the segment connecting the foci.

Before stating the problem we have to determine the parameter α as well as the values j_1 and j_2 corresponding to the surfaces S_1 and S_2, respectively, of the system under consideration. Given the radii of the two spheres r_1 and r_2 and the eccentricity ε, we have:

$$\frac{\alpha}{\sinh j_1} = r_1 \tag{4}$$

$$\frac{\alpha}{\sinh j_2} = r_2 \tag{5}$$

$$\alpha \coth j_1 - \alpha \coth j_2 = \varepsilon . \tag{6}$$

The solution of the system of equations (4), (5) and (6) gives that:

$$\alpha = \frac{\sqrt{(r_1^2 - r_2^2 - \varepsilon^2)^2 - 4\varepsilon^2 r_2^2}}{2\varepsilon} \tag{7}$$

$$e^{-j_1} = \sqrt{\left(\frac{\alpha}{r_1}\right)^2 + 1} - \frac{\alpha}{r_1} \tag{8}$$

$$e^{-j_2} = \sqrt{\left(\frac{\alpha}{r_2}\right)^2 + 1} - \frac{\alpha}{r_2}. \tag{9}$$

The parameters α, j_1, j_2 describe uniquely the system of two spheres with given eccentricity ε and radii r_1, r_2.

The actual region V, occupied by the elastic material corresponds to the domain

$$V = \{(\vartheta, j, \varphi) : j \in (j_1, j_2), \ \ \vartheta \in [0, \pi], \ \ \varphi \in [0, 2\pi]\}$$

and forms a bispherical shell.

3. Analysis of the Harmonic Motion

Let us consider the system under consideration executes harmonic motion. The displacement field $\mathbf{u}(\mathbf{r})$ in the skull region satisfies, after suppressing the time-harmonic dependence, the time independent equation of elasticity

$$c_s^2 \nabla^2 \mathbf{u}(\mathbf{r}) + (c_p^2 - c_s^2)\nabla(\nabla \cdot \mathbf{u}(\mathbf{r})) + \omega^2 \mathbf{u}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in V \tag{10}$$

where ω is the eigenfrequency (rads/sec) and $c_p = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}$, $c_s = \left(\frac{\mu}{\rho}\right)^{1/2}$ are the velocities of the longitudinal and transverse waves, respectively, λ, μ are the Lamè constants and ρ is the skull density.

The displacement field is supposed to satisfy the boundary conditions

$$\mathbf{Tu}(\mathbf{r}) = \mathbf{0}, \quad r \in S_1, S_2 \tag{11}$$

where

$$\mathbf{T} = 2\mu\hat{\mathbf{n}} \cdot \nabla + \lambda\hat{\mathbf{n}}\nabla \cdot + \mu\hat{\mathbf{n}} \times \nabla \times$$

is the stress operator on every surface having $\hat{\mathbf{n}}$ as its exterior unit normal vector.

Following Helmholtz decomposition we lead to the conclusion that the displacements field can be respresented as the superposition of an irrotational longitudinal field $\mathbf{u}^p(\mathbf{r})$ and a solenoidal transverse one $\mathbf{u}^s(\mathbf{r})$. These fields can be represented through scalar functions satisfying Helmholtz equation. More precisely

$$\mathbf{u}^p(\mathbf{r}) \in \left\{ \nabla\Phi_p(\mathbf{r}): \ (\Delta + k_p^2)\Phi_p(\mathbf{r}) = 0 \right\} \tag{12}$$

$$\mathbf{u}^s(\mathbf{r}) \in \left\{ \nabla\Phi_s(\mathbf{r}) \times (\mathbf{r} - \alpha\hat{\mathbf{z}}), \ \nabla \times \left[\nabla\Phi_s(\mathbf{r}) \times (\mathbf{r} - \alpha\hat{\mathbf{z}}) \right]: \ (\nabla^2 + k_s^2)\Phi_s(\mathbf{r}) = 0 \right\} \tag{13}$$

where $k_s = \frac{\omega}{c_s}$, $k_p = \frac{\omega}{c_p}$ are the wave numbers of the transverse and longitudinal waves, respectively.

Consequently, the equation of elasticity leads to the scalar Helmholtz equation through Helmholtz decomposition.

We note that in (12) and (13) any combination of $\mathbf{r}$ and constant vectors can be used instead of $\mathbf{r}-\alpha\hat{\mathbf{z}}$. This specific choice is based on the property of $\mathbf{r}-\alpha\hat{\mathbf{z}}$ to be, uniformly for every sphere $j = const$, a good approximation of the unit exterior normal vector (especially for large j).

The Helmholtz equation in bispherical coordinates does not admit R-separability of coordinates. However, when the eccentricity is considered to be small, Helmholtz equation can be solved appropriately.

Equations (12), (13) indicate that the determination of the elastic fields is based on the solution of the scalar Helmholtz equation

$$\nabla^2 u(\mathbf{r}) + k^2 u(\mathbf{r}) = \mathbf{0}, \tag{14}$$

where u is a scalar function.

We express the Laplacian in bispherical coordinates

$$\nabla^2 = \frac{(\cosh j - \cos\vartheta)^3}{\alpha^2}\left\{\begin{array}{l}\dfrac{\partial}{\partial j}\left(\dfrac{1}{\cosh j - \cos\vartheta}\dfrac{\partial}{\partial j}\right) + \dfrac{1}{\sin\vartheta}\dfrac{\partial}{\partial\vartheta}\left(\dfrac{\sin\vartheta}{\cosh j - \cos\vartheta}\dfrac{\partial}{\partial\vartheta}\right) + \\ \dfrac{1}{\cosh j - \cos\vartheta}\dfrac{1}{\sin^2\vartheta}\dfrac{\partial^2}{\partial\varphi^2}\end{array}\right\}. \tag{15}$$

We assume that equation (14) accepts a solution of the form:

$$u(\mathbf{r}) = \sqrt{\cosh j - \cos\vartheta}\, J(j)\Theta(\vartheta)\Phi(\varphi). \tag{16}$$

Inserting (16) in (14) we find that

$$\frac{J''(j)}{J(j)}+\frac{\Theta''(\vartheta)}{\Theta(\vartheta)}+\cot\vartheta\frac{\Theta'(\vartheta)}{\Theta(\vartheta)}+\frac{1}{\sin^2\vartheta}\frac{\Phi''(\varphi)}{\Phi(\varphi)}-\frac{1}{4}+\frac{k^2\alpha^2}{(\cosh j-\cos\vartheta)^2}=0. \tag{17}$$

Note that the last term in (17) is responsible for lack of R-separability.

We examine now how the term $\frac{\alpha^2}{(\cosh j-\cos\vartheta)^2}$ is affected by the eccentricity ε. For the limiting case of small eccentricity (the spheres tend to become concentric) equations (7), (8), (9) give

$$\alpha=\frac{r_1^2-r_2^2}{2\varepsilon} \tag{18}$$

$$e^{-j_1}=\frac{r_1\varepsilon}{r_1^2-r_2^2},\quad e^{-j_2}=\frac{r_2\varepsilon}{r_1^2-r_2^2}. \tag{19}$$

For $j_1\le j\le j_2$ and $\cosh j>>\cos\vartheta$ we can make the approximation

$$\frac{\alpha^2}{(\cosh j-\cos\vartheta)^2}\cong\frac{\alpha^2}{\cosh j^2},$$

then equation (17) is R-separable and following separation of variables we get

$$u(\mathbf{r})\in\begin{Bmatrix} j_n(k2\alpha e^{-j}) \\ y_n(k2\alpha e^{-j}) \end{Bmatrix}\times P_n^m(\cos\vartheta)\times e^{im\varphi} \tag{20}$$

where j_n, y_n stand for the spherical Bessel and Neumann functions of order n, respectively and $P_n^m(\cos\vartheta)$ are the associated Legendre functions.

Simultaneously, the equations (1), (2) and (3), relating cartesian and bispherical coordinates, take the limiting form

$$\begin{aligned} x &= (2\alpha e^{-j})\sin\vartheta\cos\varphi \\ y &= (2\alpha e^{-j})\sin\vartheta\sin\varphi \\ z &= \alpha + (2\alpha e^{-j})\cos\vartheta. \end{aligned} \tag{21}$$

Equations (21) for $r = 2\alpha e^{-j}$ represent the spherical coordinates and equations (20) constitute the solution of the Helmholtz equation in spherical coordinates.

Nevertheless, the assumption $\frac{\alpha}{\cosh j - \cos\vartheta} \approx \frac{\alpha}{\cosh j}$ corresponds to the case of concentric spheres with common center $\alpha\coth j_1 \approx \alpha\coth j_2 \approx \alpha$ as it is induced by (21) for $j = j_1$ and $j = j_2$, and so this assumption degenerates the problem to the spherical hollow system.

However, for the typical values of eccentricity occuring in the human skull system (for skull nonuniformities and the presence of facial bones) the expansion of $\frac{\alpha^2}{(\cosh j - \cos\vartheta)^2}$ in terms of powers of $\frac{\cos\vartheta}{\cos j}$ and the restriction to the first two terms, instead of the first alone, is a very good approximation.

Adopting the approximation

$$\frac{\alpha^2}{(\cosh j - \cos\vartheta)^2} \cong \frac{\alpha^2}{\cosh j^2}\left(1 + 2\frac{\cos\vartheta}{\cosh j}\right) \cong (2\alpha e^{-j})^2\left(1 + 4\frac{\cos\vartheta}{e^j}\right), \tag{22}$$

equation (14) becomes

$$\frac{\partial^2 \hat{u}}{\partial j^2} + \frac{\partial^2 \hat{u}}{\partial \vartheta^2} + \cot\vartheta \frac{\partial \hat{u}}{\partial \vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 \hat{u}}{\partial \varphi^2} - \frac{1}{4}\hat{u} + \frac{4k^2\alpha^2}{e^{2j}}\hat{u} + \frac{16k^2\alpha^2}{e^{3j}}\cos\vartheta\hat{u} = 0, \tag{23}$$

where

$$u(j,\vartheta,\varphi) = \sqrt{\cosh j - \cos\vartheta}\,\hat{u}(j,\vartheta,\varphi) \tag{24}$$

which does not R-separate coordinates, but it accepts a specific treatment. It is known that in the case of concentric spheres the eigenfrequency spectrum does not depend on the azimuthal coordinate φ [6]. Similarly, because our main interest is the examination of the shifting of the eigenfrequency spectrum from the one of the concentric spheres case, we are looking for solutions independent of φ.

In addition, following Lagrange method arguments and based on completeness of Legendre functions in the space of square - integrable functions in the interval [-1,1] we represent the solution of (23) in the form

$$\hat{u} = \sum_n \hat{G}_n(j) P_n(\cos\vartheta), \tag{25}$$

where summation extends, a priori, over $\mathbf{N}^*$.

Replacing equation (25) in (23), using the fact that $P_n(\cos\vartheta)$ satisfy Legendre equation and the relation

$$\cos\vartheta P_n(\cos\vartheta) = \frac{n}{2n+1}P_{n-1}(\cos\vartheta) + \frac{n+1}{2n+1}P_{n+1}(\cos\vartheta), \tag{26}$$

we get that

$$\sum_n (Z_n \hat{G}_n) P_n + \frac{16k^2\alpha^2}{e^{3j}} \sum_n \left[\frac{n}{2n+1} P_{n-1} \hat{G}_n + \frac{n+1}{2n+1} P_{n+1} \hat{G}_n \right] = 0, \tag{27}$$

where the arguments of the participating functions for simplicity have been omitted and Z_n is the operator

$$Z_n = \frac{\partial^2}{\partial j^2} - \left(n + \frac{1}{2} \right)^2 + \frac{4k^2\alpha^2}{e^{2j}}. \tag{28}$$

Taking into account the orthogonality of Legendre functions $P_n(\cos\vartheta)$ we conclude that for each n there exists, induced by (27), the following solution of equation (23),

$$\hat{u}^{(n)} = \hat{G}_{n-1}^{(n)}(j) P_{n-1}(\cos\vartheta) + \hat{G}_n^{(n)}(j) P_n(\cos\vartheta) + \hat{G}_{n+1}^{(n)}(j) P_{n+1}(\cos\vartheta) \tag{29}$$

which mixes the central and the two consecutive Legendre functions, where the functions $\hat{G}_{n-1}^{(n)}$, $\hat{G}_n^{(n)}$, $\hat{G}_{n+1}^{(n)}$ satisfy the following equations

$$Z_n \hat{G}_n^{(n)} = -\frac{16k^2\alpha^2}{e^{3j}} \left[\frac{n+1}{2n+3} \hat{G}_{n+1}^{(n)} + \frac{n}{2n-1} \hat{G}_{n-1}^{(n)} \right] \tag{30}$$

$$Z_{n+1} \hat{G}_{n+1}^{(n)} = -\frac{16k^2\alpha^2}{e^{3j}} \frac{n+1}{2n+1} \hat{G}_n^{(n)} \tag{31}$$

$$Z_{n-1} \hat{G}_{n-1}^{(n)} = -\frac{16k^2\alpha^2}{e^{3j}} \frac{n}{2n+1} \hat{G}_n^{(n)}. \tag{32}$$

Since the quantity $2\alpha e^{-j}$ constitutes the radial coordinate in the limit $\frac{\cos\vartheta}{e^j} \to 0$, where the spherical geometry appears as limiting case of the bispherical one, it consists of an idicative characteristic dimension of the bispherical shell for $j_1 \le j \le j_2$. In addition, given that $\frac{|\cos\vartheta|}{e^{j}} \le e^{-j}$ and that the term $\frac{\alpha}{\cosh j - \cos\vartheta}$ is expanded in terms of $\left(\frac{\cos\vartheta}{e^{j}}\right)^{n}$, $n = 0,1,2,\ldots$, the term e^{-j} is the characteristic quantity specifying the order of the desirable approximation.

