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The Number of Spanning Trees in K_n-Complements of Quasi-Threshold Graphs

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Abstract. In this paper we examine the classes of graphs whose K_n-complements are trees or quasi-threshold graphs and derive formulas for their number of spanning trees; for a subgraph H of K_n, the K_n-complement of H is the graph $K_n - H$ which is obtained from K_n by removing the edges of H. Our proofs are based on the complement spanning-tree matrix theorem, which expresses the number of spanning trees of a graph as a function of the determinant of a matrix that can be easily constructed from the adjacency relation of the graph. Our results generalize previous results and extend the family of graphs of the form $K_n - H$ admitting formulas for the number of their spanning trees.

Key words. Spanning trees, Complement spanning-tree matrix theorem, Trees, Quasi-threshold graphs, Combinatorial problems, Networks

1. Introduction

We consider finite undirected graphs with no loops or multiple edges. Let G be such a graph on n vertices. A *spanning tree* of G is an acyclic $(n-1)$-edge subgraph; note that it is connected and spans G. Let K_n denote the complete graph on n vertices. If H is a subgraph of K_n, then $K_n - H$ is defined to be the graph obtained from K_n by removing the edges of H; the graph $K_n - H$ is called the K_n-*complement* of H. Note that, if H has n vertices, then $K_n - H$ coincides with the graph $\overline{H}$, the *complement* of H.

The problem of calculating the number of spanning trees of a graph is an important, well-studied problem. Deriving formulas for different types of graphs can prove to be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical consequences related to network reliability [2, 4, 13, 18].

Thus, for both theoretical and practical purposes, we are interested in deriving formulas for the number of spanning trees of classes of graphs of the form $K_n - H$. Many cases have already been examined. For example there exist formulas for the cases when H is a pairwise disjoint set of edges [20], when it is a star

[17], when it is a complete graph [1], when it is a path [5], when it is a cycle [5], when it is a multi-star [3, 16, 22], and so on (see Berge [1] for an exposition of the main results).

The purpose of this paper is to derive formulas regarding the number of spanning trees of the graph $G = K_n - H$ in the cases where H is (i) a *tree* on k vertices, $k \leq n$, and (ii) a *quasi-threshold* graph (or QT-graph for short) on p vertices, $p \leq n$. A QT-graph is a graph that contains no induced subgraph isomorphic to P_4 or C_4, the path or cycle on four vertices [7, 12, 15, 21]. Our proofs are based on a classic result known as the *complement spanning-tree matrix* theorem [19], which expresses the number of spanning trees of a graph G as a function of the determinant of a matrix that can be easily constructed from the adjacency relation (adjacency matrix, adjacency lists, etc.) of the graph G. Calculating the determinant of the complement spanning-tree matrix seems to be a promising approach for computing the number of spanning trees of families of graphs of the form $K_n - H$, where H posses an inherent symmetry (see [1, 3, 5, 16, 22, 23]). In our cases, since neither trees nor quasi-threshold graphs possess any symmetry, we focus on their structural and algorithmic properties. Indeed, both trees and quasi-threshold graphs possess properties that allow us to efficiently use the complement spanning-tree matrix theorem; trees are characterized by simple structures and quasi-threshold graphs are characterized by a unique tree representation [10, 15] (see Section 2). We compute the number of spanning trees of these graphs using standard techniques from linear algebra and matrix theory on their complement spanning-tree matrices.

Various important classes of graphs are trees, including paths, stars and multi-stars. Moreover, the class of quasi-threshold graphs contains the classes of perfect graphs known as threshold graphs and complete split (or, c-split) graphs (see Remark 4.1) [6,8]. Thus, the results of this paper generalize previous results and extend the family of graphs of the form $K_n - H$ having formulas regarding the number of spanning trees.

The paper is organized as follows. In Section 2 we establish the notation and related terminology and we present background results. In particular, we show structural properties for the class of quasi-threshold graphs and define a unique tree representation of such graphs. In Sections 3 and 4 we present the results obtained for the graphs $K_n - T$ and $K_n - Q$, respectively, where T is a tree and Q is a quasi-threshold graph. Finally, in Section 5 we conclude the paper and discuss possible future extensions.

