DATA MINING
LECTURE 12

Link Analysis Ranking

PageRank -- Random walks

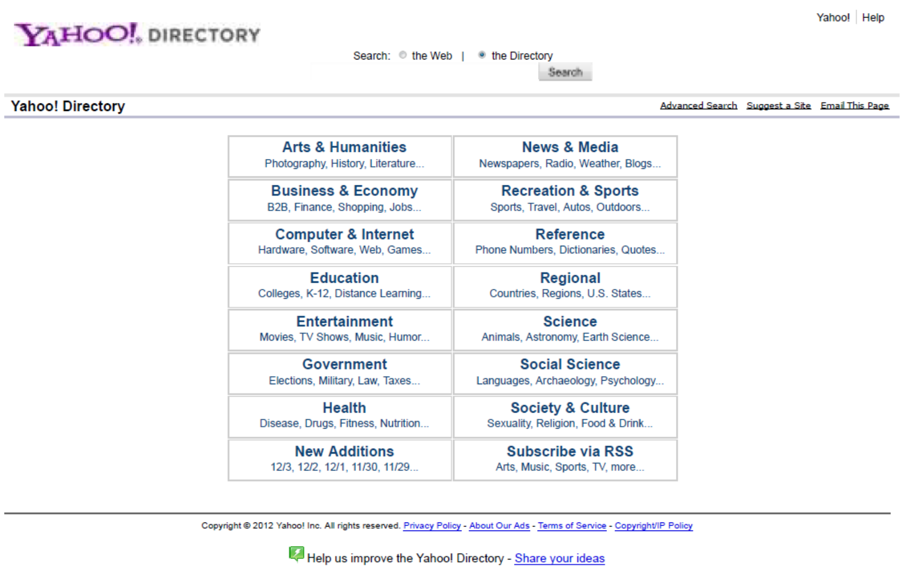
HITS

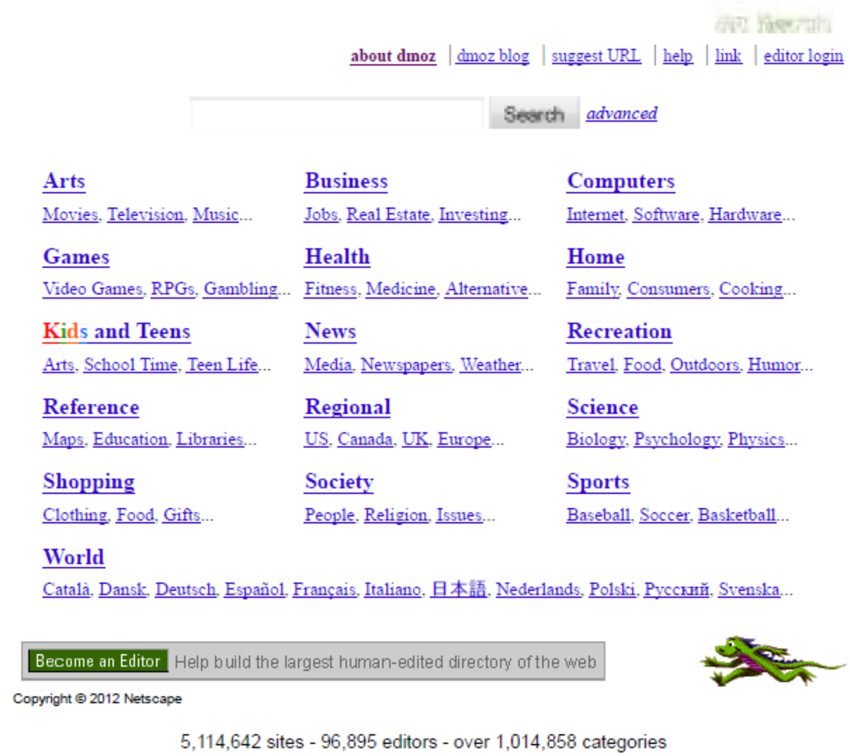
Network Science

- A number of complex systems can be modeled as networks (graphs).
  - The Web
  - (Online) Social Networks
  - Biological systems
  - Communication networks (internet, email)
  - The Economy
- We cannot truly understand such complex systems unless we understand the underlying network.
  - Everything is connected, studying individual entities gives only a partial view of a system
- Data mining for networks is a very popular area
  - Applications to the Web is one of the success stories for network data mining.

How to organize the web

- First try: Manually curated Web Directories





How to organize the web

- Second try: Web Search
  - Information Retrieval investigates:
    - Find relevant docs in a small and trusted set e.g., Newspaper articles, Patents, etc. (“needle-in-a-haystack”)
    - Limitation of keywords (synonyms, polysemy, etc)
  - But: Web is huge, full of untrusted documents, random things, web spam, etc.

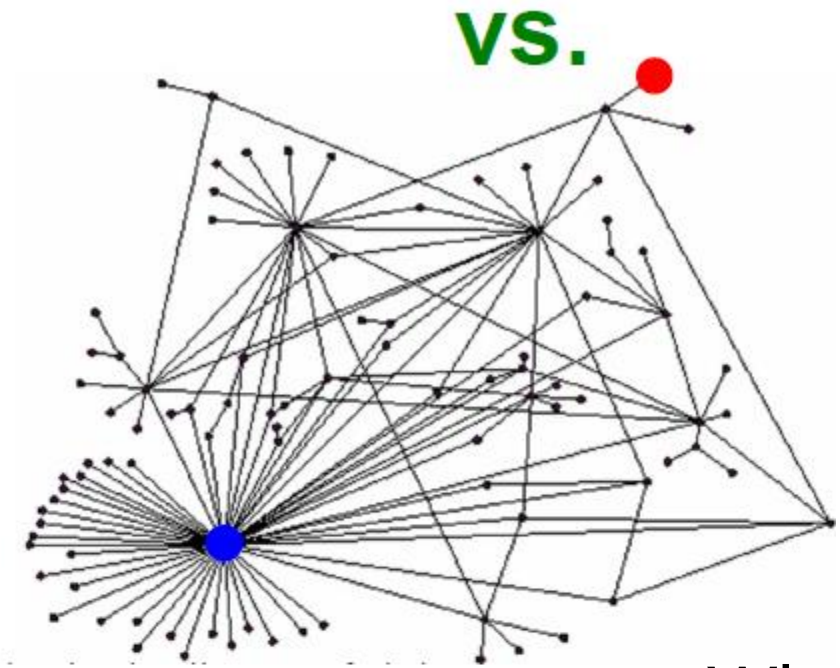
  - Everyone can create a web page of high production value
  - Rich diversity of people issuing queries
  - Dynamic and constantly-changing nature of web content

How to organize the web

- Third try (the Google era): using the web graph
  - Sift from relevance to authoritativeness
  - It is not only important that a page is relevant, but that it is also important on the web
- For example, what kind of results would we like to get for the query “greek newspapers”?

Link Analysis

- Not all web pages are equal on the web



What is the simplest way to measure importance of a page on the web?

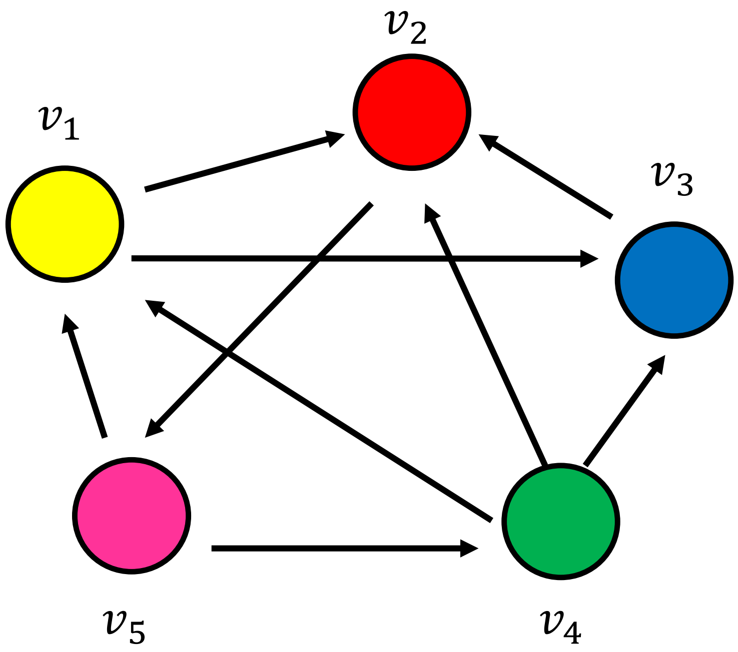
Link Analysis Ranking

- Use the graph structure in order to determine the relative importance of the nodes
  - Applications: Ranking on graphs (Web, Twitter, FB, etc)

- Intuition: An edge from node p to node q denotes endorsement
  - Node p endorses/recommends/confirms the authority/centrality/importance of node q
  - Use the graph of recommendations to assign an authority value to every node

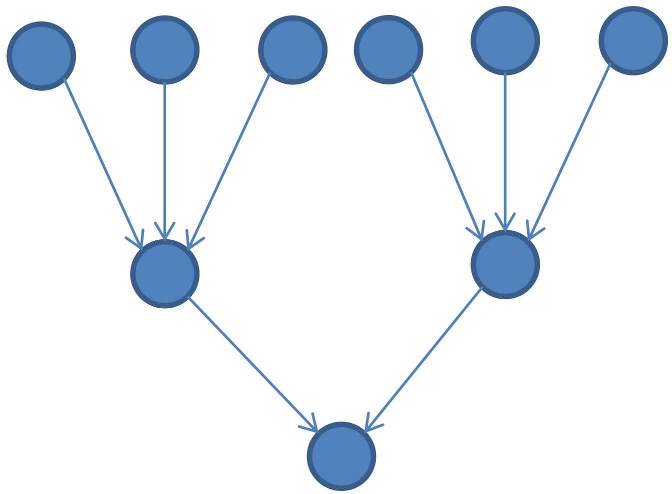
Rank by Popularity

- Rank pages according to the number of incoming edges (in-degree, degree centrality)



1. **Red Page**
2. **Yellow Page**
3. **Blue Page**
4. **Purple Page**
5. **Green Page**

Popularity



- It is not important only how many link to you, but how important are the people that link to you.
- Good authorities are pointed by good authorities
  - Recursive definition of importance

