On the Dynamic Characteristics of the Human Skull

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Abstract: In this work an attempt is made to study the dynamic characteristics of human dry skull. The analysis is based on the three-dimensional theory of elasticity and the representation of the displacement field in terms of the Navier eigenvectors. The frequency equation was solved numerically and the results obtained are fairly good compared to the experimental ones.

1. INTRODUCTION

Cranial biomechanics has drawn the attention of many researchers as a consequence of the observation that fatalities resulting from accidents involve injury to the head. The information about the physical processes and material response involved is of enormous importance to neurosurgeons and also in the design and construction of protective environments. For the analysis of human body processes there are two types of models in use: physical ones based on experimental analysis and mathematical ones based on theoretical analysis.

The mathematical modelling of the cranial system has received considerable attention in recent years and this is evidenced by the number of publications [1-9,...]. The human skull is a complex structure made up of several bones each with its own unique internal and external geometry. In analytical investigations geometrical approximations and mainly the geometry of spherical and prolate spheroidal shells have been used [2,3,4]. A parametric study of head models by utilising an analytically based numerical technique was presented in Ref. 10. The mechanical properties of cranial bone were studied in Ref. 11. As it is obvious for the collision and head injury analyses we need accurate dynamic response characteristics of the cranial system. In Ref. 12 an experimental investigation was presented to identify the dynamic characteristics of freely vibrating human skulls.

In the present investigation, adopting the approximation of the human skull by hollow sphere, an attempt is made to present an analysis for the vibrational characteristics of the human skull. The mathematical analysis is based on the three-dimensional theory of elasticity and the representation of the displacement field in terms of the Navier eigenvectors [13]. It is noted that an exact study of the dynamic characteristics of a closed spherical shell for arbitrary values of its thickness is desirable not only from the theoretical point of view as well as for the purpose of the human skull dynamic characteristics but also to estimate the range of applicability of the various shell theories [14,15,16]. In the framework of the present analysis the frequency equation was solved numerically and results for the eigenfrequencies and mode shapes of the system considered are presented in tables and graphs. It is important to note that the results obtained are in agreement with the analogous ones by using shell theories [17], in the range of their validity, as well as, with experimental results cited in Ref. 12.

2. PROBLEM FORMULATION

For motions in homogeneous isotropic, elastic solids, the displacement vector field **u** satisfies the equation:

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$$\mu \nabla^2 \boldsymbol{u}(\boldsymbol{r},t) + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u}(\boldsymbol{r},t) = \rho \frac{\partial^2 \boldsymbol{u}(\boldsymbol{r},t)}{\partial t^2}, \quad \boldsymbol{r} \in V \tag{1}$$

where λ, μ are Lamė's constants, ρ is the mass density, ∇ is the usual del operator, and t is the time.

In what follows we shall discuss the free vibrations problem of a hollow sphere in the framework of three-dimensional theory of elasticity (Fig. 1).

