# DATA MINING
# LINK ANALYSIS RANKING

PageRank – Random walks

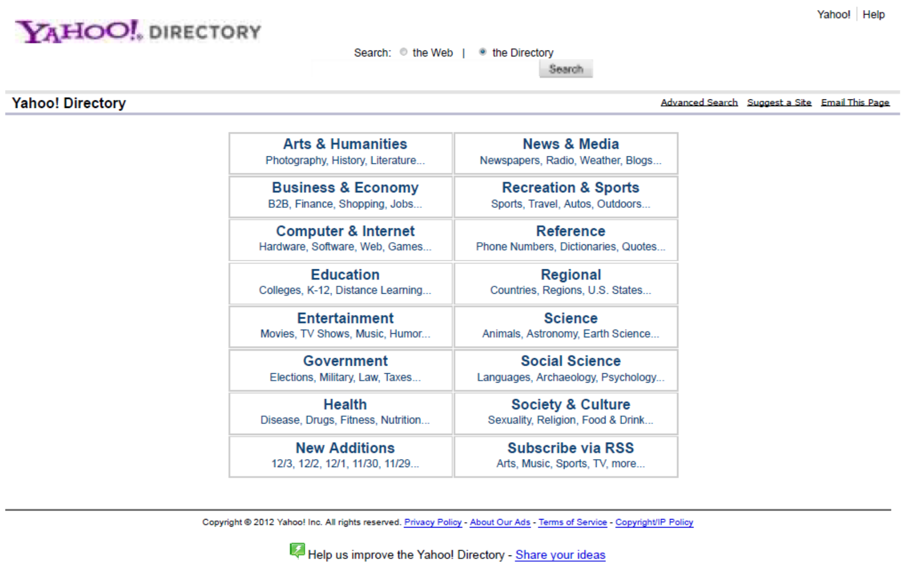
HITS

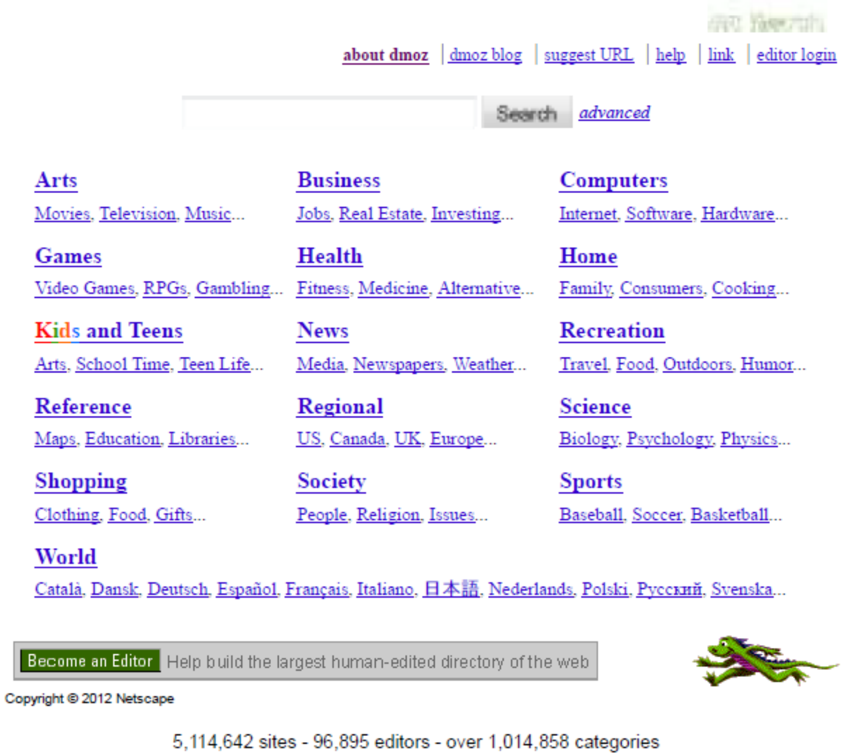
# Network Science

- A number of complex systems can be modeled as networks (graphs).
  - The Web
  - (Online) Social Networks
  - Biological systems
  - Communication networks (internet, email)
  - The Economy
- We cannot truly understand such complex systems unless we understand the underlying network.
  - Everything is connected, studying individual entities gives only a partial view of a system
- Data mining for networks is a very popular area
  - Application to the Web is one of the success stories for network data mining.

# A case study: Searching the web

- First try: Manually curated Web Directories



# A case study: Searching the web

- Second try: Web Search
  - Information Retrieval investigates:
    - Find relevant docs in a small and trusted set e.g., Newspaper articles, Patents, etc. (“needle-in-a-haystack”)
    - Limitation of keywords (synonyms, polysemy, etc)
  - But: Web is huge, full of untrusted documents, random things, web spam, etc.
    - Everyone can create a web page of high production value
    - Rich diversity of people issuing queries
    - Dynamic and constantly-changing nature of web content

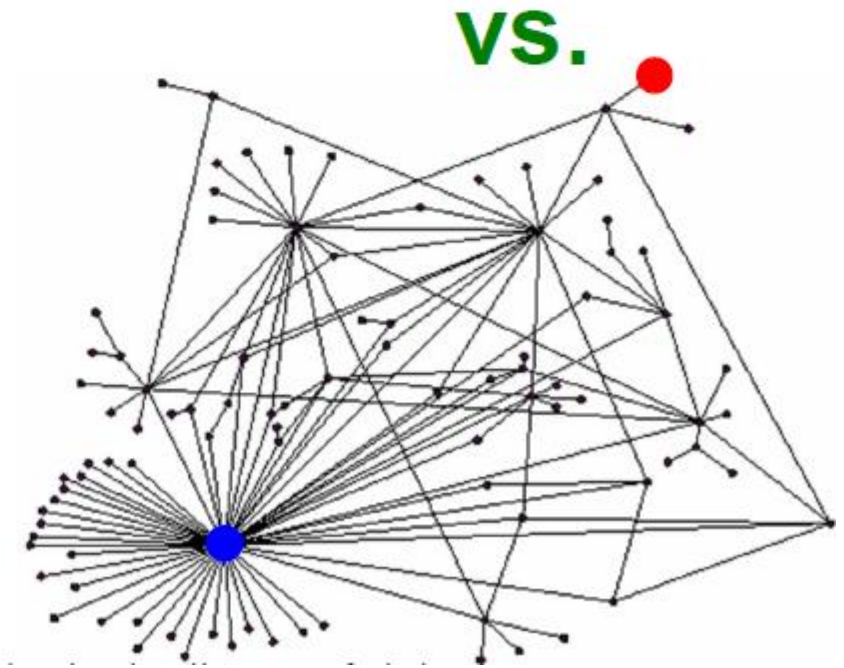
# A case study: Searching the web

- Third try (the Google era): using the web graph
  - Sift from relevance to authoritativeness
  - It is not only important that a page is relevant, but that it is also important on the web

- For example, what kind of results would we like to get for the query “game of thrones”?

# Link Analysis Ranking

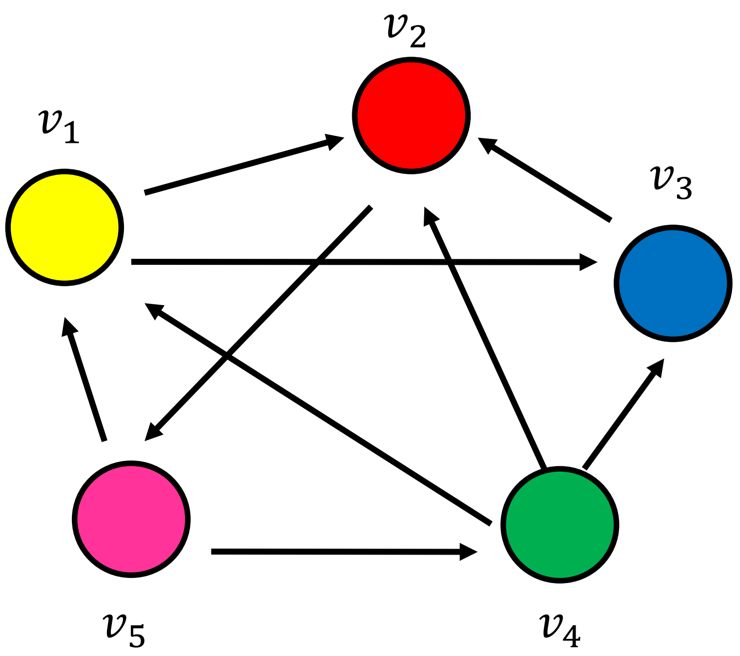
- Use the graph structure to determine the relative importance of the nodes
  - Applications: Ranking on graphs (Web, Twitter, FB, etc)
- Intuition: An edge from node p to node q denotes endorsement
  - Node p endorses/recommends/confirms the authority/centrality/importance of node q
  - Use the graph of recommendations to assign an authority value to every node




What is the simplest way to measure importance of a page on the web?

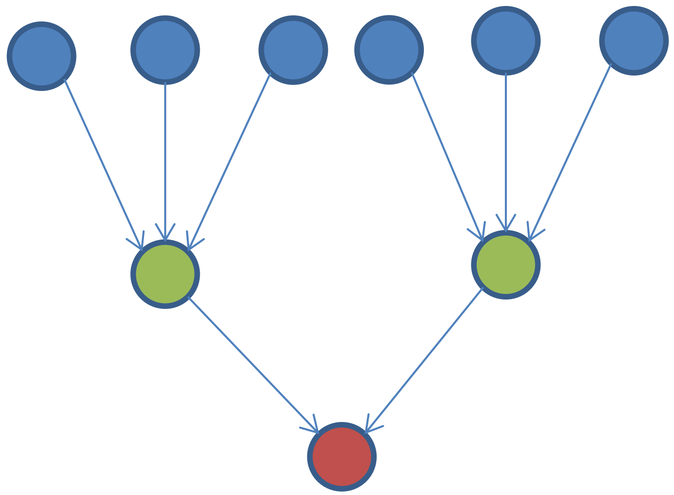
# Rank by Popularity

- Rank pages according to the number of incoming edges (in-degree, degree centrality)




1. Red Page
2. Yellow Page
3. Blue Page
4. Purple Page
5. Green Page

# Popularity



- It is not important only how many link to you, but how important are the people that link to you.
- Good authorities are pointed by good authorities
  - Recursive definition of importance

# PAGERANK

# PageRank

- Good authorities should be pointed by good authorities
  - The value of a node is the value of the nodes that point to it.
- How do we implement that?
  - Assume that we have a unit of authority to distribute to all nodes.
  - Node $i$ gets a fraction $w_i$ of that authority weight
  - Each node distributes the authority value they have to their neighbors
  - The authority value of each node is the sum of the authority fractions it collects from its neighbors.

$$w_i = \sum_{j \to i} \frac{1}{|N_{out}(j)|} w_j$$

Recursive definition

# Example

$$w_i = \sum_{j \to i} \frac{1}{|N_{out}(j)|} w_j$$

$w_1 = 1/3\ w_4 + 1/2\ w_5$

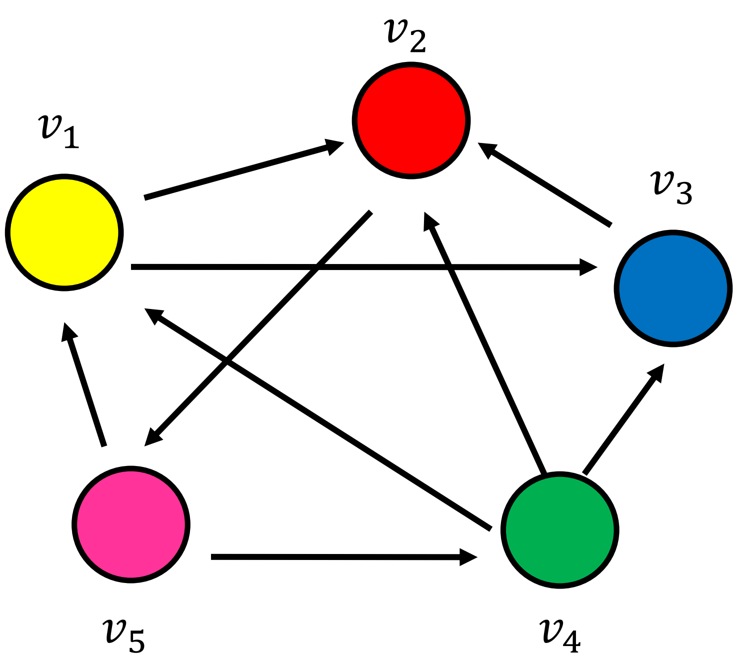
$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$

$w_1 + w_2 + w_3 + w_4 + w_5 = 1$




We can obtain the weights by solving this system of equations

# Computing PageRank weights

- A simpler way to compute the weights is by iteratively updating the weights using the equations
- PageRank Algorithm

Initialize all PageRank weights to $w_i^0 = \frac{1}{n}$

Repeat:

$$w_i^t = \sum_{j \to i} \frac{1}{|N_{out}(j)|} w_j^{t-1}$$

Until the weights do not change

- This process converges

# Example

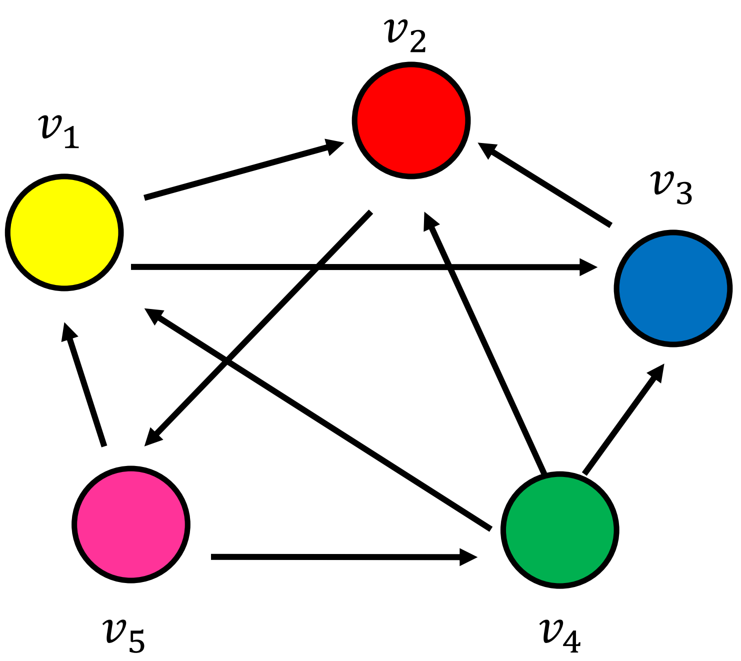
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$




| | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ |
|---|---|---|---|---|---|
| t=0 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| t=1 | 0.16 | 0.36 | 0.16 | 0.1 | 0.2 |
| t=2 | 0.13 | 0.28 | 0.11 | 0.1 | 0.36 |
| t=3 | 0.22 | 0.22 | 0.1 | 0.18 | 0.28 |
| t=4 | 0.2 | 0.27 | 0.17 | 0.14 | 0.22 |

Think of the weight as a fluid: there is constant amount of it in the graph, but it moves around until it stabilizes