PAGERANK

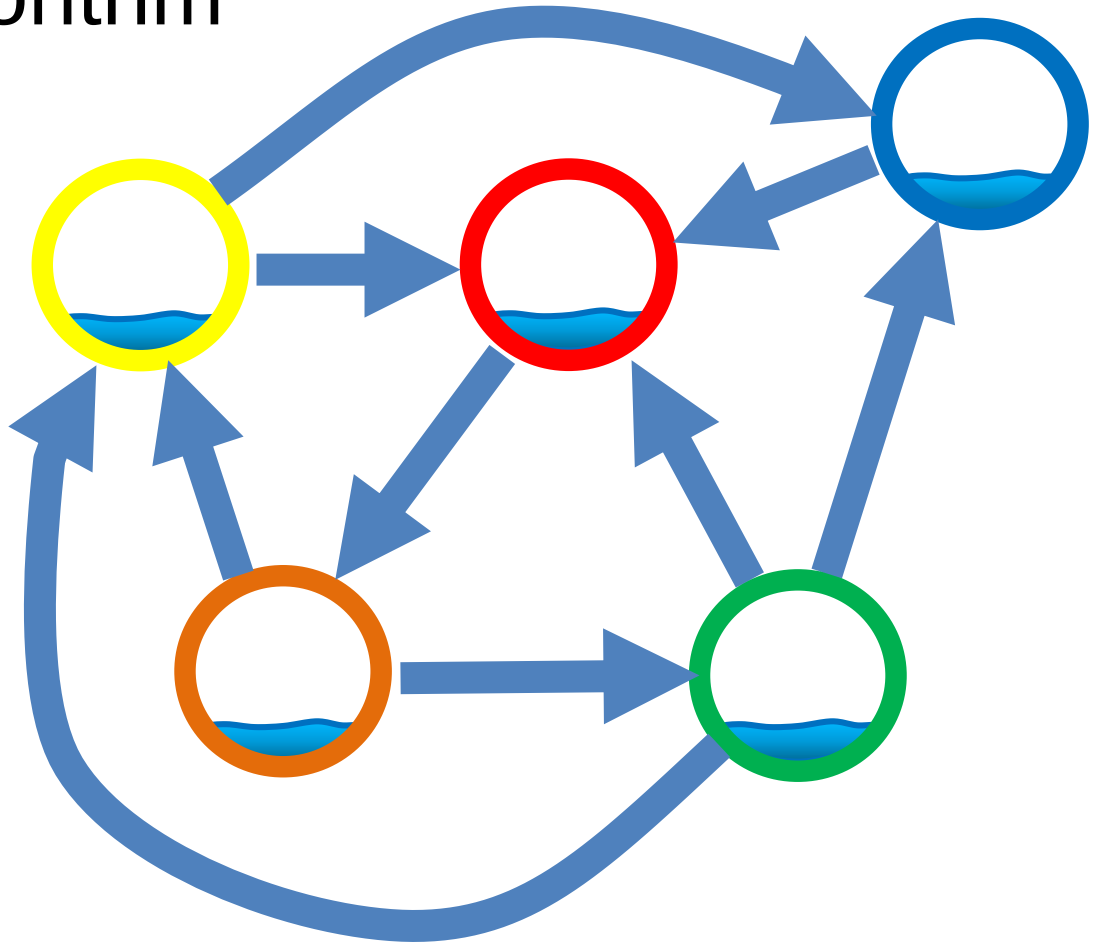
PageRank

- Good authorities should be pointed by good authorities
  - The value of a node is the value of the nodes that point to it.
- How do we implement that?
  - Assume that we have a unit of authority to distribute to all nodes.
    - Initially each node gets $\frac{1}{n}$ amount of authority
  - Each node distributes the authority value they have to their neighbors
  - The authority value of each node is the sum of the authority fractions it collects from its neighbors.

The PageRank algorithm

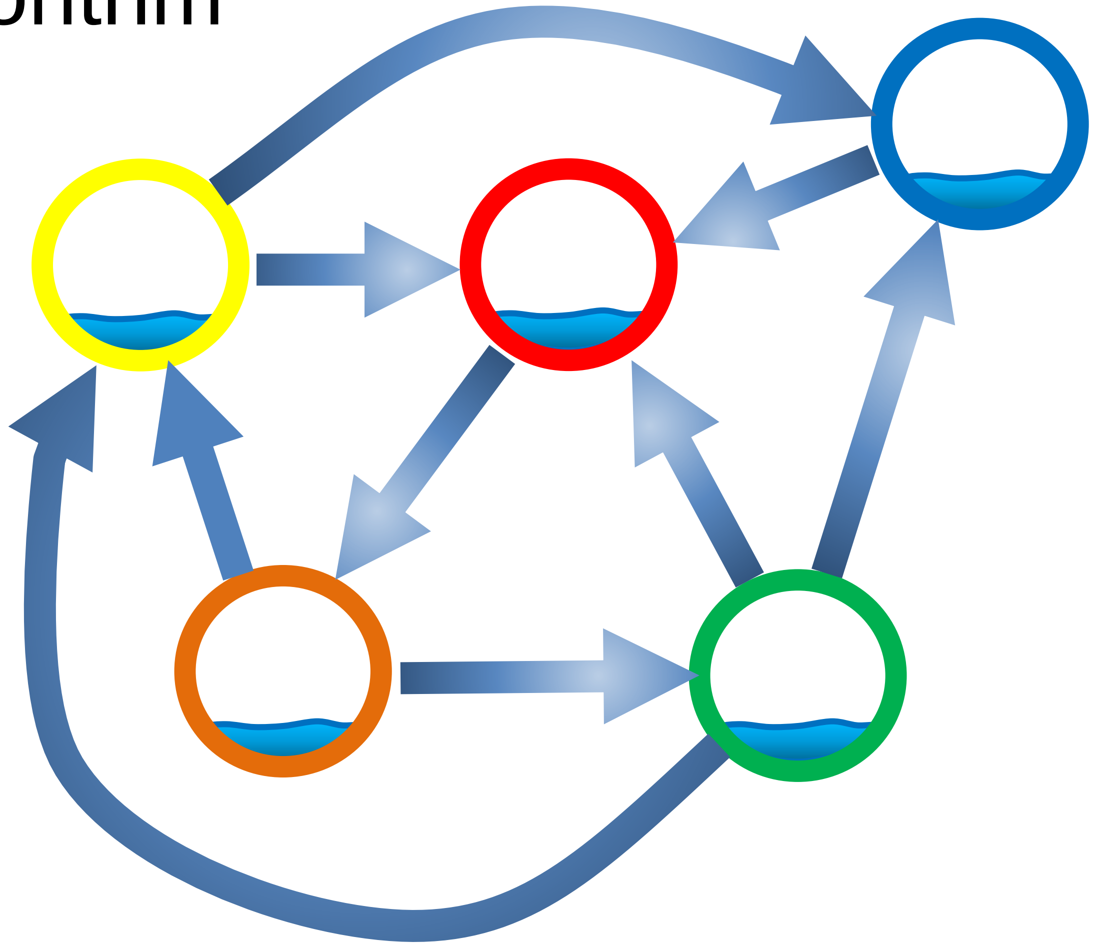
Think of the nodes in the graph as containers of capacity of 1 liter.

We distribute a liter of liquid equally to all containers



The PageRank algorithm

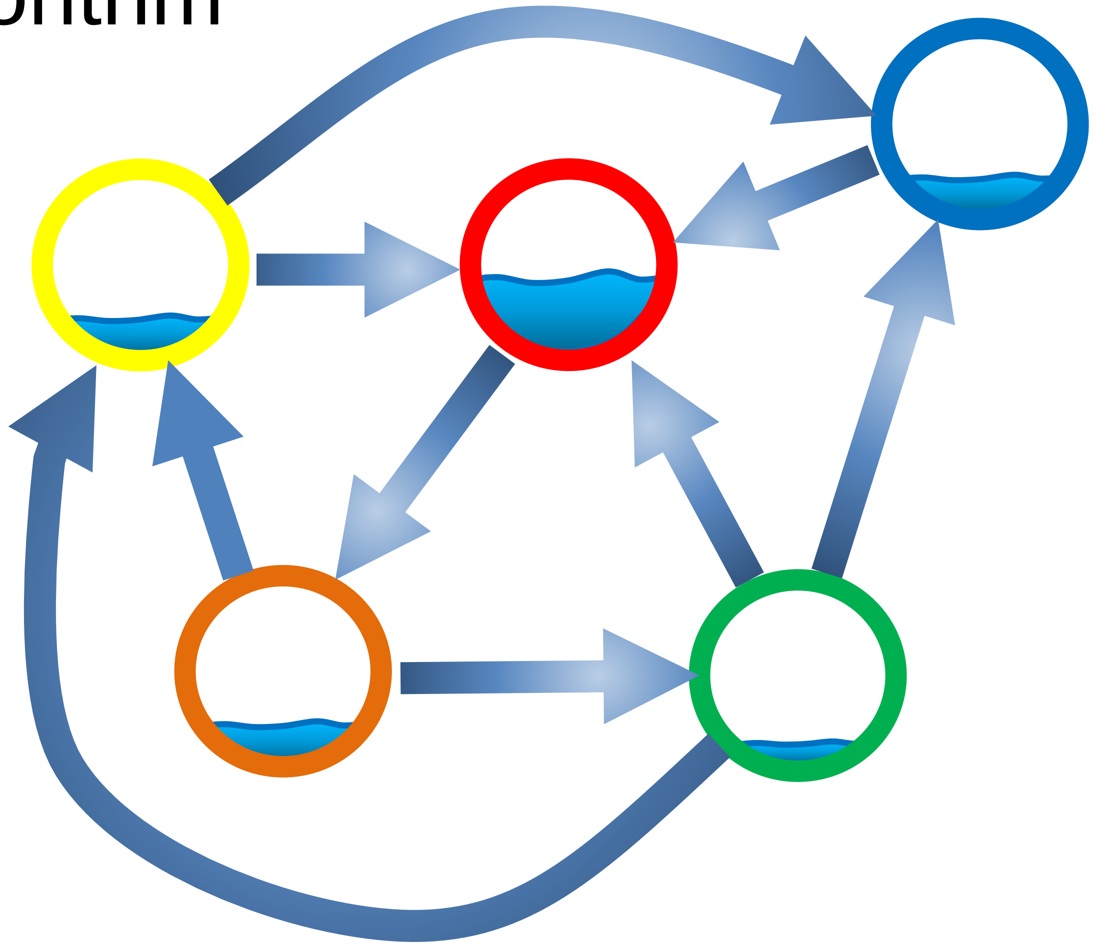
The edges act like pipes that transfer liquid between nodes.



The PageRank algorithm

The edges act like pipes that transfer liquid between nodes.

The contents of each node are distributed to its neighbors.



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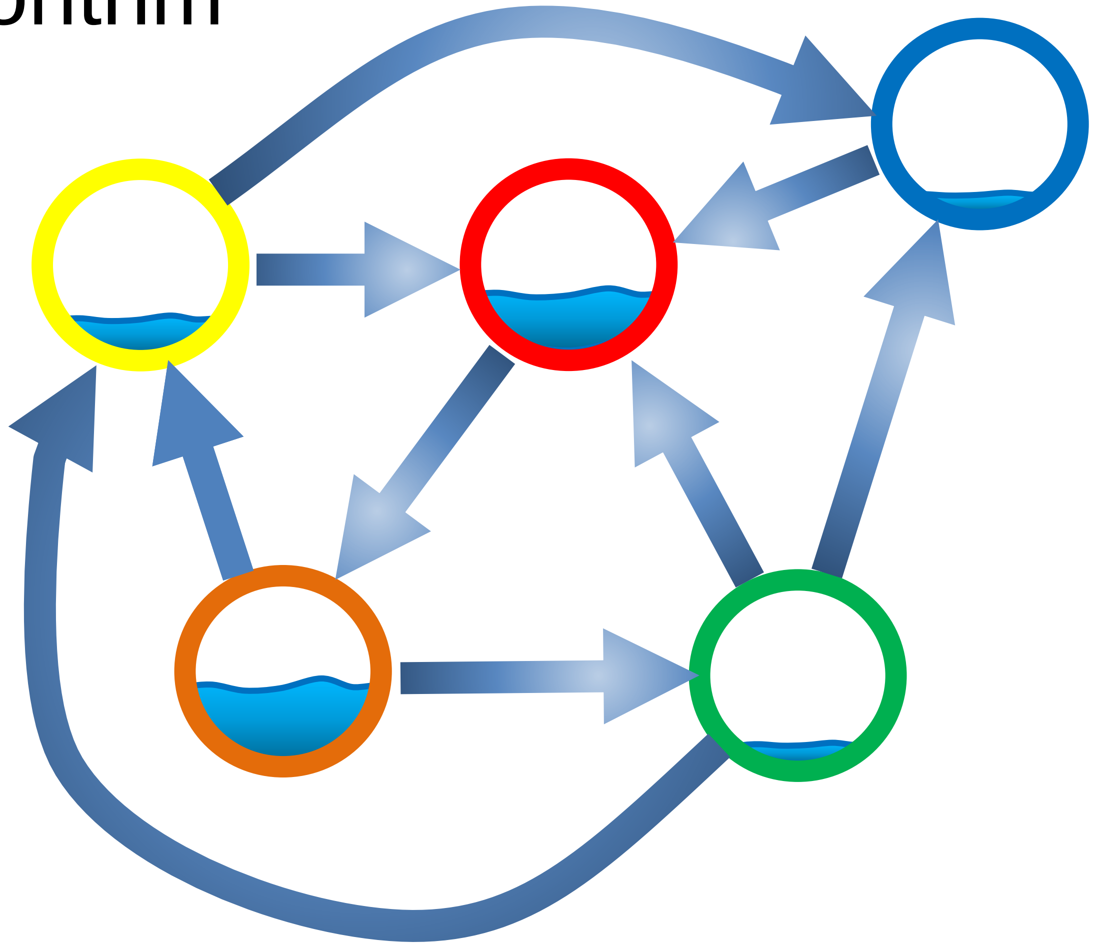

