The Effect of Geometry on the Dynamic Characteristics of the Human Skull

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THE EFFECT OF GEOMETRY ON THE DYNAMIC CHARACTERISTICS OF THE HUMAN SKULL

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SUMMARY

In the present work we discuss the role of small deviation of the spherical to spheroidal geometry on the frequency spectrum of the human dry-skull. The analysis is based on the three dimensional theory of elasticity, complex analysis techniques and the construction of the Navier eigenvectors for the problem under discussion.

1. INTRODUCTION

The analysis of processes in the human body is based on physical and mathematical models. The former type needs experimental analysis while the latter type of models is based on theoretical analysis. As far as human dry-skull is concerned we mention the important experimental investigation of Khalil et al. [1] where results for the dynamic characteristics of freely vibrating human skulls are cited. In Ref. [2] a theoretical analysis for the dynamic characteristics of the skull was presented and the results obtained were favourable compared with those of Ref. [1]. The analysis in [2] was based on the three - dimensional theory of elasticity, the representation of the displacement field of the skull in terms of the Navier eigenvectors and the assumption of the spherical geometry.

Due to the lack of the sphericity of the skull, Goldsmith [3] suggested that a human skull should be better represented by a prolate spheroid with the ratio of major to minor axes about 4/3. Adopting the suggestion of Ref. [3] Mirsa et al. [4] studied the dynamic response of a head-neck system to an impulsive load. Khalil and Hubbard [5] presented a parametric study of head response by finite element modelling representing the human skull by a single-layer spherical shell, an oval shell consisting of spherical caps and a cone frustum and a three-layer spherical shell. From their analysis they revealed that the

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load spatial distribution strongly influenced skull strains and the load required to initiate skull fracture while the other parameters produced small effects on the models responses.

In the present work our interest is concentrated to the study of the sensitivity of the frequency spectrum of the human dry - skull as the spherical geometry gives continuously place to the spheroidal one. From the physical point of view the eccentricity of the skull can be considered sufficiently small [3]. This assumption leads to spheroids not extremely prolated. The description of the problem is based on the three-dimensional theory of elasticity and the mathematical treatment on complex analysis techniques, allowing suitable space transformations and better flexibility of the geometrical approximations, the construction of the Navier eigenvectors for the problem under discussion and the procedure followed in Ref. [2]

The resulting frequency equation has been solved numerically. From the results of the present analysis we lead to the conclusion that the deviation of the spherical geometry to spheroidal one with sufficiently small eccentricity does not appreciable influence the frequency spectrum of the human dry-skull.

2. PROBLEM FORMULATION

We consider the cylindrical coordinate system (u,φ,v) to describe the system of two spheroidal surfaces $S_i, i=0,1$ which have common focus γ and semiaxes (α_i,β_i), $i=0,1$ (Fig. 1). The space V between S_0 and S_1 is supposed to be filled with a linear isotropic material with Lamé coefficients λ and μ and mass density ρ.

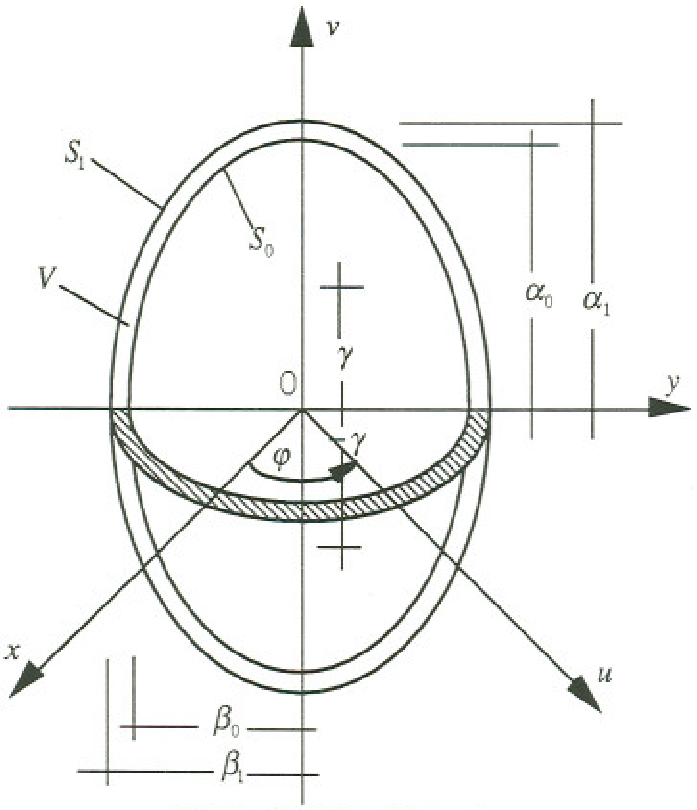

Fig. 1. Problem Geometry

In what follows we shall discuss the free vibration problem of the system under discussion (Fig. 1). The well known Navier frequency equation is given as:

$$c_s^2\nabla^2\boldsymbol{u}(\boldsymbol{r})+(c_p^2-c_s^2)\nabla(\nabla\cdot\boldsymbol{u}(\boldsymbol{r}))+\omega^2\boldsymbol{u}(\boldsymbol{r})=\boldsymbol{0},\ \ \boldsymbol{r}\in V, \tag{1}$$

where $u(r)$ is the displacement field, ω is the angular frequency measured in radians/sec, $c_P = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}$, $c_S = \left(\frac{\mu}{\rho}\right)^{1/2}$, and ∇ is the usual del operator.