In what follows we decided to keep the first two terms of the expansion and ignore terms proportional to $(e^{-j})^{n}$ with $n \ge 2$. We note that considering only the first term in the expansion of $\frac{\alpha}{\cosh j - \cos\vartheta}$ we find that

$$\hat{u}(\mathbf{r}) \in \begin{Bmatrix} J_{n+\frac{1}{2}}(k(2\alpha e^{-j})) \\ Y_{n+\frac{1}{2}}(k(2\alpha e^{-j})) \end{Bmatrix} \times P_n(\cos\vartheta) \tag{33}$$

in accordance with (20) for $m = 0$, where J_n, Y_n are the Bessel and Neumann functions, respectively.

For zeroth order approximation we have

$$\hat{G}_{n+1}^{(n),0} = G_{n-1}^{(n),0} = 0$$

and

$$\hat{G}_{n}^{(n),0} \in \begin{Bmatrix} J_{n+\frac{1}{2}}(k(2\alpha e^{-j})) \\ Y_{n+\frac{1}{2}}(k(2\alpha e^{-j})) \end{Bmatrix}. \tag{34}$$

Thus, in the first order approximation we have

$$
\begin{aligned}
&\hat{G}_n^{(n)}(j) = \hat{G}_n^{(n),0} + O(e^{-j}) \\
&\hat{G}_{n-1}^{(n)}(j) = O(e^{-j}) \\
&\hat{G}_{n+1}^{(n)}(j) = O(e^{-j}).
\end{aligned}
\tag{35}
$$

Omitting terms with order greater than one, and taking into account that the operator Z_n does not alter the order of the function on which acts we can transform the system of equations (30) - (32) as follows

$$
Z_n \hat{G}_n^{(n)} = 0 \tag{36}
$$

$$
Z_{n+1} \hat{G}_{n+1}^{(n)} = -\frac{16k^2\alpha^2}{e^{3j}} \frac{n+1}{2n+1} \hat{G}_n^{(n),0} \tag{37}
$$

$$
Z_{n-1} \hat{G}_{n-1}^{(n)} = -\frac{16k^2\alpha^2}{e^{3j}} \frac{n}{2n+1} \hat{G}_n^{(n),0}. \tag{38}
$$

Consequently,

$$
\hat{G}_n^{(n)}(j) = \hat{J}_n^l(j) = \left\{ \begin{matrix} J_{n+\frac{1}{2}}(k2\alpha e^{-j}), & l=1 \\ Y_{n+\frac{1}{2}}(k2\alpha e^{-j}), & l=2 \end{matrix} \right\} \tag{39}
$$

and

$$
\left.
\begin{aligned}
&\hat{G}_{n+1}^{(n)}(j) = c_1 \hat{J}_{n+1}^1(j) + c_2 \hat{J}_{n+1}^2(j) + \text{partial solution of (37)} \\
&\hat{G}_{n-1}^{(n)}(j) = c_3 \hat{J}_{n-1}^1(j) + c_4 \hat{J}_{n-1}^2(j) + \text{partial solution of (38)}
\end{aligned}
\right\} \tag{40}
$$

The terms $c_1 \hat{J}_{n+1}^1(j) + \hat{J}_{n+1}^2(j)$ and $c_3 \hat{J}_{n-1}^1(j) + c_4 \hat{J}_{n-1}^2(j)$ are solutions to the corresponding homogeneous equation to (37) (resp. (38)) and combined with $P_{n+1}(\cos\vartheta)$ (resp. $P_{n-1}(\cos\vartheta)$) give terms which can be incorporated to $\hat{u}^{(n+1)}$ solution (resp. $\hat{u}^{(n-1)}$) in $\hat{G}_{n+1}^{(n+1)}$ (resp. $\hat{G}_{n-1}^{(n-1)}$) term.

Thus, in order to find $\hat{G}^{(n)}_{n\pm1}(j)$ we have to determine a partial solution for every one of the equations (37) and (38) for the cases $l=1,2$ (actually we have to solve four equations).

We describe the solution procedure of eq. (37) with $l=1$. The other cases are treated similarly. More precicely we have the differential equation:

$$Z_{n+1}\hat{G}^{(n)}_{n+1,1} = -\frac{16k^2\alpha^2}{e^{3j}}\frac{n+1}{2n+1}\hat{J}^1_n(j). \tag{41}$$

Following Lagrange method we assume that

$$\hat{G}^{(n)}_{n+1}(j) = A(j)J_{n+\frac{3}{2}}(k2\alpha e^{-j}) + B(j)Y_{n+\frac{3}{2}}(k2\alpha e^{-j}) \tag{42}$$

where $A(j)$, $B(j)$ functions to be determined. This leads to the system

$$J_{n+\frac{3}{2}}(k2\alpha e^{-j})A'(j) + Y_{n+\frac{3}{2}}(k2\alpha e^{-1})B'(j) = 0$$

$$\frac{d}{dj}J_{n+\frac{3}{2}}(k2\alpha e^{-j})A'(j) + \frac{d}{dj}Y_{n+\frac{3}{2}}(k2\alpha e^{-j})B'(j) = -\frac{16k^2\alpha^2}{e^{3j}}\frac{n+1}{2n+1}J_{n+\frac{1}{2}}(k2\alpha e^{-j}).$$

By using that

$$\begin{vmatrix} J_{n+\frac{3}{2}}(k2\alpha e^{-j}) & Y_{n+\frac{3}{2}}(k2\alpha e^{-j}) \\ \frac{d}{dj}J_{n+\frac{3}{2}}(k2\alpha e^{-j}) & \frac{d}{dj}Y_{n+\frac{3}{2}}(k2\alpha e^{-j}) \end{vmatrix} = -\frac{2}{\pi}$$

the solution of the previous system results in the following expression for $\hat{G}^{(n)}_{n+1,1}(j)$,

$$\hat{G}^{(n)}_{n+1,1}(j) = -8\pi k^2\alpha^2 \frac{n+1}{2n+1}\left\{\begin{array}{l} J_{n+\frac{3}{2}}(k2\alpha e^{-j})\int^{j} J_{n+\frac{1}{2}}(k2\alpha e^{-j'})Y_{n+\frac{3}{2}}(k2\alpha e^{-j'})e^{-3j'}dj' \\ -Y_{n+\frac{3}{2}}(k2\alpha e^{-j})\int^{j} J_{n+\frac{1}{2}}(k2\alpha e^{-j'})J_{n+\frac{3}{2}}(k2\alpha e^{-j'})e^{-3j'}dj' \end{array}\right\}. \quad (43)$$

The integrals are handled by passing to spherical Bessel functions. Indeed, the second integral becomes

$$\int^{j} J_{n+\frac{1}{2}}(k2\alpha e^{-j'})J_{n+\frac{3}{2}}(k2\alpha e^{-j'})e^{-3j'}dj' =$$

$$\frac{2}{\pi}\int^{j} j_n(k2\alpha e^{-j'})j_{n+1}(k2\alpha e^{-j'})k2\alpha e^{-j'}e^{-3j'}dj' =$$

$$-\frac{1}{4\pi k^3\alpha^3}\int^{\omega} j_n(\omega')j_{n+1}(\omega')\omega'^3\,d\omega'$$

where $\omega = k2\alpha e^{-j}$.

On the other hand

$$I = \int^{\omega} j_n(\omega')j_{n+1}(\omega')\omega'^3\,d\omega'$$

$$= -\int^{\omega} \omega'^{2n+4}\left(\frac{1}{\omega'}\frac{d}{d\omega'}\right)^n\left(\frac{\sin\omega'}{\omega'}\right)\left(\frac{1}{\omega'}\frac{d}{d\omega'}\right)^{n+1}\left(\frac{\sin\omega'}{\omega'}\right)d\omega'$$

$$= -\omega^3 j_n^2(\omega) + \int^{\omega} \omega'^{2n+4}\left(\frac{1}{\omega'}\frac{d}{d\omega'}\right)^{n+1}\left(\frac{\sin\omega'}{\omega'}\right)\left(\frac{1}{\omega'}\frac{d}{d\omega'}\right)^n\left(\frac{\sin\omega'}{\omega'}\right)d\omega'$$

$$+\int^{\omega}(2n+3)\omega'^{2n+2}\left[\left(\frac{1}{\omega'}\frac{d}{d\omega'}\right)^n\left(\frac{\sin\omega'}{\omega'}\right)\right]^2 d\omega'$$

or

$$2I = -\omega^3 j_n^2(\omega) + (2n+3)\int^{\omega} \omega'^2 j_n^2(\omega')d\omega' = -\omega^3 j_n^2(\omega) + (2n+3)\frac{\pi}{2}\int^{\omega} \omega' J^2_{n+\frac{1}{2}}(\omega')d\omega'.$$

In addition we have [10]

$$\int^{\omega} \omega' J^2_{n+\frac{1}{2}}(\omega')d\omega' = \frac{\omega^2}{2}\left\{ J^2_{n+\frac{1}{2}}(\omega) - J_{n-\frac{1}{2}}(\omega) J_{n+\frac{3}{2}}(\omega) \right\}. \tag{44}$$

Consequently

$$I = -\frac{1}{2}\omega^3 j_n^2(\omega) + (2n+3)\frac{\omega^3}{4}\left\{ j_n^2(\omega) - j_{n-1}(\omega) j_{n+1}(\omega) \right\}$$
$$= \frac{1}{4}\omega^3\left[(2n+1) j_n^2(\omega) - (2n+3) j_{n+1}(\omega) j_{n-1}(\omega)\right].$$

We infer that

$$\int^{j} J_{n+\frac{1}{2}}(k2\alpha e^{-j'}) J_{n+\frac{3}{2}}(k2\alpha e^{-j'}) e^{-3j'} dj' =$$
$$-\frac{1}{4\pi k^3\alpha^3}\frac{1}{4}\omega^3\left[(2n+1) j_n^2(\omega) - (2n+3) j_{n+1}(\omega) j_{n-1}(\omega)\right].$$

Following the same arguments and proving similarly to (44) that

$$\int^{\omega} \omega' J_n(\omega') Y_n(\omega') d\omega' = \frac{\omega^2}{4}\left[2J_n(\omega)Y_n(\omega) - J_{n-1}(\omega)Y_{n+1}(\omega) - J_{n+1}(\omega)Y_{n-1}(\omega)\right]$$

we find that the first integral of (43) to be

$$\int^{j} J_{n+\frac{1}{2}}(k2\alpha e^{-j'})Y_{n+\frac{3}{2}}(k2\alpha e^{-j'})e^{-3j'}dj' =$$

$$\frac{1}{4\pi k^3\alpha^3}\left\{\frac{\omega^3}{6}+\frac{\omega^3}{2}j_n(\omega)y_n(\omega)-\frac{2n+3}{2}\int^{\omega}\omega'^2\, j_n(\omega')y_n(\omega')d\omega'\right\}.$$

The equation (43) takes the form

$$\hat{G}^{(n)}_{n+1,1} = -\sqrt{\frac{2}{\pi}}\omega^{\frac{1}{2}}\frac{n+1}{2n+1}y_{n+1}(\omega)e^{-j}\left[(2n+1)\omega^2 j_n^2(\omega)-(2n+3)\omega^2 j_{n+1}(\omega)j_{n-1}(\omega)\right]$$
$$-\sqrt{\frac{2}{\pi}}\omega^{\frac{1}{2}}4\frac{n+1}{2n+1}j_{n+1}(\omega)e^{-j}\left\{\begin{aligned}&\frac{\omega^2}{6}+\frac{\omega^2}{2}j_n(\omega)y_n(\omega)\\&-\frac{2n+3}{8}\left[2\omega^2 j_n(\omega)y_n(\omega)-\omega^2 j_{n-1}(\omega)y_{n+1}(\omega)-\omega^2 j_{n+1}(\omega)y_{n-1}(\omega)\right]\end{aligned}\right\}.$$

Similarly, we can define the functions $\hat{G}^{(n)}_{n+1,2}$, $\hat{G}^{(n)}_{n-1,1}$, $\hat{G}^{(n)}_{n-1,2}$.