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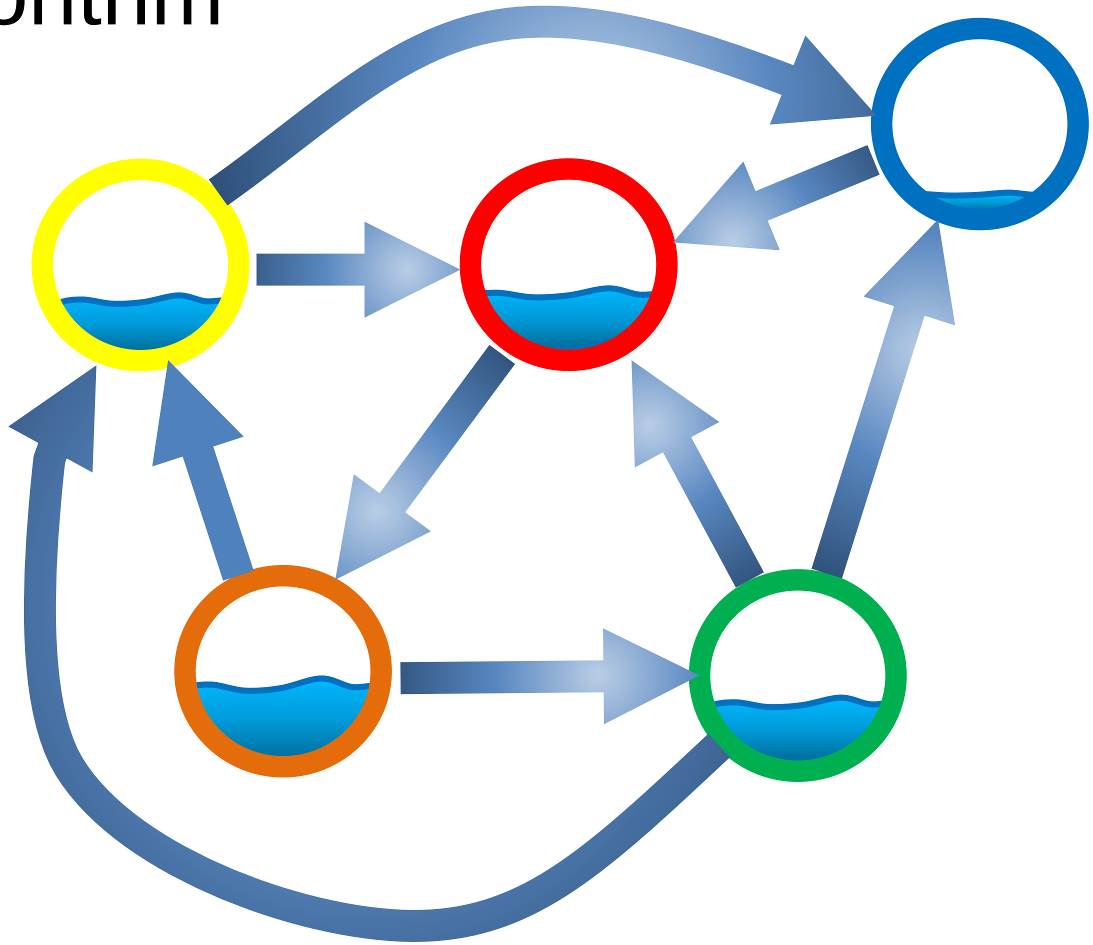
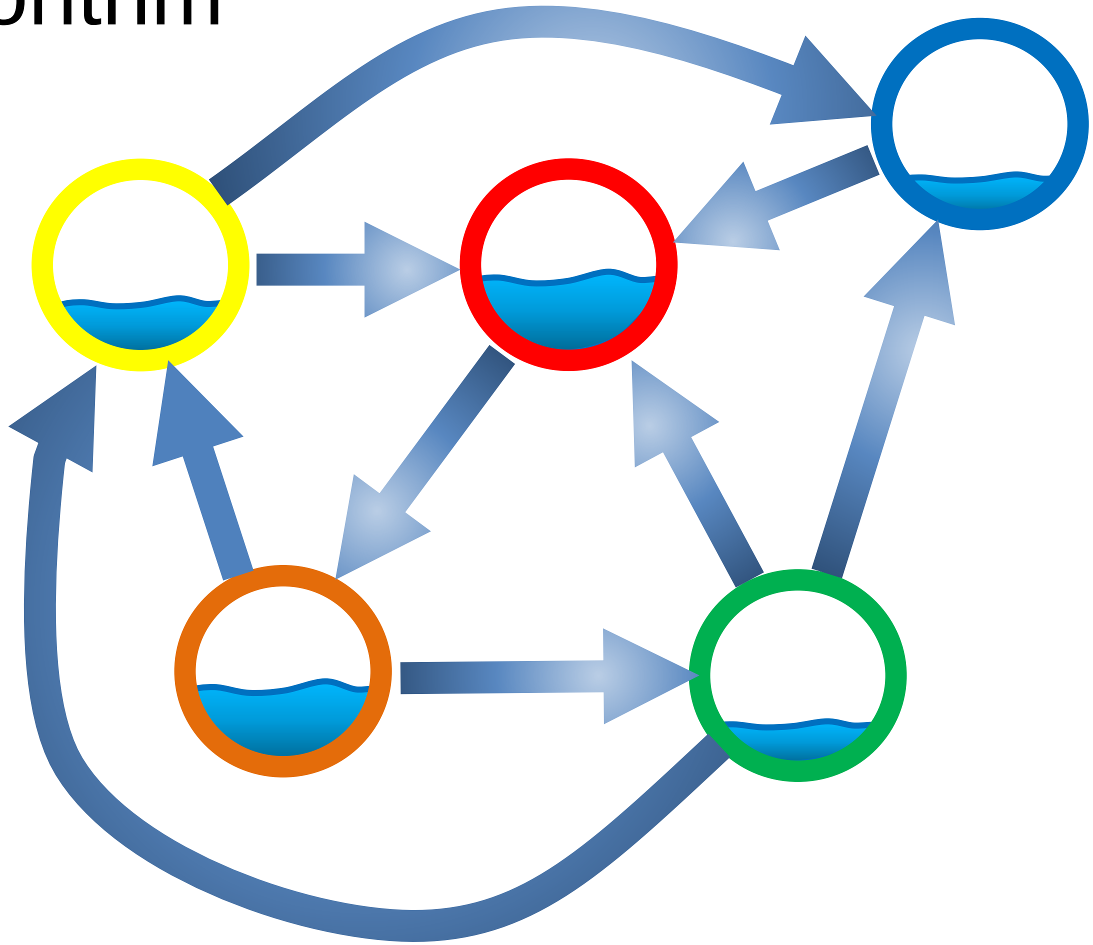


The PageRank algorithm

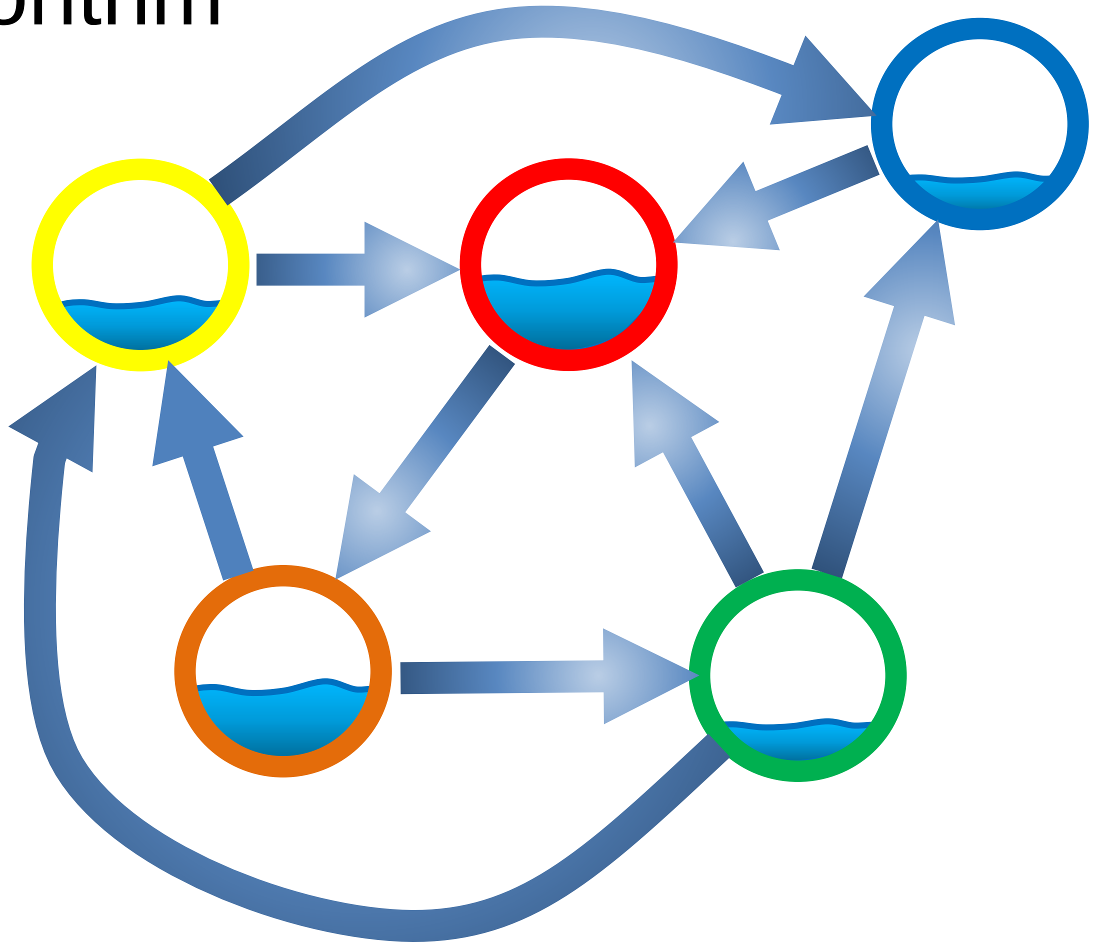
The system will reach an equilibrium state where the amount of liquid in each node remains constant.



The PageRank algorithm

The amount of liquid in each node determines the importance of the node.

Large quantity means large incoming flow from nodes with large quantity of liquid.