# Example

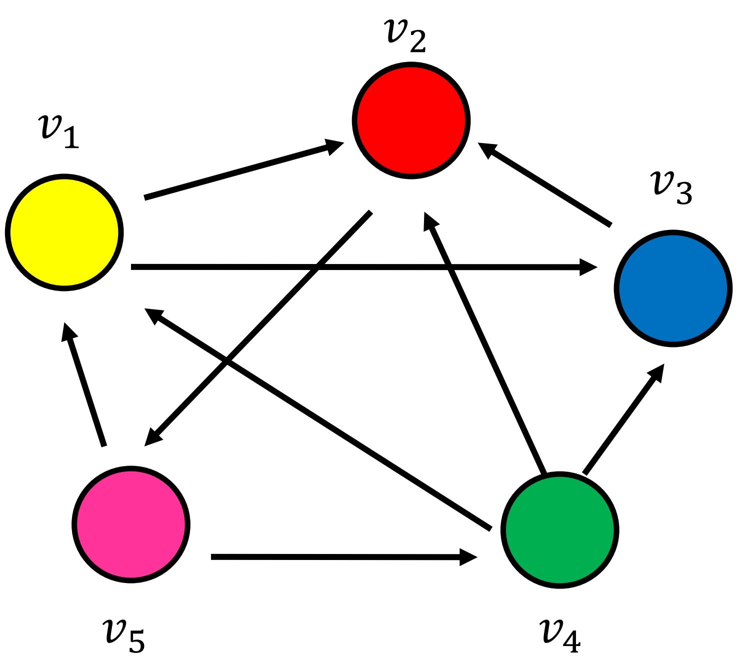
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



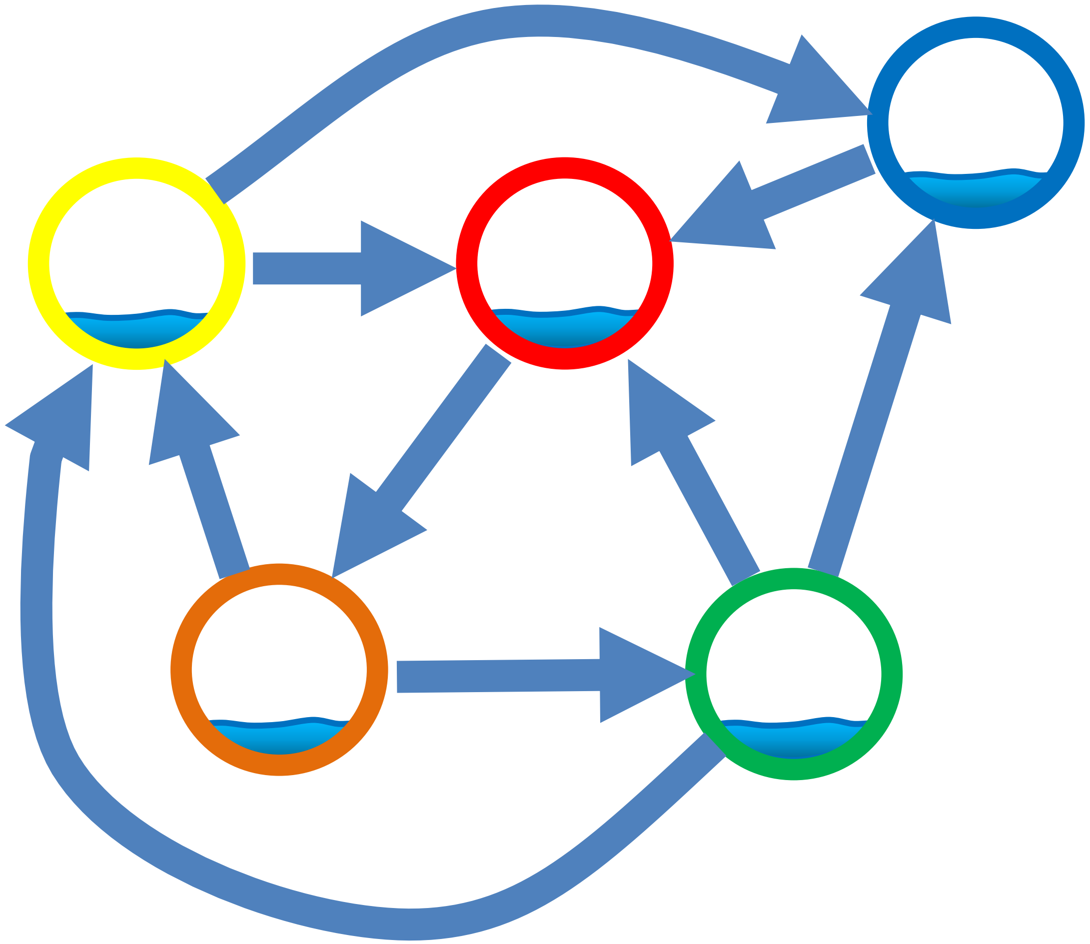

| | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ |
|---|---|---|---|---|---|
| t=25 | 0.18 | 0.27 | 0.13 | 0.13 | 0.27 |

Think of the weight as a fluid: there is constant amount of it in the graph, but it moves around until it stabilizes

# The PageRank algorithm

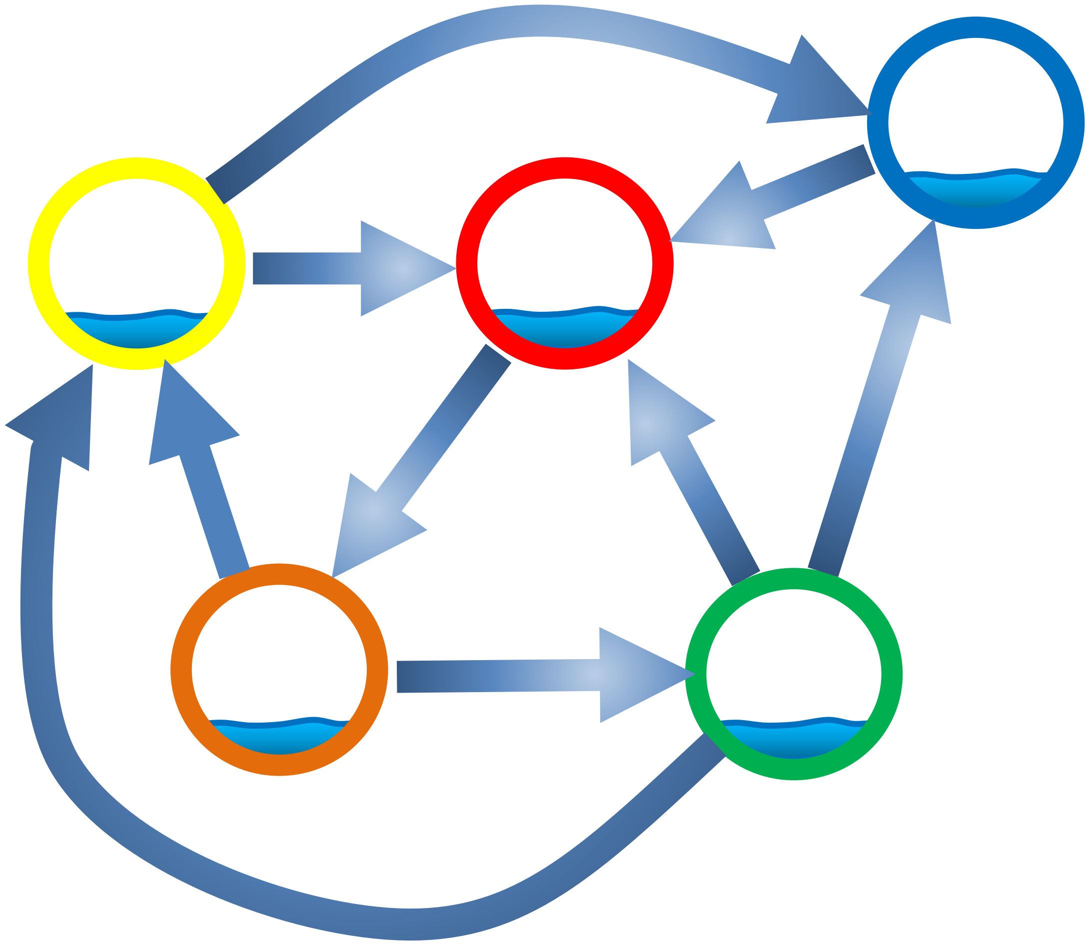
Think of the nodes in the graph as containers of capacity of 1 liter.

We distribute a liter of liquid equally to all containers



# The PageRank algorithm

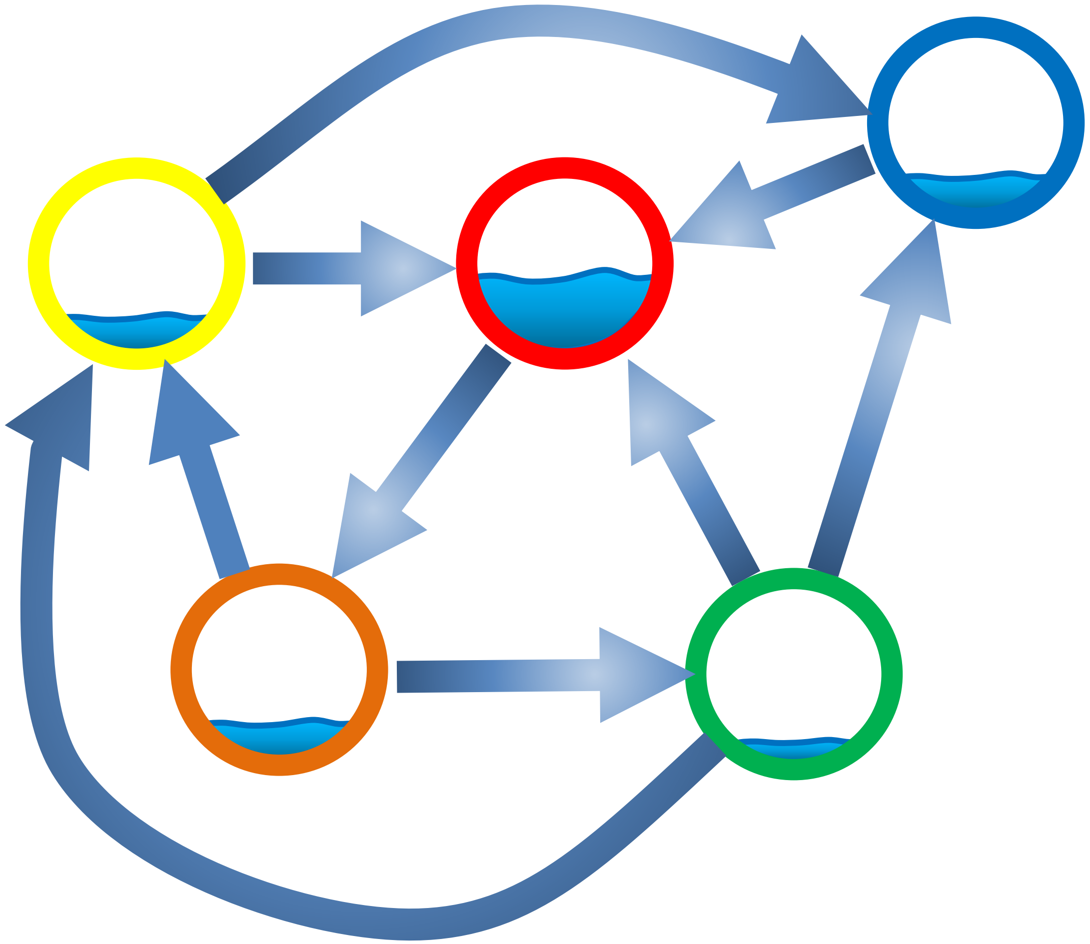
The edges act like pipes that transfer liquid between nodes.



# The PageRank algorithm

The edges act like pipes that transfer liquid between nodes.

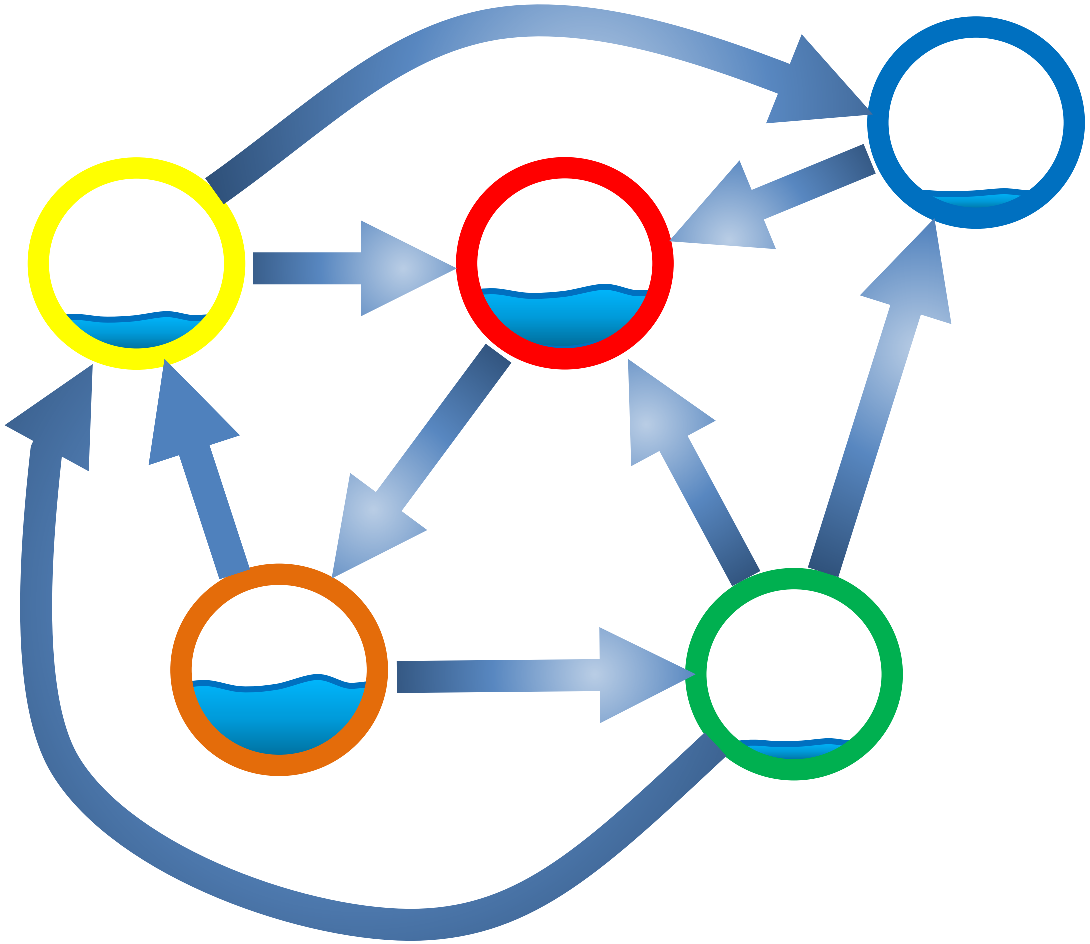
The contents of each node are distributed to its neighbors.