2. Definitions and Background Results

We consider finite undirected graphs with no loops or multiple edges. Let G be such a graph; then $V(G)$ and $E(G)$ denote the set of vertices and of edges of G respectively. The *neighborhood* $N(x)$ of a vertex $x \in V(G)$ is the set of all the vertices of G that are adjacent to x. The *closed neighborhood* of x is defined as $N[x] := \{x\} \cup N(x)$.

Let G be a graph on n vertices. The *complement spanning-tree matrix* A of the graph G is defined as follows:

$$A_{i,j} = \begin{cases} 1 - \frac{d_i}{n} & \text{if } i = j, \\ \frac{1}{n} & \text{if } i \neq j \quad \text{and} \quad (i,j) \text{ is not an edge of } G, \\ 0 & \text{otherwise,} \end{cases}$$

where d_i is the number of edges incident to vertex u_i in the complement of G; that is, d_i is the degree of the vertex u_i in $\overline{G}$. It has been shown [19] that the number of spanning trees $\tau(G)$ of G is given by

$$\tau(G) = n^{n-2} \det(A).$$

In the case where $G = K_n$, we have that $\det(A) = 1$; *Cayley's tree formula* [9] states that $\tau(K_n) = n^{n-2}$.

We next provide characterizations and structural properties of QT-graphs and show that such a graph has a unique tree representation. The following lemma follows immediately from the definition of $G[S]$ as the subgraph of G induced by the subset S of the vertex set $V(G)$.

Lemma 2.1 ([10, 15]). *If G is a QT-graph, then for every subset $S \subseteq V(G)$, $G[S]$ is also a QT-graph.*

The following theorem provides important properties for the class of QT-graphs. For convenience, we define

$$\text{cent}(G) = \{x \in V(G) \mid N[x] = V(G)\}.$$

Theorem 2.1 ([10, 15]). *Let G be an undirected graph.*

(i) *G is a QT-graph if and only if every connected induced subgraph $G[S]$, $S \subseteq V(G)$, satisfies* $\text{cent}(G[S]) \neq \emptyset$.
(ii) *G is a QT-graph if and only if $G[V(G) - \text{cent}(G)]$ is a QT-graph.*
(iii) *Let G be a connected QT-graph. If $V(G) - \text{cent}(G) \neq \emptyset$, then $G[V(G) - \text{cent}(G)]$ contains at least two connected components.*

Let G be a connected QT-graph. Then $V_1 := \text{cent}(G)$ is not an empty set by Theorem 2.1. Put $G_1 := G$, and $G[V(G) - V_1] = G_2 \cup G_3 \cup \cdots \cup G_r$, where each G_i is a connected component of $G[V(G) - V_1]$ and $r \geq 3$. Then since each G_i is an induced subgraph of G, G_i is also a QT-graph, and so let $V_i := \text{cent}(G_i) \neq \emptyset$ for $2 \leq i \leq r$. Since each connected component of $G_i[V(G_i) - \text{cent}(G_i)]$ is also a QT-graph, we can continue this procedure until we get an empty graph. Then we finally obtain the following partition of $V(G)$:

$$V(G) = V_1 + V_2 + \cdots + V_k, \text{where } V_i = \text{cent}(G_i).$$

Moreover we can define a partial order $\preceq$ on $\{V_1, V_2, \ldots, V_k\}$ as follows:

$$V_i \preceq V_j \ \text{ if } \ V_j \subseteq V(G_i).$$

It is easy to see that the above partition of $V(G)$ possesses the following properties.

Theorem 2.2 ([10, 15]). *Let G be a connected QT-graph, whose vertex set V(G) has been partitioned into sets $V_1, V_2, \ldots, V_k$, where V_1 :=cent(G). Then, the following properties hold:*

(P1) If $V_i \preceq V_j$, then every vertex of V_i and every vertex of V_j are joined by an edge of G.
(P2) For every $V_j, \mathrm{cent}(G[\{\bigcup V_i \mid V_i \preceq V_j\}]) = V_j$.
(P3) For every two V_s and V_t such that $V_s \preceq V_t$, $G[\{\bigcup V_i \mid V_s \preceq V_i \preceq V_t\}]$ is a complete graph. Moreover, for every maximal element V_t of $(\{V_i\}, \preceq)$, $G[\{\bigcup V_i \mid V_1 \preceq V_i \preceq V_t\}]$ is a maximal complete subgraph of G.
(P4) Every edge with both endpoints in V_i is a free edge; an edge (x, y) is called free if $N[x] = N[y]$.
(P5) Every edge with one endpoint in V_i and the other endpoint in V_j, where $V_i \neq V_j$, is a semi-free edge; an edge (x, y) is called semi-free if either $N[x] \subset N[y]$ or $N[x] \supset N[y]$.