PageRank

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$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

w_v: the PageRank value of node v

Recursive definition

Example

$w_1 = 1/3\ w_4 + 1/2\ w_5$

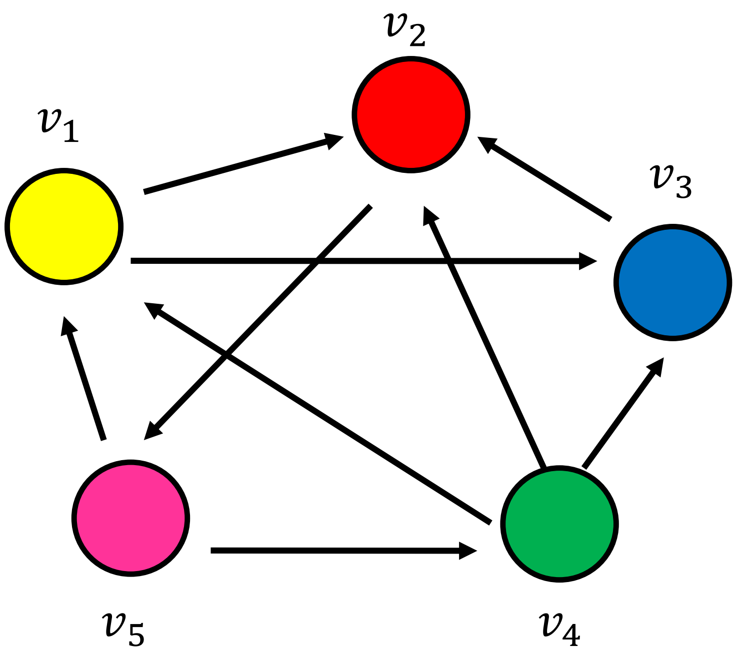
$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$

$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$



Computing PageRank weights

- A simple way to compute the weights is by iteratively updating the weights
- PageRank Algorithm

> Initialize all PageRank weights to $\frac{1}{n}$
>
> Repeat:
>
> $$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$
>
> Until the weights do not change

- This process converges

Example

$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

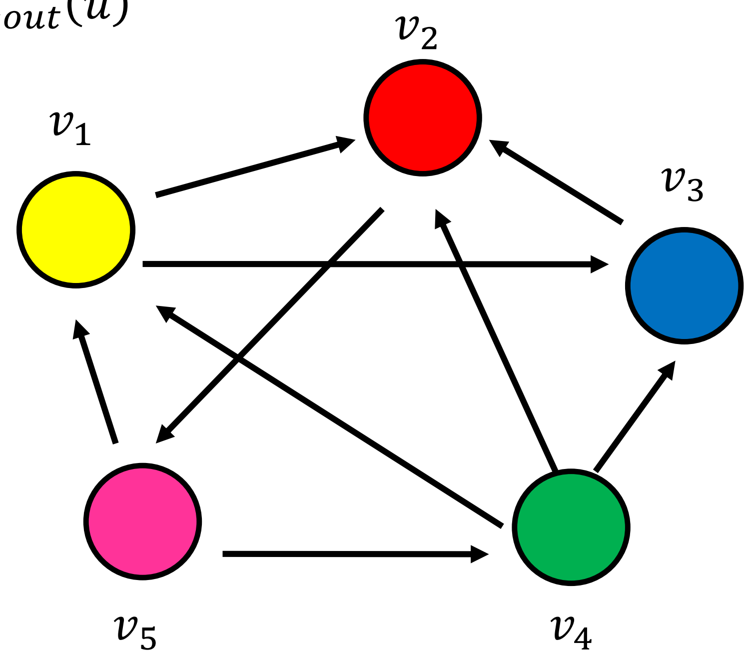
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



	w_1	w_2	w_3	w_4	w_5
t=0	0.2	0.2	0.2	0.2	0.2
t=1	0.16	0.36	0.16	0.1	0.2
t=2	0.13	0.28	0.11	0.1	0.36
t=3	0.22	0.22	0.1	0.18	0.28
t=4	0.2	0.27	0.17	0.14	0.22

Think of the weight as a fluid: there is constant amount of it in the graph, but it moves around until it stabilizes

Example

$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

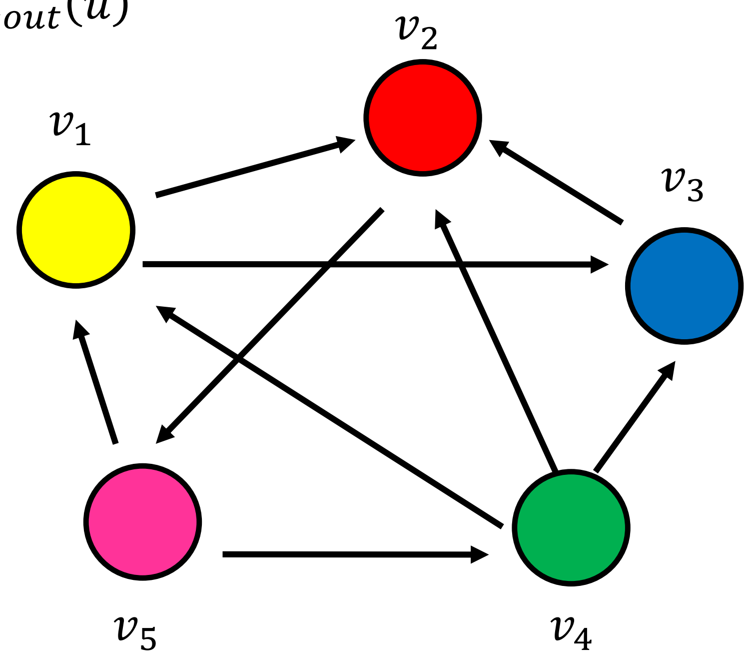
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



	w_1	w_2	w_3	w_4	w_5
t=25	0.18	0.27	0.13	0.13	0.27

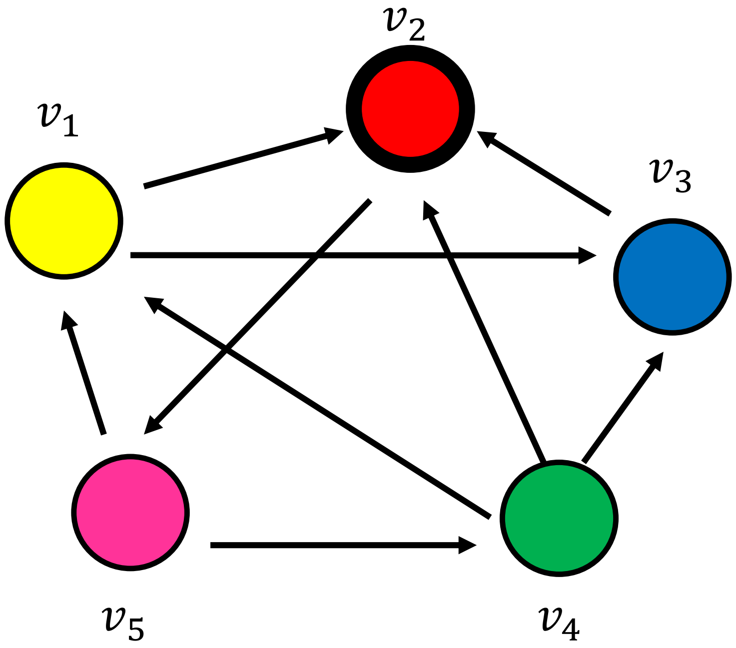
Think of the weight as a fluid: there is constant amount of it in the graph, but it moves around until it stabilizes

Random Walks on Graphs

- The algorithm defines a random walk on the graph
- Random walk:
  - Start from a node chosen uniformly at random with probability $\frac{1}{n}$.
    - Pick one of the outgoing edges uniformly at random
    - Move to the destination of the edge
    - Repeat.
- The Random Surfer model
  - Users wander on the web, following links.