Consequently, we lead to the solutions

$$u^{(n),l} = \sqrt{\cosh j-\cos\vartheta}\sum_{i=-1}^{1}\hat{G}^{(n)}_{n+i,l}(j)P_{n+i}(\cos\vartheta). \tag{45}$$

Omitting terms of order greater than one in the expansion of $\dfrac{\cos\vartheta}{\cosh j}$, we finally obtain

$$u^{(n),l} = \left[G^{(n)}_{n-1,l}-\frac{n}{2n+1}e^{-j}G^{(n)}_{n,l}\right]P_{n-1}+G^{(n)}_{n,l}P_n+\left[G^{(n)}_{n+1,l}-\frac{n+1}{2n+1}e^{-j}G^{(n)}_{n,l}\right]P_{n+1} \tag{46}$$

with

$$G^{(n)}_{n,l} = \begin{cases} j_n(\omega), & l=1\\ y_n(\omega), & l=2\end{cases} \tag{47}$$

and

$$G^{(n)}_{n+1,1}(\omega)=-y_{n+1}(\omega)\frac{n+1}{2n+1}e^{-j}\left[(2n+1)\omega^2 j_n^2(\omega)-(2n+3)\omega^2 j_{n+1}(\omega)j_{n-1}(\omega)\right]$$
$$-4j_{n+1}(\omega)\frac{n+1}{2n+1}e^{-j}\left\{\begin{array}{l}\frac{\omega^2}{6}+\frac{\omega^2}{2}j_n(\omega)y_n(\omega)-\\ \frac{2n+3}{8}\omega^2\left[2j_n(\omega)y_n(\omega)-j_{n-1}(\omega)y_{n+1}(\omega)-j_{n+1}(\omega)y_{n-1}(\omega)\right]\end{array}\right\} \quad (48)$$

$$G^{(n)}_{n+1,2}(\omega)=j_{n+1}(\omega)\frac{n+1}{2n+1}e^{-j}\left[(2n+1)\omega^2 y_n^2(\omega)-(2n+3)\omega^2 y_{n+1}(\omega)y_{n-1}(\omega)\right]$$
$$+4y_{n+1}(\omega)\frac{n+1}{2n+1}e^{-j}\left\{\begin{array}{l}\frac{\omega^2}{6}+\frac{\omega^2}{2}j_n(\omega)y_n(\omega)-\\ \frac{2n+3}{8}\omega^2\left[2j_n(\omega)y_n(\omega)-j_{n-1}(\omega)y_{n+1}(\omega)-j_{n+1}(\omega)y_{n-1}(\omega)\right]\end{array}\right\} \quad (49)$$

$$G^{(n)}_{n-1,1}(\omega)=-y_{n-1}(\omega)\frac{n}{2n+1}e^{-j}\left[(2n-1)\omega^2 j_{n-1}^2(\omega)-(2n+1)\omega^2 j_n(\omega)j_{n-2}(\omega)\right]$$
$$-4j_{n-1}(\omega)\frac{n}{2n+1}e^{-j}\left\{\begin{array}{l}\frac{\omega^2}{6}+\frac{\omega^2}{2}j_{n-1}(\omega)y_{n-1}(\omega)-\\ \frac{2n+1}{8}\omega^2\left[2j_{n-1}(\omega)y_{n-1}(\omega)-j_{n-2}(\omega)y_n(\omega)-j_n(\omega)y_{n-2}(\omega)\right]\end{array}\right\} \quad (50)$$

$$G^{(n)}_{n-1,2}(\omega)=j_{n-1}(\omega)\frac{n}{2n+1}e^{-j}\left[(2n-1)\omega^2 y_{n-1}^2(\omega)-(2n+1)\omega^2 y_n(\omega)y_{n-2}(\omega)\right]$$
$$+4y_{n-1}(\omega)\frac{n}{2n+1}e^{-j}\left\{\begin{array}{l}\frac{\omega^2}{6}+\frac{\omega^2}{2}j_{n-1}(\omega)y_{n-1}(\omega)-\\ \frac{2n+1}{8}\omega^2\left[2j_{n-1}(\omega)y_{n-1}(\omega)-j_{n-2}(\omega)y_n(\omega)-j_n(\omega)y_{n-2}(\omega)\right]\end{array}\right\}. \quad (51)$$

After the determination of the solutions $u^{(n),l}$ of the Helmholtz equation, we construct the Navier eigenfunctions

$$\begin{aligned}\mathbf{L}_n^l(\mathbf{r}') &= \nabla u^{(n),l}(\mathbf{r}';k_p)\\ \mathbf{M}_n^l(\mathbf{r}') &= \nabla u^{(n),l}(\mathbf{r}';k_s)\times(\mathbf{r}-\alpha\hat{\mathbf{z}})\\ \mathbf{N}_n^l(\mathbf{r}') &= \nabla\times\mathbf{M}_n^l(\mathbf{r}')\end{aligned} \quad (52)$$

The operator ∇ in bispherical coordinates has the form

$$\nabla = \frac{1}{\alpha}(\cosh j - \cos\vartheta)(\hat{j}\frac{\partial}{\partial j} + \hat{\vartheta}\frac{\partial}{\partial\vartheta}) \qquad (53)$$

and must be applied appropriately (as Eq. (52) defines) on the function $u^{(n),l}$ given by (46) to produce Navier eigenfunctions. The analytical procedure is very complex. However, the determination of the Navier eigenfunctions is an intermediate step in order to satisfy the boundary conditions.

Indeed, the completeness of Navier eigenfunctions allows for the following representation of the elastic field

$$\mathbf{u}(\mathbf{r}') = \sum_{n=0}^{+\infty}\sum_{l=1}^{2}\left[\alpha_n^l \mathbf{L}_n^l(\mathbf{r}') + \beta_n^l \mathbf{M}_n^l(\mathbf{r}') + \gamma_n^l \mathbf{N}_n^l(\mathbf{r}')\right]. \qquad (54)$$

The boundary conditions (11) impose that

$$\mathbf{Tu}(\mathbf{r}') = \sum_{n=0}^{+\infty}\sum_{l=1}^{2}\left[\alpha_n^l \mathbf{TL}_n^l(\mathbf{r}') + \beta_n^l \mathbf{TM}_n^l(\mathbf{r}') + \gamma_n^l \mathbf{TN}_n^l(\mathbf{r}')\right] = 0, \quad \mathbf{r}' \in S_1, S_2. \qquad (55)$$

Therefore, we have to determine the functions $\mathbf{TL}_n^l(\mathbf{r}')$, $\mathbf{TM}_n^l(\mathbf{r}')$ and $\mathbf{TN}_n^l(\mathbf{r}')$.

Taking the normal unit vector $\hat{\mathbf{n}}$ on the surfaces S_1, S_2 to be $-\hat{j}$ we find easily that

$$\mathbf{TL}_n^l(\mathbf{r}') = -\left[2\mu\hat{j}\cdot\nabla\mathbf{L}_n^l(\mathbf{r}') - \lambda k_p^2 \hat{j} u_n^{(n),l}(\mathbf{r}')\right] \qquad (56)$$

$$\mathbf{TM}_n^l(\mathbf{r}') = -\left[2\mu\hat{j}\cdot\nabla\mathbf{M}_n^l(\mathbf{r}') + \mu\hat{j}\times\mathbf{N}_n^l(\mathbf{r}')\right] \qquad (57)$$

$$\mathbf{TN}_n^l(\mathbf{r}') = -\left[2\mu\hat{j}\cdot\nabla\mathbf{N}_n^l(\mathbf{r}') + \mu k_s^2 \hat{j}\times\mathbf{M}_n^l(\mathbf{r}')\right]. \qquad (58)$$

We define the functions

$$
\begin{aligned}
A^{(n)}_{n-1,j}(j;k_i) &= G^{(n)}_{n-1,l}(j;k_i) - \frac{n}{2n+1} e^{-j} G^{(n)}_{n,l}(j;k_i) \\
A^{(n)}_{n,l}(j;k_i) &= G^{(n)}_{n,l}(j;k_i) \\
A^{(n)}_{n+1,l}(j;k_i) &= G^{(n)}_{n+1,l}(j;k_i) - \frac{n+1}{2n+1} e^{-j} G^{(n)}_{n,l}(j;k_i)
\end{aligned}
\tag{59}
$$

where $k_i,\ \ i=p,s$ indicate the longitudinal and transverse wave numbers, respectively.

For simplicity in what follows we denote $A^{(n),p}_{n+i,l} = A^{(n)}_{n+i,l}(j;k_p)$ and $A^{(n),s}_{n+i,l} = A^{(n)}_{n+i,l}(j;k_s)$ for $i=-1,0,1;\ \ l=1,2$.

Finally, we lead to the following expressions

$$
\mathbf{TL}^l_n = -2\mu\hat{j}\left\{
\begin{aligned}
&\left\{
\begin{aligned}
&\left[-k_p^2 \sum_{i=-1}^{1} A^{(n),p}_{n+i,l} P_{n+i}(\cos\vartheta) - \frac{4}{r^2}\cos\vartheta e^{-j}\left[\frac{\partial A^{(n),p}_{n,l}}{\partial j} + A^{(n),p}_{n,l} n(n+1)\right] P_n(\cos\vartheta)\right. \\
&\left. + \frac{1}{r^2}\sum_{i=-1}^{1}\left[2\frac{\partial}{\partial j}A^{(n),p}_{n+i,l} + (n+i)(n+i+1)A^{(n),p}_{n+i,l}\right] P_{n+i}(\cos\vartheta)\right]
\end{aligned}
\right\} \\
&+\hat{\vartheta}\left\{
\begin{aligned}
&\frac{2}{r^2}\sin\vartheta e^{-j}\frac{\partial}{\partial j}A^{(n),p}_{n,l}P_n(\cos\vartheta) + \frac{1}{r^2}\sum_{i=-1}^{1}\left[\frac{\partial}{\partial j}A^{(n),p}_{n+i,l} + A^{(n),p}_{n+i,l}\right]\frac{\partial}{\partial\vartheta}P_{n+i}(\cos\vartheta) \\
&-\frac{2}{r^2}\cos\vartheta e^{-j}\left[2\frac{\partial}{\partial j}A^{(n),p}_{n,l} + A^{(n),p}_{n,l}\right]\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta)
\end{aligned}
\right\}
\end{aligned}
\right\}
$$

$$
+ \lambda k_p^2 \hat{j} \sum_{i=-1}^{1} A^{(n),p}_{n+i,l} P_{n+i}(\cos\vartheta),
\tag{60}
$$

$$\mathbf{TM}_n^l = -2\mu\hat{\varphi}\left\{\begin{array}{l} -\frac{1}{r}\sum_{i=-1}^{1}\frac{\partial}{\partial j}A_{n+i,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_{n+i}(\cos\vartheta) \\ +\frac{3\cos\vartheta}{r}e^{-j}\frac{\partial}{\partial j}A_{n,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) - k_s^2 r\sin\vartheta e^{-j}A_{n,l}^{(n),s}P_n(\cos\vartheta) \\ +\frac{n(n+1)}{r}\sin\vartheta e^{-j}A_{n,l}^{(n),s}P_n(\cos\vartheta) - \frac{1}{r}\cos\vartheta e^{-j}A_{n,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) \end{array}\right\}$$

$$+2\mu\hat{\varphi}\frac{1}{r}\left\{\sum_{i=-1}^{1}A_{n+i,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_{n+i}(\cos\vartheta) - 2\cos\vartheta e^{-j}A_{n,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta)\right\}$$

$$+\mu k_s^2 A_{n,l}^{(n),s}P_n(\cos\vartheta)r\sin\vartheta e^{-j}\hat{\varphi}, \tag{61}$$