# The PageRank algorithm

The edges act like pipes that transfer liquid between nodes.

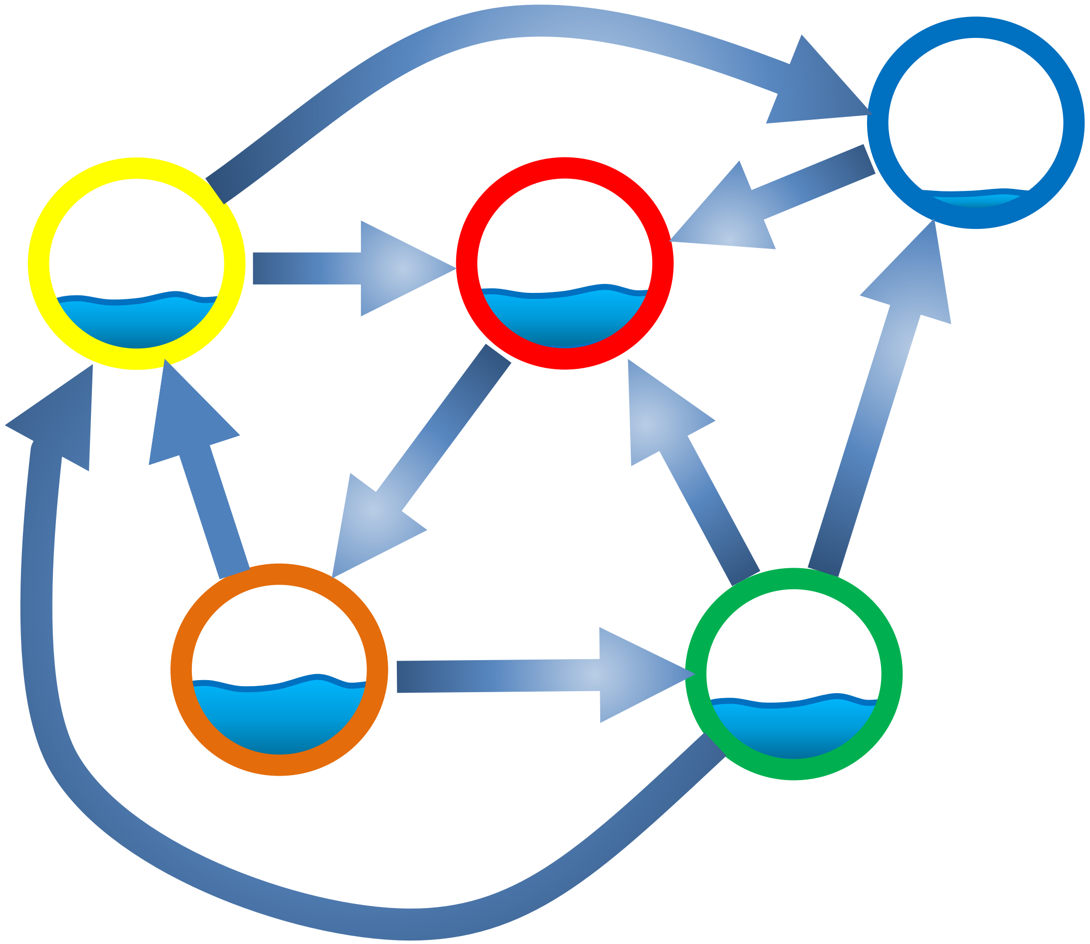
The contents of each node are distributed to its neighbors.



# The PageRank algorithm

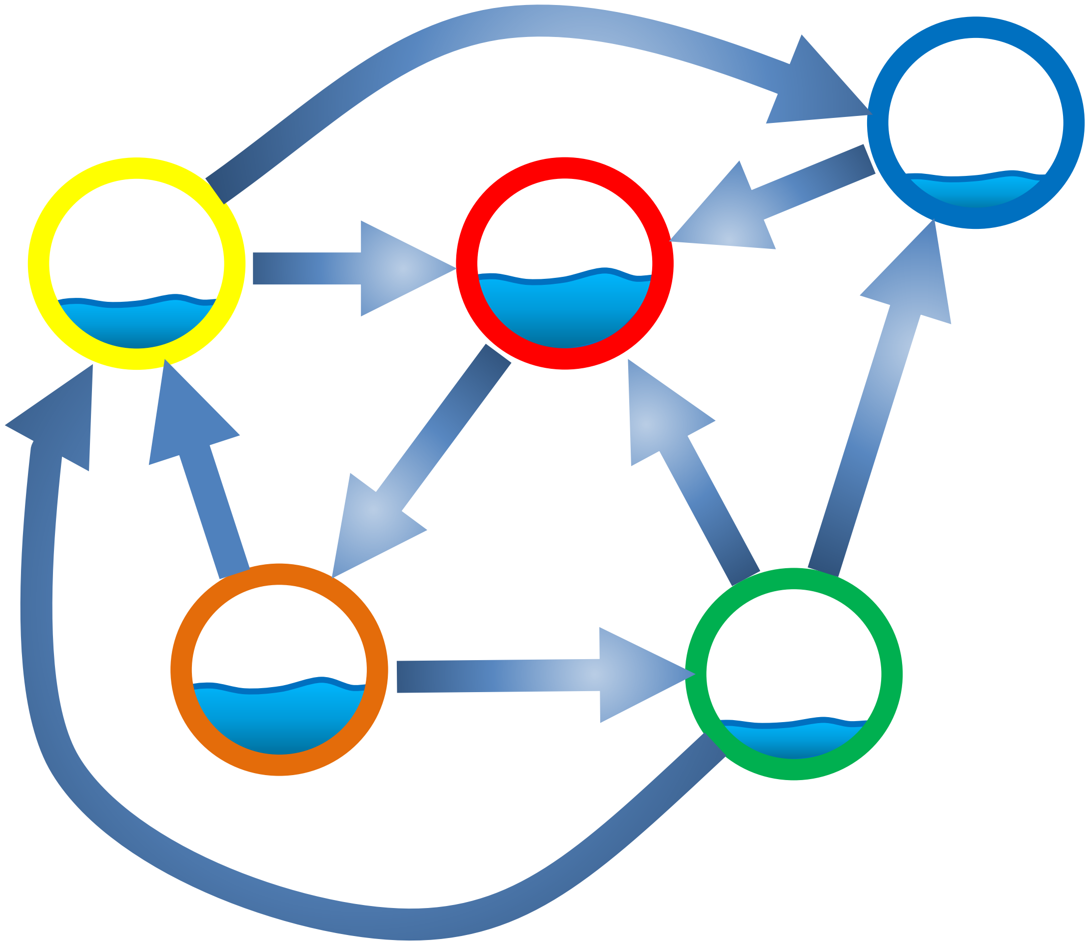
The edges act like pipes that transfer liquid between nodes.

The contents of each node are distributed to its neighbors.



# The PageRank algorithm

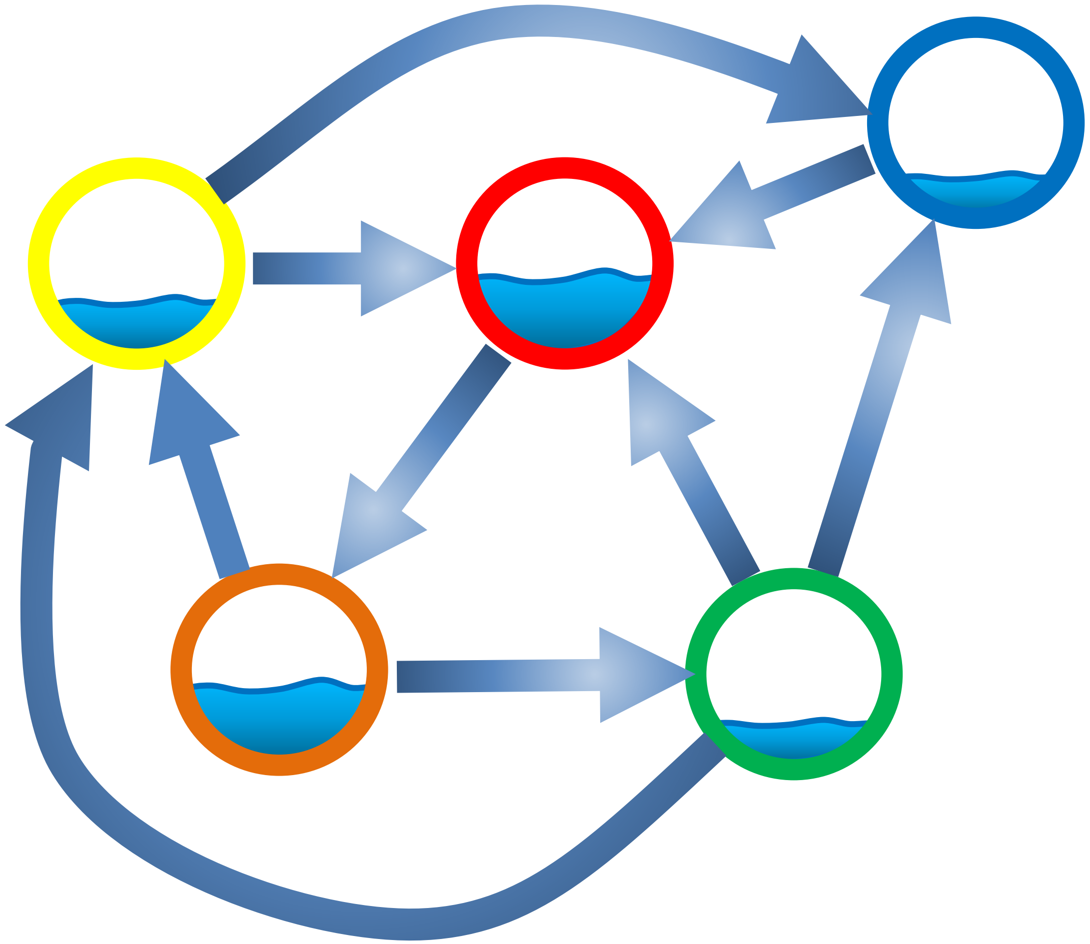
The system will reach an equilibrium state where the amount of liquid in each node remains constant.



# The PageRank algorithm

The amount of liquid in each node determines the importance of the node.

Large quantity means large incoming flow from nodes with large quantity of liquid.



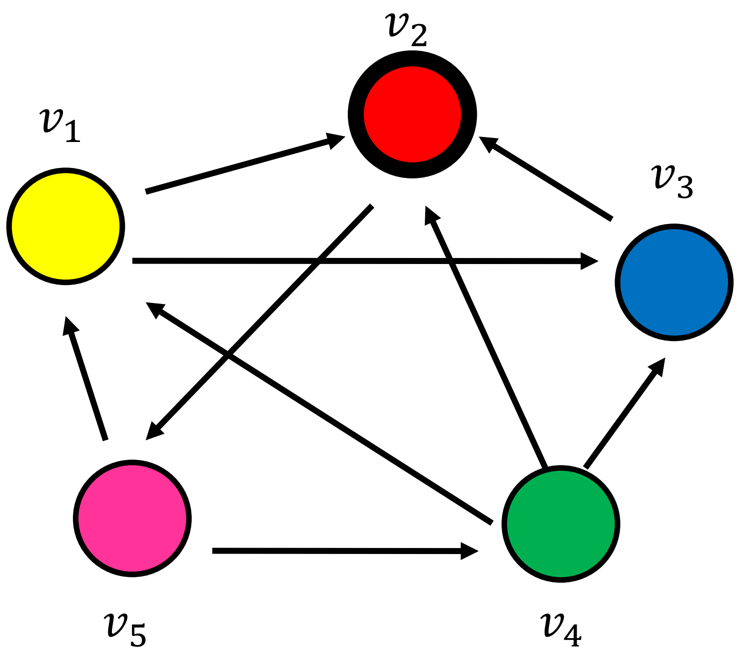
# Random Walks on Graphs

- The algorithm defines a random walk on the graph

- Random walk:
  - Start from a node chosen uniformly at random with probability $\frac{1}{n}$.
    - Pick one of the outgoing edges uniformly at random
    - Move to the destination of the edge
    - Repeat.