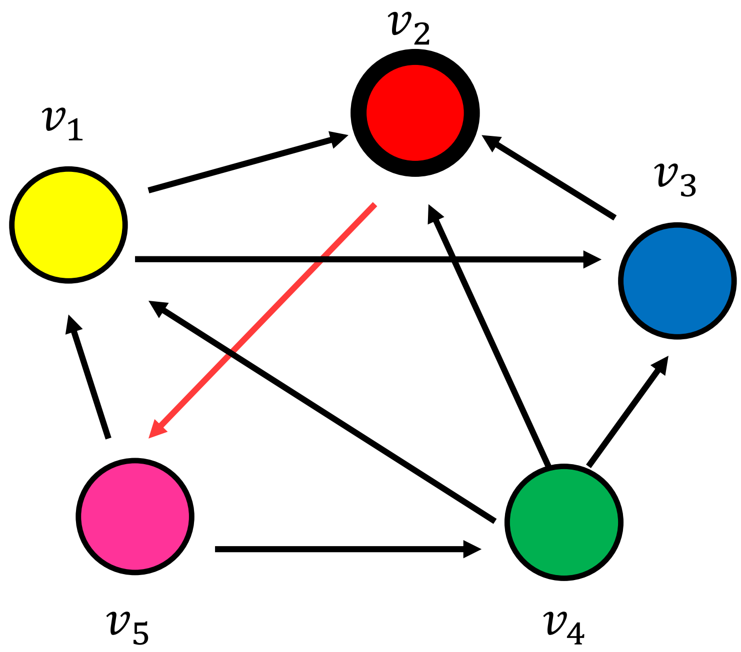
Example

- Step 0



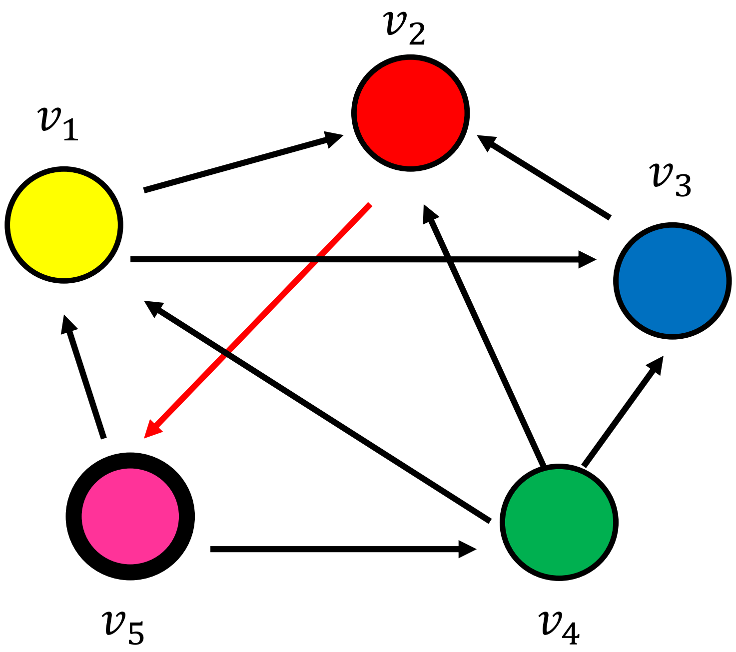
Example

- Step 0



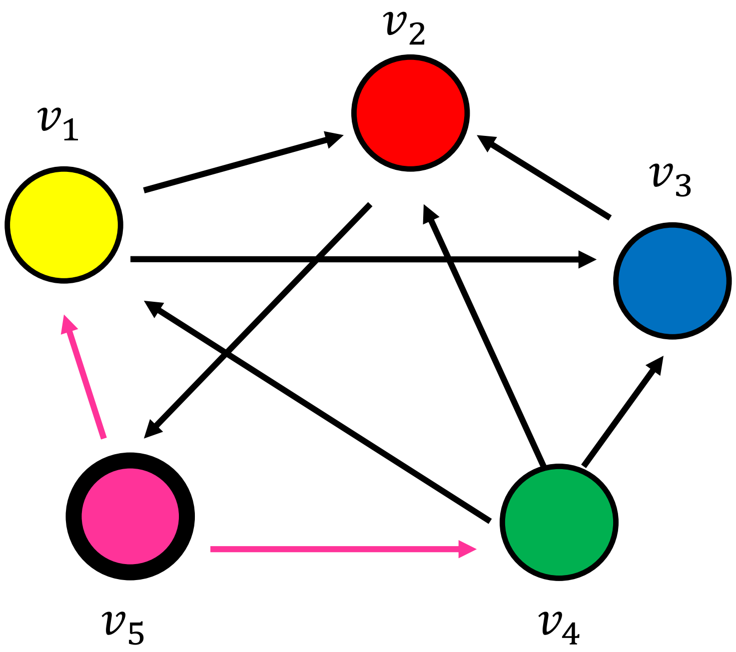
Example

- Step 1



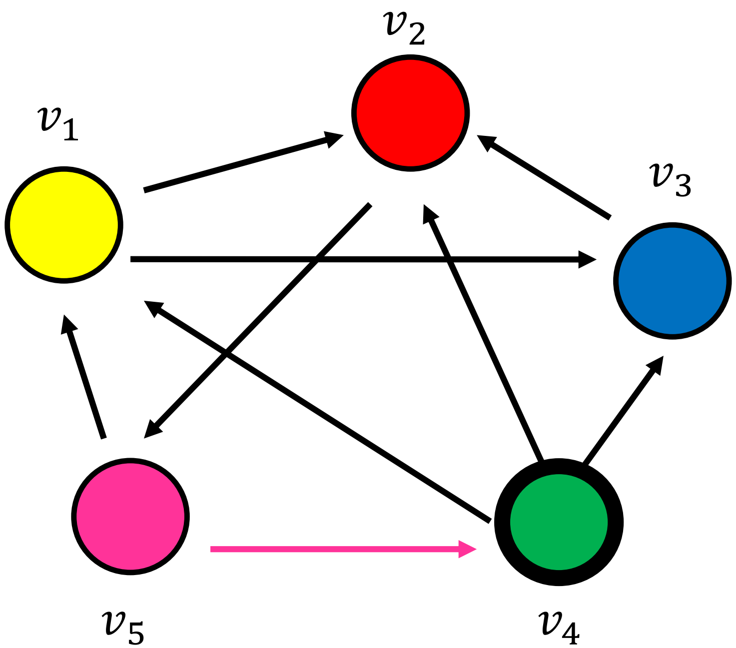
Example

- Step 1



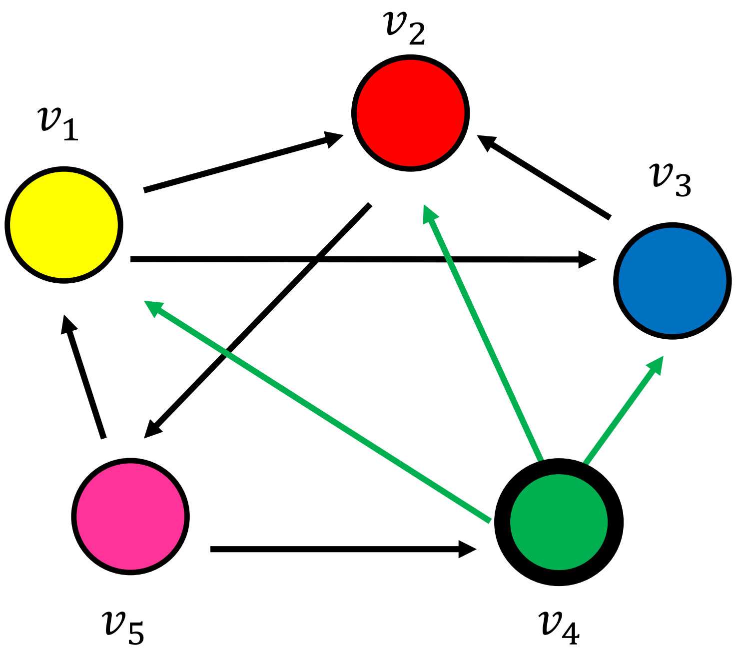
Example

- Step 2



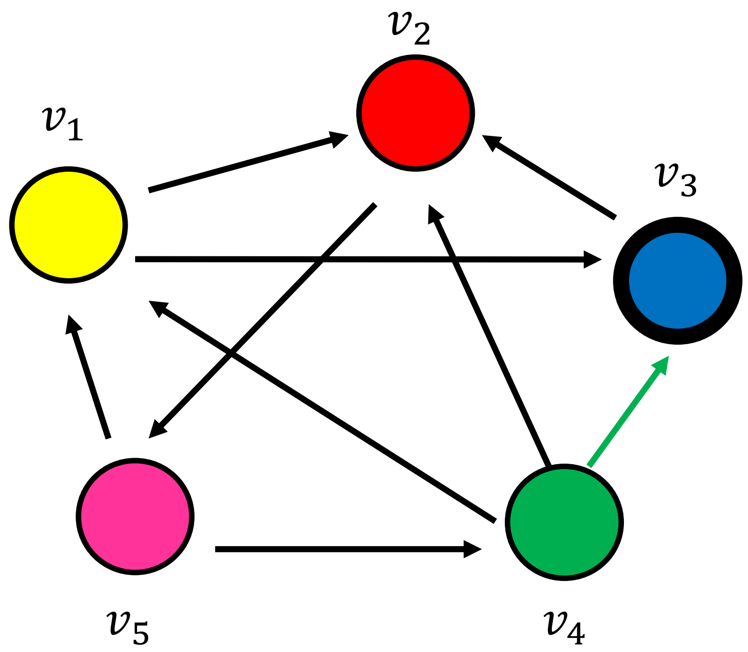
Example

- Step 2



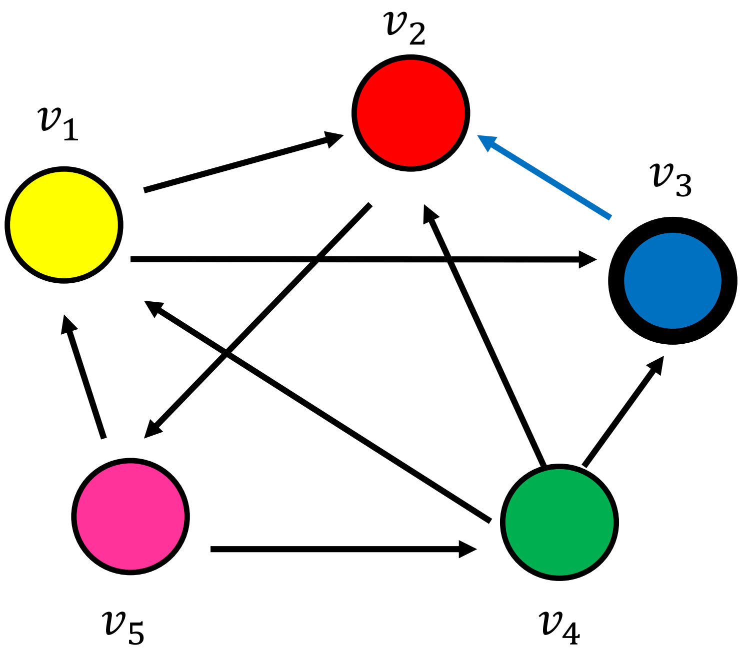
Example

- Step 3



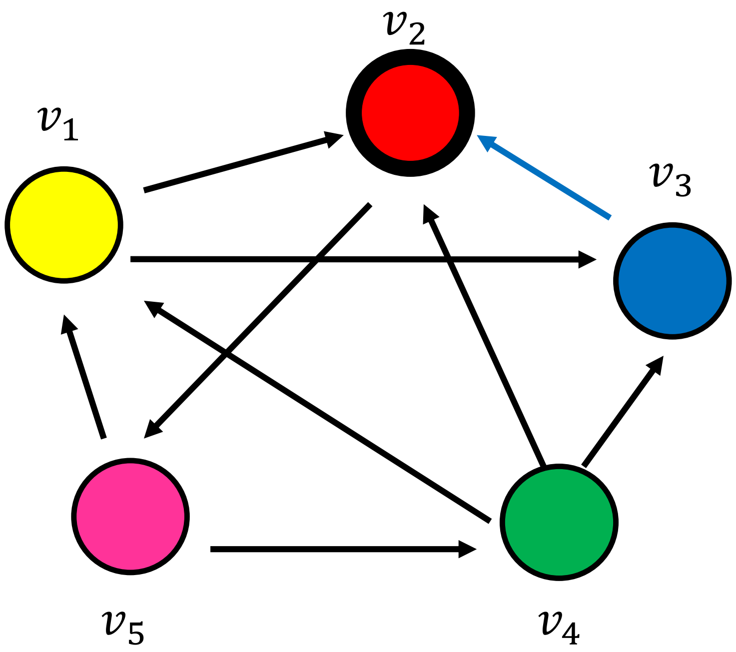
Example

- Step 3



Example

- Step 4…



Random walk

- Question: what is the probability p_i^t of being at node i after t steps?

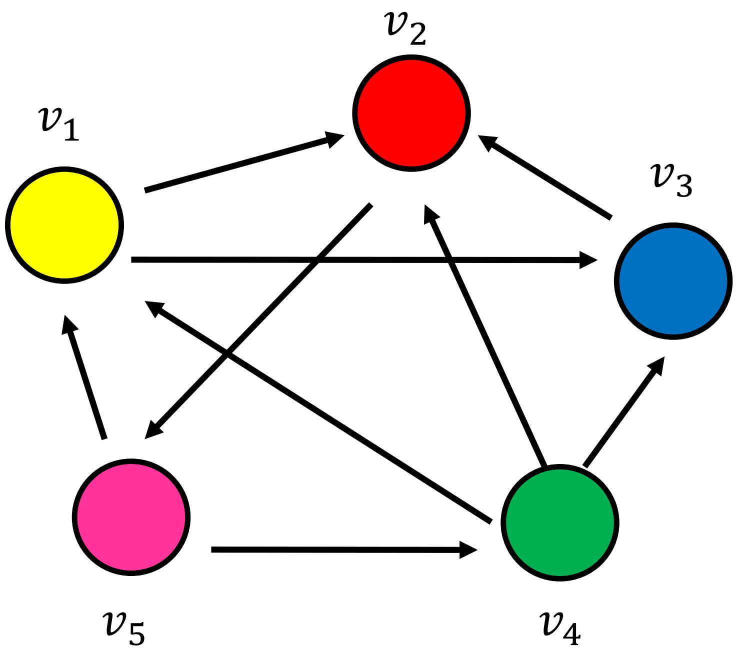
$$p_1^0 = \frac{1}{5} \qquad p_1^t = \frac{1}{3}p_4^{t-1} + \frac{1}{2}p_5^{t-1}$$

$$p_2^0 = \frac{1}{5} \qquad p_2^t = \frac{1}{2}p_1^{t-1} + p_3^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_3^0 = \frac{1}{5} \qquad p_3^t = \frac{1}{2}p_1^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_4^0 = \frac{1}{5} \qquad p_4^t = \frac{1}{2}p_5^{t-1}$$

$$p_5^0 = \frac{1}{5} \qquad p_5^t = p_2^{t-1}$$



The equations are the same as those for the PageRank computation

Markov chains

- A Markov chain describes a discrete time stochastic process over a set of states
$$S = \{s_1, s_2, \dots, s_n\}$$
according to a transition probability matrix $P = \{P_{ij}\}$
  - P_{ij} = probability of moving to state j when at state i

- Matrix P has the property that the entries of all rows sum to 1
$$\sum_j P[i,j] = 1$$
A matrix with this property is called stochastic

- State probability distribution: The vector $p^t = (p_i^t, p_2^t, \dots, p_n^t)$ that stores the probability of being at state s_i after t steps

- Memorylessness property: The next state of the chain depends only at the current state and not on the past of the process (first order MC)
  - Higher order MCs are also possible

- Markov Chain Theory: After infinite steps the state probability vector converges to a unique distribution if the chain is irreducible and aperiodic