The results of Theorem 2.2 provide structural properties for the class of QT-graphs. We shall refer to the structure that meets the properties of Theorem 2.2 as the *cent-tree* of the graph G and denote it by $T_c(G)$. The cent-tree is a rooted tree with root V_1; every node V_i of the tree $T_c(G)$ is either a leaf or has at least two children. Moreover, $V_s \preceq V_t$ if and only if V_s is an ancestor of V_t in $T_c(G)$.

3. Trees

Let T be a tree on k vertices. In the following construction we view T as an ordered, rooted tree: one vertex $r \in V(T)$ is specified as the root and the children of each vertex are given an ordering (the root is not considered a leaf if it has one child). We partition the vertex set of the graph T, in the following manner:

We set $T_1 := T$ and let leaves(T_1) be the set of leaves of the tree T_1. Then $V_1 :=$ leaves(T_1) is not an empty set. We delete the leaves of the tree T_1 and let T_2 be the resulting tree. We set $V_2 :=$ leaves(T_2) and we continue this procedure until we get an empty tree. Then, we finally obtain the following partition of $V(T)$:

$$V(T) = V_1 + V_2 + \cdots + V_h,$$

where

$$V_i = \text{leaves}(T_i), \ T_{i+1} = T_i - \text{leaves}(T_i), \quad \text{and} \quad T_1 = T.$$

We call this partition the *st-partition* of the tree T.

We consider the vertex sets $V_1, V_2, \ldots, V_h$ of the *st*-partition of a graph T as ordered sets; we here adopt the left-to-right ordering within T. Denote by $V_i^{-1}(u_j)$ the position of the vertex u_j in the ordered set V_i.

We label the vertices of T from 1 to k in the order that they appear in the ordered sets $V_1, V_2, \ldots, V_h$. More precisely, if ℓ_i and ℓ_j denote the labels of the vertices u_i and u_j, respectively, then $\ell_i < \ell_j$ if and only if either both vertices u_i and u_j belong to the same vertex set V_p and $V_p^{-1}(u_i) < V_p^{-1}(u_j)$ or vertices u_i and u_j belong to different vertex sets V_p and V_q, respectively, and $p < q$. This labeling defines a vertex ordering of T; we call it the *st-labeling* of T.

Let $\ell_1, \ell_2, \ldots, \ell_k$ be the labels taken by the *st*-labeling of the tree T. For every vertex u_i of T, we define the vertex set $\mathrm{ch}(i) \subseteq V(T)$ as follows:

$$\mathrm{ch}(i) = \{u_j \in V(T) | u_j \in N(u_i) \quad \text{and} \quad \ell_i > \ell_j\}.$$

Hereafter, we shall also use i to denote the vertex u_i of T, $1 \le i \le k$. Note that $i \in V(T)$ is a leaf if and only if $\mathrm{ch}(i) = \emptyset$. Given a rooted tree T, we recursively define the following function L on $V(T)$:

$$L(i) = \begin{cases} a_i & \text{if } \ i \ \text{ is a leaf ,} \\ a_i - b^2 \displaystyle\sum_{j \in \mathrm{ch}(i)} \frac{1}{L(j)} & \text{otherwise ,} \end{cases}$$

where $a_i = 1 - d_i b$ and $b = 1/n$; recall that $n \ge k$ and d_i is the degree of the vertex i in T. We call L the *st-function* of T; hereafter, we use L_i to denote $L(i)$, $1 \le i \le k$.