We assume that the surfaces $S_i, i = 0,1$ are stress free, that is:

$$\boldsymbol{Tu}(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in S_0, S_1 \tag{2}$$

where $\boldsymbol{T}$ stands for the stress operator

$$\boldsymbol{T} = 2\mu\hat{\boldsymbol{r}} \cdot \nabla + \lambda\hat{\boldsymbol{r}}\nabla \cdot + \mu\hat{\boldsymbol{r}} \times \nabla \times \tag{3}$$

and $\hat{\boldsymbol{r}}$ is the unit outward normal vector on $S_i, i = 0,1$.

We note that the problem described by equation (1) and the boundary conditions (2) is a well - posed mathematical problem.

3. PROBLEM SOLUTION - FREQUENCY EQUATION

The mathematical analysis of the present problem can be based on the construction of the Navier vector eigenfunctions in spheroidal coordinates and the procedure followed in Ref. [2].

In the present work we are interesting in studying the "sensitivity" of the frequency spectrum of the cranial system as the spherical geometry gives continously place to the spheroidal one, procedure described by the crucial parameter γ, the eccentricity. More specifically to study how a perturbation in geometry influences the dynamical characteristics of the system. Thus, from the physical point of view the eccentricity, γ, can be considered sufficiently small, assumption leading to spheroids not extremely prolated. This leads us to avoid the introduction of spheroidal coordinates where we have to deal with the extremely difficult problems of the construction of Navier eigenvectors, as well as, the satisfaction of the boundary conditions on the discontinuity surfaces.

Under these remarks, the method we present here is based on rather complex analysis techniques allowing suitable space transformations and better flexibility to the above geometrical approximations.

We consider a plane $\varphi =$const. which intersects the system of the two spheroids. The section is a system of two ellipses with common focus γ and semiaxes (α_i, β_i), $i = 0,1$. In addition to the space of ellipses (space - 1) we consider the space of the spheres (space - 2), where we have two concentric spheres of radius $r_i = \alpha_i + \beta_i$, $i = 0,1$.

Introducing the cylindrical coordinates system (R, φ, z) and taking the intersection with the plane $\varphi =$const. we get the system of two circles with radius r_i, $i = 0,1$ (Fig. 2).

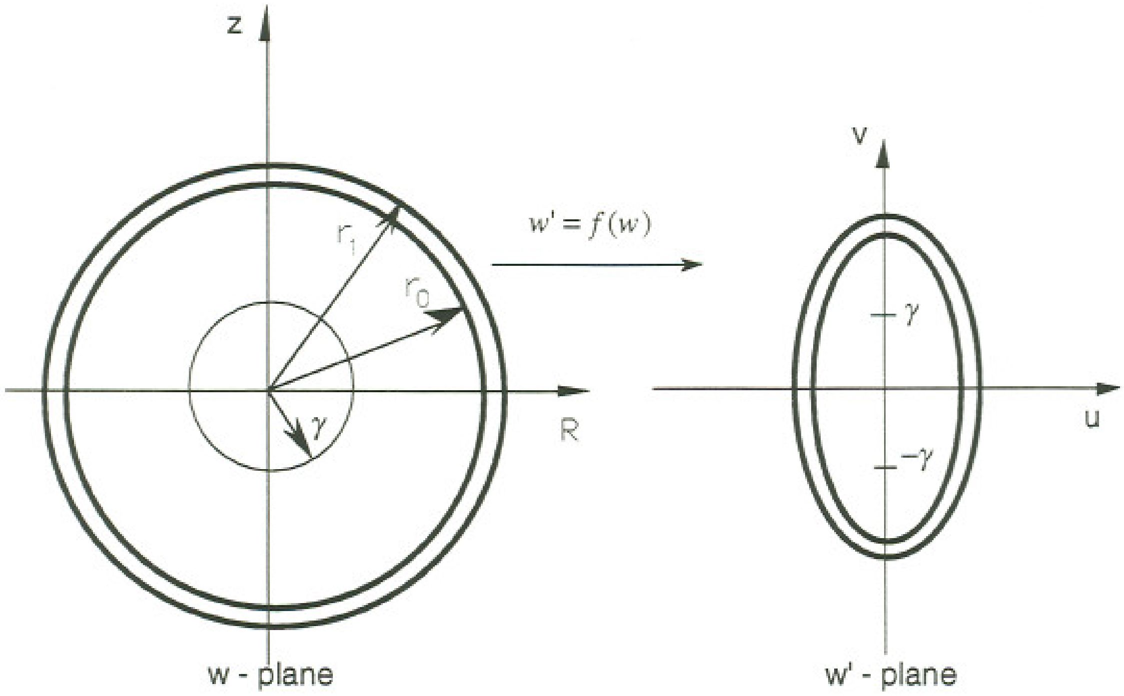

Figure 2: Conformal Mapping

The space of the circles can be transformed conformally onto the space of ellipses through the Joukowski transform,

$$w' = f(w) = \frac{1}{2}\left(w - \frac{\gamma^2}{w}\right) \tag{4}$$

where

$$w' = u + iv, \qquad w = R + iz \tag{5}$$

More precisely, every circle of radius $r \neq \gamma$ in w-plane is transformed to the ellipse of semiaxes $\frac{1}{2}\left(r + \frac{\gamma^2}{r}\right)$, $\frac{1}{2}\left(r - \frac{\gamma^2}{r}\right)$ in w' -plane, while the circle $r = \gamma$ is transformed to the segment connecting the two focuses.