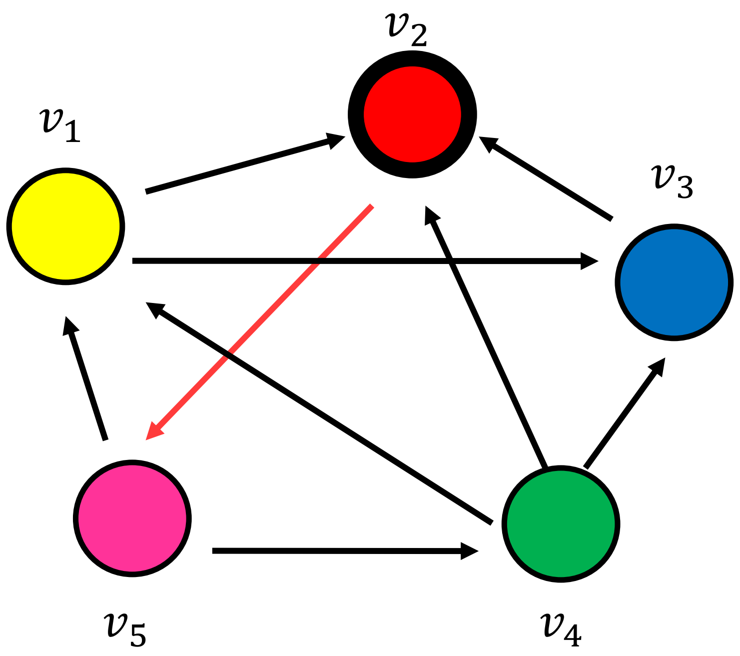
# Example

- Step 0



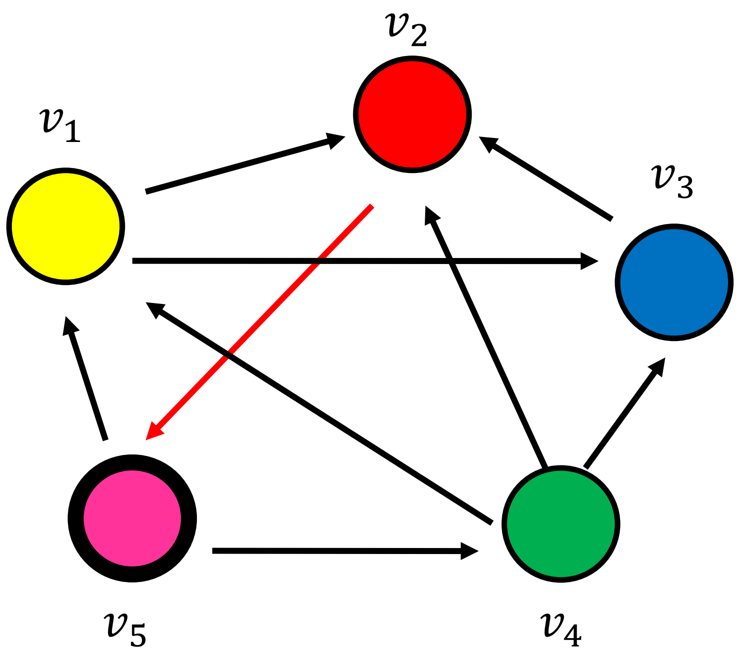
# Example

- Step 0



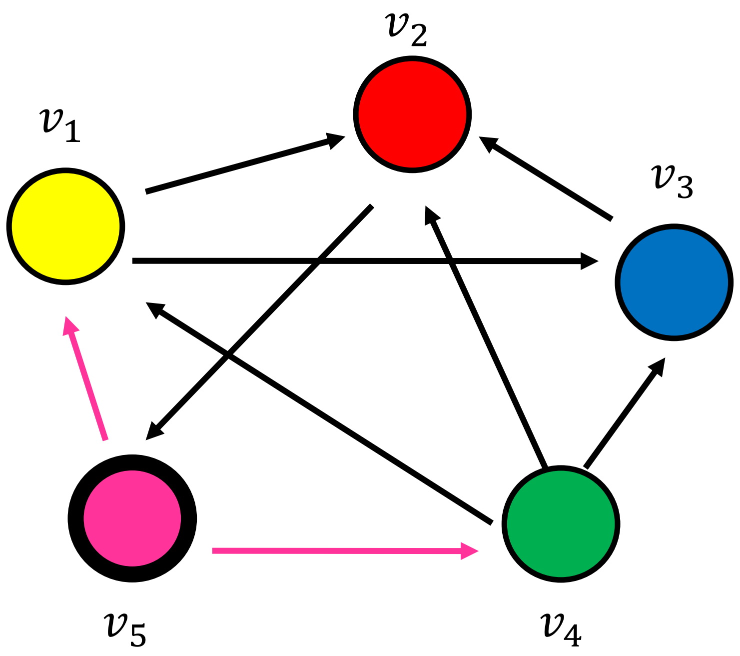
# Example

- Step 1



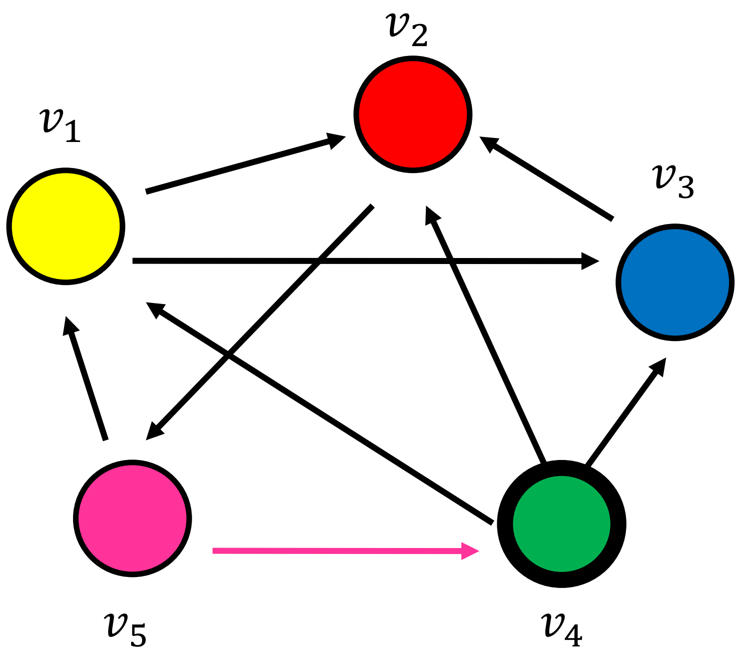
# Example

- Step 1



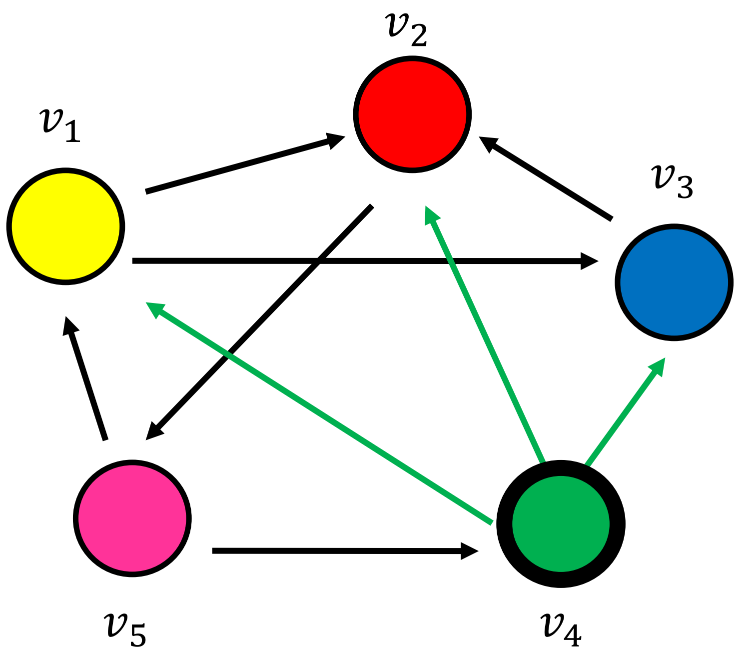
# Example

- Step 2



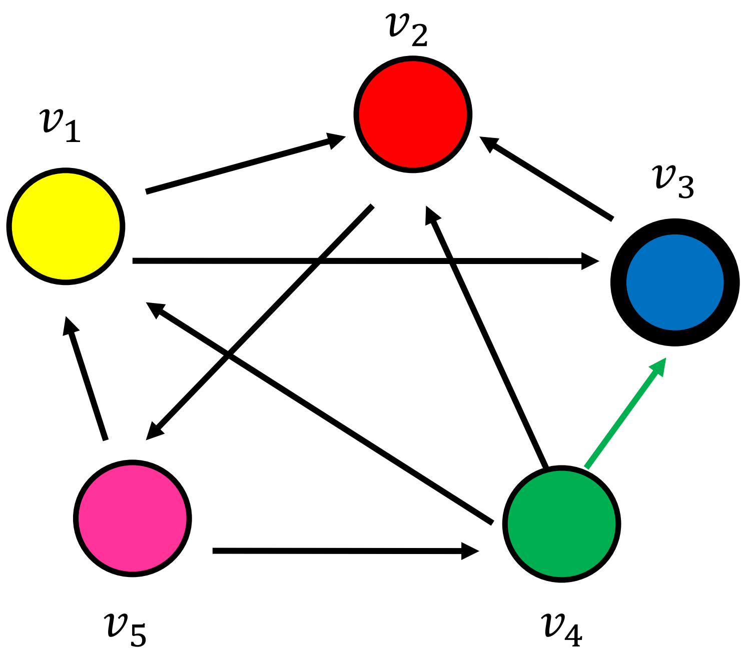
# Example

- Step 2



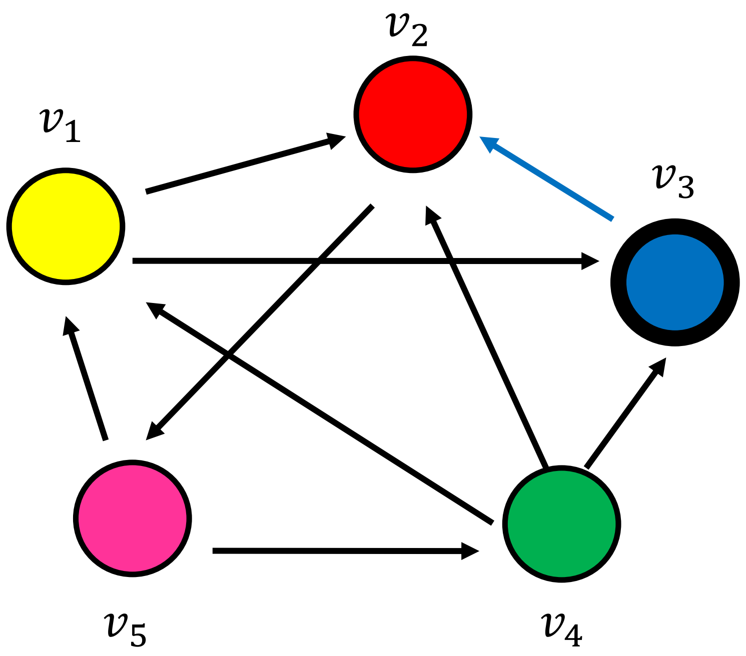
# Example

- Step 3



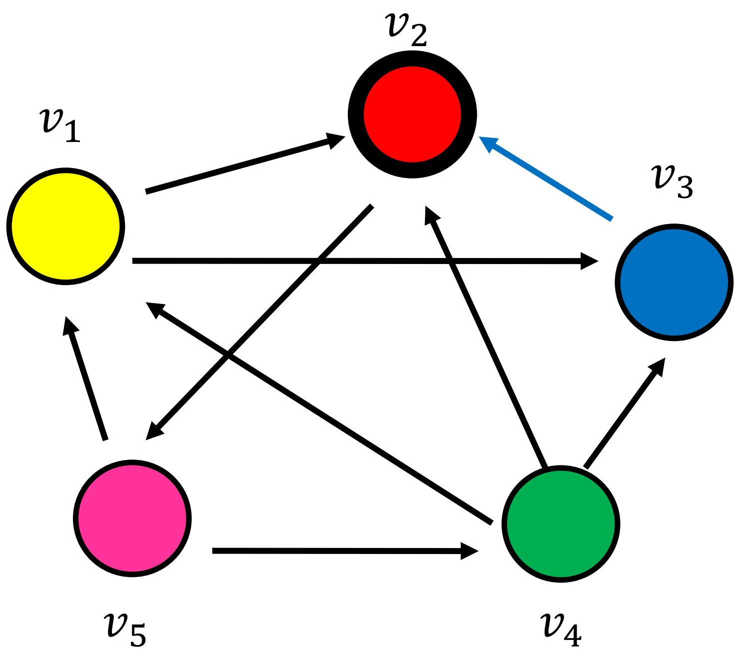
# Example

- Step 3



# Example

- Step 4…



# Random walk

- Question: what is the probability $p_i^t$ of being at node $i$ after $t$ steps?

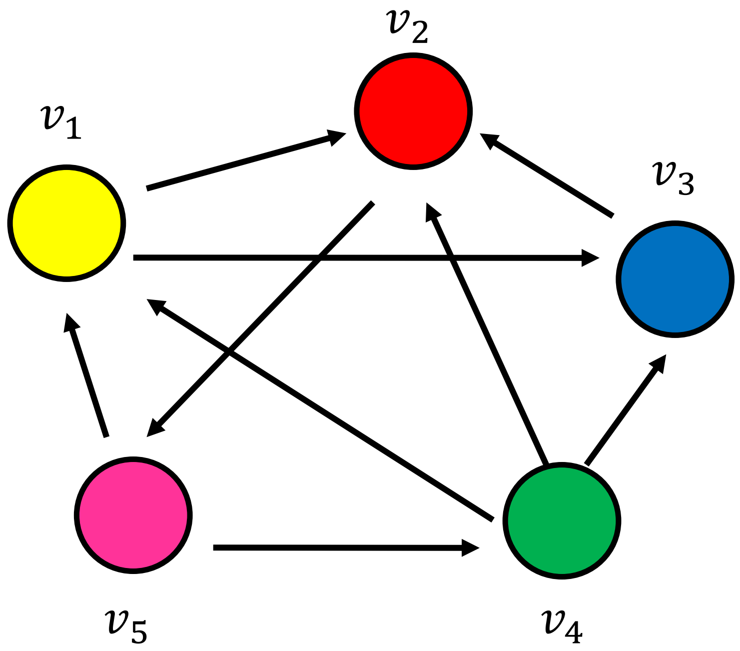
$$p_1^0 = \frac{1}{5} \qquad p_1^t = \frac{1}{3} p_4^{t-1} + \frac{1}{2} p_5^{t-1}$$

$$p_2^0 = \frac{1}{5} \qquad p_2^t = \frac{1}{2} p_1^{t-1} + p_3^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_3^0 = \frac{1}{5} \qquad p_3^t = \frac{1}{2} p_1^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_4^0 = \frac{1}{5} \qquad p_4^t = \frac{1}{2} p_5^{t-1}$$

$$p_5^0 = \frac{1}{5} \qquad p_5^t = p_2^{t-1}$$



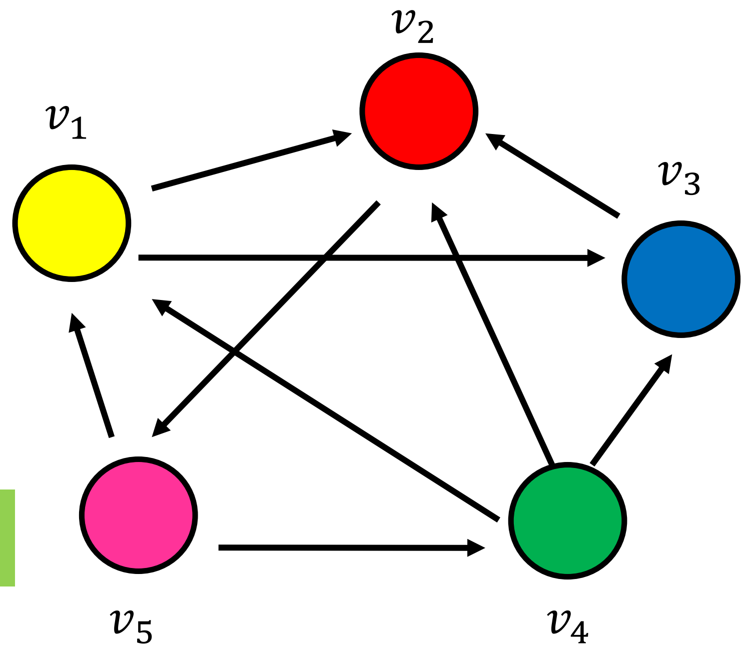

$$p_i^t = \sum_{j \to i} \frac{1}{|N_{out}(j)|} p_j^{t-1}$$

The equations are the same as those for the PageRank iterative computation

# Random walk

- At convergence:

$$p_i = \sum_{j \to i} \frac{1}{|N_{out}(j)|} p_j$$




We get the same equation as for PageRank

The PageRank of node $i$ is the probability that the random walk is at node $i$ after a very large (infinite) number of steps

# Markov chains

- A Markov chain describes a discrete time stochastic process over a set of states

$$S = \{s_1, s_2, \ldots, s_n\}$$

according to a transition probability matrix $P = \{P_{ij}\}$

  - $P_{ij}$ = probability of moving from state $i$ to state $j$

- Matrix $P$ has the property that the entries of all rows sum to 1

$$\sum_j P[i,j] = 1$$

A matrix with this property is called stochastic

# Markov chains

- The stochastic process proceeds in steps and moves between the states:
  - State probability distribution: The vector $p^t = (p_1^t, p_2^t, \dots, p_n^t)$ that stores the probability distribution of being at state $s_i$ after $t$ steps
- Memorylessness property: The next state of the chain depends only on the current state and not on the past of the process (first order MC)
  - Higher order MCs are also possible
- We can compute the vector $p^t$ at step $t$ using a vector-matrix multiplication

$$p^t = p^{t-1}P$$

# Stationary distribution

- The stationary distribution of a random walk with transition matrix $P$, is a probability distribution $\pi$, such that $\pi = \pi P$