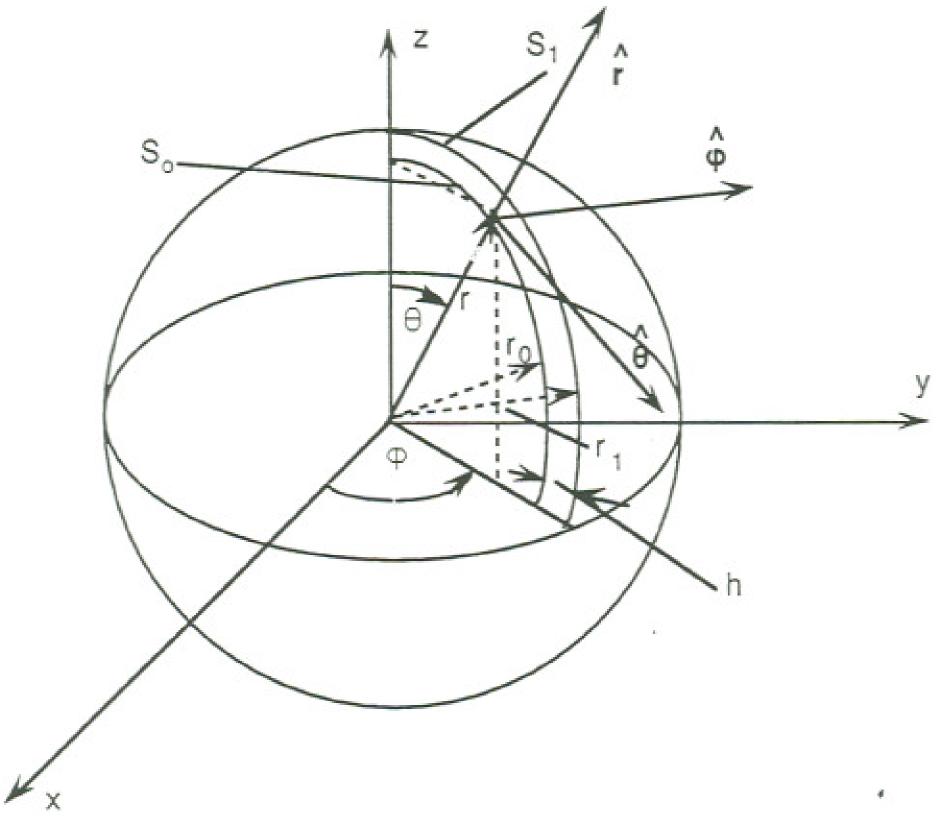

Figure 1: Problem Geometry

For the problem under discussion we assume that:

$$\boldsymbol{u}(\boldsymbol{r},t) = \mathrm{Re}[\boldsymbol{u}_o(\boldsymbol{r})e^{i\omega t}] \tag{2}$$

where ω denotes the angular frequency measured in radians/sec and $i = \sqrt{-1}$.

Replacing (2) into the equation of motion we obtain the well-known Navier frequency equation:

$$\mu \nabla^2 \boldsymbol{u}_o(\boldsymbol{r}) + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u}_o(\boldsymbol{r}) + \rho\omega^2 \boldsymbol{u}_o(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in V_1. \tag{3}$$

Introducing the dimensionless variables

$$r' = \frac{r}{r_1}, \Omega = \frac{\omega r_1}{c_p}$$

into (3) we lead to

$$c'^2_s \nabla'^2 u_o(r') + (1 - c'^2_s)\nabla' \nabla' \cdot u_o(r') + \Omega^2 u_o(r') = 0, \tag{4}$$

$$c'^2_s \nabla'^2 \boldsymbol{u}_o(\boldsymbol{r}') + (1 - c'^2_s)\nabla' \nabla' \cdot \boldsymbol{u}_o(\boldsymbol{r}') + \Omega^2 \boldsymbol{u}_o(\boldsymbol{r}') = 0, \quad (4)$$

where

$$\nabla' = r_1 \nabla, c'_s = \frac{c_s}{c_p}, c_s = (\frac{\mu}{\rho})^{1/2}, c_p = (\frac{\lambda + 2\mu}{\rho})^{1/2}.$$

The interaction of the system considered with its own environment enters the mathematical formulation through the boundary conditions on the surfaces S_0 and S_1(Fig. 1).

We assume that the surfaces S_i, i=0,1 are stress free, that is

$$\begin{aligned} &\boldsymbol{T}\boldsymbol{u}_o(\boldsymbol{r}') = 0, \qquad \boldsymbol{r}' = \boldsymbol{r}'_o \\ &and \\ &\boldsymbol{T}\boldsymbol{u}_o(\boldsymbol{r}') = 0, \qquad \boldsymbol{r}' = \boldsymbol{r}'_1 \end{aligned} \quad (5)$$

where

$$\boldsymbol{T} = 2\mu' \hat{\boldsymbol{r}} \cdot r_1 \nabla' + \lambda' \hat{\boldsymbol{r}} r_1 \nabla' \cdot + \mu' \hat{\boldsymbol{r}} \times r_1 \nabla' \quad (6)$$

stands for the dimensionless surface traction operator in V, $\hat{\boldsymbol{r}}$ is the unit outward normal vector on S_i, i=0,1 and $(\lambda', \mu') = (\lambda/\mu, 1)$.

We note that the problem described by the equation of motion, (4), and the boundary conditions (5) is a well-posed mathematical problem.

3. FREQUENCY EQUATION

For the solution of the problem under discussion we adopted a method based on the representation of the displacement field $\boldsymbol{u}_o(\boldsymbol{r})$ in terms of the Navier eigenvectors. As it is well known, Navier eigenvectors are a result of the Helmholtz decomposition and constitute a complete set of vector functions in the space of solutions of time-independent Navier equation.

The Navier eigenvectors have the following form:

$$\begin{aligned} &L_n^{m,l}(r') = \dot{g}_n^l(\Omega r') P_n^m(\hat{r}) + \sqrt{n(n+1)} \frac{g_n^l(\Omega r')}{\Omega r'} B_n^m(\hat{r}) \\ &M_n^{m,l}(r') = \sqrt{n(n+1)} g_n^l(k'_s r') C_n^m(\hat{r}) \\ &N_n^{m,l}(r') = n(n+1) \frac{g_n^l(k'_s r')}{k'_s r'} P_n^m(\hat{r}) + \sqrt{n(n+1)} \left\{ \dot{g}_n^l(k'_s r') + \frac{g_n^l(k'_s r')}{k'_s r'} \right\} B_n^m(\hat{r}) \\ &n = 0,1,2,......, \quad l = 1,2, \quad m = -n, -n+1, ..., n-1, n, \quad r \in V \end{aligned} \quad (7)$$

where

$$k'_s = \frac{\Omega}{c'_s}, \qquad \dot{g}_n^l(z) = \frac{d}{dz}(g_n^l) \quad (8)$$

and $g_n^1(z)$ and $g_n^2(z)$ represent the spherical Bessel functions of the first, $j_n(z)$, and second kind, $y_n(z)$, respectively.