Now, instead of solving the problem in space - 1 we solve it in space - 2 and then come back to the spheroidal space by suitable use of the transformation connecting the w, w' - planes whose complete rotation around the two axes z and v, respectively, gives rise to the two dual spaces.

The solution $\boldsymbol{u}(\boldsymbol{r})$ of the Navier equation (1) can be expanded in a complete set of Navier eigenfunctions [6] given as follows,

$$\boldsymbol{L}(\boldsymbol{r}) = \nabla\Phi_p(\boldsymbol{r}) \tag{6}$$

$$\boldsymbol{M}(\boldsymbol{r}) = \nabla\Phi_s(\boldsymbol{r}) \times \boldsymbol{r} \tag{7}$$

$$\boldsymbol{N}(\boldsymbol{r}) = \nabla \times \left\{\nabla\Phi_s(\boldsymbol{r}) \times \boldsymbol{r}\right\}, \tag{8}$$

where Φ_l, $l = p, s$ satisfy the Helmholtz equation

$$\nabla^2 \Phi_l + k_l^2 \Phi_l = 0, \quad k_l = \frac{\omega}{c_l}. \tag{9}$$

We note that under the transformation $(u, \varphi, v) \leftrightarrow (R, \varphi, z)$ we consider the function $\Phi_l(u, \varphi, v) = \tilde{\Phi}_l(R, \varphi, z)$. The function $\tilde{\Phi}_l$ lives in an "easier" space but it does not obey yet the Helmholtz equation, that is

$$(\nabla^2 + k^2)\Phi = 0 \quad \Rightarrow$$

$$|f'(\omega)|^2 \left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{u^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial v^2} + k^2 \right) \Phi = 0 \Rightarrow$$

$$\left\{ \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2} + |f'(\omega)|^2 \left(k^2 + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{u^2}\frac{\partial^2}{\partial \varphi^2} \right) \right\} \tilde{\Phi} = 0 \tag{10}$$

where the complex analysis result

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2} \right) \tilde{\Phi} = |f'(\omega)|^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \tag{11}$$

has been used and the subscript "$_l$" was suppressed.

Introducing spherical coordinates (r, ϑ, φ) in space - 2 we get the following auxiliary relations

$$|f'(\omega)|^2 = \frac{1}{4}\left(1 + \frac{\gamma^4}{r^4}\right) - \frac{1}{2}\frac{\gamma^2}{r^2}\cos 2\vartheta$$

$$\frac{1}{u}\frac{\partial}{\partial u} = \frac{1}{|f'(\omega)|^2}\frac{1}{r\sin\vartheta}\left\{ \frac{1 + \frac{\gamma^2}{r^2}}{1 - \frac{\gamma^2}{r^2}} \sin\vartheta \frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta \frac{\partial}{\partial \vartheta} \right\}$$

$$\frac{1}{u^2}\frac{\partial^2}{\partial \varphi^2} = 4\frac{1}{r^2(1 - \frac{\gamma^2}{r^2})^2 \sin^2\vartheta}\frac{\partial^2}{\partial \varphi^2}$$

and the equation (10) becomes

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{1}{r^2}\cot\vartheta \frac{\partial}{\partial \vartheta} + k^2 \frac{1}{4}\left(1 + \frac{\gamma^4}{r^4}\right) - \right.$$

$$-\frac{k^2}{2}\frac{\gamma^2}{r^2}\cos 2\vartheta + \frac{1}{r}\frac{2\frac{\gamma^2}{r^2}}{1-\frac{\gamma^2}{r^2}}\frac{\partial}{\partial r} + \frac{1+\frac{\gamma^4}{r^4}-2\frac{\gamma^2}{r^2}\cos 2\vartheta}{\left(1-\frac{\gamma^2}{r^2}\right)^2 r^2 \sin^2\vartheta}\frac{\partial^2}{\partial \varphi^2}\Bigg\}\tilde{\Phi} = 0. \tag{11}$$

Following separation of variables we assume that:

$$\tilde{\Phi}(r,\vartheta,\varphi) = R(r)F(\varphi)\Theta(\vartheta). \tag{12}$$

Replacing (12) into the equation (11) we obtain

$$F''(\varphi) - \lambda F(\varphi) = 0 \tag{13}$$

$$r^2 R''(r) + \frac{2r}{1-\frac{\gamma^2}{r^2}}R'(r) + \left\{\frac{k^2}{4}r^2\left(1+\frac{\gamma^4}{r^4}\right) + \frac{\lambda}{r^2}\frac{4\frac{\gamma^2}{r^2}}{1-\frac{\gamma^2}{r^2}} - \alpha\right\}R(r) = 0 \tag{14}$$

$$\Theta''(\vartheta) + \cot\vartheta\,\Theta'(\vartheta) + \lambda\frac{1}{\sin^2\vartheta}\Theta(\vartheta) - \frac{k^2}{2}\gamma^2\cos 2\vartheta\,\Theta(\vartheta) + \alpha\Theta(\vartheta) = 0 \tag{15}$$

where λ and α are separation constants.