$$\mathbf{TN}_n^l = -2\mu\frac{1}{r^2}\hat{j}\left\{\begin{array}{l} -\sum_{i=-1}^{1}\frac{\partial}{\partial j}A_{n+i,l}^{(n),s}(n+i)(n+i+1)P_{n+i}(\cos\vartheta) \\ +\frac{\partial}{\partial j}\left(e^{-j}A_{n,l}^{(n),s}\right)\left[\sin\vartheta\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) + n(n+1)\cos\vartheta P_n(\cos\vartheta)\right] \\ -\frac{\partial}{\partial j}\left(e^{-j}\frac{\partial}{\partial j}A_{n,l}^{(n),s}\right)\left(\sin\vartheta\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) + 2\cos\vartheta P_n(\cos\vartheta)\right) \\ +4\cos\vartheta e^{-j}\frac{\partial}{\partial j}A_{n,l}^{(n),s}n(n+1)P_n(\cos\vartheta) \end{array}\right\}$$

$$-2\mu\frac{1}{r^2}\hat{\vartheta}\left\{\begin{array}{l} -\sum_{i=-1}^{1}\frac{\partial^2}{\partial j^2}A_{n+i,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_{n+i}(\cos\vartheta) + \frac{\partial^2}{\partial j^2}\left(e^{-j}A_{n,l}^{(n),s}\right)\cos\vartheta\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) \\ +\frac{\partial^2}{\partial j^2}\left(e^{-j}\frac{\partial}{\partial j}A_{n,l}^{(n),s}\right)\sin\vartheta P_n(\cos\vartheta) + 4\cos\vartheta e^{-j}\frac{\partial^2}{\partial j^2}A_{n,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) \end{array}\right\}$$

$$-\mu k_s^2\hat{\vartheta}\left\{\begin{array}{l} -\sum_{i=-1}^{1}A_{n+i,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_{n+i}(\cos\vartheta) + \cos\vartheta e^{-j}A_{n,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) \\ +\sin\vartheta e^{-j}\frac{\partial}{\partial j}A_{n,l}^{(n),s}P_n(\cos\vartheta) \end{array}\right\}$$

$$+2\mu\frac{1}{r^2}\hat{j}\left\{\begin{array}{l} \sum_{i=-1}^{1}A_{n+i,l}^{(n),s}(n+i)(n+i+1)P_{n+i}(\cos\vartheta) + \\ e^{-j}\left[-\sin\vartheta\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) - \cos\vartheta n(n+1)P_n(\cos\vartheta)\right]A_{n,l}^{(n),s} \\ +e^{-j}\frac{\partial}{\partial j}A_{n,l}^{(n),s}\left[2\cos\vartheta P_n(\cos\vartheta) + \sin\vartheta\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta)\right] \end{array}\right\}$$

$$+2\mu\frac{1}{r^2}\hat{\vartheta}\left\{\begin{array}{l} -\sum_{i=-1}^{1}A_{n+i,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_{n+i}(\cos\vartheta) + e^{-j}\cos\vartheta A_{n,l}^{(n),s}\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta) \\ +e^{-j}\sin\vartheta\frac{\partial}{\partial j}A_{n,l}^{(n),s}P_n(\cos\vartheta) \end{array}\right\}$$

$$+4\frac{\mu}{r^2}e^{-j}\hat{\vartheta}A_{n,l}^{(n),s}\left[\cos\vartheta\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta)+n(n+1)\sin\vartheta P_n(\cos\vartheta)\right]$$
$$-4\frac{\mu}{r^2}e^{-j}\hat{j}A_{n,l}^{(n),s}n(n+1)\cos\vartheta P_n(\cos\vartheta)+4\frac{\mu}{r^2}e^{-j}\hat{\vartheta}A_{n,l}^{(n),s}\cos\vartheta\frac{\partial}{\partial\vartheta}P_n(\cos\vartheta). \qquad (62)$$

Inserting (60), (61), (62) in (55) and exploiting the orthogonality of Legendre functions, we lead to the following linear algebraic system $((6\times(n'+1))\times(6\times(n'+1))$, where n' indicates the truncation level)

$$\begin{bmatrix} \mathbf{D}_0^{(0)} & \mathbf{D}_1^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{D}_0^{(1)} & \mathbf{D}_1^{(1)} & \mathbf{D}_2^{(1)} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{D}_1^{(2)} & \mathbf{D}_2^{(2)} & \mathbf{D}_3^{(2)} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_2^{(3)} & \mathbf{D}_3^{(3)} & \mathbf{D}_4^{(3)} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \dots \\ \dots \end{bmatrix} = \mathbf{0} \qquad (63)$$

where

$$\mathbf{x}_i = \left[\alpha_i^1,\ \alpha_i^2,\ \beta_i^1,\ \beta_i^2,\ \gamma_i^1,\ \gamma_i^2\right]^T$$

and the matrices $\mathbf{D}_n^{(n)}$, $\mathbf{D}_{n-1}^{(n)}$, $\mathbf{D}_{n+1}^{(n)}$ are 6 x 6 and their elements are given in Appendices A, B, C, respectively.

The system of equations (63) can be written in compact form as

$$\mathbf{Dx} = \mathbf{0}. \qquad (64)$$

In order the system (64) to have nontrivial solutions we must have

$$\det(\mathbf{D}) = 0. \qquad (65)$$

Condition (65) provides the characteristic (frequency) equation, the roots of which are the eigenfrequencies ω_k of the system under discussion.

4. Numerical Results and Discussion

The frequency equation (65) has been solved numerically by using a matrix determinant computation routine along with a bisection method. The dimension of the matrix $\boldsymbol{D}$ depends on the value of n', which ensures convergence of ω_k, $k=1,2,\ldots,20$. The computation of $\omega_k=\omega_k(\varepsilon,n')$ has been done by using an iterative procedure and the results obtained are cited in Table 1. We observe that the value of n' for convergence is strongly dependent on the values of ε, $n'_{conv.}=n'(\varepsilon)$. In the case under discussion, for fixed values of ε, the computational procedure was repeated until $\left\|\omega_k^{(n')}-\omega_k^{(n'+1)}\right\|\approx O(10^{-4})$, $k=1,2,\ldots,20$. It is noted that the accuracy of the bisection method used is of the order of $O(10^{-8})$ and the computing time needed is an increasing function of n'.

The elements of $\boldsymbol{D}$ are functions of ω (see Appendices A, B and C) and therefore they have to be computed for every different value of it. This fact requires the computation of Spherical Bessel functions of the first and second kind as well as their derivatives. In our computations we have used Seed's method [11] for the computation of Bessel functions of fractional order and their derivatives by using the definitions of spherical Bessel functions for integer n'. We note that recursive relations, although they offer flexibility and fast computations, are not helpful for high values of n' and values of the argument close to zero.

The numerical computations for the system under discussion have been performed by using the following properties for the dry skull:

$E=6.5x10^9\,N/m^2$, $\nu=0.25$, $\rho=2.1326x10^3\,Kg/m^3$, where E and ν denote the Young's modulus and Poisson's ratio, respectively.

The geometry of the base case (two concentric spheres, Ref. 6) is defined by:

$r_1 = 0.082\ m,\ r_2 = 0.076\ m.$

Taking into account the morphology of the human skull we can consider eccentricies ε varying as

$\varepsilon \in [0.0,\ 0.004\ m].$

In our computations the determination of the eccentricity is given by a new parameter δ which gives the eccentricity as a percentage of the inner sphere radius, i.e.

$\varepsilon = \delta r_2.$

The effect of the eccentricity on the frequency spectrum of the human dry skull is given in Table 2. We observe that the deviation from the base case introduces a new pattern of eigenfrequencies and shift of the existing egenfrequencies. The new eigenfrequencies are close to zero but for larger values of δ those eigenfrequencies enter the frequency spectrum (see Fig 2a and Fig2b).

The variation of ω_k with inner sphere radius r_2 is shown in Table 3. We observe that the inner radius (or the thickness of the skull) is a vital parameter in the determination of the eigenfrequency spectrum and its deviation from the base case. Table 5 gives the frequency spectrum of the system for Poisson's ratio $\nu = 0.2$, $\nu = 0.25$ and $\nu = 0.3$. As the Poisson's ratio becomes smaller the effect of the eccentricity on the shifting of the eigenfrequency spectrum becomes more significant and the new eigenfrequencies entering the spectrum are larger.

From the results obtained we lead to the following:

- the eccentricity of the human skull plays an important role on the dynamic characteristics of the human head
- facial bones and thickness nonuniformities should be considered in detailed models of the system

Such a detailed morphology model for the human head is under preparation and it will appear in a future communication.

Table 1: Convergence of ω_k, $k = 1,2,...,22$.
$(E = 6.5x10^9 N/m^2,\ \nu = 0.25,\ \rho = 2.1326x10^3 Kg/m^3,\ r_1 = 0.082m,\ r_2 = 0.076m,\ \varepsilon = 0.01r_2)$

No.	$n'=0$	$n'=1$	$n'=2$	$n'=3$	$n'=4$	$n'=5$	$n'=6$	$n'=7$	$n'=8$	$n'=9$	$n'=10$	$n'=11$	$n'=12$	$n'=13$	$n'=14$
1		0.0603	0.0830	0.1994	0.2922	0.3519	0.3519	0.3519	0.3519	0.3519	0.3519	0.3519	0.3519	0.3519	0.3519
2		0.1396	0.1422	0.4341	0.3250	0.3881	0.4824								
3		0.4407	0.3435			0.4928	0.4938	0.4938	0.4938	0.4938	0.4938	0.4938	0.4938	0.4938	0.4938
4					0.5459	0.5460	0.5460	0.5460	0.5460	0.5460	0.5460	0.5460	0.5460	0.5460	0.5460
5					0.5810			0.5761	0.6633						0.7092
6				0.7478	0.7417	0.7492	0.7492	0.7492	0.7492		0.7492	0.7492	0.7492	0.7492	0.7492
7		0.8352									0.8345	0.9170	0.9979		
8			1.1296			1.0187	1.0244	1.0125	1.0126	1.0126	1.0126	1.0126	1.0126	1.0126	1.0126
9						1.1657								1.0772	1.1551
10			1.3076	1.2072		1.2481	1.2183	1.2232	1.2231	1.2232	1.2232	1.2232	1.2232	1.2232	1.2232
11	1.5479	1.6370		1.6504		1.5714	1.6262	1.5926	1.5899	1.5901	1.5901	1.5901	1.5901	1.5901	1.5901
12			1.7586		1.7571	1.7057	1.7344	1.7311	1.7382	1.7380	1.7380	1.7380	1.7380	1.7380	1.7380
13				1.8146	1.8359	1.7990	1.7757	1.8274	1.8144	1.8158	1.8156	1.8156	1.8156	1.8156	1.8156
14		1.9593			1.8525		1.8390	1.9228	1.8549	1.8557	1.8561	1.8561	1.8561	1.8561	1.8561
15				2.0802				1.9771	2.0406	2.2095	2.1801	2.1810	2.1811	2.1811	2.1811
16							2.3428	2.1797	2.1356	2.4627	2.4472	2.4060	2.4067	2.4067	2.4067
17				2.5282	2.3400	2.4428		2.2288	2.3436	2.5636	2.5630	2.5469	2.5397	2.5398	2.5398
18			2.6488		2.6511	2.5636	2.6043	2.5504	2.5536			2.6462	2.5925	2.5929	2.5929
19											2.7673	2.7162	2.7167	2.7161	2.7161
20									2.7857					2.7554	2.7560
21			2.9101	2.8927	2.8434	2.8825	2.8788	2.8678	2.8891	2.8830	2.8853	2.8811	2.8812	2.8813	2.8813
22					2.9252			2.9305							2.9398

Table 2: Variation of ω_k, $k = 1,2,...,18$ with $\varepsilon = \delta r_2$, $\delta \in [0, 0.01]$.
$(E = 6.5x10^9 N/m^2,\ \nu = 0.25,\ \rho = 2.1326x10^3 Kg/m^3,\ r_1 = 0.082m,\ r_2 = 0.076m.)$

No	$\delta = 0$	$\delta \to 0$	$\delta = 5x10^{-6}$	$\delta = 1x10^{-5}$	$\delta = 1x10^{-4}$	$\delta = 1x10^{-3}$	$\delta = 0.01$
1		0.0101	0.0165	0.0474	0.0870	0.2099	0.3479
2		0.0147	0.0332	0.0644	0.1074	0.2240	0.3519
3		0.0273	0.0447	0.0746	0.1549	0.3360	0.4938
4		0.0333	0.0527	0.1108	0.2150	0.3478	0.5460
5		0.0495	0.0782	0.1522	0.2398	0.3532	
6		0.0681	0.1076	0.1904	0.3540	0.3809	
7		0.0864	0.1350		0.4835	0.4335	
8					0.5880	0.6385	
9	0.7083	0.7083	0.7083	0.7084	0.7150	0.7070	0.7083
10	0.8522	0.8522	0.8524	0.8528	0.8845	0.8693	0.7491
11	0.9486	0.9486	0.9499	0.9498	1.0152	1.0566	1.0126
12	1.0632	1.0632	1.0634	1.0644	1.0504	1.1087	1.0829
13	1.1971 1.2184	1.1971 1.2184	1.1971 1.2187	1.1971 1.2194	1.1971	1.1972	1.2232 1.2318
14	1.4211	1.4211	1.4213	1.4220	1.3074 1.4976	1.3595 1.4374	
15	1.5479	1.5479	1.5479	1.5479	1.5479	1.5439	1.5901
16	1.6702	1.6702	1.6703	1.6708	1.7289	1.6355 1.6994	1.7380
17	1.8924	1.8924	1.8924	1.8927	1.9001	1.8840 1.8910	1.8156 1.8561
18	1.9614	1.9614	1.9615	1.9618	2.0031	2.0753	2.1811

Figure 2a: Variation of ω_k, $k = 9,\dots,18$ with $\varepsilon = \delta r_2$, $\delta \in [0, 0.01]$.
$E = 6.5x10^9 N/m^2$, $\nu = 0.25$, $\rho = 2.1326x10^3 Kg/m^3$, $r_1 = 0.082m$, $r_2 = 0.076m$..