The functions $P_n^m(\hat{r})$, $B_n^m(\hat{r})$ and $C_n^m(\hat{r})$ defined on the unit sphere, are the vector spherical harmonics introduced by Hansen in spherical polar coordinates (r, ϑ, φ), and are given as follows

$$B_n^m(\hat{r}) = \frac{1}{\sqrt{n(n+1)}}\left\{\hat{\vartheta}\frac{\partial}{\partial\vartheta} + \hat{\varphi}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right\}Y_n^m(\hat{r}) \tag{10}$$

$$C_n^m(\hat{r}) = \frac{1}{\sqrt{n(n+1)}}\left\{\hat{\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \hat{\varphi}\frac{\partial}{\partial\vartheta}\right\}Y_n^m(\hat{r}) \tag{11}$$

where $\hat{\vartheta}$ and $\hat{\varphi}$ are the unit vectors in ϑ and φ - directions respectively, $Y_n^m(\hat{r}) = P_n^m(\cos\vartheta)e^{im\varphi}$ are the spherical harmonics and $P_n^m(\cos\vartheta)$ are the well known Legendre functions.

We assume that the displacement vector field $\boldsymbol{u}_o(\boldsymbol{r})$ has the representation

$$\boldsymbol{u}_o(\boldsymbol{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{l=1}^{2}\left\{\alpha_n^{m,l}\boldsymbol{L}_n^{m,l}(\boldsymbol{r}) + \beta_n^{m,l}\boldsymbol{M}_n^{m,l}(\boldsymbol{r}) + \gamma_n^{m,l}\boldsymbol{N}_n^{m,l}(\boldsymbol{r})\right\}. \tag{12}$$

The spherical polar coordinates of the real part of $\boldsymbol{u}_o(\boldsymbol{r})$ are given as

$$u_{or} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{l=1}^{2}\left\{\begin{array}{l}\alpha_n^l\dot{g}_n^l(k'_p, r')\cos(m\varphi)P_n^m(\cos\vartheta) + \\ \gamma_n^l(n(n+1))\dfrac{g_n^l(k'_s r')}{k'_s r'}\cos(m\varphi)P_n^m(\cos\vartheta)\end{array}\right\}$$

$$u_{o\vartheta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{l=1}^{2}\left\{\begin{array}{l}\alpha_n^l\dfrac{g_n^l(\Omega r')}{\Omega r'}\cos(m\varphi)\dfrac{\partial}{\partial\vartheta}P_n^m(\cos\vartheta) \\ -\beta_n^l g_n^l(k'_s r')\dfrac{m\sin(m\varphi)}{\sin\vartheta}P_n^m(\cos\vartheta) \\ +\gamma_n^l(\dot{g}_n^l(k'_s r') + \dfrac{g_n^l(k'_s r')}{k'_s r'})\cos(m\varphi)\dfrac{\partial}{\partial\vartheta}P_n^m(\cos\vartheta)\end{array}\right\} \tag{12a}$$

$$u_{o\varphi} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{l=1}^{2}\left\{\begin{array}{l}-\alpha_n^l\dfrac{g_n^l(\Omega r')}{\Omega r'}\dfrac{m\sin(m\varphi)}{\sin\vartheta}P_n^m(\cos\vartheta) \\ -\beta_n^l g_n^l(k'_s r')\cos(m\varphi)\dfrac{\partial}{\partial\vartheta}P_n^m(\cos\vartheta) \\ -\gamma_n^l(\dot{g}_n^l(k'_s r') + \dfrac{g_n^l(k'_s r')}{k'_s r'})\dfrac{m\sin(m\varphi)}{\sin\vartheta}P_n^m(\cos\vartheta)\end{array}\right\}.$$