Since periodicity over the angle φ is required we have

$\lambda = -m^2, \; \mathrm{m} = 0,1,2,...$

and

$F(\varphi) = e^{im\varphi}$.

Substituting $x = \cos\vartheta$, $P(x) = \Theta(\vartheta)$ and $\lambda = -m^2$ into equation (15) we get

$$(1-x^2)P''(x) - 2xP'(x) + (\alpha + k^2\frac{\gamma^2}{2} - \frac{m^2}{1-x^2} - k^2\gamma^2 x^2)P(x) = 0. \tag{15a}$$

For the case $\gamma = 0$ the system (14), (15a) takes the form

$$r^2 R''(r) + 2rR'(r) + (\frac{k^2 r^2}{4} - \alpha)R(r) = 0$$

$$(1-x^2)P''(x) - 2xP'(x) + (\alpha - \frac{m^2}{1-x^2})P(x) = 0$$

which is the one corresponding to the spherical geometry.

In this case $\alpha = n(n+1)$, $n = m, \mathrm{m}+1,...$ and the solutions of the above system are:

$R_n(r) \in \left\{ j_n(\frac{kr}{2}),\; y_n(\frac{kr}{2}) \right\}$ and $P(x) = P_n^m(x)$, as it was expected, where $j_n(s)$ and $y_n(s)$ are the spherical Bessel functions of the first and second kind, respectively, and P_n^m are the Legendre functions.

In the general case $\gamma \neq 0$ the equations (14) and (15a) can be solved by Frobenius method but this treatment leads to very complicated and unuseful for the present analysis expressions.

For the purpose of our analysis we supposed that γ can be considered to be "sufficiently" small; actually we need the ratio $\gamma^2/r^2 << 1$. Given that $\alpha_0 + \beta_0 \leq r \leq \alpha_1 + \beta_1$ and $\gamma = (\alpha_i^2 - \beta_i^2)^{1/2}$, $i = 0,1$, we find that

$$\left(\frac{1 - \frac{\beta_1}{\alpha_1}}{1 + \frac{\beta_1}{\alpha_1}} \right) \leq \frac{\gamma^2}{r^2} \leq \left(\frac{1 - \frac{\beta_0}{\alpha_0}}{1 + \frac{\beta_0}{\alpha_0}} \right).$$

We note that the typical case for the cranial systems is characterised by the ratio $\beta_0/\alpha_0 = 0.95$. But even in the extreme case where $\beta_0/\alpha_0 = 0.90$ we have $\gamma^2/r^2 \leq 0.0526$ fact justifying the adopted assumption. Taking into account that $\gamma^2/r^2 << 1$ the equation (14) is simplified as follows

$$r^2 R''(r) + 2rR'(r) + \left\{ \left(\frac{kr}{2} \right)^2 - \alpha - \frac{4m^2\gamma^2}{r^4} \right\} R(r) = 0, \qquad (16)$$

while equation (15a) remains unaltered.

In spherical geometry we showed [2], by using three dimensional theory of elasticity, that the eigenfrequencies of the dry-skull do not depend on the azimuthial wave number m although the eigenmodes do and therefore as far as these eigenfrequencies shifting is concerned, as γ varies, we can restrict our analysis to the case $m = 0$.

The equations (16) and (15a) for $m = 0$ become

$$r^2 R''(r) + 2rR'(r) + \left\{ \left(\frac{kr}{2} \right)^2 - n(n+1) \right\} R(r) = 0 \qquad (17)$$

$$(1 - x^2)P''(x) - 2xP'(x) + \left\{ n(n+1) + \frac{k^2\gamma^2}{2} - k^2\gamma^2x^2 \right\} P(x) = 0, \qquad (18)$$

where we have replaced the separation constant $\alpha = n(n+1)$, $n = 0,1,2,...$ as it is imposed by the asymptotic case $\gamma \to 0$.

It is easily verified that the equation (17) has the solutions

$$R_n(r) \in \left\{ j_n\left(\frac{kr}{2} \right), y_n\left(\frac{kr}{2} \right), n = 0,1,2,... \right\}.$$

The solutions of equation (18), taking advantage of the completeness of the Legendre polynomials is expressed in the form

$$\hat{P}_n(x;k\gamma) = \sum_{l=0}^{+\infty} C_l(n,k\gamma)P_l(x). \tag{19}$$

Replacing (19) into equation (18) we lead to the following recurrence relation for the coefficients $C_l(n,k\gamma)$:

$$\left[\left\{n(n+1)+\frac{k^2\gamma^2}{2}-l(l+1)\right\}-k^2\gamma^2\left\{\frac{l^2}{(2l+1)(2l-1)}+\frac{(l+1)^2}{(2l+1)(2l+3)}\right\}\right]C_l(n,k\gamma)=$$

$$k^2\gamma^2\frac{(l+2)(l+1)}{(2l+5)(2l+3)}C_{l+2}(n,k\gamma)+k^2\gamma^2\frac{(l-1)l}{(2l-3)(2l-1)}C_{l-2}(n,k\gamma). \tag{20}$$

Given that

$$C_l(n,0)=\delta_{l\,n},$$

the appropriate normalisation of the coefficients of (19) is obtained by demanding

$$C_n(n,k\gamma)=1.$$

From the previous analysis we lead to the conclusion that, under the assumption $\gamma^2/r^2 << 1$ and putting $m=0$, the solutions of the equation (11) are

$$\tilde{\Phi}_n(r,\vartheta,\varphi)=\tilde{\Phi}_n(r,\vartheta)=R_n\left(\frac{kr}{2}\right)\hat{P}_n(\cos\vartheta;k\gamma). \tag{21}$$

Now, we are in the position to produce the Navier eigenfunctions.