- The stationary distribution is an eigenvector of matrix $P$
  - the principal left eigenvector of P – stochastic matrices have maximum eigenvalue 1

- Markov Chain Theory: The random walk converges to a unique stationary distribution independent of the initial vector under some conditions

# Random walks

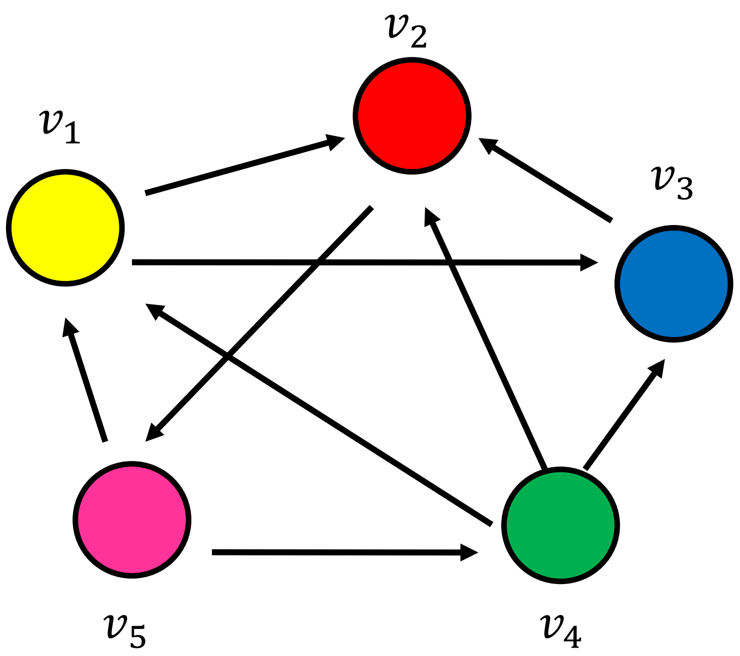
- Markov Chains are equivalent to random walks
  - The set of states $S$ is the set of nodes of the graph $G$
  - The transition probability matrix is the probability that we follow an edge from one node to another

$$P[i,j] = \frac{1}{|\mathrm{N}_{out}(i)|}$$

# The Pagerank random walk and Markov Chain

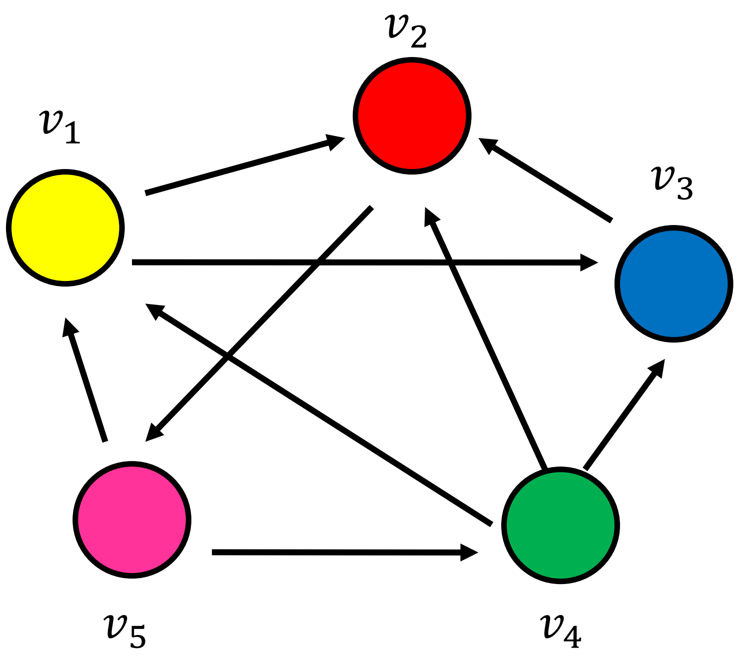
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



# The Pagerank random walk and Markov Chain

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$




$$p_1^t = \frac{1}{3} p_4^{t-1} + \frac{1}{2} p_5^{t-1}$$

$$p_2^t = \frac{1}{2} p_1^{t-1} + p_3^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_3^t = \frac{1}{2} p_1^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_4^t = \frac{1}{2} p_5^{t-1}$$

$$p_5^t = p_2^{t-1}$$

$$p^t = p^{t-1} P$$

# Computing the stationary distribution

- The Power Method, same as the PageRank computation

> Initialize $p^0$ to some distribution
> Repeat
> $\quad p^t = p^{t-1}P$
> Until convergence

- After many iterations $p^t \to \pi$ regardless of the initial vector $p^0$ if the graph is strongly connected, and not bipartite.

- Power method because it computes $p^t = p^0 P^t$

- The rate of convergence is determined by the second eigenvalue $\lambda_2$

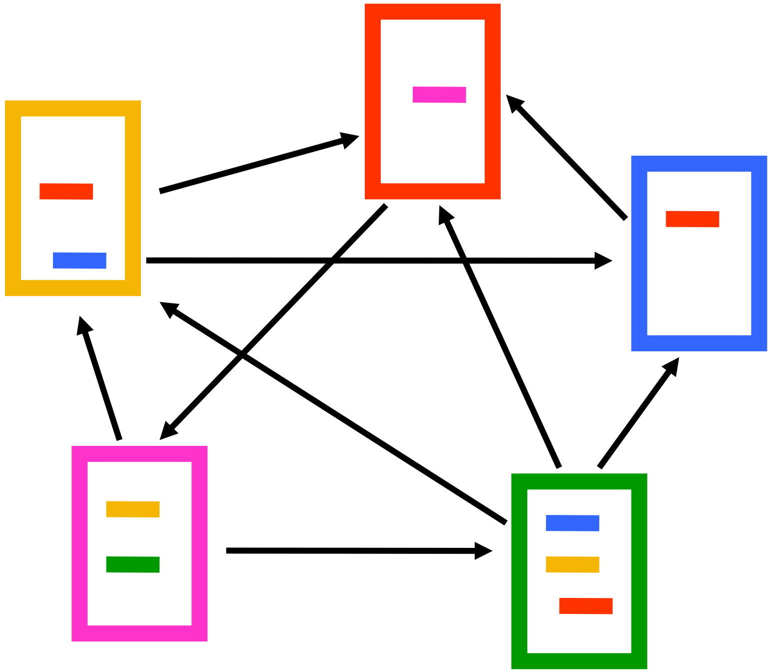
# The stationary distribution

- $\pi$ is the left eigenvector of transition matrix $P$
- $\pi(i)$: the probability of being at node $i$ after very large (infinite) number of steps
- $\pi(i)$: the fraction of times that the random walk visited state $i$ as $t \rightarrow \infty$
- $\pi = p^0 P^\infty$, where $P$ is the transition matrix, $p^0$ the original vector
  - $P(i,j)$: probability of going from $i$ to $j$ in one step
  - $P^2(i,j)$: probability of going from $i$ to $j$ in two steps (sum of probabilities of all paths of length 2)
  - $P^\infty(i,j) = \pi(j)$: probability of going from $i$ to $j$ in infinite steps – starting point does not matter.

# The PageRank random walk

- Vanilla random walk
  - make the adjacency matrix stochastic and run a random walk

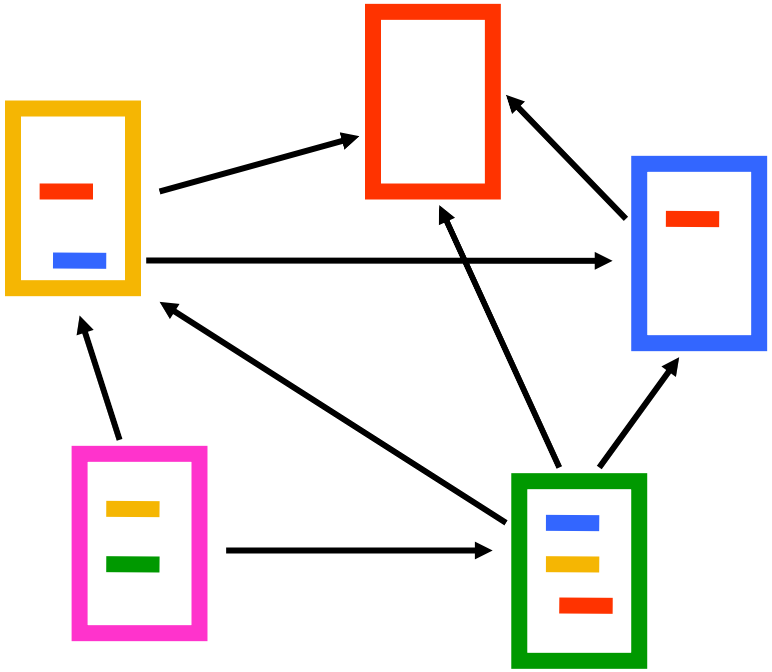
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



# The PageRank random walk

- What about sink nodes?
  - what happens when the random walk moves to a node without any outgoing inks?

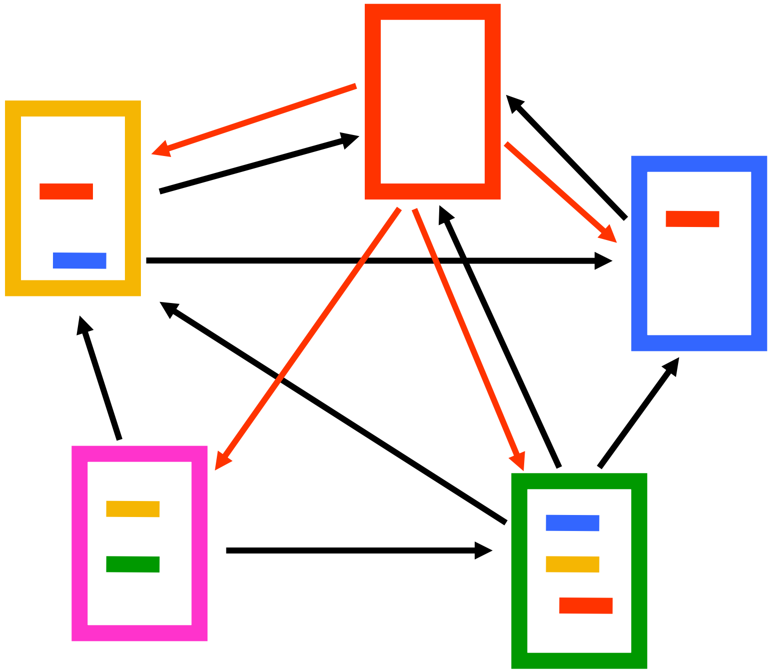
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



# The PageRank random walk

- Replace these row vectors with a vector $u$
  - typically, the uniform vector

$$P' = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



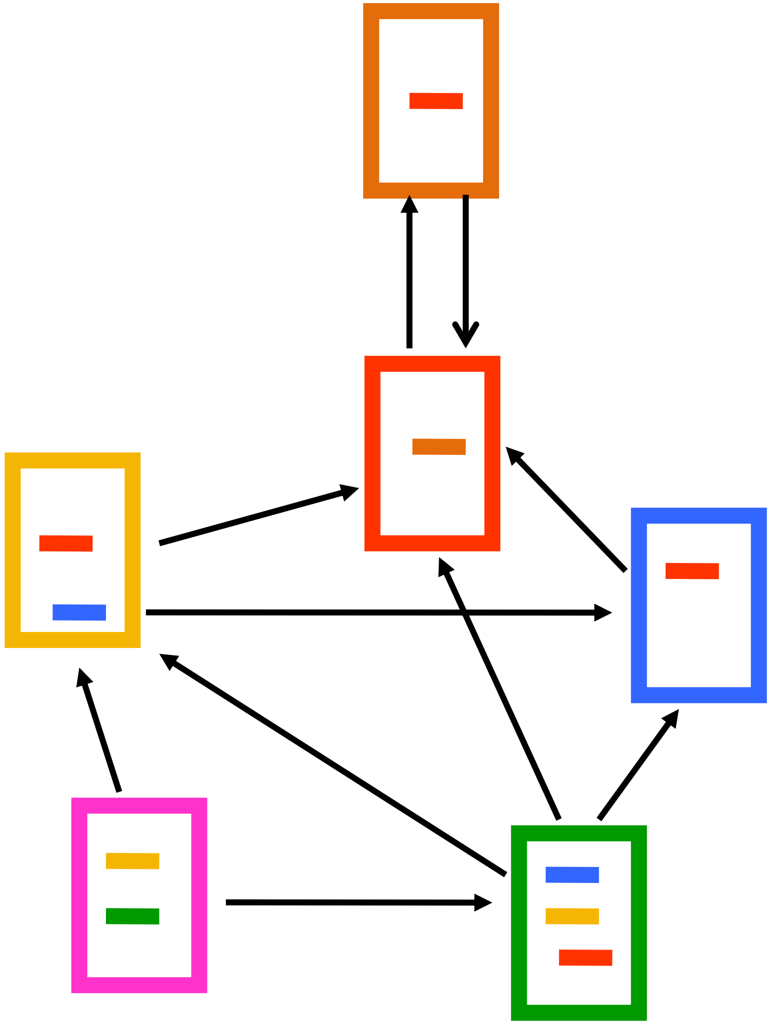
Outer product

$$P' = P + du^T \qquad d = \begin{cases} 1 & \text{if i is sink} \\ 0 & \text{otherwise} \end{cases}$$

$u$: The jump vector

# The PageRank random walk

- What about loops?
  - Spider traps



# The PageRank random walk

- At every step with (fixed) probability $\alpha$ perform a random jump to a node selected according the distribution vector $u$
  - Typically, to a uniform vector
- You can think of the random jump as a restart of the random walk

$$P'' = (1-\alpha)\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + \alpha \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

$P'' = (1-\alpha)P' + \alpha \mathbf{1} u^T$, where **1** is the vector of all 1s