Random walks

- Random walks on graphs correspond to Markov Chains
  - The set of states S is the set of nodes of the graph G
  - The transition probability matrix is the probability that we follow an edge from one node to another

$$P[i,j] = \frac{1}{\mathrm{d}_{out}(i)}$$

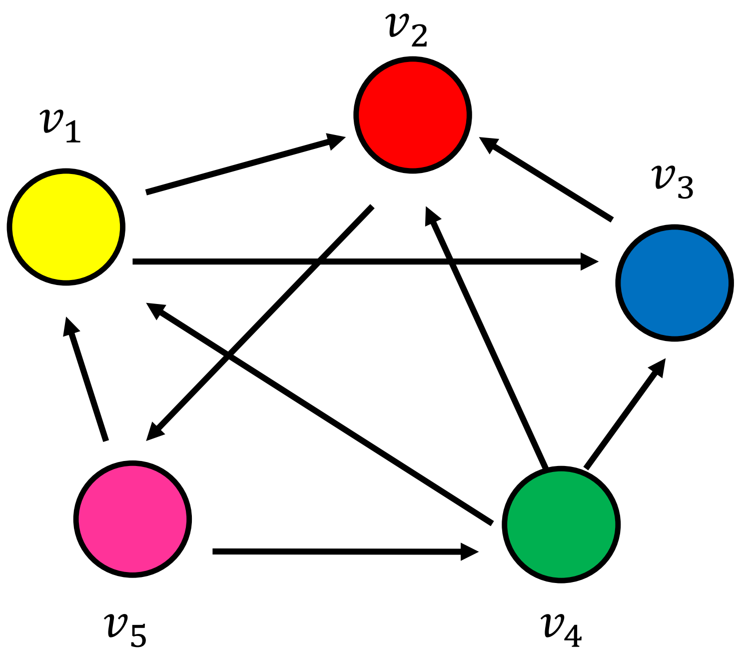
- We can compute the vector p^t at step t using a vector-matrix multiplication

$$p^{t+1} = p^t P$$

An example

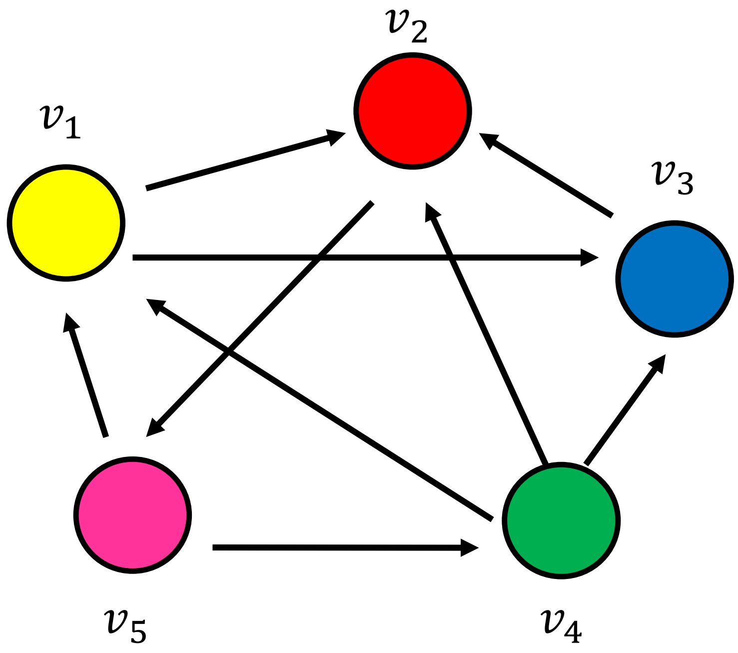
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



An example

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



$$p_1^t = \frac{1}{3} p_4^{t-1} + \frac{1}{2} p_5^{t-1}$$

$$p_2^t = \frac{1}{2} p_1^{t-1} + p_3^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_3^t = \frac{1}{2} p_1^{t-1} + \frac{1}{3} p_4^{t-1}$$

$$p_4^t = \frac{1}{2} p_5^{t-1}$$

$$p_5^t = p_2^{t-1}$$

Stationary distribution

- The stationary distribution of a random walk with transition matrix P, is a probability distribution π, such that $\pi = \pi P$
- The stationary distribution is an eigenvector of matrix P
  - the principal left eigenvector of P – stochastic matrices have maximum eigenvalue 1
- The probability π_i is the fraction of times that we visited state i as $t \to \infty$
- Markov Chain Theory: The random walk converges to a unique stationary distribution independent of the initial vector if the graph is strongly connected, and not bipartite.

Computing the stationary distribution

- The Power Method

> Initialize p^0 to some distribution
> Repeat
> $p^t = p^{t-1}P$
> Until convergence

- After many iterations $\mathrm{p}^{\mathrm{t}} \rightarrow \pi$ regardless of the initial vector p^0
- Power method because it computes $p^t = p^0 P^t$

- Rate of convergence
  - determined by the second eigenvalue λ_2

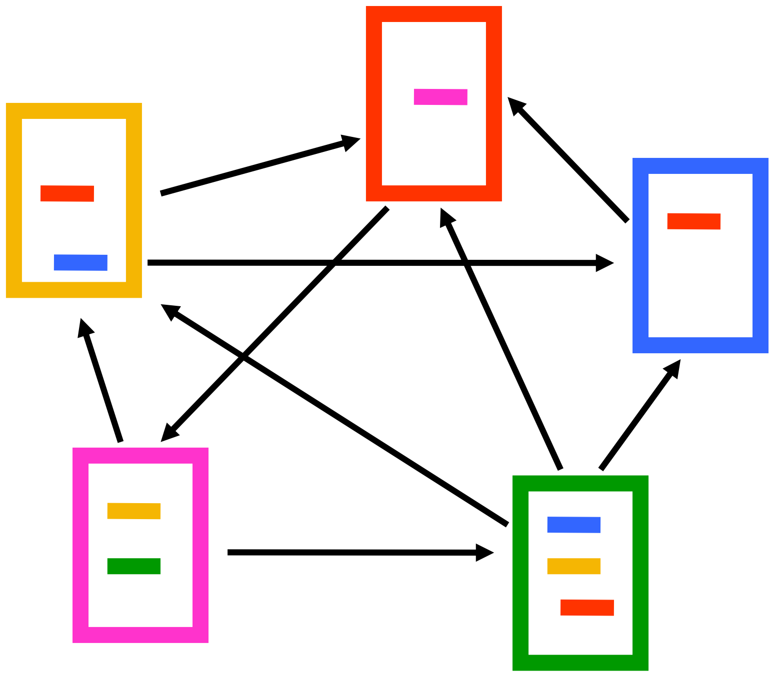
The stationary distribution

- What is the meaning of the stationary distribution π of a random walk?
- $\pi(i)$: the probability of being at node i after very large (infinite) number of steps
- π is the left eigenvector of transition matrix P
- $\pi = p_0 P^{\infty}$, where P is the transition matrix, p_0 the original vector
  - $P(i, j)$: probability of going from i to j in one step
  - $P^2(i, j)$: probability of going from i to j in two steps (probability of all paths of length 2)
  - $P^{\infty}(i, j) = \pi(j)$: probability of going from i to j in infinite steps – starting point does not matter.

The PageRank random walk

- Vanilla random walk
  - make the adjacency matrix stochastic and run a random walk

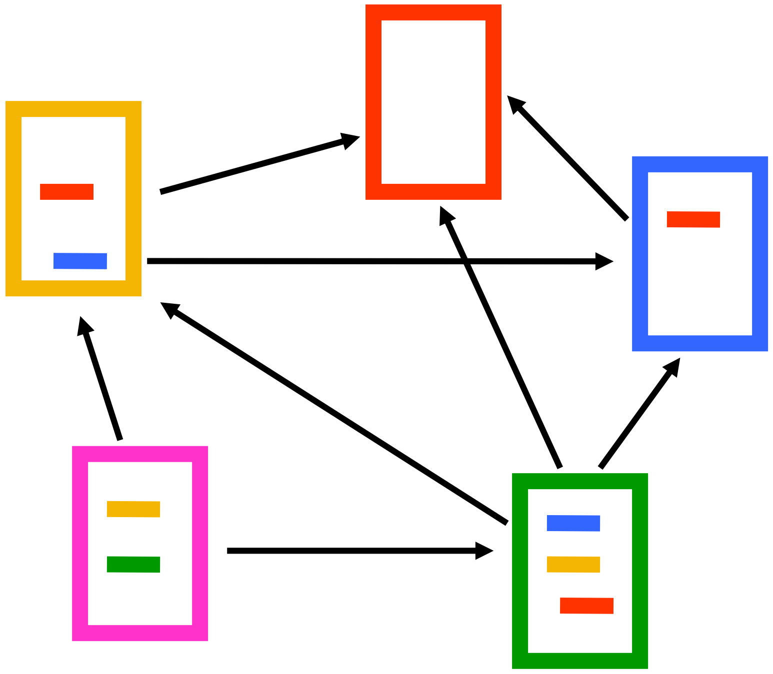
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- What about sink nodes?
  - what happens when the random walk moves to a node without any outgoing inks?

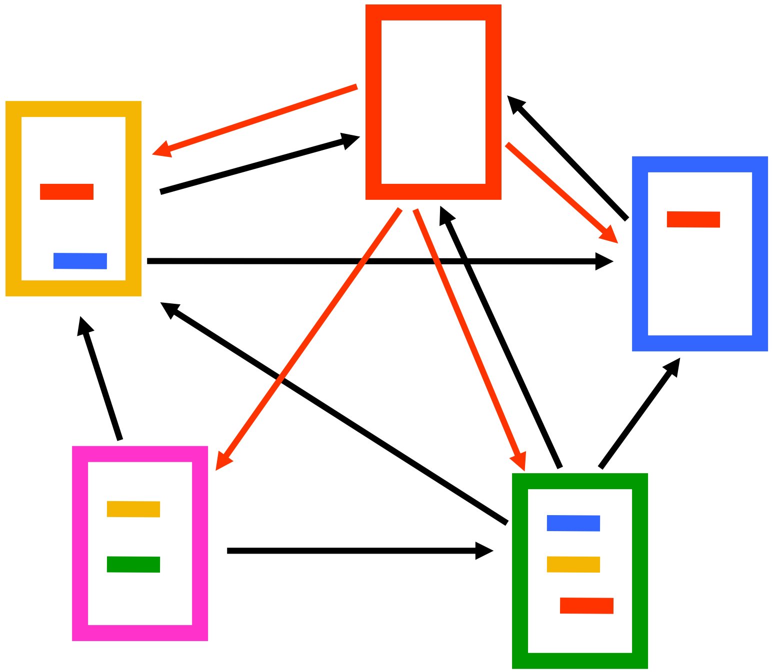
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- Replace these row vectors with a vector v
  - typically, the uniform vector

$$P' = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

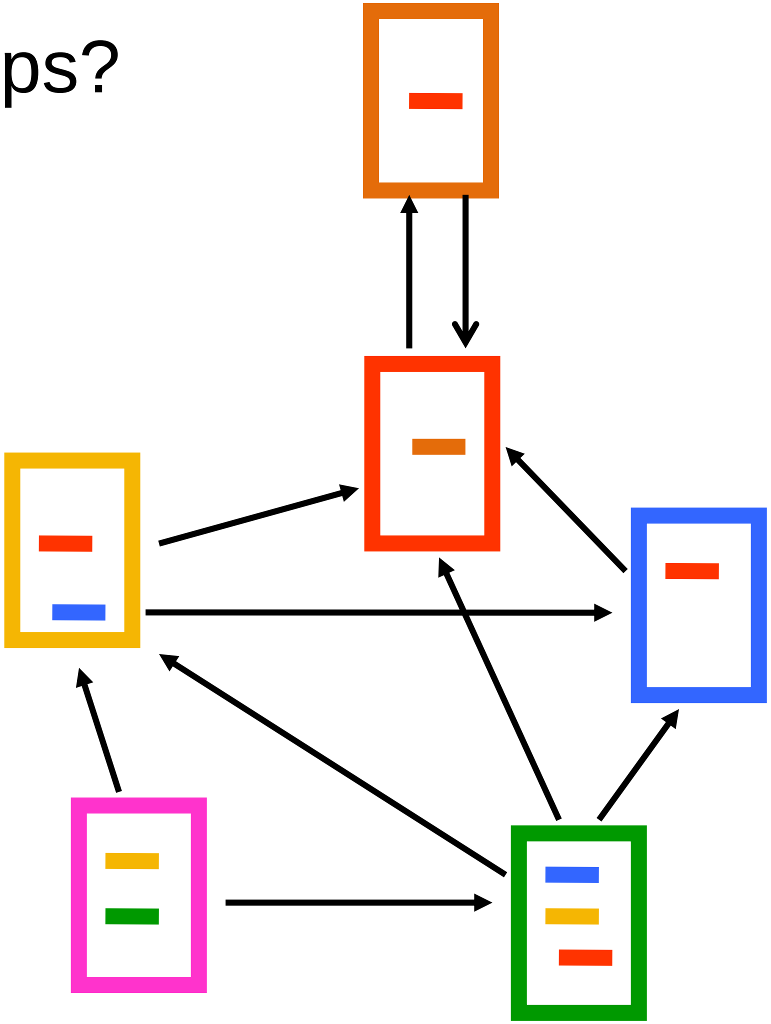


$P' = P + dv^T$

$$d = \begin{cases} 1 & \text{if i is sink} \\ 0 & \text{otherwise} \end{cases}$$