The problem now is reduced to the determination of the coefficients $\alpha_n^{m,l}$, $\beta_n^{m,l}$ and $\gamma_n^{m,l}$. Since the expression (12) satisfies the equation of motion, (4), it remains to ask the boundary conditions (5) to be satisfied. We note that the presence of the surface traction operator T in the boundary conditions (5) requires to know how this operator acts on the Navier eigenvectors. After tedious and extended manipulations we obtained the following relations

$$\begin{aligned}\boldsymbol{T}\boldsymbol{L}_n^{m,l}(\hat{r}) = &-\left[\frac{4\mu'}{r'}\dot{g}_n^l(\Omega r') + 2\mu'\Omega(1 - \frac{n(n+1)}{\Omega^2 r'^2})g_n^l(\Omega r') + \lambda'\Omega g_n^l(\Omega r')\right]\boldsymbol{P}_n^m(\hat{r}) \\ &+2\mu'\sqrt{n(n+1)}\left[\frac{\dot{g}_n^l(\Omega r')}{r'} - \frac{g_n^l(\Omega r')}{\Omega r'^2}\right]\boldsymbol{B}_n^m(\hat{r})\end{aligned} \tag{13}$$

$$TM_n^{m,l}(\hat{r}) = \mu' \sqrt{n(n+1)} \left[k'_s \dot{g}_n^l(k'_s r') - \frac{1}{r'} g_n^l(k'_s r') \right] C_n^m(\hat{r}) \tag{14}$$

$$\begin{aligned} TN_n^{m,l}(\hat{r}) &= 2\mu' n(n+1) \left[\frac{\dot{g}_n^l(k'_s r')}{r'} - \frac{g_n^l(k'_s r')}{k'_s r'^2} \right] P_n^m(\hat{r}) \\ &+ \mu' \sqrt{n(n+1)} \left[-2\frac{\dot{g}_n^l(k'_s r')}{r'} - k'_s g_n^l(k'_s r') + 2\frac{n(n+1)-1}{k'_s r'^2} g_n^l(k'_s r') \right] B_n^m(\hat{r}). \end{aligned} \tag{15}$$

Inserting (12) into the expression for the boundary conditions (5), and taking into account the relations (13-15) as well as the advantage of the orthogonality arguments for the vector spherical harmonic functions we conclude that for every specific pair of integers (n,m) (with $|m| \leq n$) the 6-coefficients involved in the expression (12) satisfy a linear algebraic homogeneous system with six equations.

This system in matrix form is given as

$$\begin{bmatrix} A_n^1(r'_1) & A_n^2(r'_1) & 0 & 0 & D_n^1(r'_1) & D_n^2(r'_1) \\ 0 & 0 & C_n^1(r'_1) & C_n^2(r'_1) & 0 & 0 \\ B_n^1(r'_1) & B_n^2(r'_1) & 0 & 0 & E_n^1(r'_1) & E_n^2(r'_1) \\ A_n^1(r'_o) & A_n^2(r'_o) & 0 & 0 & D_n^1(r'_o) & D_n^2(r'_o) \\ 0 & 0 & C_n^1(r'_o) & C_n^2(r'_o) & 0 & 0 \\ B_n^1(r'_o) & B_n^2(r'_o) & 0 & 0 & E_n^1(r'_o) & E_n^2(r'_o) \end{bmatrix} \begin{bmatrix} \alpha_n^{m,1} \\ \alpha_n^{m,2} \\ \beta_n^{m,1} \\ \beta_n^{m,2} \\ \gamma_n^{m,1} \\ \gamma_n^{m,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

or in compact form

$$\underline{\underline{D}}\,\underline{x} = \underline{0} \tag{17}$$

where

$$A_n^l(r') = -\left[\frac{4}{r'} \dot{g}_n^l(\Omega r') + 2\Omega\left(1 - \frac{n(n+1)}{\Omega^2 r'^2}\right) g_n^l(\Omega r') + \lambda' \Omega g_n^l(\Omega r') \right]$$

$$B_n^l(r') = 2\sqrt{n(n+1)} \left[\frac{1}{r'} \dot{g}_n^l(\Omega r') - \frac{g_n^l(\Omega r')}{\Omega r'^2} \right]$$

$$C_n^l(r') = \sqrt{n(n+1)} \left[k'_s \dot{g}_n^l(k'_s r') - \frac{1}{r'} g_n^l(k'_s r') \right]$$

$$D_n^l(r') = 2n(n+1) \left[\frac{\dot{g}_n^l(k'_s r')}{r'} - \frac{g_n^l(k'_s r')}{k'_s r'^2} \right]$$

$$E_n^l(r') = \sqrt{n(n+1)} \left[-2\frac{\dot{g}_n^l(k'_s r')}{r'} - k'_s g_n^l(k'_s r') + 2\frac{n(n+1)-1}{k'_s r'^2} g_n^l(k'_s r') \right]$$

In order for the system (17) to have a non-trivial solution the following condition has to be satisfied

$$\det(\underline{\underline{D}}) = 0. \tag{18}$$

This condition provides the characteristic (frequency) equation, the roots of which are the eigenfrequency coefficients Ω of the system under discussion.