We consider the graph $G = K_n - T$, where T is a tree on k vertices. We first assign to each vertex of the graph G a label from 1 to n so that the vertices with degree $n-1$ obtain the smallest labels; that is, we label the vertices with degree $n-1$ from 1 to $n-k$. We label all the other vertices with degree less than $n-1$ from $n-k+1$ to n according to the *st*-labeling of T. Notice that the vertices with degree less than $n-1$ induce the graph $\overline{T}$ in G.

Then, we form the complement spanning-tree matrix A of the graph G; it has the following form:

$$A = \begin{bmatrix} I_{n-k} & \\ & B \end{bmatrix},$$

where the submatrix B concerns those vertices of the graph $K_n - T$ that have degree less than $n-1$; throughout the paper, empty entries in matrices or determinants represent zeros. Let

$$
\begin{aligned}
V_1 &= (u_1, u_2, \ldots, u_\ell),\\
V_2 &= (u_{\ell+1}, u_{\ell+2}, \ldots, u_s),\\
V_3 &= (u_{s+1}, u_{s+2}, \ldots, u_r),\\
&\vdots\\
V_h &= (u_k)
\end{aligned}
$$

be the vertex sets of the *st*-partition of T; recall that the vertices $u_1, u_2, \ldots, u_k$ of $K_n - T$ have degrees less than $n-1$. Thus, B is a $k \times k$ matrix having the following structure:

$$
B = \begin{bmatrix}
a_1 & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & a_\ell & & & & & & & \\
& & & a_{\ell+1} & & & (b)_{j,i} & & & \\
& & & & \ddots & & & & & \\
& & & & & a_s & & & & \\
& & & & & & a_{s+1} & & & \\
& & & (b)_{i,j} & & & & \ddots & & \\
& & & & & & & & a_r & \\
& & & & & & & & & \ddots \\
& & & & & & & & & & a_k
\end{bmatrix}, \qquad (1)
$$

where, according to the definition of the complement spanning-tree matrix, $a_i = 1 - d_i b$, and the entries $(b)_{i,j}$ and $(b)_{j,i}$ of the off-diagonal positions (i,j) and (j,i) are both b if $j \in \mathrm{ch}(i)$ and 0 otherwise, $1 \le j \le i \le k$. Note that $b = 1/n$ and d_i is the degree of the vertex i in T.

Starting from the upper left part of the matrix, the first ℓ rows of the matrix correspond to the ℓ vertices of the set V_1; the next $s - \ell$ rows correspond to the vertices of the set V_2, and so forth. The last row corresponds to the root of T.

From the form of the matrix A, we see that $\det(A) = \det(B)$. Thus, we focus on the computation of the determinant of matrix B.

In order to compute the determinant $\det(B)$, we start by multiplying each column i, $1 \le i \le \ell$, of the matrix B by $-b/a_i$ and adding it to the column j if $(b)_{i,j} = b$ $(i < j \le k)$. This makes all the strictly upper-diagonal entries $(b)_{i,j}$, that is, for $i < j \le \ell$, into zeros. Now expand in terms of rows $1, 2, \ldots, \ell$, getting

$$\det(B) = \prod_{i=1}^{\ell} L_i \begin{vmatrix} f_{\ell+1}^{\ell} & & & & & & \\ & \ddots & & & & & \\ & & f_s^{\ell} & & (b)_{j,i} & & \\ & & & f_{s+1}^{\ell} & & & \\ & & & & \ddots & & \\ & & (b)_{i,j} & & & f_r^{\ell} & \\ & & & & & & \ddots \\ & & & & & & & f_k^{\ell} \end{vmatrix} = \prod_{i=1}^{\ell} L_i \ \det(B'),$$

where

$L_i = a_i$, for $1 \le i \le \ell$, since the vertices $1, 2, \ldots, \ell$ are leaves of T, and

$$f_t^{\ell} = a_t - b^2 \sum_{\substack{i \in \mathrm{ch}(t) \\ 1 \le i \le \ell}} \frac{1}{L_i}, \quad \text{for} \quad \ell + 1 \le t \le k.$$