The PageRank random walk

- What about loops?
  - Spider traps



The PageRank random walk

- Add a random jump to vector v with prob α
  - Typically, to a uniform vector
  - Guarantees irreducibility, convergence
- You can think of the random jump as a restart of the random walk

$$P'' = (1-\alpha)\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + \alpha \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

$P'' = (1-\alpha)P' + \alpha u v^T$, where u is the vector of all 1s

Random walk with restarts

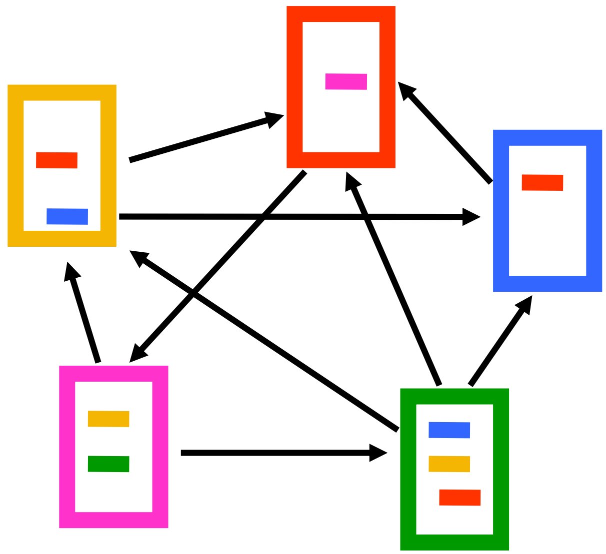
PageRank algorithm [BP98]

- Rank according to the stationary distribution

$$w_v = (1-\alpha)\sum_{u\to v}\frac{1}{d_{out}(u)}w_u + \alpha\frac{1}{n}$$

  - $\alpha = 0.15$ in most cases

- The Random Surfer model
  - Start with a random page
  - With probability α follow one of the links in the page
  - With probability $1-\alpha$ restart from a random page



1. **Red Page**
2. **Purple Page**
3. **Yellow Page**
4. **Blue Page**
5. **Green Page**

Stationary distribution with random jump

- If v is the jump vector

$$p^0 = v$$
$$p^1 = (1-\alpha)p^0 P + \alpha v = (1-\alpha)vP + \alpha v$$
$$p^2 = (1-\alpha)p^1 P + \alpha v = (1-\alpha)^2 vP^2 + (1-\alpha)\alpha vP + \alpha v$$
$$p^2 = (1-\alpha)p^2 P + \alpha v = (1-\alpha)^3 vP^3 + (1-\alpha)^2 \alpha vP^2 + +(1-\alpha)\alpha vP + \alpha v$$
$$\vdots$$
$$p^\infty = \alpha v + (1-\alpha)\alpha vP + (1-\alpha)^2 \alpha vP^2 + \cdots = \alpha(I-(1-\alpha)P)^{-1}$$

- Explanation: When you start a random walk:
  - With probability α you will restart immediately
  - With probability $(1-\alpha)\alpha$ you will do one step and then restart
  - With probability $(1-\alpha)^2\alpha$ you will do two steps and then restart
  - Etc…
- Conclusion: you are not likely to walk very far
  - On average the random walk restarts every $1/\alpha$ steps

Stationary distribution with random jump

- With the random jump the shorter paths are more important, since the weight decreases exponentially
  - This changes the stationary distribution. When starting from some node x, nodes close to x have higher probability
- Jump/Restart vector:
  - If v is not uniform, we can bias the random walk towards the nodes that are close to v
  - Personalized Pagerank:
    - Always restart to some node x
      - E.g., the home page of a user
  - Topic-Specific Pagerank
    - Restart to nodes about a specific topic
      - E.g., Greek pages, University home pages
      - Anti-spam

Random walks on undirected graphs

- For undirected graphs, the stationary distribution is proportional to the degrees of the nodes
  - Thus in this case a random walk is the same as degree popularity
- This is no longer true if we do random jumps
  - Now the short paths play a greater role, and the previous distribution does not hold.

Pagerank implementation

- Store the graph in adjacency list, or list of edges
- Keep current pagerank values and new pagerank values
- Go through edges and update the values of the destination nodes.
- Repeat until the difference (L_1 or L_∞ difference) is below some small value ε.

A (Matlab-friendly) PageRank algorithm

- Performing vanilla power method is now too expensive – the matrix is not sparse

$q^0 = v$

$t = 1$

repeat

$\quad q^t = (P'')^T q^{t-1}$

$\quad \delta = \|q^t - q^{t-1}\|$

$\quad t = t + 1$

until $\delta < \varepsilon$

Efficient computation of $y = (P'')^T x$

$$y = (1-\alpha) P^T x$$
$$\beta = \|x\|_1 - \|y\|_1$$
$$y = y + \beta v$$

P = normalized adjacency matrix

$P' = P + dv^T$, where d_i is 1 if i is sink and 0 o.w.

$P'' = (1-\alpha)P' + \alpha uv^T$, where u is the vector of all 1s

Pagerank history

- Huge advantage for Google in the early days
  - It gave a way to get an idea for the value of a page, which was useful in many different ways
    - Put an order to the web.
  - After a while it became clear that the anchor text was probably more important for ranking
  - Also, link spam became a new (dark) art
- Flood of research
  - Numerical analysis got rejuvenated
  - Huge number of variations
  - Efficiency became a great issue.
  - Huge number of applications in different fields
    - Random walk is often referred to as PageRank.

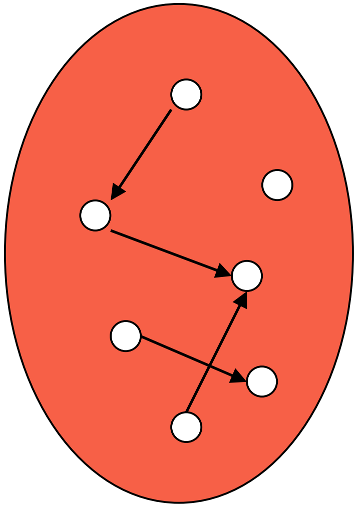
THE HITS ALGORITHM

The HITS algorithm

- Another algorithm proposed around the same time as Pagerank for using the hyperlinks to rank pages
  - Kleinberg: then an intern at IBM Almaden
  - IBM never made anything out of it