4. NUMERICAL RESULTS - DISCUSSION

The frequency equation (18) has been solved numerically and for this purpose a matrix determinant computation routine was used for different frequency coefficients Ω, along with a bisection method to refine steps close to its roots. The root finding algorithm is followed by an LU decomposition and back-substitution routine to determine the eigenvector $\underline{x}$ whose elements are used for the computation of the corresponding displacement components. Given that for every particular n we get independent systems, we deduce that finally the eigenvectors are obtained by (12a) without the external summation over n.

The computations were made for material properties analogous to those of human skull [11]:

$$E = 1.379 \times 10^9 \ (N/m^2),\ \nu = 0.20\text{-}0.35,\ \rho = 2.0 \times 10^3 \ (kg/m^3),$$
$$r_1 = 0.040\text{-}0.082 \ m,\ r_o = 0.036\text{-}0.08199 \ m.$$

In Table 1 are cited the first three roots of (18) for different n (n order of Bessel function). From these results we observe that we have degeneracy, that is the frequency $\omega = 1.8924 c_p / r_1$ corresponds to two different mode shapes which are shown in Figure 1 for n=1 and n=3, respectively.

Table 1 : Eigenfrequency coefficients Ω_n for r_1=0.082 m, r_o=0.076 m and n=0,1, ...,8

	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8
1	0	0							
2			0.7083						
3				0.8522					
4					0.9486				
5						1.063			
6			1.1971						
7							1.2184		
8								1.4210	
9	1.5479								
10									1.6695
11		1.8924		1.8924					
12					2.5324				
13			2.6253						
14						3.1672			
15				3.5229					
16							3.7855		
17								4.3983	
18					4.4651				
19									5.0077
20						5.4218			
21							6.3843		
22								7.3490	
23									8.3143

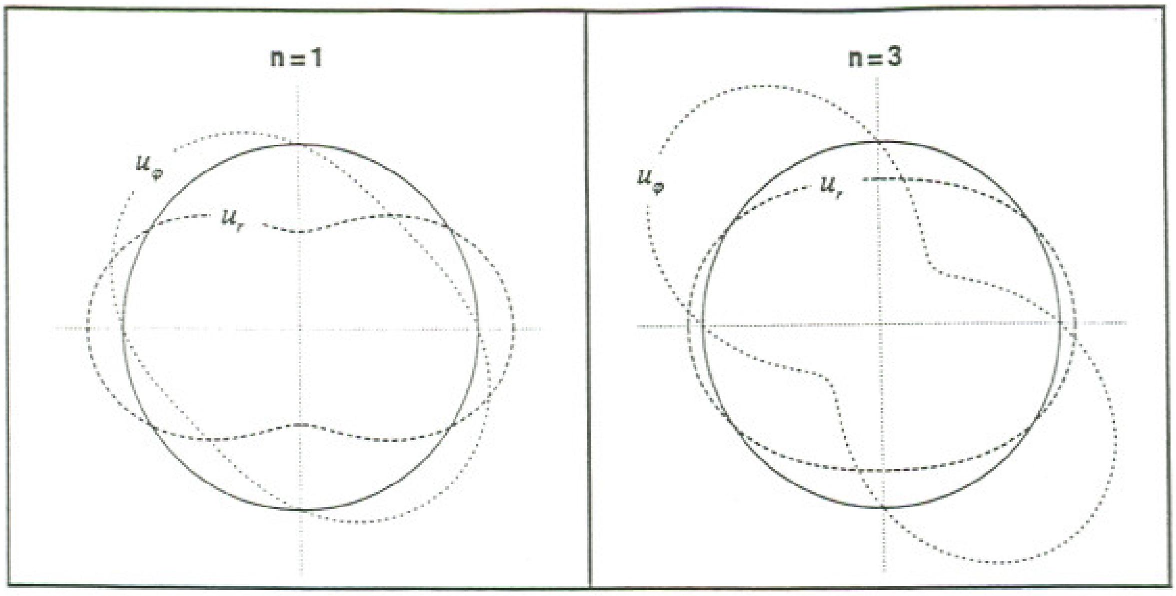

Figure 1: u_r and u_φ for Ω=1.8924, $\vartheta = \pi/2$ and n=1,3

As it is obvious the dynamic characteristics of the human skull are influenced by its geometry as well as from its own material properties. The variation of $f = \omega/2\pi$ (in Hz) with $(r_1 - r_o)/r_1$ = const. is given as

$$f = \delta_m / r_1$$

where the first nine values of δ_m are cited in Table 2.

Table 2: Values of δ_m for E=1.379x10^{9}N/m^2, v=0.25, ρ=2.0X10^3 Kg/m^3, $(r_1 - r_o)/r_1$ = 0.0732.

m	1	2	3	4	5	6	7	8	9
δ_m	102.56	121.46	134.80	153.92	173.32	173.88	204.88	224.08	240.48

The variations of min Ω_n, n=1,2,3 with h/r_1, $h = r_1 - r_o$, and Poisson's Ratio v are shown in Figures 2 and 3 respectively.