We observe that the $(k-\ell) \times (k-\ell)$ matrix B' has a structure similar to that of the initial matrix B; see Eq. (1). Thus, for the computation of its determinant $\det(B')$, we follow a similar simplification; that is, we start by multiplying each column i, $1 \le i \le s - \ell$, of the matrix B' by $-b/f_i^{\ell}$ and adding it to the column j if $(b)_{i,j} = b$ $(s < j \le k)$. Then, we obtain

$$\det(B) = \prod_{i=1}^{\ell} L_i \prod_{i=\ell+1}^{s} L_i \begin{vmatrix} f_{s+1}^{s} & & & & \\ & \ddots & & (b)_{j,i} & \\ & & f_r^{s} & & \\ & (b)_{i,j} & & \ddots & \\ & & & & f_k^{s} \end{vmatrix} = \prod_{i=1}^{s} L_i \ \det(B''),$$

where

$$L_i = f_i^{\ell}, \quad \text{for} \quad \ell + 1 \le i \le s, \text{and}$$

$$f_t^{s} = a_t - b^2 \sum_{\substack{i \in \mathrm{ch}(t) \\ 1 \le i \le s}} \frac{1}{L_i}, \ \text{for} \quad s + 1 \le t \le k.$$

The matrix B'' also has structure similar to that of the initial matrix B; see Eq. (1). It differs only on the smaller size and on the diagonal values. Thus, continuing in the same fashion we can finally show that

$$\det(B) = \prod_{i=1}^{k} L_i,$$

where L is the *st*-function of T and k is the number of vertices of T.

Thus, based on the formula that gives the number $\tau(G)$ of the spanning trees of the graph $G = K_n - T$ and the fact that $\det(A) = \det(B)$, we obtain the following result.

Theorem 3.1. *Let T be a tree on k vertices, $k \leq n$, and let L be the st-function on T. The number of spanning trees of the graph $G = K_n - T$ is equal to*

$$\tau(G) = n^{n-2} \prod_{i=1}^{k} L_i.$$

Remark 3.1. We point out that Theorem 3.1 provides a simple linear-time algorithm for computing the number of spanning trees of the graph $G = K_n - T$, where T is a tree on k vertices, $k \leq n$; that is, for a graph on n vertices and m edges the algorithm runs in $O(n + m)$ time. Note that the time complexity is measured according to the uniform cost criterion; under the uniform cost criterion each instruction requires one unit of time and each register requires one unit of space.□

4. Quasi-threshold Graphs

In this section, we derive a formula for the number of the spanning trees of the graph $K_n - Q$, where Q is a quasi-threshold graph.

Let Q be a QT-graph on p vertices and let $V_1, V_2, \ldots, V_k$ be the nodes of its cent-tree $T_c(Q)$ containing $p_1, p_2, \ldots, p_k$ vertices, respectively. We let d_i denote the degree of an arbitrary vertex of the node V_i. Recall that all the vertices $u \in V(Q)$ of a node V_i have the same degree. In Fig. 1 we show a cent-tree of a QT-graph on 12 vertices. Nodes V_3 and V_{10} contain two vertices, while all the other contain one vertex. The degree of a vertex in node V_3 is 4.

We next form the submatrix B of the complement spanning-tree matrix A of the graph $K_n - Q$ based on the structure of the cent-tree $T_c(Q)$, as well as on the st-partition of $T_c(Q)$.

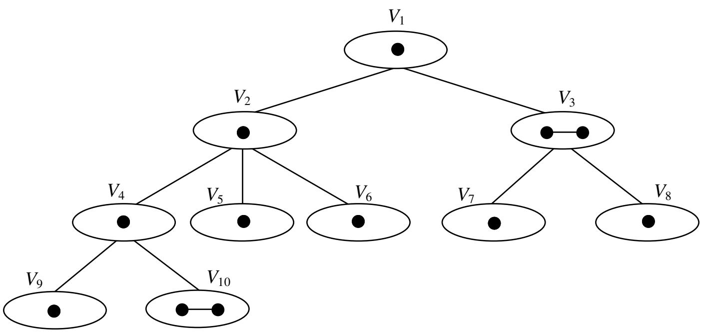

Fig. 1. A cent-tree $T_c(Q)$ of a QT-graph on 12 vertices

Let $V_1', V_2', \ldots, V_h'$ be the node sets of the *st*-partition of $T_c(Q)$. More precisely, the nodes of the $T_c(Q)$ are partitioned in the following sets:

$$\begin{aligned} V_1' &= V_1, \ldots, V_\ell, \\ V_2' &= V_{\ell+1}, \ldots, V_s, \\ &\vdots \\ V_h' &= V_k. \end{aligned}$$

Then, we label the vertices of the graph Q from $n-p+1$ to n as follows: First, we label the vertices in V_1 from $(n-p)+1$ to $(n-p)+p_1$; next, we label the vertices in V_2 from $(n-p)+p_1+1$ to $(n-p)+p_1+p_2$; finally, we label the vertices in V_k.