$\alpha$: jump/restart probability

Random walk with restarts

# The PageRank weights

- For the PageRank weights we have

$$p_i = (1-\alpha)\sum_{j \to i} \frac{1}{|N_{out}(j)|} p_j + \alpha u_i$$

  - $\alpha = 0.15$ in most cases

- In matrix-vector terms, if $p$ is the stationary distribution:

$$p^T = p^T(1-\alpha)P + \alpha u^T$$

- Solving for $p$:

$$p^T = \alpha u^T (I - (1-\alpha)P)^{-1}$$

# Stationary distribution with random jump

- If $u$ is the jump vector

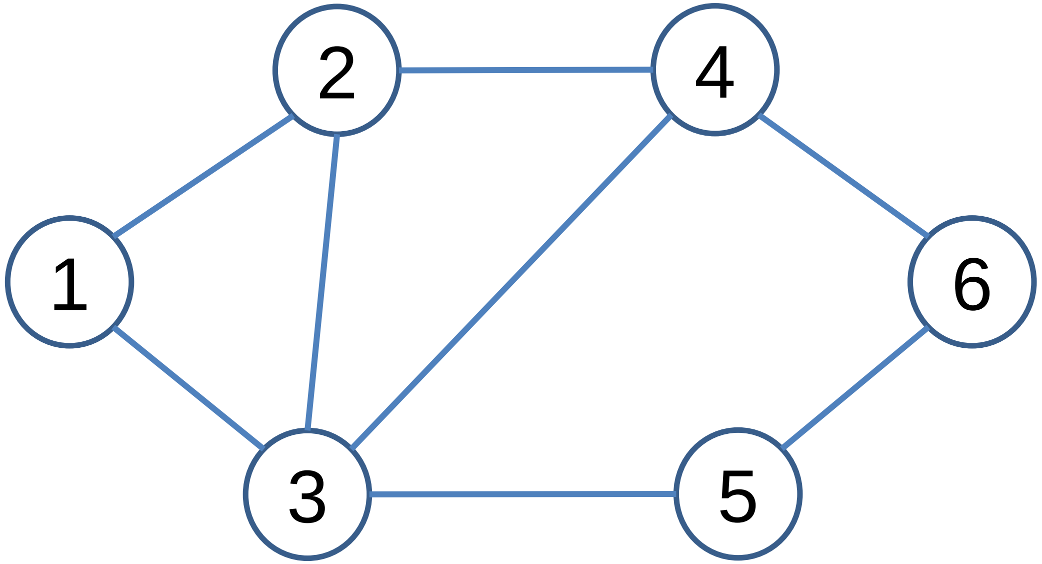
$$p^0 = u$$
$$p^1 = (1-\alpha)p^0 P + \alpha u = (1-\alpha)uP + \alpha u$$
$$p^2 = (1-\alpha)p^1 P + \alpha u = (1-\alpha)^2 uP^2 + (1-\alpha)\alpha uP + \alpha u$$
$$p^3 = (1-\alpha)p^2 P + \alpha u = (1-\alpha)^3 uP^3 + (1-\alpha)^2 \alpha uP^2 + (1-\alpha)\alpha uP + \alpha u$$
$$p^k = (1-\alpha)^k uP^k + (1-\alpha)^{k-1}\alpha uP^{k-1} + \cdots + (1-\alpha)\alpha uP + \alpha u$$
$$p^\infty = \alpha u + (1-\alpha)\alpha uP + (1-\alpha)^2 \alpha uP^2 + \cdots = \alpha(I-(1-\alpha)P)^{-1}u$$

- Explanation: From the last step trace the last restart :
  - With probability $\alpha$ we just restarted in the last step
  - With probability $(1-\alpha)\alpha$ we restarted one step before and then did a random walk step
  - With probability $(1-\alpha)^2\alpha$ we restarted two steps before and then did two random walk steps
  - Etc…
- Conclusion: you are not likely to walk very far
  - The probability that you did $k$ steps after the last restart $(1-\alpha)^k$ drops exponentially with $k$
  - When (re)starting from some node $x$, nodes close to $x$ have higher probability
  - On average the random walk restarts every $1/\alpha$ steps

# Random walks with restarts

- Restart vector:
  - If $u$ is not uniform, we can bias the random walk towards the nodes that are close to the restart nodes
- Personalized Pagerank:
  - Always restart to some node $x$, e.g., the home page of a user
- Topic-Specific Pagerank
  - Restart to nodes about a specific topic, e.g., Greek pages, University home pages
    - Anti-spam
- Random Walks with restarts is a general technique for measuring closeness on graphs.

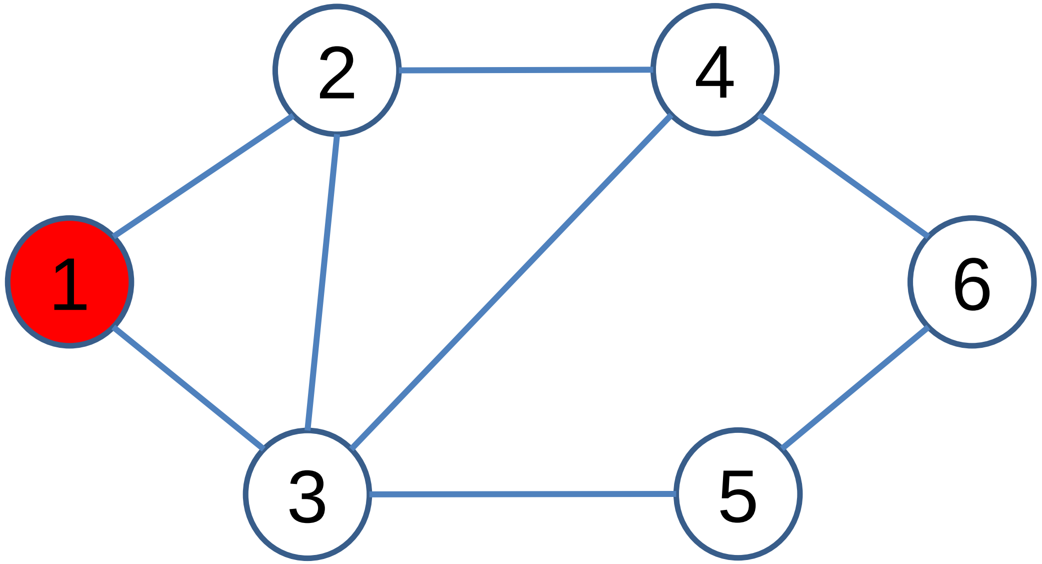
# Personalized Pagerank Example




- Global Pagerank vector (uniform jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$)

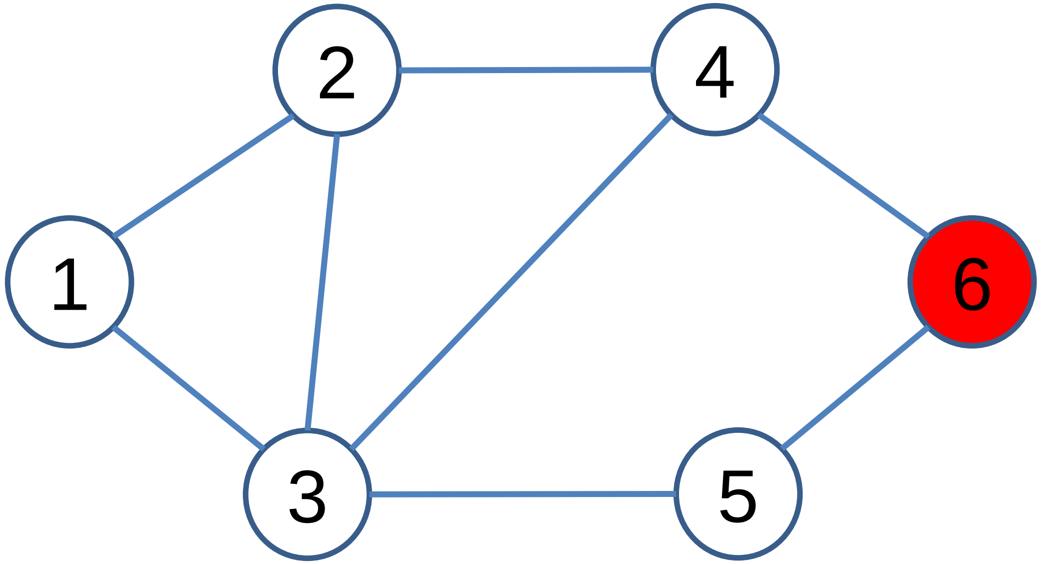
$$[0.13, 0.18, 0.24, 0.18, 0.13, 0.13]$$

# Personalized Pagerank Example




- Global Pagerank vector (jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$) :
  $[0.13, 0.18, 0.24, 0.18, 0.13, 0.13]$
- Personalized Pagerank for node 1 (jump vector $[1,0,0,0,0,0]$):
  $[0.26, 0.20, 0.24, 0.14, 0.08, 0.07]$

# Personalized Pagerank Example




- Global Pagerank vector (jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$) :

$$[0.13, 0.18, 0.24, 0.18, 0.13, 0.13]$$

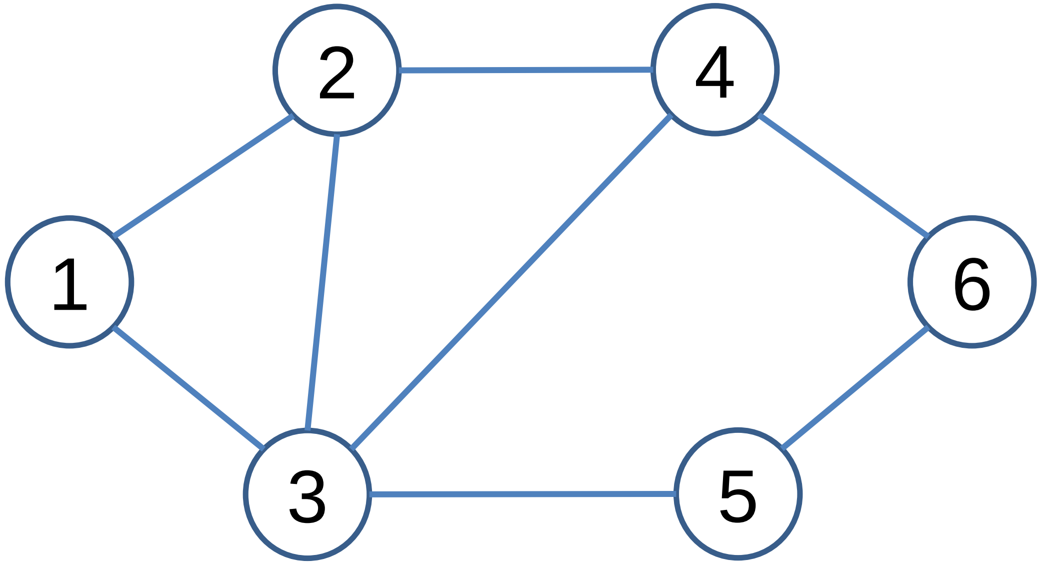
- Personalized Pagerank from node 1 (jump vector $[1,0,0,0,0,0]$):

$$[0.26, 0.20, 0.24, 0.14, 0.08, 0.07]$$

- Personalized Pagerank for node 6 (jump vector $[0,0,0,0,0,1]$):

$$[0.07, 0.13, 0.19, 0.19, 0.15, 0.27]$$

# Personalized Pagerank Example




With $a = 0.5$

- Global Pagerank vector (jump vector $\left[\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right]$) :
  $$[0.14, 0.17, 0.21, 0.18, 0.15, 0.15]$$
- Personalized Pagerank from node 1 (jump vector $[1,0,0,0,0,0]$):
  $$[0.55, 0.17, 0.18, 0.05, 0.03, 0.02]$$
- Personalized Pagerank for node 6 (jump vector $[0,0,0,0,0,1]$):
  $$[0.02, 0.04, 0.07, 0.16, 0.15, 0.56]$$

# Random walks on undirected graphs

- For undirected graphs, the stationary distribution is proportional to the degrees of the nodes
  - In this case a random walk is the same as degree popularity

- This is no longer true if we do random jumps
  - Now the short paths play a greater role, and the previous distribution does not hold.

# Pagerank implementation

- Store the graph in adjacency list, or list of edges
- Keep current pagerank values and new pagerank values
- Go through edges and update the values of the destination nodes.
- Repeat until the difference ($L_1$ or $L_\infty$ difference) is below some small value ε.

# A (Matlab/Numpy-friendly) PageRank algorithm

- Performing vanilla power method is now too expensive – the matrix is not sparse

$q^0 = u$

$t = 1$

repeat

$\quad q^t = (P'')^T q^{t-1}$

$\quad \delta = \|q^t - q^{t-1}\|$

$\quad t = t + 1$

until $\delta < \varepsilon$

Efficient computation of $y = (P'')^T x$

$$y = (1-\alpha)P^T x$$

$$\beta = \|x\|_1 - \|y\|_1$$

$$y = y + \beta u$$

$P$ = normalized adjacency matrix

$P' = P + du^T$, where $d_i$ is 1 if $i$ is sink and 0 o.w.

$P'' = (1-\alpha)P' + \alpha \mathbf{1} u^T$, where $\mathbf{1}$ is the vector of all 1s

# Pagerank history

- Huge advantage for Google in the early days
  - It gave a way to get an idea for the value of a page, which was useful in many different ways
    - Put an order to the web.
  - After a while it became clear that the anchor text was probably more important for ranking
  - Also, link spam became a new (dark) art
- Flood of research
  - Numerical analysis got rejuvenated
  - Huge number of variations
  - Efficiency became a great issue.
  - Huge number of applications in different fields
    - Random walk is often referred to as PageRank.

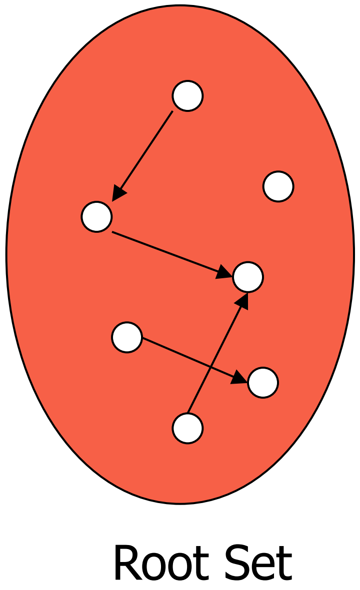
# THE HITS ALGORITHM

# The HITS algorithm

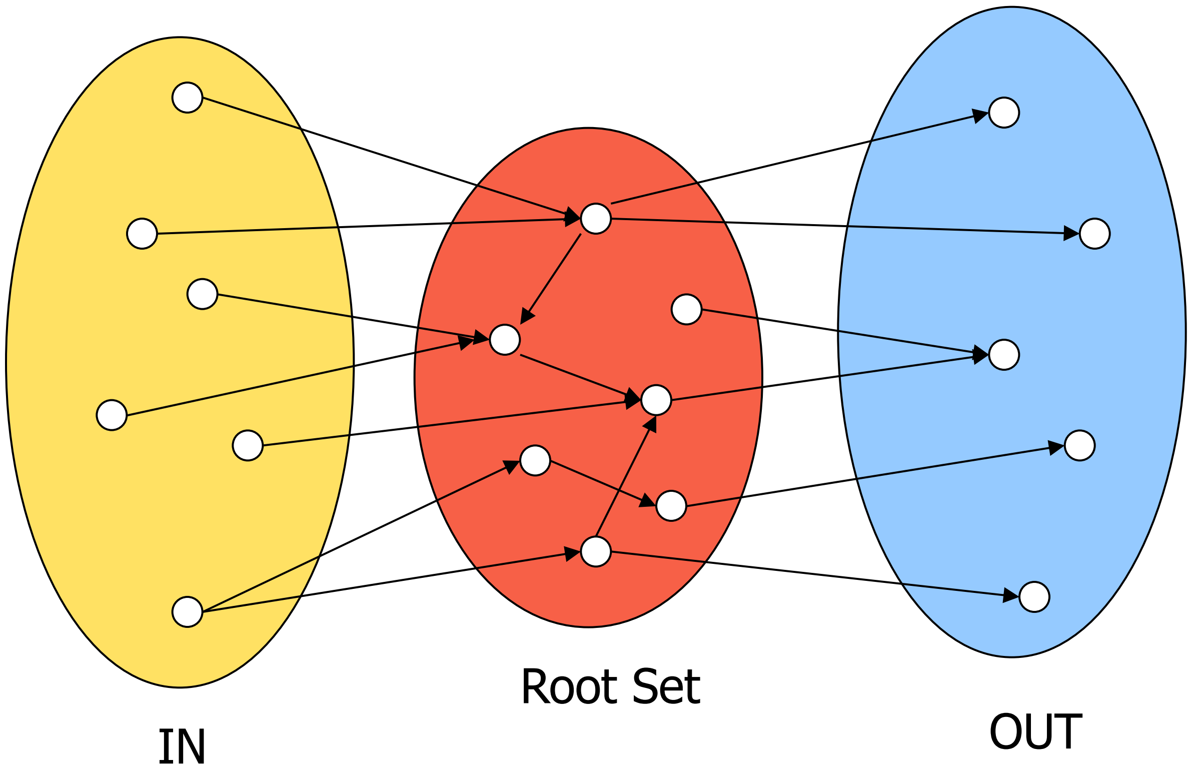
- Another algorithm proposed around the same time as Pagerank for using the hyperlinks to rank pages
  - Kleinberg: then an intern at IBM Almaden
  - IBM never made anything out of it