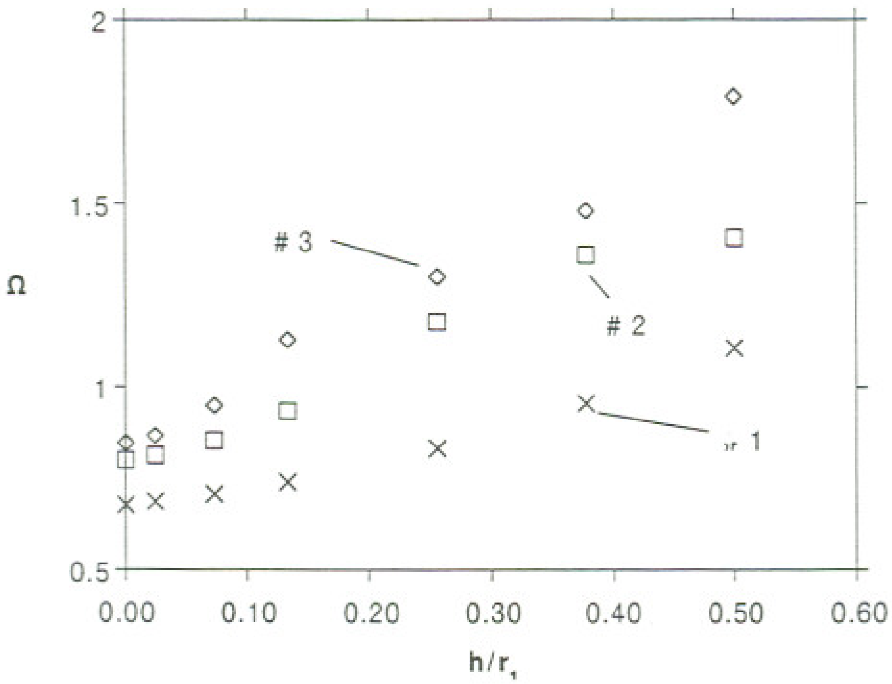

Figure 2: Variation of Ω with h/r_1.

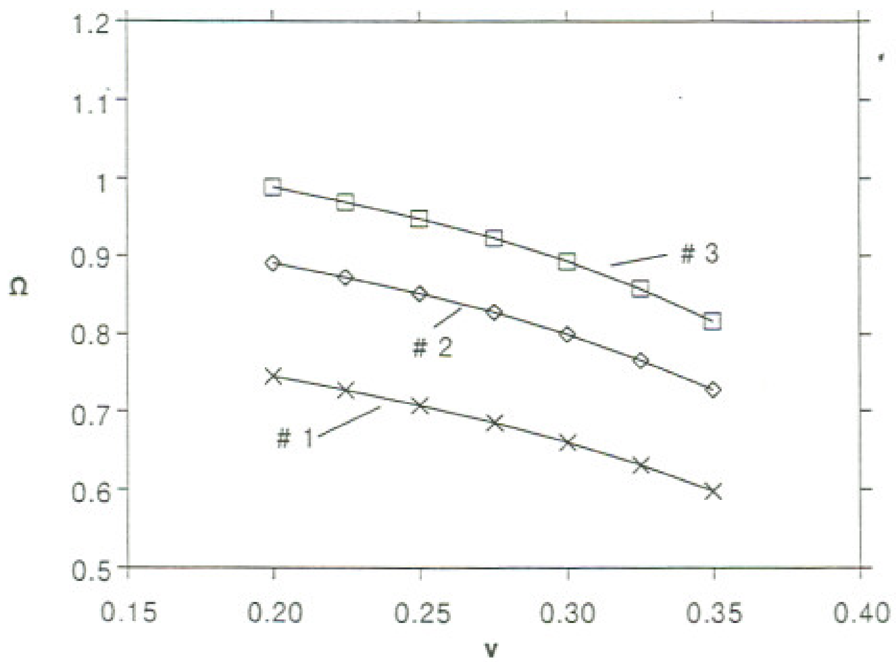

Figure 3: Variation of Ω with v.

From the literature [15] it is known that for $h/r_1 \leq 0.10$ the improved theory for spherical shells [16] is quite satisfactory for thin shells. Beyond this the coupling effects with thickness stretch mode become important and any shell theory should include this effect for better results. For the sake of comparison we present in Table 3 the results of our analysis which testify the previous conclusions.