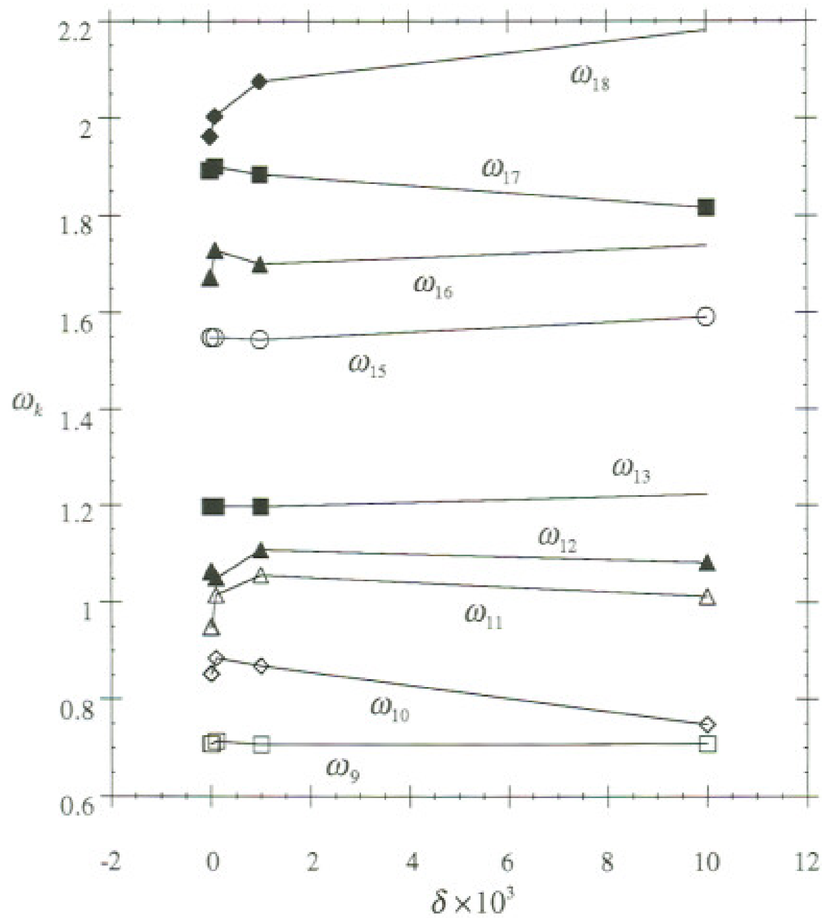

Figure 2b: Variation of $\omega_k, k = 1,\dots,4$ with $\varepsilon = \delta r_2$, $\delta \in [0,0.01]$.
$E = 6.5x10^9 N/m^2$, $\nu = 0.25$, $\rho = 2.1326x10^3 Kg/m^3$, $r_1 = 0.082m$, $r_2 = 0.076m$..

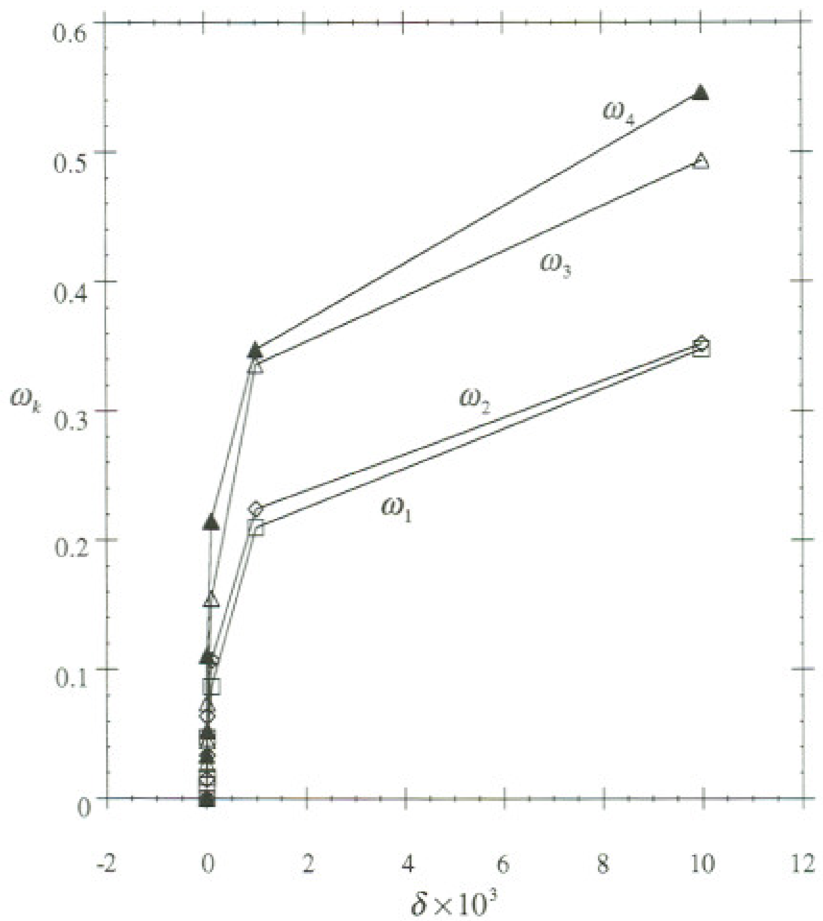

Table 3: Variation of ω_k, $k = 1,2,\ldots,16$ with $r_2 \in [0.041m,\ 0.076m]$ for $\varepsilon = 0$ and $\varepsilon = 7.6\times10^{-4}$.
$(E = 6.5x10^9 N/m^2,\ \nu = 0.25,\ \rho = 2.1326x10^3 Kg/m^3,\ r_1 = 0.082m)$

	$r_2 = 0.076m$		$r_2 = 0.070m$		$r_2 = 0.060m$		$r_2 = 0.050m$		$r_2 = 0.041m$	
No.	$\delta = 0$	$\delta = 0.01$	$\delta = 0$	$\delta = 0.01086$	$\delta = 0$	$\delta = 0.01267$	$\delta = 0$	$\delta = 0.0152$	$\delta = 0$	$\delta = 0.0184$
1		0.3479		0.3461		0.3332		0.3245		0.3251
2		0.3519		0.4297		0.3922		0.3689		0.3818
3		0.4938		0.6136		0.6220		0.3830		0.4974
4		0.5460	0.7482	0.8186		0.8194		0.5441		0.7738
5	0.7083	0.7083	0.9532	1.0894	0.8417	0.8486		0.7717		0.8366
6	0.8522	0.7491	1.1725	1.1431		0.8722	0.9688	1.0480	1.1066	1.0872
7	0.9486	1.0126	1.2394	1.4612	1.2061	1.1386		1.0739		1.3745
8	1.0632	1.0829	1.4668	1.5429	1.3071	1.2668		1.2741	1.4061	1.3847
9	1.1971	1.2232	1.6112	1.6648		1.3950	1.3661	1.3369		1.4698
10	1.2184	1.2318		1.6771	1.6557	1.4964	1.5113	1.4481	1.7928	1.8074
11	1.4211	1.5901		1.7579	1.7327	1.8997		1.6867	2.0365	1.9887
12	1.5479	1.7380	1.8365	1.8236	2.0557	2.0308	1.8793	1.8481		1.9944
13	1.6702	1.8156		1.9607	2.0650	2.1826	2.1211	2.1001	2.1300	2.1406
14	1.8924	1.8561	2.2673	2.0948	2.1922	2.1930	2.1364	2.1142	2.2016	2.1898
15	1.9614	2.1811	2.6288	2.3249	2.7676	2.5285	2.1516	2.1626	2.5152	2.8664
16	2.2896	2.4067	2.7067	2.3969	2.7844	2.7018	2.8102	2.2382	2.7827	2.8897

Table 4: Variation of ω_k, $k=1,2,...,15$ for different Poisson's ratios ($\delta=0.00$, $\delta=0.01$).
($E=6.5x10^9 N/m^2$, $\rho=2.1326x10^3 Kg/m^3$, $r_1=0.082m$, $r_2=0.076m$.)

	$\nu=0.20$		$\nu=0.25$		$\nu=0.30$	
No.	$\delta=0.00$	$\delta=0.01$	$\delta=0.00$	$\delta=0.01$	$\delta=0.00$	$\delta=0.01$
1		0.3286		0.3479		0.3258
2		0.3733		0.3519		0.4572
3		0.5238		0.4938		0.5452
4		0.6982		0.5460		
5	0.7452	0.8287	0.7083	0.7083	0.6609	0.6625
6	0.8914	1.0740	0.8522	0.7491	0.8000	1.1547
7	0.9888	1.2157	0.9486	1.0126	0.8942	1.5092
8	1.1052	1.2513	1.0632	1.0829	1.0052	1.6546
9	1.4714	1.6612	1.1971	1.2232	1.1083	1.6809
10	1.5573	1.7963	1.2184	1.2318	1.1549	1.7344
11	1.7273	1.9257	1.4211	1.5901	1.3495	2.0333
12	1.9041	1.9537	1.5479	1.7380	1.5131	2.2860
13	2.0076	2.2844	1.6702	1.8156	1.5880	2.4045
14	2.0271	2.5117	1.8924	1.8561	1.7524	2.5152
15	2.3656	2.5536	1.9614	2.1811	1.8495	2.5223

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References

[1] Charalambopoulos A., Dassios G., Fotiadis D.I., and Massalas C.V., On the Dynamic Characteristics of the Human Skull - Brain System, Mathematical and Computer Modelling, accepted (1996).

[2] Charalambopoulos A., Fotiadis D.I., and Massalas C.V., Frequency Spectrum of the Human Head - Neck System, Int. J. Engng. Sci., accepted (1996).

[3] Misra J.C., Axisymmetric Vibration of Human Skull - Brain System, Ingenieur - Archiv 47, 11-19 (1978).

[4] Misra J.C. and Chakravarty S., Stability of Fluid - Filled Spheroidal and Spherical Shells - Mathematical Models for the Human Head, Acta Mechanica 44, 159-168 (1982).

[5] Misra J.C. and Chakravarty S., A Free - Vibration Analysis for the Human Cranial System, J. Biomechanics 15, 635-645 (1982).

[6] Charalambopoulos A., Dassios G., Fotiadis D.I., Kostopoulos V., and Massalas C.V., On the Dynamic Characteristics of the Human Skull, Int. J. Engng. Sci. 34(12), 1339-1348 (1996).

[7] Khalil T.B., and Viano D.C., Experimental Analysis of the Vibrational Characteristics of the Human Skull, J. of Sound and Vibration 63(3), 351 - 376 (1979).

[8] Charalambopoulos A., Fotiadis D.I., and Massalas C.V., The Effect of Geometry on the Dynamic Characteristics of the Human Skull, Int. J. Engng. Sci., accepted (1996).

[9] Moon P. and Spencer D.G., Field Theory for Engineers, D. Van Nostrand Company, (1961).

[10] Bell W.W. , Special Functions for Scientists and Engineers, D. Van Nostrand Company, (1968).

[11] Press W.H., Teukolsky S.A., Vettering W.T. and Flannery B. P., Numerical Recipes: The Art of Scientific Computation, 2nd Ed., Cambridge University Press, Cambridge (1992).