Following equation (6) we obtain

$$\begin{aligned}
\boldsymbol{L}(\boldsymbol{r}) &= \nabla\Phi_p(\boldsymbol{r}) = \hat{\boldsymbol{u}}\frac{\partial}{\partial u}\Phi_p(\boldsymbol{r})+\hat{\boldsymbol{v}}\frac{\partial}{\partial v}\Phi_p(\boldsymbol{r}) = \\
&= \hat{\boldsymbol{u}}\left\{\frac{\partial r}{\partial u}\frac{\partial}{\partial r}\tilde{\Phi}_p(\boldsymbol{r})+\frac{\partial \vartheta}{\partial u}\frac{\partial}{\partial \vartheta}\tilde{\Phi}_p(\boldsymbol{r})\right\}+ \\
&+\hat{\boldsymbol{v}}\left\{\frac{\partial r}{\partial u}\frac{\partial}{\partial r}\Phi_p(\boldsymbol{r})+\frac{\partial \vartheta}{\partial v}\frac{\partial}{\partial \vartheta}\tilde{\Phi}_p(\boldsymbol{r})\right\} \\
&= \frac{1}{2|f'(\omega)|^2}\frac{\partial}{\partial r}\tilde{\Phi}_p(\boldsymbol{r})\left\{\hat{\boldsymbol{u}}\left(1+\frac{\gamma^2}{r^2}\right)\sin\vartheta+\hat{\boldsymbol{v}}\left(1-\frac{\gamma^2}{r^2}\right)\cos\vartheta\right\}+ \\
&+\frac{1}{2|f'(\omega)|^2}\frac{1}{r}\frac{\partial}{\partial \vartheta}\tilde{\Phi}_p(\boldsymbol{r})\left\{\hat{\boldsymbol{u}}\left(1-\frac{\gamma^2}{r^2}\right)\cos\vartheta-\hat{\boldsymbol{v}}\left(1+\frac{\gamma^2}{r^2}\right)\sin\vartheta\right\}.
\end{aligned} \tag{22}$$

where $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ are the unit vectors in u and v - directions, respectively.

Taking into account that $\gamma^2/r^2 << 1$, the expression (22) becomes

$$L(r) = 2\hat{r}\frac{\partial}{\partial r}\tilde{\Phi}_p(r) + 2\hat{\vartheta}\frac{1}{r}\frac{\partial}{\partial \vartheta}\tilde{\Phi}_p(r) \tag{23}$$

where $\hat{r}$ and $\hat{\vartheta}$ are the unit vectors in r and ϑ - directions, respectively.

Dividing (23) by the constant wave number k_p and replacing $\tilde{\Phi}_p(r)$ by the expression (21) we infer that

$$L_n^l(r) = \dot{g}_n^l(k_p\frac{r}{2})P_n(\hat{r},k_p\gamma) + 2\frac{g_n^l(k_p\frac{r}{2})}{k_p r}B_n(\hat{r},k_p\gamma)\hat{\vartheta}, \tag{24}$$

$n = 0,1,2,\ldots,\quad l = 1,2$

where

$$P_n(\hat{r},k\gamma) = \hat{r}\hat{P}_n(\cos\vartheta) = \hat{r}\sum_{m=0}^{+\infty}C_m(n,k\gamma)P_m(\cos\vartheta)$$

$$B_n(\hat{r},k\gamma) = \frac{d}{d\vartheta}\hat{P}_n(\cos\vartheta) = \sum_{m=0}^{+\infty}C_m(n,k\gamma)\frac{d}{d\vartheta}P_m(\cos\vartheta)$$

$$\dot{g}_n^l(s) = \frac{d}{ds}(g_n^l)$$

and $g_n^1(s)$, $g_n^2(s)$ represent the spherical Bessel functions of the first, $j_n(s)$, and second kind, $y_n(s)$, respectively.