Thus, based on the above labeling of the vertices of the QT-graph Q, we can easily construct the matrix B of the graph $K_n - Q$; it is a $p \times p$ matrix and has the following form:

$$B = \begin{bmatrix} M_1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & M_\ell & & & & & & & & \\ & & & M_{\ell+1} & & & & [b]_{j,i} & & & \\ & & & & \ddots & & & & & & \\ & & & & & M_s & & & & & \\ & & & & & & M_{s+1} & & & & \\ & & & [b]_{i,j} & & & & \ddots & & & \\ & & & & & & & & M_r & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & M_k \end{bmatrix}, \tag{2}$$

where M_i is a $p_i \times p_i$ submatrix of the form

$$M_i = \begin{bmatrix} a_i & b & \cdots & b \\ b & a_i & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a_i \end{bmatrix},$$

and the entries $[b]_{i,j}$ and $[b]_{j,i}$ of the off-diagonal positions (i,j) and (j,i), respectively, of matrix B correspond to $p_i \times p_j$ and $p_j \times p_i$ submatrices with all their elements b's if node V_j is a descendant of node V_i in $T_c(Q)$ and zeros otherwise, $1 \le j \le i \le k$. Recall that $a_i = 1 - d_i b$, where d_i is the degree of an arbitrary vertex in node V_i of $T_c(Q)$, and $b = 1/n$.

In order to compute the determinant of the matrix B we first simplify the determinants of the matrices M_i, $1 \le i \le k$. We multiply the last row of the matrix M_i by -1 and add it to the first $p_i - 1$ rows of the matrix M_i, $1 \le i \le k$. Then we add the first $p_i - 1$ columns of the matrix M_i to the last column of the matrix M_i, $1 \le i \le k$, and we obtain

$$\det(M_i) = \begin{vmatrix} a_i - b & & & \\ & a_i - b & & \\ & & \ddots & \\ b & b & & a_i - b + p_i b \end{vmatrix} = (a_i - b)^{p_i - 1}(a_i - (1 - p_i)b).$$

It now suffices to substitute the above value in the determinant of matrix B. We point out that after simplifying the determinant of matrices M_i only the diagonal and the last row of each matrix M_i have nonzero entries; the diagonal has nonzero entries since $d_i < n - 1$. Thus, we have

$$\det(B) = \prod_{i=1}^{k} p_i (a_i - b)^{p_i - 1} \det(D), \tag{3}$$

where

$$D = \begin{bmatrix} \sigma_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \sigma_\ell & & & & & & & \\ & & & \sigma_{\ell+1} & & & (b)_{j,i} & & & \\ & & & & \ddots & & & & & \\ & & & & & \sigma_s & & & & \\ & & & & & & \sigma_{s+1} & & & \\ & & & (b)_{i,j} & & & & \ddots & & \\ & & & & & & & & \sigma_r & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & \sigma_k \end{bmatrix} \tag{4}$$

is a $k \times k$ matrix with diagonal elements $\sigma_i = (a_i - (1 - p_i)b)/p_i$, $1 \le i \le k$, and the entry $(b)_{i,j}$ of the off-diagonal position (i, j) is b if node V_j is a descendant of node V_i in $T_c(Q)$ and 0 otherwise, $1 \le j \le i \le k$.

We observe that if we set $p_i = 1$ in matrix D, $1 \le i \le k$, then D is equal to the submatrix B of the graph $K_n - Q$, where Q is a graph of a special type; it is a QT-graph on k vertices possessing the property that each node of its cent-tree $T_c(Q)$ contains a single vertex; see Fig. 2.