Table 3: Eigenfrequency coefficients Ω_n for n=0,1,...,8
h/r_1 =0.10, ν=0.30

n	Present Analysis	Shell Theory	
		Improved [17]	Classical Theory
0	0	0	0
1	0	0	0
2	0.7072	0.707	0.708
3	0.8683	0.870	0.873
4	0.9992	1.004	1.014
5	1.1687	1.174	1.202
6	1.3953	1.399	1.462
7	1.6791	1.682	1.798
8	2.0134	2.014	2.208

The authors of Ref. 12 introduced an experimental method to identify the dynamic characteristics (resonant frequencies and mode shapes) of the human dry skull. For the purpose of comparison with the results of the present analysis we selected two cases of Ref. 12 which are referred to as skull I and skull II. Skull I and its characteristics correspond to those of a 50th percentile adult male, while those of skull II correspond to a 5th percentile adult female. The equivalence of geometry used in our computations is as follows:

Skull I: r_0=0,0691m, r_1=0.07735m
Skull II: r_0=0.0602 m, r_1=0.06845 m

The results obtained as well as the analogous experimental ones of Ref. 12 are presented in Table 4 and graphically in Figure 4.

Table 4: Comparison with Experimental Results

	Skull I		Skull II	
	Experiment[12]	Present Analysis	Experiment[12]	Present Analysis
1	1385	1331 (n=2)	1641	1525 (n=2)
2	1786	1649 (n=3)	2344	1906 (n=3)
3	1903	1911 (n=4)	2969	2265 (n=4)
4	2449	2248 (n=2,5)	3477	2561 (n=2)
5	2857	2772 (n=6)	4453	2752 (n=5)
6	3386	2922 (n=0)	5000	3323 (n=0)
7	3523	3371 (n=6)		3386 (n=6)
8	3845	3577 (n=1,3)		4063 (n=1,3)
9	4069	4063 (n=8)		4157 (n=7)
10	4245	4812 (n=4)		5015 (n=8)
11	4636	4962 (n=2)		5459 (n=4)
12		5911 (n=5)		5628 (n=2)
13		6655 (n=3)		6812 (n=5)
14		7170 (n=6)		

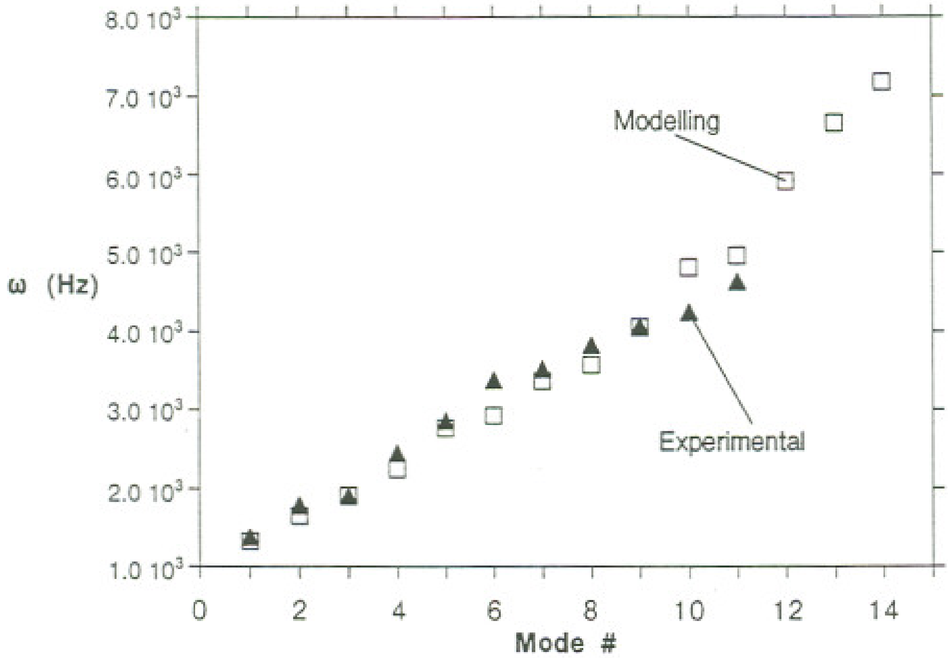

Figure 4: Skull I: Comparison with Experimental Results

For the case of skull I the results obtained are in good agreement with the experimental ones. The first eigenfrequency predicted by our analysis, f_1=1.331 Hz a shifting about 4 % with respect to the experimental one f_{1exp}=1.385 Hz appears, while in f_{11}=4962 Hz a shifting about 6.5 % in the opposite direction from the experimental one, f_{11exp}=4636 Hz. In the case of skull II the measured frequencies are predicted by the present analysis. However, in the interval [1525, 5015] more eigenfrequencies exist than the presented in Ref. 12. This fact needs more study from experimental and theoretical point of view.