In a similar manner we find the remaining Navier eigenfunctions as

$$M_n^l(r) = g_n^l(k_s\frac{r}{2})\left\{-B_n(\hat{r},k_s\gamma)\hat{\varphi}\right\}, \tag{25}$$

$$N_n^l(r) = 2n(n+1)\frac{g_n^l(k_s\frac{r}{2})}{k_s r}P_n(\hat{r},k\gamma) + \left\{\dot{g}_n^l(k_s\frac{r}{2}) + 2\frac{g_n^l(k_s\frac{r}{2})}{k_s r}\right\}B_n(\hat{r},k_s\gamma)\hat{\vartheta}. \tag{26}$$

The displacement field can be represented in the most general form as follows

$$u(r) = \sum_{n=0}^{+\infty}\sum_{l=1}^{2}\left\{\alpha_n^l L_n^l(r) + \beta_n^l M_n^l(r) + \gamma_n^l N_n^l(r)\right\} \tag{27}$$

and the stress field is obtained by applying the stress operator T on $u(r)$. After tedious and lengthy calculations, we find that

$$\mathbf{T}'\mathbf{u}(\mathbf{r}') = \sum_{n=0}^{\infty}\sum_{l=1}^{2}\left\{\begin{array}{l}\left[\alpha_n^l A_n^l(r') + \gamma_n^l D_n^l(r')\right]\mathbf{P}_n(\hat{\mathbf{r}}, k'_p\, \gamma') + \\ \left[\alpha_n^l B_n^l(r') + \gamma_n^l E_n^l(r')\right] B_n(\hat{\mathbf{r}}, k'_s\, \gamma')\hat{\vartheta} + \\ \beta_n^l C_n^l(r')\left[-B_n(\hat{\mathbf{r}}, k'_s\, \gamma')\hat{\varphi}\right]\end{array}\right\}, \tag{28}$$

where

$$A_n^l(r') = -\left[\frac{4}{(\frac{r'}{2})}\mu'\,\dot{g}_n^l(k'_p\frac{r}{2}) + 2\mu' k'_p(1 - \frac{n(n+1)}{(k'_p\frac{r}{2})^2})g_n^l(k'_p\frac{r}{2}) + \lambda' k'_p g_n^l(k'_p\frac{r}{2})\right] \tag{29a}$$

$$B_n^l(r') = 2\mu'\left[\frac{1}{(\frac{r'}{2})}\dot{g}_n^l(k'_p\frac{r'}{2}) - \frac{g_n^l(k'_p\frac{r'}{2})}{k'_p(\frac{r'}{2})^2}\right] \tag{29b}$$

$$C_n^l(r') = \mu'\left[k'_s\,\dot{g}_n^l(k'_s\frac{r'}{2}) - \frac{1}{(\frac{r'}{2})}g_n^l(k'_s\frac{r'}{2})\right] \tag{29c}$$

$$D_n^l(r') = 2n(n+1)\mu'\left[\frac{\dot{g}_n^l(k'_s\frac{r'}{2})}{(\frac{r'}{2})} - \frac{g_n^l(k'_s\frac{r'}{2})}{k'_s(\frac{r'}{2})^2}\right] \tag{29d}$$

$$E_n^l(r') = \mu'\left[-2\frac{\dot{g}_n^l(k'_s\frac{r'}{2})}{(\frac{r'}{2})} - k'_s\,g_n^l(k'_s\frac{r'}{2}) + 2\frac{n(n+1)-1}{k'_s(\frac{r'}{2})^2}g_n^l(k'_s\frac{r'}{2})\right]. \tag{29e}$$

and $(\cdot)'$ denotes nondimensionalised quantities as follows:

$$\mathbf{r}' = \frac{\mathbf{r}}{r_1},\quad \Omega = \frac{\omega r_1}{c_p},\quad \gamma' = \frac{\gamma}{r_1},$$

$$\nabla' = r_1\nabla, c'_s = \frac{c_s}{c_p}, k'_p = \frac{\Omega}{c'_p}, k'_s = \frac{\Omega}{c'_s}.$$

The boundary conditions (2) lead to the equations

$$\sum_{n=0}^{+\infty}\sum_{l=1}^{2}\left[\alpha_n^l A_n^l(r'_i)\sum_{q=0}^{+\infty}C_q(n,k'_p\,\gamma') + \gamma_n^l D_n^l(r'_i)\sum_{q=0}^{+\infty}C_q(n,k'_s\,\gamma')\right]P_q(\cos\vartheta) = 0 \tag{30a}$$

$$\sum_{n=0}^{+\infty}\sum_{l=1}^{2}\left[\alpha_n^l B_n^l(r'_i)\sum_{q=0}^{+\infty}C_q(n,k'_p\gamma')+\gamma_n^l E_n^l(r'_i)\sum_{q=0}^{+\infty}C_q(n,k'_s\gamma')\right]\sin\vartheta P'_q(\cos\vartheta)=0 \quad (30b)$$

$$\sum_{n=0}^{+\infty}\sum_{l=1}^{2}\beta_n^l C_n^l(r'_i)\sum_{q=0}^{+\infty}C_q(n,k'_s\gamma')\sin\vartheta P'_q(\cos\vartheta)=0, \quad i=0,1 \quad (30c)$$