# Query dependent input

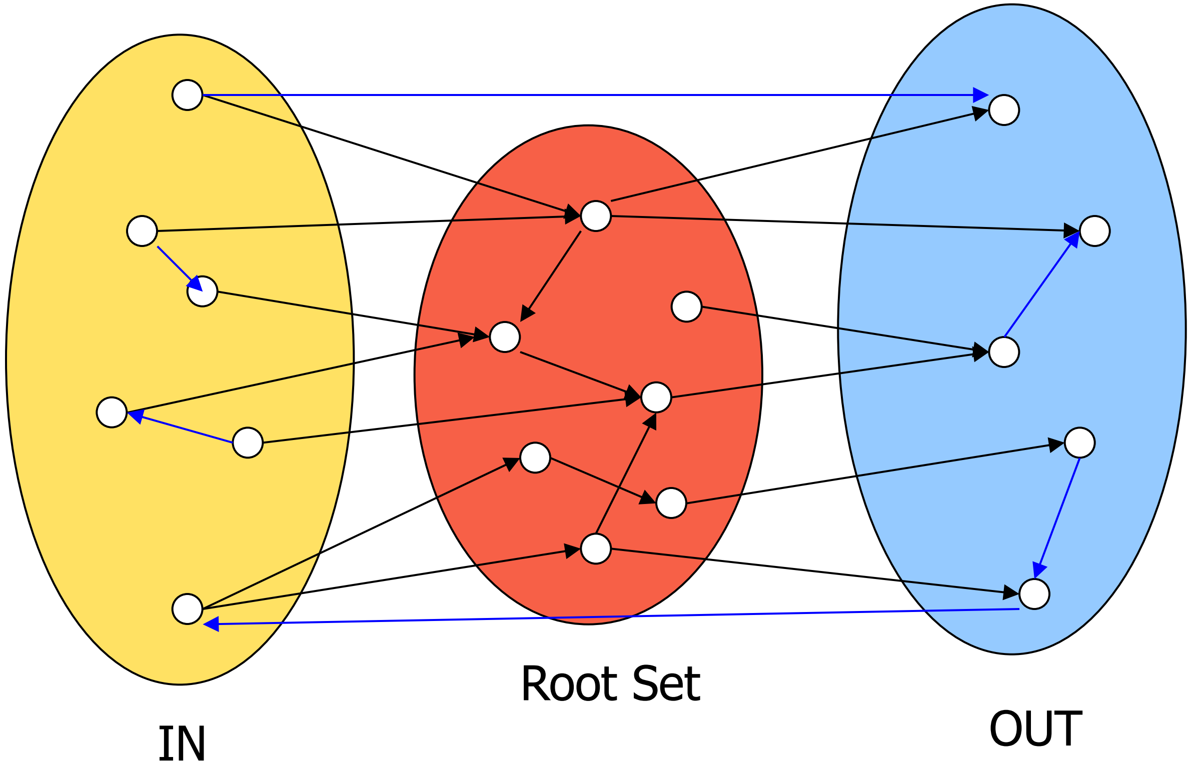
Root set obtained from a text-only search engine



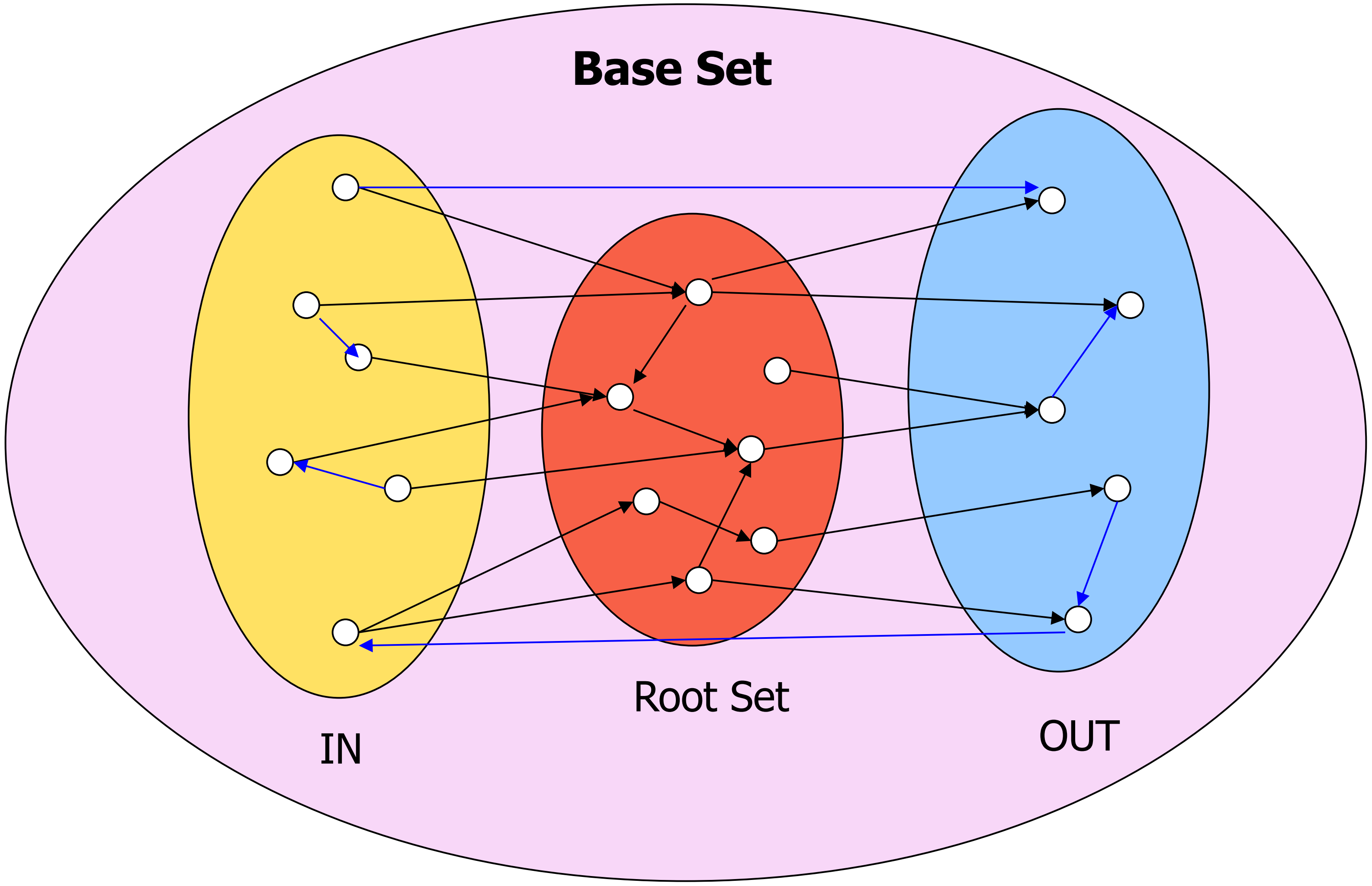
# Query dependent input



# Query dependent input

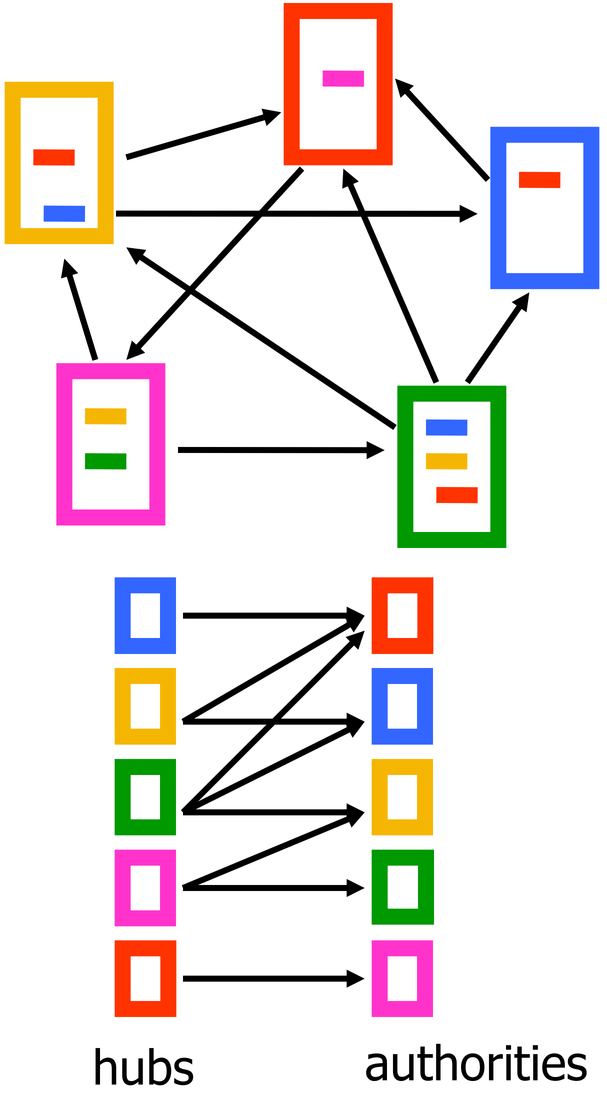


# Query dependent input



# Hubs and Authorities [K98]

- Authority is not necessarily transferred directly between authorities
- Pages have double identity
  - hub identity
  - authority identity
- Good hubs point to good authorities
- Good authorities are pointed by good hubs



# Hubs and Authorities

- Two kind of weights:
  - Hub weight
  - Authority weight

- The hub weight is the sum of the authority weights of the authorities pointed to by the hub

- The authority weight is the sum of the hub weights that point to this authority.

# HITS Algorithm

- Initialize all weights to 1.
- Repeat until convergence
  - $O$ operation : hubs collect the weight of the authorities

$$h_i^t = \sum_{j:i \to j} a_j^{t-1}$$

  - $I$ operation: authorities collect the weight of the hubs

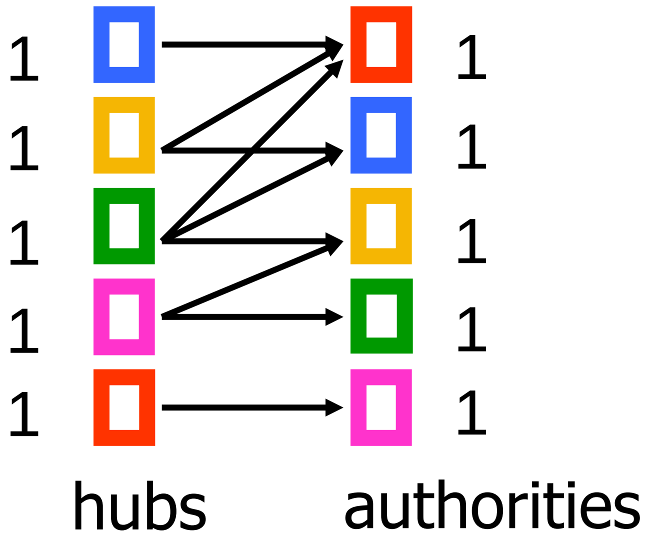
$$a_i^t = \sum_{j:j \to i} h_j^{t-1}$$

  - Normalize weights under some norm

The order of updates does not matter after many iterations.

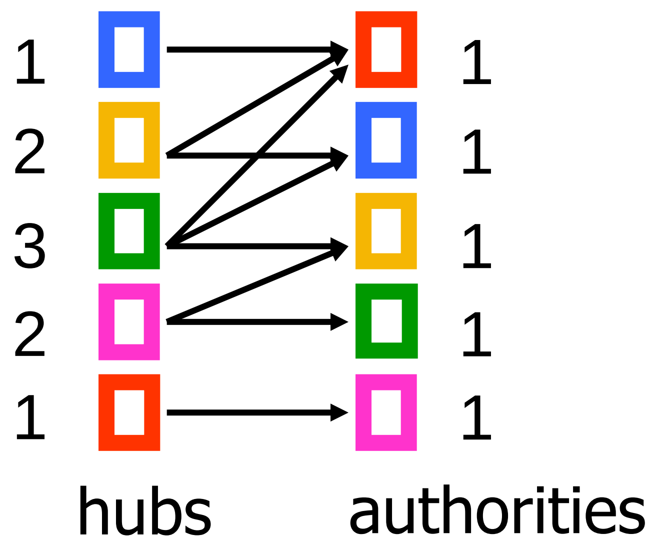
# Example

Initialize



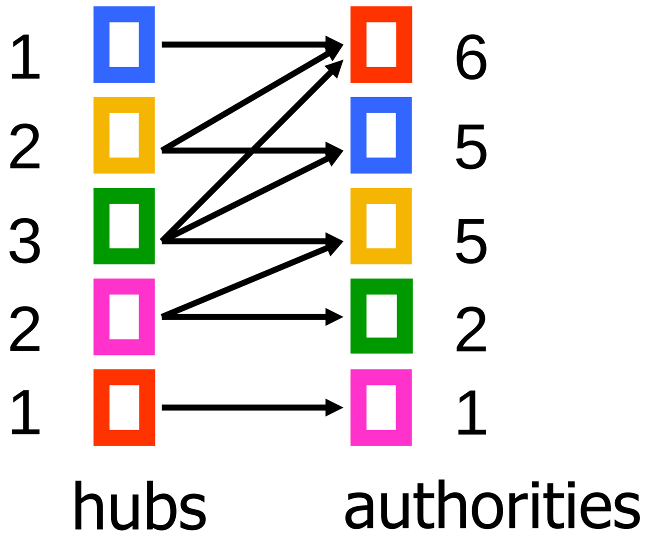
# Example

Step 1: O operation



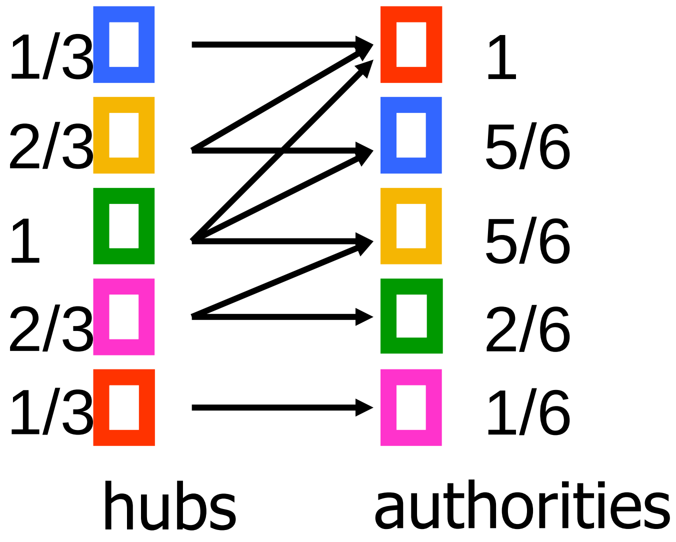
# Example

Step 1: I operation



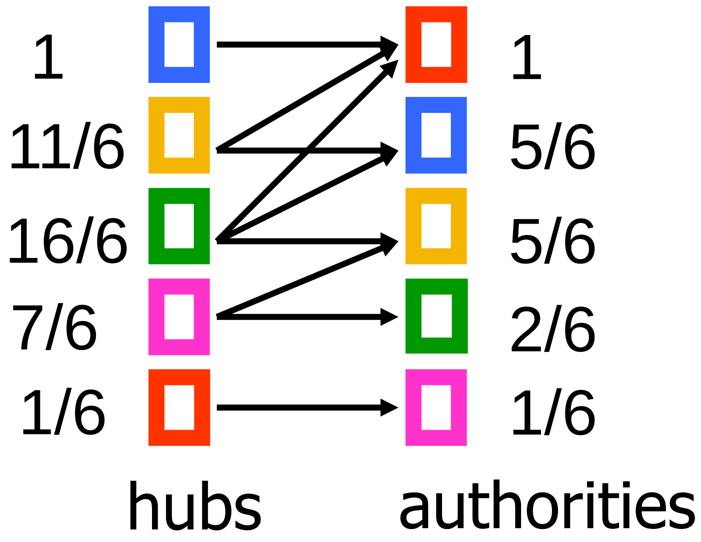
# Example

Step 1: Normalization (Max norm)