APPENDIX A:

Elements of the diagonal block matrices $\mathbf{D}_n^{(n)}$:

$$d(1,1) = -k_p^2 A_{n,1}^{(n),p}(j_2) + \frac{2}{r_2^2}\frac{\partial}{\partial j} A_{n,1}^{(n),p}(j_2) + \frac{n(n+1)}{r_2^2} A_{n,1}^{(n),p}(j_2) - \frac{\lambda k_p^2}{2} A_{n,1}^{(n),p}(j_2),$$

$$d(1,2) = -k_p^2 A_{n,2}^{(n),p}(j_2) + \frac{2}{r_2^2}\frac{\partial}{\partial j} A_{n,2}^{(n),p}(j_2) + \frac{n(n+1)}{r_2^2} A_{n,2}^{(n),p}(j_2) - \frac{\lambda k_p^2}{2} A_{n,1}^{(n),p}(j_2),$$

$$d(1,3) = 0,\ \ d(1,4) = 0,$$

$$d(1,5) = \left(-\frac{1}{r_2^2}\frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_2) - \frac{1}{r_2^2} A_{n,1}^{(n),s}(j_2) \right) n(n+1),$$

$$d(1,6) = \left(-\frac{1}{r_2^2}\frac{\partial}{\partial j} A_{n,2}^{(n),s}(j_2) - \frac{1}{r_2^2} A_{n,2}^{(n),s}(j_2) \right) n(n+1),$$

$$d(2,1) = -k_p^2 A_{n,1}^{(n),p}(j_1) + \frac{2}{r_1^2}\frac{\partial}{\partial j} A_{n,1}^{(n),p}(j_1) + \frac{n(n+1)}{r_1^2} A_{n,1}^{(n),p}(j_1) - \frac{\lambda k_p^2}{2} A_{n,1}^{(n),p}(j_1),$$

$$d(2,2) = -k_p^2 A_{n,2}^{(n),p}(j_1) + \frac{2}{r_1^2}\frac{\partial}{\partial j} A_{n,2}^{(n),p}(j_1) + \frac{n(n+1)}{r_1^2} A_{n,2}^{(n),p}(j_1) - \frac{\lambda k_p^2}{2} A_{n,1}^{(n),p}(j_1),$$

$$d(2,3) = 0,\ \ d(2,4) = 0,$$

$$d(2,5) = \left(-\frac{1}{r_1^2}\frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_1) - \frac{1}{r_1^2} A_{n,1}^{(n),s}(j_1) \right) n(n+1),$$

$$d(2,6) = \left(-\frac{1}{r_1^2}\frac{\partial}{\partial j} A_{n,2}^{(n),s}(j_1) - \frac{1}{r_1^2} A_{n,2}^{(n),s}(j_1) \right) n(n+1),$$

$$d(3,1) = 0,\ \ d(3,2) = 0,$$

$$d(3,3) = \frac{1}{r_2}\frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_2) + \frac{1}{r_2} A_{n,1}^{(n),s}(j_2),\ \ d(3,4) = \frac{1}{r_2}\frac{\partial}{\partial j} A_{n,2}^{(n),s}(j_2) + \frac{1}{r_2} A_{n,2}^{(n),s}(j_2),$$

$$d(3,5) = 0,\ \ d(3,6) = 0,$$

$$d(4,1) = 0,\ \ d(4,2) = 0,$$

$$d(4,3) = \frac{1}{r_1}\frac{\partial}{\partial j} A_{n,1}^{(n),s}(j_1) + \frac{1}{r_1} A_{n,1}^{(n),s}(j_1),\ \ d(4,4) = \frac{1}{r_1}\frac{\partial}{\partial j} A_{n,2}^{(n),s}(j_1) + \frac{1}{r_1} A_{n,2}^{(n),s}(j_1),$$

$$d(4,5) = 0,\ d(4,6) = 0,$$

$$d(5,1) = -\frac{1}{r_2^2}\left(\frac{\partial}{\partial j}A_{n,1}^{(n),p}(j_2) + A_{n,1}^{(n),p}(j_2)\right),\ d(5,2) = -\frac{1}{r_2^2}\left(\frac{\partial}{\partial j}A_{n,2}^{(n),p}(j_2) + A_{n,2}^{(n),p}(j_2)\right),$$

$$d(5,3) = 0,\ d(5,4) = 0,$$

$$d(5,5) = \frac{1}{r_2^2}\left(\frac{\partial^2}{\partial j^2}A_{n,1}^{(n),s}(j_2)\right) + \frac{k_s^2}{2}A_{n,1}^{(n),s}(j_2),\ d(5,6) = \frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}A_{n,2}^{(n),s}(j_2) + \frac{k_s^2}{2}A_{n,2}^{(n),s}(j_2),$$

$$d(6,1) = -\frac{1}{r_1^2}\left(\frac{\partial}{\partial j}A_{n,1}^{(n),p}(j_1) + A_{n,1}^{(n),p}(j_1)\right),\ d(6,2) = -\frac{1}{r_1^2}\left(\frac{\partial}{\partial j}A_{n,2}^{(n),p}(j_1) + A_{n,2}^{(n),p}(j_1)\right),$$

$$d(6,3) = 0,\ d(6,4) = 0,$$

$$d(6,5) = \frac{1}{r_1^2}\left(\frac{\partial^2}{\partial j^2}A_{n,1}^{(n),s}(j_1)\right) + \frac{k_s^2}{2}A_{n,1}^{(n),s}(j_1),\ d(6,6) = \frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}A_{n,2}^{(n),s}(j_1) + \frac{k_s^2}{2}A_{n,2}^{(n),s}(j_1)$$

APPENDIX B:

Elements of the lower diagonal block matrices $\mathbf{D}_{n-1}^{(n)}$:

$$d(1,1) = -k_p^2 A_{n,1}^{(n-1),p}(j_2) - \frac{4}{r_2^2}e^{-j_2}\left(\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),p}(j_2) + n(n-1)A_{n-1,1}^{(n-1),p}(j_2)\right)\frac{n}{2n-1}$$
$$+\frac{1}{r_2^2}\left[2\frac{\partial}{\partial j}A_{n,1}^{(n-1),p}(j_2) + n(n+1)A_{n,1}^{(n-1),p}(j_2)\right] - \frac{\lambda k_p^2}{2}A_{n,1}^{(n-1),p}(j_2),$$

$$d(1,2) = -k_p^2 A_{n,2}^{(n-1),p}(j_2) - \frac{4}{r_2^2}e^{-j_2}\left(\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),p}(j_2) + n(n-1)A_{n-1,2}^{(n-1),p}(j_2)\right)\frac{n}{2n-1}$$
$$+\frac{1}{r_2^2}\left[2\frac{\partial}{\partial j}A_{n,2}^{(n-1),p}(j_2) + n(n+1)A_{n,2}^{(n-1),p}(j_2)\right] - \frac{\lambda k_p^2}{2}A_{n,2}^{(n-1),p}(j_2),$$

$$d(1,3) = 0,\ \ d(1,4) = 0,$$

$$d(1,5) = -\frac{1}{r_2^2}\frac{\partial}{\partial j}A_{n,1}^{(n-1),s}(j_2)n(n+1) + \frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\right)\frac{n(n-1)}{2n-1}$$
$$\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\right)(n-1)\frac{n^2}{2n-1} - \frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_2)\right)\frac{n(n-1)}{2n-1}$$
$$-\frac{2}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_2)\right)\frac{n}{2n-1} - \frac{1}{r_2^2}A_{n,1}^{(n-1),s}(j_2)n(n+1) + \frac{1}{r_2^2}e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\frac{n(n-1)}{2n-1}$$
$$+\frac{1}{r_2^2}e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\frac{n^2(n-1)}{2n-1} - \frac{2}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_2)\frac{n}{2n-1} - \frac{1}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_2)\frac{n(n-1)}{2n-1}$$
$$+\frac{2}{r_2^2}e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\frac{n^2(n-1)}{2n-1} + \frac{4}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_2)\frac{n^2(n-1)}{2n-1},$$

$$d(1,6)=-\frac{1}{r_2^2}\frac{\partial}{\partial j}A_{n,2}^{(n-1),s}(j_2)n(n+1)+\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\right)\frac{n(n-1)}{2n-1}$$

$$\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\right)(n-1)\frac{n^2}{2n-1}-\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\right)\frac{n(n-1)}{2n-1}$$

$$-\frac{2}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\right)\frac{n}{2n-1}-\frac{1}{r_2^2}A_{n,2}^{(n-1),s}(j_2)n(n+1)+\frac{1}{r_2^2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n(n-1)}{2n-1}$$

$$+\frac{1}{r_2^2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n^2(n-1)}{2n-1}-\frac{2}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\frac{n}{2n-1}-\frac{1}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\frac{n(n-1)}{2n-1}$$

$$+\frac{2}{r_2^2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n^2(n-1)}{2n-1}+\frac{4}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\frac{n^2(n-1)}{2n-1},$$

$$d(2,1)=-k_p^2A_{n,1}^{(n-1),p}(j_1)-\frac{4}{r_1^2}e^{-j_1}\left(\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),p}(j_1)+n(n-1)A_{n-1,1}^{(n-1),p}(j_1)\right)\frac{n}{2n-1}$$

$$+\frac{1}{r_1^2}\left[2\frac{\partial}{\partial j}A_{n,1}^{(n-1),p}(j_1)+n(n+1)A_{n,1}^{(n-1),p}(j_1)\right]-\frac{\lambda k_p^2}{2}A_{n,1}^{(n-1),p}(j_1),$$

$$d(2,2)=-k_p^2A_{n,2}^{(n-1),p}(j_1)-\frac{4}{r_1^2}e^{-j_1}\left(\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),p}(j_1)+n(n-1)A_{n-1,2}^{(n-1),p}(j_1)\right)\frac{n}{2n-1}$$

$$+\frac{1}{r_1^2}\left[2\frac{\partial}{\partial j}A_{n,2}^{(n-1),p}(j_1)+n(n+1)A_{n,2}^{(n-1),p}(j_1)\right]-\frac{\lambda k_p^2}{2}A_{n,2}^{(n-1),p}(j_1),$$

$$d(2,3)=0,\ \ d(2,4)=0,$$

$$d(2,5)=-\frac{1}{r_1^2}\frac{\partial}{\partial j}A_{n,1}^{(n-1),s}(j_1)n(n+1)+\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\right)\frac{n(n-1)}{2n-1}$$

$$\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\right)(n-1)\frac{n^2}{2n-1}-\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\right)\frac{n(n-1)}{2n-1}$$

$$-\frac{2}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\right)\frac{n}{2n-1}-\frac{1}{r_1^2}A_{n,1}^{(n-1),s}(j_1)n(n+1)+\frac{1}{r_1^2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n(n-1)}{2n-1}$$

$$+\frac{1}{r_1^2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n^2(n-1)}{2n-1}-\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\frac{n}{2n-1}-\frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\frac{n(n-1)}{2n-1}$$

$$+\frac{2}{r_1^2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n^2(n-1)}{2n-1}+\frac{4}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\frac{n^2(n-1)}{2n-1},$$

$$d(2,6)=-\frac{1}{r_1^2}\frac{\partial}{\partial j}A_{n,2}^{(n-1),s}(j_1)n(n+1)+\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\right)\frac{n(n-1)}{2n-1}$$

$$\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\right)(n-1)\frac{n^2}{2n-1}-\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\right)\frac{n(n-1)}{2n-1}$$

$$-\frac{2}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\right)\frac{n}{2n-1}-\frac{1}{r_1^2}A_{n,2}^{(n-1),s}(j_1)n(n+1)+\frac{1}{r_1^2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{n(n-1)}{2n-1}$$

$$+\frac{1}{r_1^2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{n^2(n-1)}{2n-1}-\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\frac{n}{2n-1}-\frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\frac{n(n-1)}{2n-1}$$

$$+\frac{2}{r_1^2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{n^2(n-1)}{2n-1}+\frac{4}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\frac{n^2(n-1)}{2n-1},$$

$d(3,1)=0,\ d(3,2)=0,$

$$d(3,3)=-\frac{3}{r_2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_2)\frac{n-1}{2n-1}-\frac{3}{2}k_s^2r_2e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\frac{1}{2n-1}+$$

$$\frac{1}{r_2}e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\frac{n(n-1)}{2n-1}-\frac{1}{r_2}\frac{\partial}{\partial j}A_{n,1}^{(n-1),s}(j_2)-\frac{1}{r_2}A_{n,1}^{(n-1),s}(j_2)$$

$$-\frac{2}{r_2}e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\frac{1}{2n-1}+\frac{1}{r_2}e^{-j_2}A_{n-1,1}^{(n-1),s}(j_2)\frac{n-1}{2n-1},$$

$$d(3,4)=-\frac{3}{r_2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\frac{n-1}{2n-1}-\frac{3}{2}k_s^2r_2e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{1}{2n-1}+$$

$$\frac{1}{r_2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n(n-1)}{2n-1}-\frac{1}{r_2}\frac{\partial}{\partial j}A_{n,2}^{(n-1),s}(j_2)-\frac{1}{r_2}A_{n,2}^{(n-1),s}(j_2)$$

$$-\frac{2}{r_2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{1}{2n-1}+\frac{1}{r_2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n-1}{2n-1},$$