Taking advantage of the orthogonality of $P_q(\cos\vartheta)$ as well as of the derivatives $P'_q(\cos\vartheta)$, with weight function $\sin^2\vartheta$, we obtain the following linear algebraic system

$$\begin{bmatrix} \underline{\underline{D}}_{11} & \underline{\underline{D}}_{12} & \underline{\underline{D}}_{13} & \underline{\underline{D}}_{14} & \cdots & \cdots \\ \underline{\underline{D}}_{21} & \underline{\underline{D}}_{22} & \underline{\underline{D}}_{23} & \cdots & \cdots & \cdots \\ \underline{\underline{D}}_{31} & \underline{\underline{D}}_{32} & \underline{\underline{D}}_{33} & \cdots & \cdots & \cdots \\ \underline{\underline{D}}_{41} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}\boldsymbol{x}=\boldsymbol{0} \quad (31)$$

where

$$\underline{\underline{D}}_{mn}=\begin{bmatrix} A_n^1(r'_1)C_m(n,k'_p\gamma') & A_n^2(r'_1)C_m(n,k'_p\gamma') & 0 & 0 & D_n^1(r'_1)C_m(n,k'_s\gamma') & D_n^2(r'_1)C_m(n,k'_s\gamma') \\ 0 & 0 & C_n^1(r'_1)C_m(n,k'_s\gamma') & C_n^2(r'_1)C_m(n,k'_s\gamma') & 0 & 0 \\ B_n^1(r'_1)C_m(n,k'_p\gamma') & B_n^2(r'_1)C_m(n,k'_p\gamma') & 0 & 0 & E_n^1(r'_1)C_m(n,k'_s\gamma') & E_n^2(r'_1)C_m(n,k'_s\gamma') \\ A_n^1(r'_o)C_m(n,k'_p\gamma') & A_n^2(r'_o)C_m(n,k'_p\gamma') & 0 & 0 & D_n^1(r'_o)C_m(n,k'_s\gamma') & D_n^2(r'_o)C_m(n,k'_s\gamma') \\ 0 & 0 & C_n^1(r'_o)C_m(n,k'_s\gamma') & C_n^2(r'_o)C_m(n,k'_s\gamma') & 0 & 0 \\ B_n^1(r'_o)C_m(n,k'_p\gamma') & B_n^2(r'_o)C_m(n,k'_p\gamma') & 0 & 0 & E_n^1(r'_o)C_m(n,k'_s\gamma') & E_n^2(r'_o)C_m(n,k'_s\gamma') \end{bmatrix}$$

for $n+m$ even,

$\underline{\underline{D}}_{mn}=\boldsymbol{0}$ for $n+m$ odd,

and

$$\boldsymbol{x}=\left[\alpha_1^1,\alpha_1^2,\beta_1^1,\beta_1^2,\gamma_1^1,\gamma_1^2,\alpha_2^1,\alpha_2^2,\ldots,\gamma_2^1,\gamma_2^2,\ldots\right]^T.$$

This system in matrix form is written as

$$\underline{\underline{D}}\,\underline{x}=\underline{0}. \quad (32)$$

In order for the system (32) to have a non-trivial solution the following condition has to be satisfied

$$\det(\underline{\underline{D}})=0. \quad (33)$$

This condition provides the characteristic (frequency) equation, the roots of which are the eigenfrequency coefficients Ω of the system under discussion. The mode shape corresponding to each Ω can be obtained by solving the system (32).

4. NUMERICAL RESULTS - DISCUSSION

The frequency equation (33) has been solved numerically and for this purpose a matrix determinant computation routine was used for different frequency coefficients Ω, along with a bisection method to refine steps close to its roots. In order to determine the eigenvector $\underline{x}$, whose elements can be used for the computation of the corresponding displacement components, the root finding algorithm is followed by an LU decomposition and back-substitution routine.

The computations were made for material properties analogous to those of human skull [2]:

$$E = 1.379x10^9 (N/m^2),\ \ \nu = 0.25,\ \ \rho = 2.0x10^3 (kg/m^3)$$

and the case of $\gamma = 0.0$ corresponds to spherical skull with dimensions

$$r_1 = 0.082 \text{ m},\ \ r_0 = 0.076 \text{ m}.$$

The results obtained by solving the equation (33) for different values of γ' are cited in Tables 1 and 2. The variation of the first three eigenfrequency coefficients, $\Omega_n = \Omega_n(\gamma')$, $n = 1,2,3$ are presented graphically in Figures 1 and 2. We note that the computations have been done by assuming that the volume defined by the surface S_0 is constant as well as the thickness of the skull (or its mass).

From the results obtained we lead to the conclusion that the influence of eccentricity, γ', in the framework of the cranial system is not important on the frequency spectrum of the dry-skull. Especially this influence is increasing for the higher frequencies.

Table 1: Eigenfrequency coefficients, Ω_n, as a function of γ'
(The brain volume and the thickness of the skull are constant)

	$\gamma' = 0.0$	$\gamma' = 0.18488$	$\gamma' = 0.26292$	$\gamma' = 0.50549$	$\gamma' = 0.68183$
1	0.7083	0.7080	0.7075	0.7056	0.7037
2	0.8522	0.8514	0.8505	0.8462	0.8422
3	0.9486	0.9466	0.9447	0.9352	0.9264
4	1.0631	1.0594	1.0555	1.0370	1.0197
5	1.1971	1.1966	1.1961	1.1746	1.1447
6	1.2184	1.2120	1.2056	1.1937	1.1913
7	1.4211	1.4117	1.4021	1.3555	1.3107
8	1.5479	1.5472	1.5466	1.5431	1.5398