It is easy to see that, if we form the submatrix B of the complement spanning-tree matrix A of $K_n - Q$, where Q is the QT-graph of Fig. 2, using an appropriate

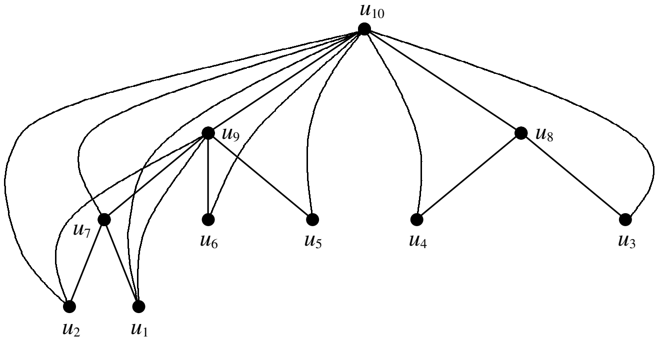

Fig. 2. A QT-graph Q on 10 vertices. Every node V_i of the cent-tree $T_c(Q)$ contains exactly one vertex

vertex labeling, that is, $\ell_2 = n - 9$, $\ell_1 = n - 8$, $\ldots$, $\ell_{10} = n$, then we obtain $D = B$. The idea now is to transform the $k \times k$ matrix D into a form similar to that of the $k \times k$ matrix B of a tree T on k vertices; see Eq. (1) in Section 3. We proceed as follows:

We first apply the following operations to each row $i = 1, 2, \ldots, k$ of the matrix D:

- We find the minimum index j such that $i < j \leq k$ and $D_{i,j} \neq 0$, and then
- we multiply the jth column by -1 and add it to the ℓth column, if $D_{i,\ell} = D_{i,j}$ and $j + 1 \leq \ell \leq k$.

Next, we apply similar operations to each column $j = 1, 2, \ldots, k$ of the matrix D:

- We find the minimum index i such that $1 \leq j < i$ and $D_{i,j} \neq 0$, and then
- we multiply the ith row by -1 and add it to the ℓth row, if $D_{\ell,j} = D_{i,j}$ and $i + 1 \leq \ell \leq k$.

Thus, we obtain

$$\det(D) = \begin{vmatrix} a'_1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & a'_\ell & & & & (b'_j)_{j,i} & & & & \\ & & & a'_{\ell+1} & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & a'_s & & & & & \\ & & (b'_j)_{i,j} & & & & a'_{s+1} & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & a'_r & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & a'_k \end{vmatrix},$$

where

$$a'_i = \begin{cases} \sigma_i & \text{if } V_i \text{ is a leaf of } T_c(Q), \\ \sigma_i + \sum_{\substack{j \in \mathrm{ch}(i) \\ \ell+1 \le j \le k}} (\sigma_j - 2b) & \text{otherwise,} \end{cases} \tag{5}$$

and

$$b'_i = \begin{cases} b & \text{if } V_i \text{ is a leaf of } T_c(Q), \\ b - \sigma_i & \text{otherwise .} \end{cases} \tag{6}$$

Note that the entry $(b'_j)_{i,j}$ in the off-diagonal position (i,j) is b'_j if node V_j is a descendant of node V_i in $T_c(Q)$ and 0 otherwise, $1 \le j \le i \le k$. Recall that $\sigma_i = (a_i - (1 - p_i)b)/p_i$; in the case where each node of the cent-tree $T_c(Q)$ contains a single vertex, we have $\sigma_i = a_i$ (in this case $p_i = 1$, for every $i = 1, 2, \ldots, k$).

It is easy to see that the structure of the resulting $k \times k$ matrix D is similar to that of the $k \times k$ matrix B of a tree; see Eq. (1) in Section 3. Thus, for the computation of the determinant $\det(D)$, we can use similar techniques.

We next define the following function ϕ on the nodes on the cent-tree of a QT-graph Q:

$$\phi(i) = \begin{cases} a'_i & \text{if } i \in V_i \text{ and } V_i \text{ is a leaf of } T_c(Q), \\ a'_i - \sum\limits_{j \in \mathrm{ch}(i)} \frac{(b'_j)^2}{\phi(j)} & \text{otherwise ,} \end{cases}$$

where a'_i and b'_i are defined in Eq. (5) and Eq. (6), respectively. We call the function ϕ the cent-function of the graph Q or, equivalently, the *cent-function* of the cent-tree $T_c(Q)$; hereafter, we use ϕ_i to denote $\phi(i)$, $1 \le i \le k$.