$d(3,5)=0,\ d(3,6)=0,$

$d(4,1)=0,\ d(4,2)=0,$

$$d(4,3)=-\frac{3}{r_1}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\frac{n-1}{2n-1}-\frac{3}{2}k_s^2r_1e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{1}{2n-1}+$$

$$\frac{1}{r_1}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n(n-1)}{2n-1}-\frac{1}{r_1}\frac{\partial}{\partial j}A_{n,1}^{(n-1),s}(j_1)-\frac{1}{r_1}A_{n,1}^{(n-1),s}(j_1)$$

$$-\frac{2}{r_1}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{1}{2n-1}+\frac{1}{r_1}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n-1}{2n-1},$$

$$d(4,4) = -\frac{3}{r_1} e^{-j_1} \frac{\partial}{\partial j} A_{n-1,2}^{(n-1),s}(j_1) \frac{n-1}{2n-1} - \frac{3}{2} k_s^2 r_1 e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{1}{2n-1} +$$

$$\frac{1}{r_1} e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{n(n-1)}{2n-1} - \frac{1}{r_1} \frac{\partial}{\partial j} A_{n,2}^{(n-1),s}(j_1) - \frac{1}{r_1} A_{n,2}^{(n-1),s}(j_1)$$

$$-\frac{2}{r_1} e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{1}{2n-1} + \frac{1}{r_1} e^{-j_1} A_{n-1,2}^{(n-1),s}(j_1) \frac{n-1}{2n-1},$$

$d(4,5) = 0,\ d(4,6) = 0,$

$$d(5,1) = \frac{2}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n-1,1}^{(n-1),p}(j_2) \frac{1}{2n-1} - \frac{1}{r_2^2} \left(\frac{\partial}{\partial j} A_{n,1}^{(n-1),p}(j_2) + A_{n,1}^{(n-1),p}(j_2) \right)$$

$$+\frac{2}{r_2^2} e^{-j_2} \left(2 \frac{\partial}{\partial j} A_{n-1,1}^{(n-1),p}(j_2) + A_{n-1,1}^{(n-1),p}(j_2) \right) \frac{n-1}{2n-1},$$

$$d(5,2) = \frac{2}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n-1,2}^{(n-1),p}(j_2) \frac{1}{2n-1} - \frac{1}{r_2^2} \left(\frac{\partial}{\partial j} A_{n,2}^{(n-1),p}(j_2) + A_{n,2}^{(n-1),p}(j_2) \right)$$

$$+\frac{2}{r_2^2} e^{-j_2} \left(2 \frac{\partial}{\partial j} A_{n-1,2}^{(n-1),p}(j_2) + A_{n-1,2}^{(n-1),p}(j_2) \right) \frac{n-1}{2n-1},$$

$d(5,3) = 0,\ d(5,4) = 0,$

$$d(5,5) = \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} A_{n,1}^{(n-1),s}(j_2) - \frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} \left(e^{-j_2} A_{n-1,1}^{(n-1),s}(j_2) \right) \frac{n-1}{2n-1}$$

$$+\frac{1}{r_2^2} \frac{\partial^2}{\partial j^2} \left(e^{-j_2} \frac{\partial}{\partial j} A_{n-1,1}^{(n-1),s}(j_2) \right) \frac{1}{2n-1} - \frac{4}{r_2^2} e^{-j_2} \frac{\partial^2}{\partial j^2} A_{n-1,1}^{(n-1),s}(j_2) \frac{n-1}{2n-1}$$

$$+\frac{k_s^2}{2} A_{n,1}^{(n-1),s}(j_2) + \frac{k_s^2}{2} e^{-j_2} A_{n-1,1}^{(n-1),s}(j_2) \frac{n-1}{2n-1} - \frac{k_s^2}{2} e^{-j_2} \frac{\partial}{\partial j} A_{n-1,1}^{(n-1),s}(j_2) \frac{1}{2n-1}$$

$$-\frac{1}{r_2^2} A_{n,1}^{(n-1),s}(j_2) + \frac{1}{r_2^2} e^{-j_2} A_{n-1,1}^{(n-1),s}(j_2) \frac{n-1}{2n-1}$$

$$-\frac{1}{r_2^2} e^{-j_2} \frac{\partial}{\partial j} A_{n-1,1}^{(n-1),s}(j_2) \frac{1}{2n-1} + \frac{2}{r_2^2} e^{-j_2} A_{n-1,1}^{(n-1),s}(j_2) \frac{n-1}{2n-1}$$

$$-\frac{2}{r_2^2} e^{-j_2} A_{n-1,1}^{(n-1),s}(j_2) \frac{1}{2n-1} + \frac{2}{r_2^2} e^{-j_2} A_{n-1,1}^{(n-1),s}(j_2) \frac{n-1}{2n-1},$$

$$d(5,6)=\frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}A_{n,2}^{(n-1),s}(j_2)-\frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\right)\frac{n-1}{2n-1}$$
$$+\frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\right)\frac{1}{2n-1}-\frac{4}{r_2^2}e^{-j_2}\frac{\partial^2}{\partial j^2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n-1}{2n-1}$$
$$+\frac{k_s^2}{2}A_{n,2}^{(n-1),s}(j_2)+\frac{k_s^2}{2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n-1}{2n-1}-\frac{k_s^2}{2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\frac{1}{2n-1}$$
$$-\frac{1}{r_2^2}A_{n,2}^{(n-1),s}(j_2)+\frac{1}{r_2^2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n-1}{2n-1}$$
$$-\frac{1}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_2)\frac{1}{2n-1}+\frac{2}{r_2^2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n-1}{2n-1}$$
$$-\frac{2}{r_2^2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{1}{2n-1}+\frac{2}{r_2^2}e^{-j_2}A_{n-1,2}^{(n-1),s}(j_2)\frac{n-1}{2n-1},$$

$$d(6,1)=\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),p}(j_1)\frac{1}{2n-1}-\frac{1}{r_1^2}\left(\frac{\partial}{\partial j}A_{n,1}^{(n-1),p}(j_1)+A_{n,1}^{(n-1),p}(j_1)\right)$$
$$+\frac{2}{r_1^2}e^{-j_1}\left(2\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),p}(j_1)+A_{n-1,1}^{(n-1),p}(j_1)\right)\frac{n-1}{2n-1},$$

$$d(6,2)=\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),p}(j_1)\frac{1}{2n-1}-\frac{1}{r_1^2}\left(\frac{\partial}{\partial j}A_{n,2}^{(n-1),p}(j_1)+A_{n,2}^{(n-1),p}(j_1)\right)$$
$$+\frac{2}{r_1^2}e^{-j_1}\left(2\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),p}(j_1)+A_{n-1,2}^{(n-1),p}(j_1)\right)\frac{n-1}{2n-1},$$

$$d(6,3)=0,\ d(6,4)=0,$$

$$d(6,5)=\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}A_{n,1}^{(n-1),s}(j_1)-\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\right)\frac{n-1}{2n-1}$$
$$+\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\right)\frac{1}{2n-1}-\frac{4}{r_1^2}e^{-j_1}\frac{\partial^2}{\partial j^2}A_{n-1,1}^{(n-1),s}(j_1)\frac{n-1}{2n-1}$$
$$+\frac{k_s^2}{2}A_{n,1}^{(n-1),s}(j_1)+\frac{k_s^2}{2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n-1}{2n-1}-\frac{k_s^2}{2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\frac{1}{2n-1}$$
$$-\frac{1}{r_1^2}A_{n,1}^{(n-1),s}(j_1)+\frac{1}{r_1^2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n-1}{2n-1}$$
$$-\frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,1}^{(n-1),s}(j_1)\frac{1}{2n-1}+\frac{2}{r_1^2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n-1}{2n-1}$$
$$-\frac{2}{r_1^2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{1}{2n-1}+\frac{2}{r_1^2}e^{-j_1}A_{n-1,1}^{(n-1),s}(j_1)\frac{n-1}{2n-1},$$

$$d(6,6)=\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}A_{n,2}^{(n-1),s}(j_1)-\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\right)\frac{n-1}{2n-1}$$

$$+\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\right)\frac{1}{2n-1}-\frac{4}{r_1^2}e^{-j_1}\frac{\partial^2}{\partial j^2}A_{n-1,2}^{(n-1),s}(j_1)\frac{n-1}{2n-1}$$

$$+\frac{k_s^2}{2}A_{n,2}^{(n-1),s}(j_1)+\frac{k_s^2}{2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{n-1}{2n-1}-\frac{k_s^2}{2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\frac{1}{2n-1}$$

$$-\frac{1}{r_1^2}A_{n,2}^{(n-1),s}(j_1)+\frac{1}{r_1^2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{n-1}{2n-1}$$

$$-\frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n-1,2}^{(n-1),s}(j_1)\frac{1}{2n-1}+\frac{2}{r_1^2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{n-1}{2n-1}$$

$$-\frac{2}{r_1^2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{1}{2n-1}+\frac{2}{r_1^2}e^{-j_1}A_{n-1,2}^{(n-1),s}(j_1)\frac{n-1}{2n-1}.$$

APPENDIX C:

Elements of the upper diagonal block matrices $\mathbf{D}_{n+1}^{(n)}$:

$$d(1,1)=-k_p^2A_{n,1}^{(n+1),p}(j_2)-\frac{4}{r_2^2}e^{-j_2}\left(\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),p}(j_2)+(n+1)(n+2)A_{n+1,1}^{(n+1),p}(j_2)\right)\frac{n+1}{2n+3}$$

$$+\frac{1}{r_2^2}\left(2\frac{\partial}{\partial j}A_{n,1}^{(n+1),p}(j_2)+n(n+1)A_{n,1}^{(n+1),p}(j_2)\right)-\frac{\lambda k_p^2}{2}A_{n,1}^{(n+1),p}(j_2),$$

$$d(1,2)=-k_p^2A_{n,2}^{(n+1),p}(j_2)-\frac{4}{r_2^2}e^{-j_2}\left(\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),p}(j_2)+(n+1)(n+2)A_{n+1,2}^{(n+1),p}(j_2)\right)\frac{n+1}{2n+3}$$

$$+\frac{1}{r_2^2}\left(2\frac{\partial}{\partial j}A_{n,2}^{(n+1),p}(j_2)+n(n+1)A_{n,2}^{(n+1),p}(j_2)\right)-\frac{\lambda k_p^2}{2}A_{n,2}^{(n+1),p}(j_2),$$

$$d(1,3)=d(1,4)=0,$$

$$d(1,5)=-\frac{1}{r_2^2}\frac{\partial}{\partial j}A_{n,1}^{(n+1),s}(j_2)n(n+1)+\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\right)\frac{(n+1)(n+2)}{(2n+3)}$$

$$+\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\right)\frac{(n+1)^2(n+2)}{2n+3}-\frac{1}{r_2^2}A_{n,1}^{(n+1),s}(j_2)n(n+1)$$

$$-\frac{1}{r_2^2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{(n+1)(n+2)}{2n+3}+\frac{1}{r_2^2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{(n+1)^2(n+2)}{2n+3}$$

$$-\frac{2}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+1}{2n+3}-\frac{1}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_2)\frac{(n+1)(n+2)}{2n+3}$$

$$+\frac{2}{r_2^2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{(n+1)^2(n+2)}{2n+3}+\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_2)\right)\frac{n(n+1)}{2n+3}$$

$$+\frac{4}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_2)\frac{(n+1)^2(n+2)}{(2n+3)},$$

$$d(1,6)=-\frac{1}{r_2^2}\frac{\partial}{\partial j}A_{n,2}^{(n+1),s}(j_2)n(n+1)+\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\right)\frac{(n+1)(n+2)}{(2n+3)}$$

$$+\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\right)\frac{(n+1)^2(n+2)}{2n+3}-\frac{1}{r_2^2}A_{n,2}^{(n+1),s}(j_2)n(n+1)$$

$$-\frac{1}{r_2^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{(n+1)(n+2)}{2n+3}+\frac{1}{r_2^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{(n+1)^2(n+2)}{2n+3}$$

$$-\frac{2}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+1}{2n+3}-\frac{1}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_2)\frac{(n+1)(n+2)}{2n+3}$$

$$+\frac{2}{r_2^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{(n+1)^2(n+2)}{2n+3}+\frac{1}{r_2^2}\frac{\partial}{\partial j}\left(e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_2)\right)\frac{n(n+1)}{2n+3}$$

$$+\frac{4}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_2)\frac{(n+1)^2(n+2)}{2n+3},$$

$$d(2,1)=-k_p^2A_{n,1}^{(n+1),p}(j_1)-\frac{4}{r_1^2}e^{-j_1}\left(\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),p}(j_1)+(n+1)(n+2)A_{n+1,1}^{(n+1),p}(j_1)\right)\frac{n+1}{2n+3}$$

$$+\frac{1}{r_1^2}\left(2\frac{\partial}{\partial j}A_{n,1}^{(n+1),p}(j_1)+n(n+1)A_{n,1}^{(n+1),p}(j_1)\right)-\frac{\lambda k_p^2}{2}A_{n,1}^{(n+1),p}(j_1),$$

$$d(2,2)=-k_p^2A_{n,2}^{(n+1),p}(j_1)-\frac{4}{r_1^2}e^{-j_1}\left(\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),p}(j_1)+(n+1)(n+2)A_{n+1,2}^{(n+1),p}(j_1)\right)\frac{n+1}{2n+3}$$

$$+\frac{1}{r_1^2}\left(2\frac{\partial}{\partial j}A_{n,2}^{(n+1),p}(j_1)+n(n+1)A_{n,2}^{(n+1),p}(j_1)\right)-\frac{\lambda k_p^2}{2}A_{n,2}^{(n+1),p}(j_1),$$