In Figure 5 mode shapes ($u_r, u_\varphi, u_\vartheta$) are presented corresponding to the first five eigenfrequencies given in Table 1 (1: n=2, Ω= 0.7083, 2: n=3, Ω= 0.8522, 3: n=4, Ω=0.9486, 4: n=5, Ω=1.063, 5: n=2, Ω= 1.1971).

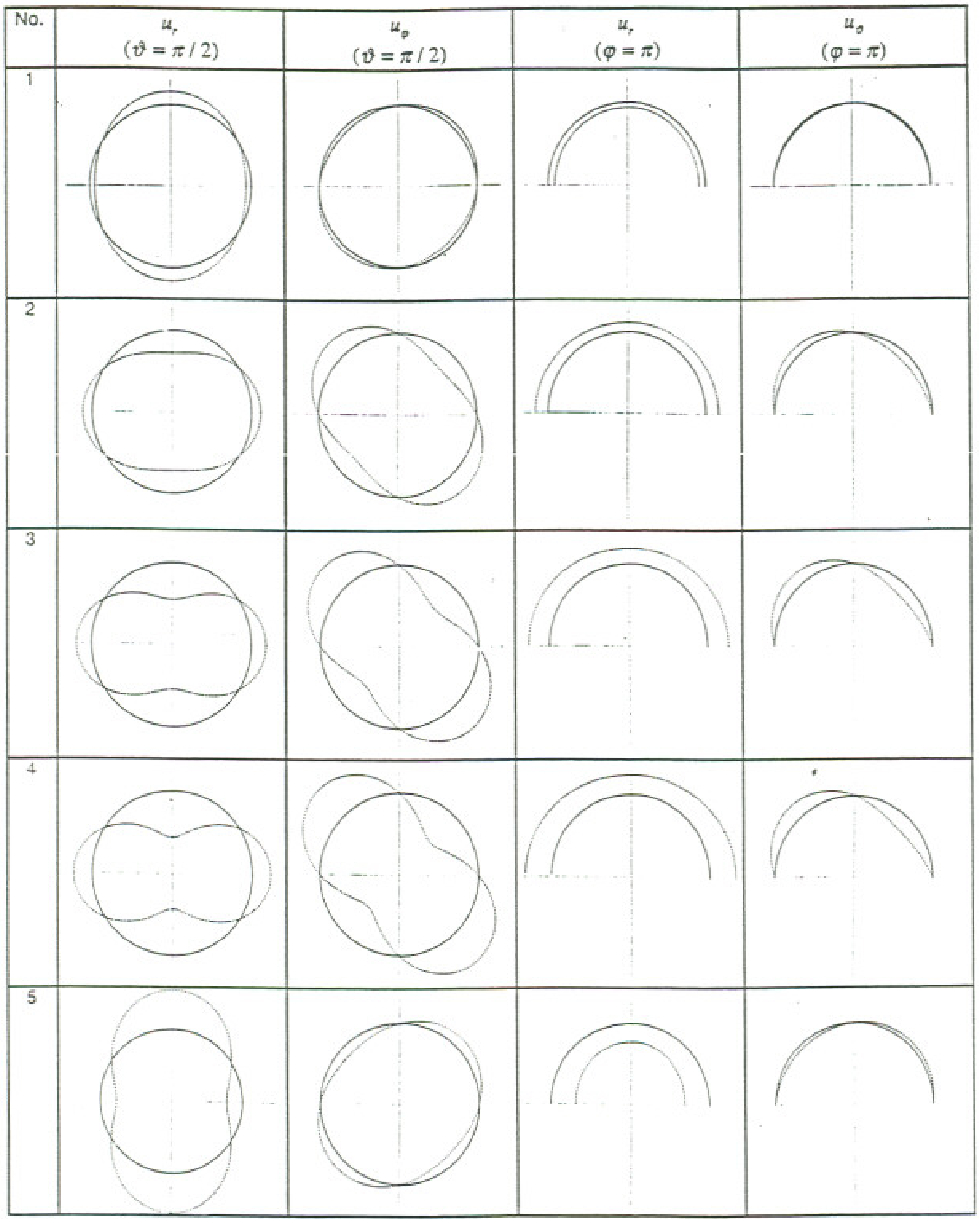

Figure 5: Mode Shapes ($u_r, u_\varphi, u_\vartheta$)

5. CONCLUDING REMARKS

In the analysis presented we dealt with the dynamic characteristics of the human dry skull. The results obtained are favourable compared with the experimental ones and this fact leads us to the conclusion that the geometry considered as well as the analysis adopted (three-dimensional theory of elasticity) could be applied to the study of the cranial system. The results of the dynamic analysis of the human skull-brain system, as well as those of head-neck system are in preparation and will be presented in the future.

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