Following the same elimination scheme as that for the computation of the determinant of the matrix B in Section 3, we obtain

$$\det(D) = \prod_{i=1}^{k} \phi_i. \tag{7}$$

Thus, the results of this section are summarized in the following theorem.

Theorem 4.1. *Let Q be a quasi-threshold graph on p vertices and let $V_1, V_2, \ldots, V_k$ be the nodes of the cent-tree of Q.* Let *ϕ be the cent-function of the graph Q. Then, the number of spanning trees of the graph $G = K_n - Q$ is equal to*

$$\tau(G) = n^{n+k-p-2} \prod_{i=1}^{k} p_i (n - d_i - 1)^{p_i - 1} \phi_i,$$

where p_i is the number of vertices of the node V_i and d_i is the degree of an arbitrary vertex in node V_i, $1 \le i \le k$.

Proof. As mentioned in Section 3, the complement spanning-tree matrix A of a graph $K_n - Q$ can be represented by

$$A = \begin{bmatrix} I_{n-p} & \\ & B \end{bmatrix},$$

where the submatrix B concerns those vertices of the graph $K_n - Q$ that have degree less than $n - 1$; these vertices induce the graph $\overline{Q}$. Since $a_i = 1 - d_i b$ and $b = 1/n$, from Eq. (3) we have

$$\det(B) = n^{k-p} \prod_{i=1}^{k} p_i (n - d_i - 1)^{p_i - 1} \det(D).$$

From the above equality and Eq. (7), we obtain

$$\det(B) = n^{k-p} \prod_{i=1}^{k} p_i (n - d_i - 1)^{p_i - 1} \phi_i.$$

The number of spanning trees $\tau(G)$ of the graph G is equal to $n^{n-2} \det(A)$. Thus, since $\det(A) = \det(B)$, the theorem follows. $\square$

Theorem 4.1 coupled with Theorem 3.1 implies a simple linear-time algorithm for computing the number of spanning trees of the graph $G = K_n - Q$, where Q is a quasi-threshold graph on p vertices, $p \leq n$ (see also Remark 3.1).

Remark 4.1. As mentioned in the introduction, the class of quasi-threshold graphs contains the class of c-split graphs (complete split graphs); recall that a graph is defined to be a c-split graph if there is a partition of its vertex set into a stable set S and a complete set K and every vertex in S is adjacent to all the vertices in K [6].

Thus, the cent-tree of a c-split graph H consists of $|S| + 1$ nodes $V_1, V_2, \ldots, V_{|S|+1}$ such that $V_1 = K$ and the nodes $V_2, V_3, \ldots, V_{|S|+1}$ are children of the root V_1; each child contains exactly one vertex $u \in S$.

Let H be a c-split graph on p vertices and let $V(H) = K + S$ be the partition of its vertex set. Then, by Theorem 4.1, we obtain that the number of spanning trees of the graph $G = K_n - H$ is given by the following closed formula:

$$\tau(G) = n^{n-p-1} (n - |K|)^{|S|-1} (n - p)^{|K|},$$

where $p = |K| + |S|$ and $p \leq n$. $\square$

5. Concluding Remarks

It is well known that the classes of quasi-threshold and threshold graphs are perfect graphs. Thus, it is reasonable to ask whether the complement spanning-tree matrix theorem can be efficiently used for deriving formulas, regarding the number of spanning trees, for other classes of perfect graphs [6].

It has been shown that the classes of perfect graphs, namely complement reducible graphs, or so-called cographs, and permutation graphs, have nice structural and algorithmic properties: a cograph admits a unique tree representation, up to isomorphism, called a cotree [11] (note that the class of cographs contain the classes of quasi-threshold and threshold graphs), while a permutation graph $G[\pi]$ can be transformed into a directed acyclic graph and, then, into a rooted tree by exploiting the inversion relation on the elements of the permutation π [14].

Based on these properties, one can work towards the investigation whether the classes of cographs and permutation graphs belong to the family of graphs that admit formulas for the number of their spanning trees.

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