$d(2,3)=0,\ d(2,4)=0,$

$$d(2,5) = -\frac{1}{r_1^2}\frac{\partial}{\partial j}A_{n,1}^{(n+1),s}(j_1)n(n+1) + \frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\right)\frac{(n+1)(n+2)}{(2n+3)}$$

$$+\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\right)\frac{(n+1)^2(n+2)}{2n+3} - \frac{1}{r_1^2}A_{n,1}^{(n+1),s}(j_1)n(n+1)$$

$$-\frac{1}{r_1^2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{(n+1)(n+2)}{2n+3} + \frac{1}{r_1^2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{(n+1)^2(n+2)}{2n+3}$$

$$-\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+1}{2n+3} - \frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_1)\frac{(n+1)(n+2)}{2n+3}$$

$$+\frac{2}{r_1^2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{(n+1)^2(n+2)}{2n+3} + \frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_1)\right)\frac{n(n+1)}{2n+3}$$

$$+\frac{4}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_1)\frac{(n+1)^2(n+2)}{2n+3},$$

$$d(2,6) = -\frac{1}{r_1^2}\frac{\partial}{\partial j}A_{n,2}^{(n+1),s}(j_1)n(n+1) + \frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\right)\frac{(n+1)(n+2)}{(2n+3)}$$

$$+\frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\right)\frac{(n+1)^2(n+2)}{2n+3} - \frac{1}{r_1^2}A_{n,2}^{(n+1),s}(j_1)n(n+1)$$

$$-\frac{1}{r_1^2}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{(n+1)(n+2)}{2n+3} + \frac{1}{r_1^2}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{(n+1)^2(n+2)}{2n+3}$$

$$-\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+1}{2n+3} - \frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_1)\frac{(n+1)(n+2)}{2n+3}$$

$$+\frac{2}{r_1^2}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{(n+1)^2(n+2)}{2n+3} + \frac{1}{r_1^2}\frac{\partial}{\partial j}\left(e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_1)\right)\frac{n(n+1)}{2n+3}$$

$$+\frac{4}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_1)\frac{(n+1)^2(n+2)}{2n+3},$$

$$d(3,1) = 0,\ d(3,2) = 0,$$

$$d(3,3) = -\frac{3}{r_2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+2}{2n+3} + \frac{3}{2}k_s^2 r_2 e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{1}{2n+3}$$

$$-\frac{1}{r_2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{(n+2)(n+1)}{2n+3} - \frac{1}{r_2}\frac{\partial}{\partial j}A_{n,1}^{(n+1),s}(j_2)$$

$$+\frac{1}{r_2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+2}{2n+3} - \frac{1}{r_2}A_{n,1}^{(n+1),s}(j_2) + \frac{2}{r_2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{1}{2n+3},$$

$$d(3,4)=-\frac{3}{r_2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+2}{2n+3}+\frac{3}{2}k_s^2 r_2 e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{1}{2n+3}$$

$$-\frac{1}{r_2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{(n+2)(n+1)}{2n+3}-\frac{1}{r_2}\frac{\partial}{\partial j}A_{n,2}^{(n+1),s}(j_2)$$

$$+\frac{1}{r_2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+2}{2n+3}-\frac{1}{r_2}A_{n,2}^{(n+1),s}(j_2)+\frac{2}{r_2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{1}{2n+3},$$

$d(3,5)=0,\ d(3,6)=0,$

$d(4,1)=0,\ d(4,2)=0,$

$$d(4,3)=-\frac{3}{r_1}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{3}{2}k_s^2 r_1 e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{1}{2n+3}$$

$$-\frac{1}{r_1}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{(n+2)(n+1)}{2n+3}-\frac{1}{r_1}\frac{\partial}{\partial j}A_{n,1}^{(n+1),s}(j_1)$$

$$+\frac{1}{r_1}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+2}{2n+3}-\frac{1}{r_1}A_{n,1}^{(n+1),s}(j_1)+\frac{2}{r_1}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{1}{2n+3},$$

$$d(4,4)=-\frac{3}{r_1}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{3}{2}k_s^2 r_1 e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{1}{2n+3}$$

$$-\frac{1}{r_1}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{(n+2)(n+1)}{2n+3}-\frac{1}{r_1}\frac{\partial}{\partial j}A_{n,2}^{(n+1),s}(j_1)$$

$$+\frac{1}{r_1}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+2}{2n+3}-\frac{1}{r_1}A_{n,2}^{(n+1),s}(j_1)+\frac{2}{r_1}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{1}{2n+3},$$

$d(4,5)=0,\ d(4,6)=0,$

$$d(5,1)=-\frac{2}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),p}(j_2)\frac{1}{2n+3}-\frac{1}{r_2^2}\left(\frac{\partial}{\partial j}A_{n,1}^{(n+1),p}(j_2)+A_{n,1}^{(n+1),p}(j_2)\right)$$

$$+\frac{2}{r_2^2}e^{-j_2}\left(2\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),p}(j_2)+A_{n+1,1}^{(n+1),p}(j_2)\right)\frac{n+2}{2n+3},$$

$$d(5,2)=-\frac{2}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),p}(j_2)\frac{1}{2n+3}-\frac{1}{r_2^2}\left(\frac{\partial}{\partial j}A_{n,2}^{(n+1),p}(j_2)+A_{n,2}^{(n+1),p}(j_2)\right)$$

$$+\frac{2}{r_2^2}e^{-j_2}\left(2\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),p}(j_2)+A_{n+1,2}^{(n+1),p}(j_2)\right)\frac{n+2}{2n+3},$$

$d(5,3)=0,\ d(5,4)=0,$

$$d(5,5) = \frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}A_{n,1}^{(n+1),s}(j_2) - \frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\right)\frac{n+2}{2n+3}$$

$$-\frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\right)\frac{1}{2n+3} - \frac{4}{r_2^2}e^{-j_2}\frac{\partial^2}{\partial j^2}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+2}{2n+3}$$

$$+\frac{k_s^2}{2}A_{n,1}^{(n+1),s}(j_2) + \frac{k_s^2}{2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+2}{2n+3} + \frac{k_s^2}{2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_2)\frac{1}{2n+3}$$

$$-\frac{1}{r_2^2}A_{n,1}^{(n+1),s}(j_2) + \frac{1}{r_2^2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+2}{2n+3} + \frac{1}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_2)\frac{1}{2n+3}$$

$$+\frac{2}{r_2^2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+2}{2n+3} + \frac{2}{r_2^2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{1}{2n+3} + \frac{2}{r_2^2}e^{-j_2}A_{n+1,1}^{(n+1),s}(j_2)\frac{n+2}{2n+3},$$

$$d(5,6) = \frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}A_{n,2}^{(n+1),s}(j_2) - \frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\right)\frac{n+2}{2n+3}$$

$$-\frac{1}{r_2^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\right)\frac{1}{2n+3} - \frac{4}{r_2^2}e^{-j_2}\frac{\partial^2}{\partial j^2}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+2}{2n+3}$$

$$+\frac{k_s^2}{2}A_{n,2}^{(n+1),s}(j_2) + \frac{k_s^2}{2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+2}{2n+3} + \frac{k_s^2}{2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_2)\frac{1}{2n+3}$$

$$-\frac{1}{r_2^2}A_{n,2}^{(n+1),s}(j_2) + \frac{1}{r_2^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+2}{2n+3} + \frac{1}{r_2^2}e^{-j_2}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_2)\frac{1}{2n+3}$$

$$+\frac{2}{r_2^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+2}{2n+3} + \frac{2}{r_2^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{1}{2n+3} + \frac{2}{r_2^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_2)\frac{n+2}{2n+3},$$

$$d(6,1) = -\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),p}(j_1)\frac{1}{2n+3} - \frac{1}{r_1^2}\left(\frac{\partial}{\partial j}A_{n,1}^{(n+1),p}(j_1) + A_{n,1}^{(n+1),p}(j_1)\right)$$

$$+\frac{2}{r_1^2}e^{-j_1}\left(2\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),p}(j_1) + A_{n+1,1}^{(n+1),p}(j_1)\right)\frac{n+2}{2n+3},$$

$$d(6,2) = -\frac{2}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),p}(j_1)\frac{1}{2n+3} - \frac{1}{r_1^2}\left(\frac{\partial}{\partial j}A_{n,2}^{(n+1),p}(j_1) + A_{n,2}^{(n+1),p}(j_1)\right)$$

$$+\frac{2}{r_1^2}e^{-j_1}\left(2\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),p}(j_1) + A_{n+1,2}^{(n+1),p}(j_1)\right)\frac{n+2}{2n+3},$$

$d(6,3) = 0,\ d(6,4) = 0,$

$$d(6,5)=\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}A_{n,1}^{(n+1),s}(j_1)-\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\right)\frac{n+2}{2n+3}$$
$$-\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\right)\frac{1}{2n+3}-\frac{4}{r_1^2}e^{-j_1}\frac{\partial^2}{\partial j^2}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+2}{2n+3}$$
$$+\frac{k_s^2}{2}A_{n,1}^{(n+1),s}(j_1)+\frac{k_s^2}{2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{k_s^2}{2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_1)\frac{1}{2n+3}$$
$$-\frac{1}{r_1^2}A_{n,1}^{(n+1),s}(j_1)+\frac{1}{r_1^2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,1}^{(n+1),s}(j_1)\frac{1}{2n+3}$$
$$+\frac{2}{r_1^2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{2}{r_1^2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{1}{2n+3}+\frac{2}{r_1^2}e^{-j_1}A_{n+1,1}^{(n+1),s}(j_1)\frac{n+2}{2n+3},$$

$$d(6,6)=\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}A_{n,2}^{(n+1),s}(j_1)-\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\right)\frac{n+2}{2n+3}$$
$$-\frac{1}{r_1^2}\frac{\partial^2}{\partial j^2}\left(e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\right)\frac{1}{2n+3}-\frac{4}{r_1^2}e^{-j_1}\frac{\partial^2}{\partial j^2}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+2}{2n+3}$$
$$+\frac{k_s^2}{2}A_{n,2}^{(n+1),s}(j_1)+\frac{k_s^2}{2}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{k_s^2}{2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_1)\frac{1}{2n+3}$$
$$-\frac{1}{r_1^2}A_{n,2}^{(n+1),s}(j_1)+\frac{1}{r_1^2}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{1}{r_1^2}e^{-j_1}\frac{\partial}{\partial j}A_{n+1,2}^{(n+1),s}(j_1)\frac{1}{2n+3}$$
$$+\frac{2}{r_1^2}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+2}{2n+3}+\frac{2}{r_1^2}e^{-j_1}A_{n+1,2}^{(n+1),s}(j_1)\frac{1}{2n+3}+\frac{2}{r_1^2}e^{-j_2}A_{n+1,2}^{(n+1),s}(j_1)\frac{n+2}{2n+3}.$$

In Appendices A, B and C we have used for simplicity the convention

$$\frac{\partial}{\partial x}f(x_1)=\frac{\partial}{\partial x}f(x)\Big|_{x=x_1}.$$