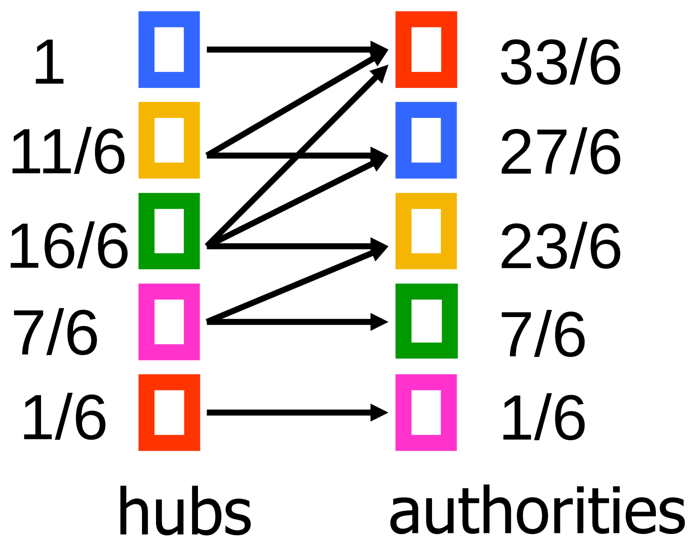
# Example

Step 2: O step



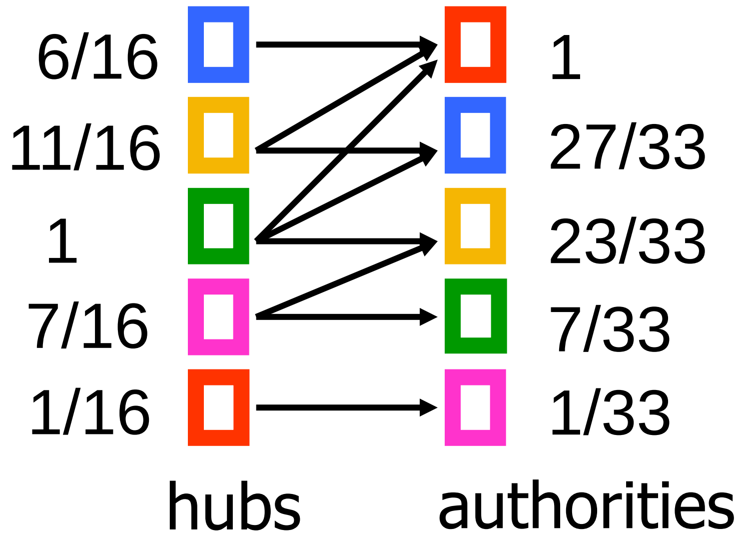
# Example

Step 2: I step



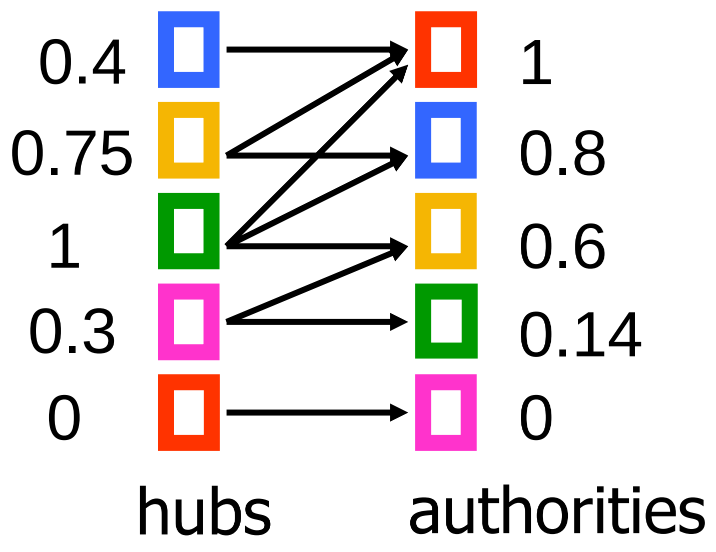
# Example

Step 2: Normalization



# Example

Convergence



# HITS and eigenvectors

- The HITS algorithm is a power-method eigenvector computation
- In vector terms
  - $a^t = A^T h^{t-1}$ and $h^t = A a^{t-1}$
  - $a^t = A^T A a^{t-1}$ and $h^t = A A^T h^{t-1}$
  - Repeated iterations will converge to the eigenvectors

- The authority weight vector $a$ is the eigenvector of $A^T A$
- The hub weight vector $h$ is the eigenvector of $A A^T$

- The vectors $a$ and $h$ are the singular vectors of the matrix A

# Singular Value Decomposition

$$A = \underset{[n\times r]}{U}\ \underset{[r\times r]}{\Sigma}\ \underset{[r\times n]}{V^T} = \begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r\end{bmatrix} \begin{bmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r\end{bmatrix} \begin{bmatrix}\vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_r\end{bmatrix}$$

- **r** : rank of matrix A
- $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ : singular values (square roots of eig-vals $AA^T$, $A^TA$)
- $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r$ : left singular vectors (eig-vectors of $AA^T$)
- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$ : right singular vectors (eig-vectors of $A^TA$)
- $$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

# Why does the Power Method work?

- If a matrix R is real and symmetric, it has real eigenvalues and eigenvectors: $(\lambda_1, w_1), \ (\lambda_2, w_2), \ \ldots, (\lambda_r, w_r)$
  - r is the rank of the matrix
  - $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|$
- For any matrix R, the eigenvectors $w_1, w_2, \ldots, w_r$ of R define a basis of the vector space
  - For any vector $x$, $Rx = \alpha_1 w_1 + a_2 w_2 + \cdots + a_r w_r$
- After t multiplications we have:

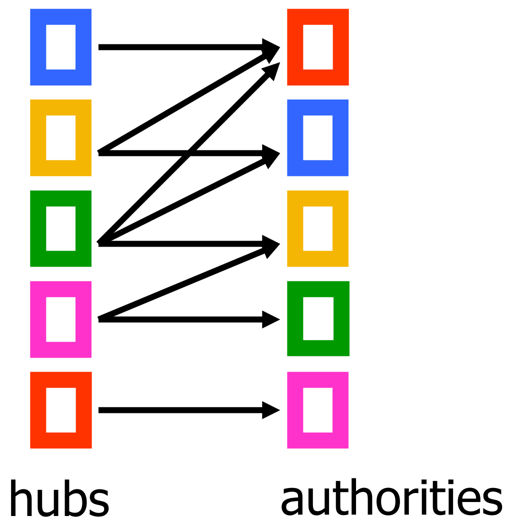
$$R^t x = \lambda_1^{t-1} \alpha_1 w_1 + \lambda_2^{t-1} a_2 w_2 + \cdots + \lambda_2^{t-1} a_r w_r$$

- Normalizing leaves only the term $w_1$.

# OTHER ALGORITHMS

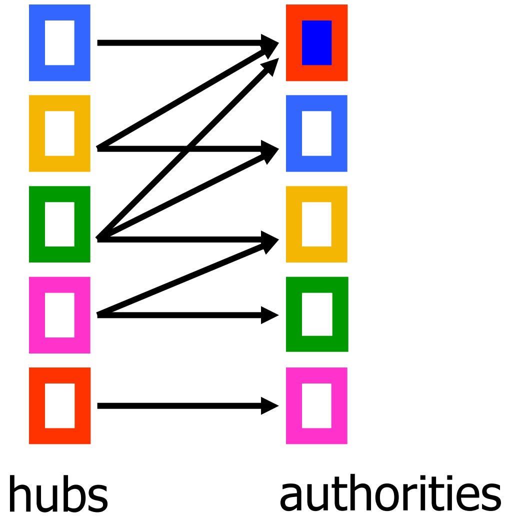
# The SALSA algorithm

- Perform a random walk on the bipartite graph of hubs and authorities alternating between the two
- What does this random walk converges to?



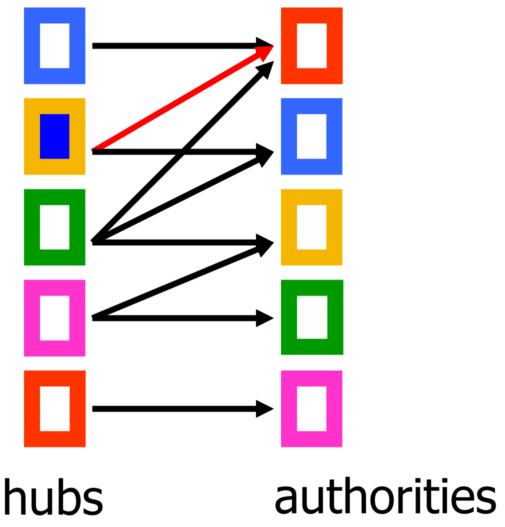
# The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority



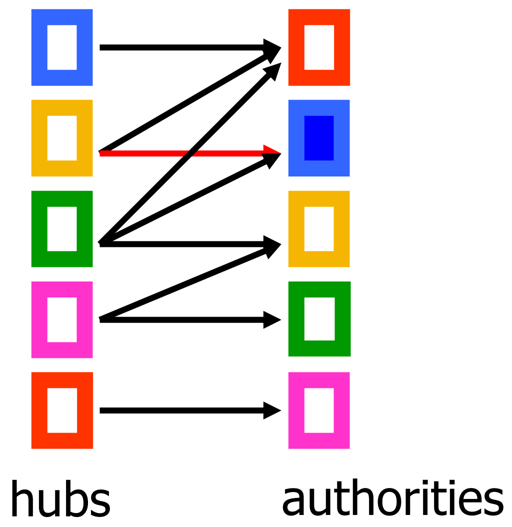
# The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority
- Choose one of the in-coming links uniformly at random and move to a hub
  - e.g. move to the yellow authority with probability 1/3



# The SALSA algorithm

- Start from an authority chosen uniformly at random
  - e.g. the red authority
- Choose one of the in-coming links uniformly at random and move to a hub
  - e.g. move to the yellow authority with probability 1/3
- Choose one of the out-going links uniformly at random and move to an authority
  - e.g. move to the blue authority with probability 1/2



# The SALSA algorithm

- Formally we have probabilities:
  - $a_i$: probability of being at authority $i$
  - $h_j$: probability of being at hub $j$
- The probability of being at authority i is computed as:

$$a_i^t = \sum_{j \in N_{in}(i)} \frac{1}{d_{out}(j)} h_j^{t-1}$$

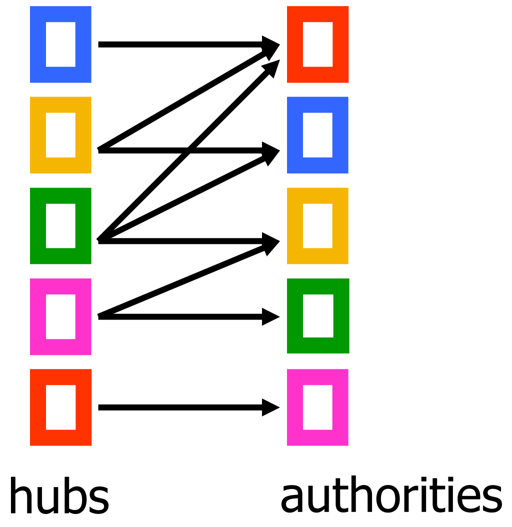
- The probability of being at hub $j$ is computed as

$$h_j^t = \sum_{i \in N_{out}(j)} \frac{1}{d_{in}(i)} a_i^{t-1}$$

- Repeated computation converges

# The SALSA algorithm

- In matrix terms
  - $A_c$ = the matrix $A$ where columns are normalized to sum to 1
  - $A_r$ = the matrix $A$ where rows are normalized to sum to 1
- The hub computation
  - $h = A_c a$
- The authority computation
  - $a = A_r^T h = A_r^T A_c a$
- In MC terms the transition matrix
  - $P = A_r A_c^T$




$$h_2 = 1/3\, a_1 + 1/2\, a_2$$

$$a_1 = h_1 + 1/2\, h_2 + 1/3\, h_3$$

# Social network analysis

- Evaluate the centrality of individuals in social networks
  - degree centrality
    - the (weighted) degree of a node
  - distance centrality
    - the average (weighted) distance of a node to the rest in the graph

$$D_c(v) = \frac{1}{\sum_{u \neq v} d(v,u)}$$

  - betweenness centrality
    - the average number of (weighted) shortest paths that use node v

$$B_c(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

# Counting paths – Katz 53

- The importance of a node is measured by the weighted sum of paths that lead to this node
- $A^m[i,j]$ = number of paths of length $m$ from $i$ to $j$
- Compute

$$P = bA + b^2A^2 + \cdots + b^mA^m + \cdots = (I - bA)^{-1} - I$$

- converges when $b < \lambda_1(A)$
- Rank nodes according to the column sums of the matrix $P$

# Bibliometrics

- Impact factor (E. Garfield 72)
  - counts the number of citations received for papers of the journal in the previous two years
- Pinsky-Narin 76
  - perform a random walk on the set of journals
  - $P_{ij}$ = the fraction of citations from journal i that are directed to journal j