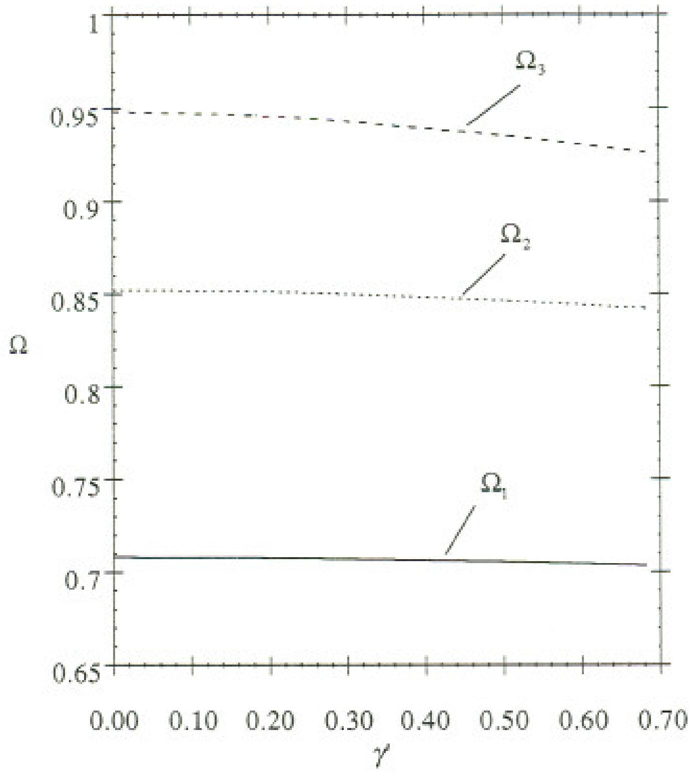

Figure 1: Variation of Ω_n with γ', $\Omega_n = \Omega_n(\gamma')$, $n = 1,2,3$.
(The brain volume and the thickness of the skull are constant).

Table 2: Eigenfrequency coefficients, Ω_n, as a function of γ' (The brain volume and the mass of the skull are constant).

	$\gamma' = 0.0$	$\gamma' = 0.18488$	$\gamma' = 0.26292$	$\gamma' = 0.50549$	$\gamma' = 0.68183$
1	0.7083	0.7081	0.7079	0.7067	0.7052
2	0.8522	0.8517	0.8513	0.8486	0.8454
3	0.9486	0.9476	0.9465	0.9405	0.9335
4	1.0631	1.0611	1.0591	1.0472	1.0336
5	1.1971	1.1968	1.1966	1.1813	1.1686
6	1.2184	1.2150	1.2115	1.1945	1.1932
7	1.4211	1.4159	1.4108	1.3813	1.3467
8	1.5479	1.5476	1.5472	1.5450	1.5425

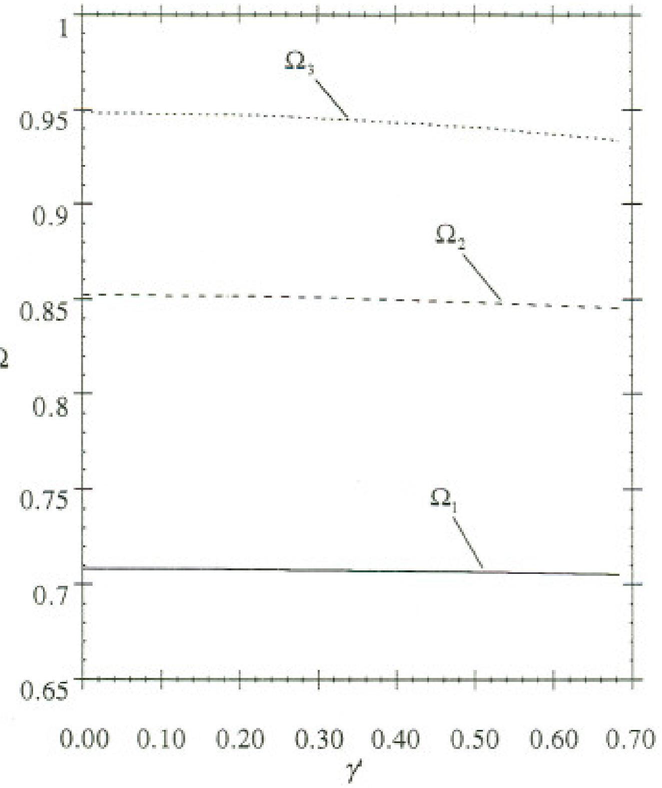

Figure 2: Variation of Ω_n with γ', $\Omega_n = \Omega_n(\gamma')$, $n = 1,2,3$.
(The brain volume and the mass of the skull are constant).

We note that $\gamma' \in [0, 0.68183]$ and $\frac{\gamma'^2}{r'^2} \in [0, 7.0592x10^{-2}]$.

5. CONCLUDING REMARKS

In the present work an attempt was made to reveal the role of the deviation from the spherical geometry to spheroidal one on the eigenfrequencies of the human dry-skull. For this purpose we proposed a mathematical analysis to avoid the introduction of spheroidal coordinates which leads to extremely difficult problem related to the construction of Navier eigenvectors. The proposed analysis is based on complex analysis techniques and the construction of Navier eigenvectors of a simplified problem. The results obtained give us the information that the frequency spectrum of the skull has immaterial influence from the eccentricity γ' and this fact consists of a message for the simplification of geometry when we deal with problems related to the response of the cranial system.

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