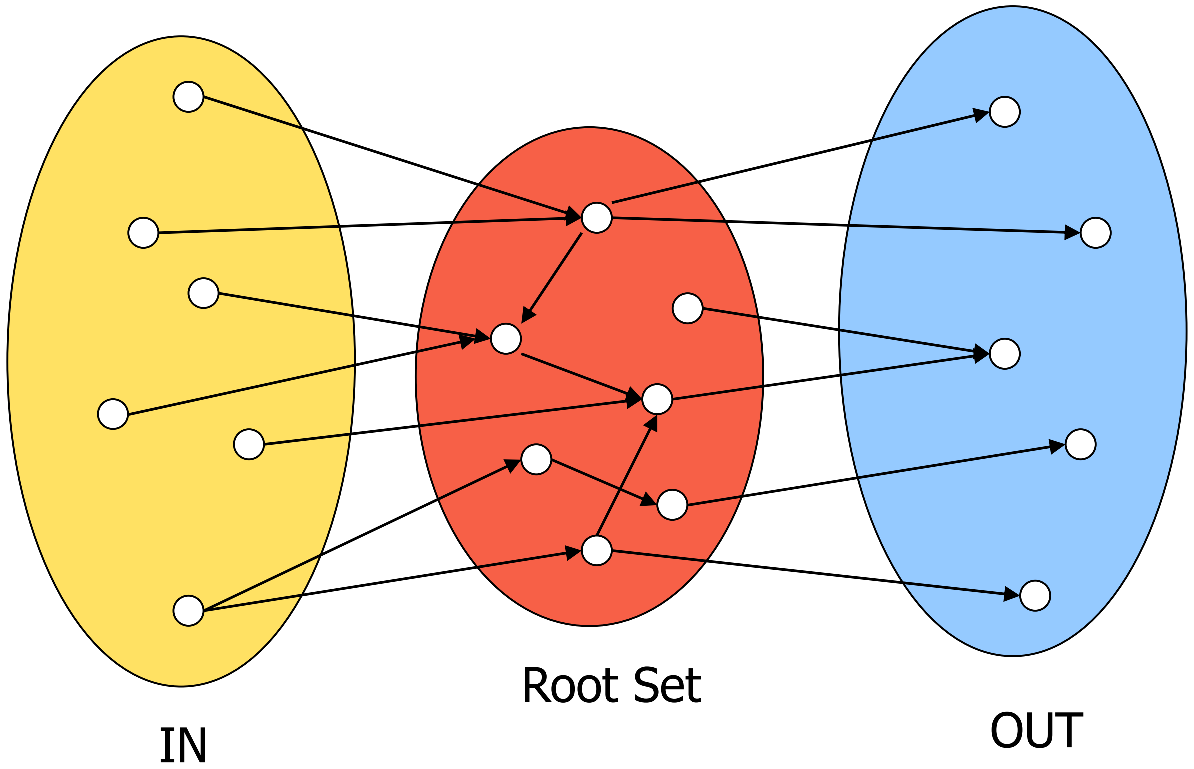
Query dependent input

Root set obtained from a text-only search engine

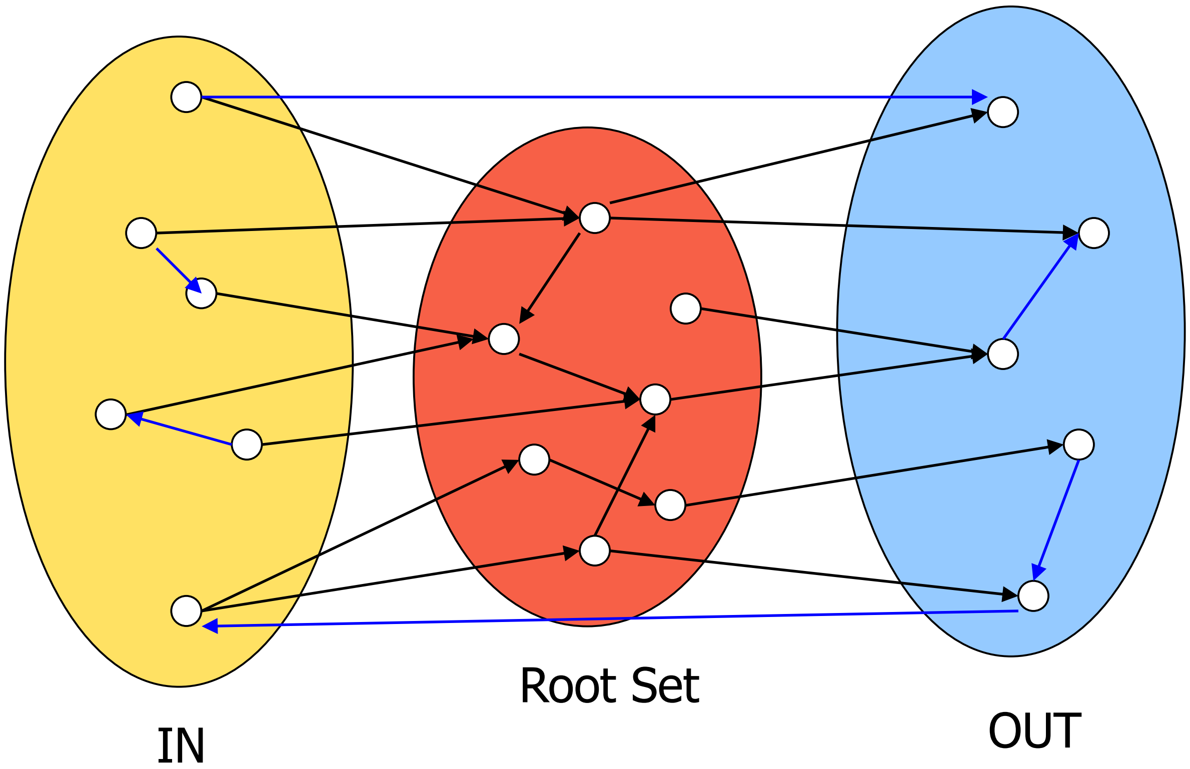


Root Set

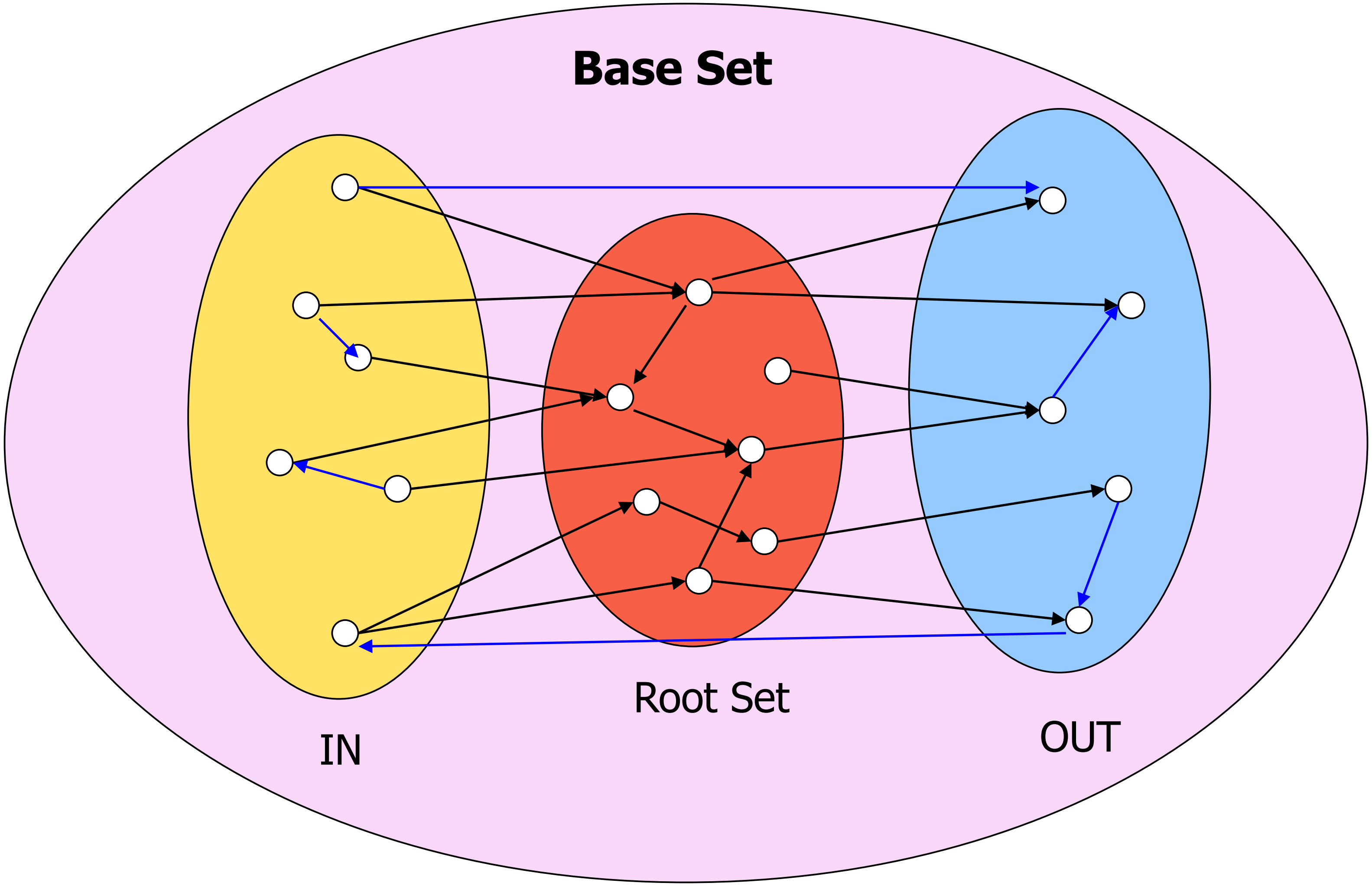
Query dependent input



Query dependent input

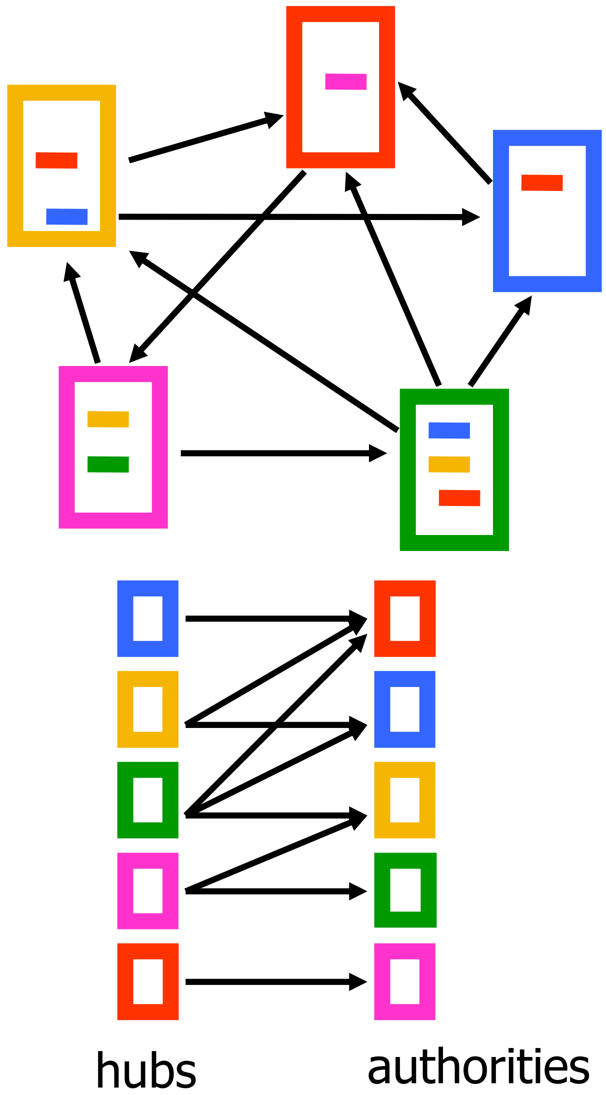


Query dependent input



Hubs and Authorities [K98]

- Authority is not necessarily transferred directly between authorities
- Pages have double identity
  - hub identity
  - authority identity
- Good hubs point to good authorities
- Good authorities are pointed by good hubs



Hubs and Authorities

- Two kind of weights:
  - Hub weight
  - Authority weight

- The hub weight is the sum of the authority weights of the authorities pointed to by the hub

- The authority weight is the sum of the hub weights that point to this authority.

HITS Algorithm

- Initialize all weights to 1.
- Repeat until convergence
  - O operation : hubs collect the weight of the authorities

$$h_i = \sum_{j:i \to j} a_j$$

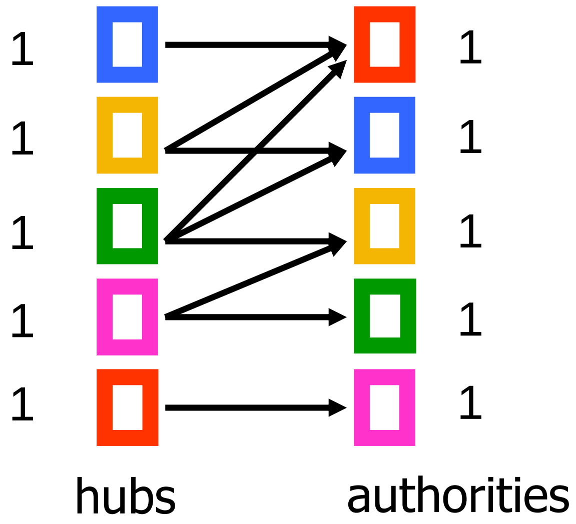
  - I operation: authorities collect the weight of the hubs

$$a_i = \sum_{j:j \to i} h_j$$

  - Normalize weights under some norm

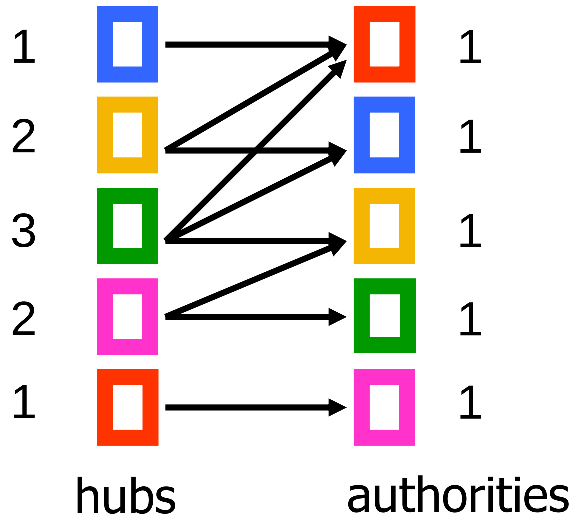
Example

Initialize



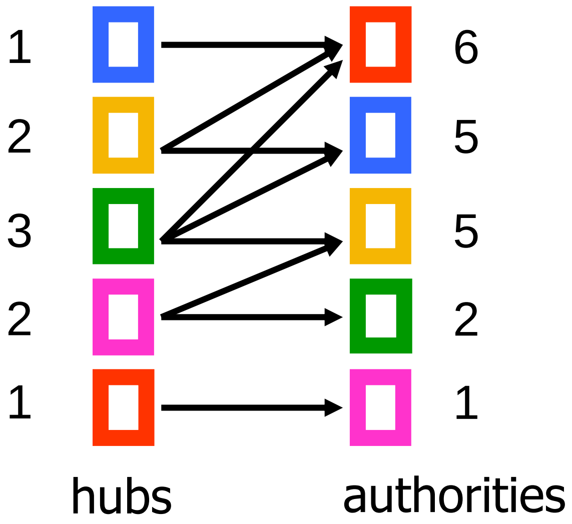
Example

Step 1: O operation



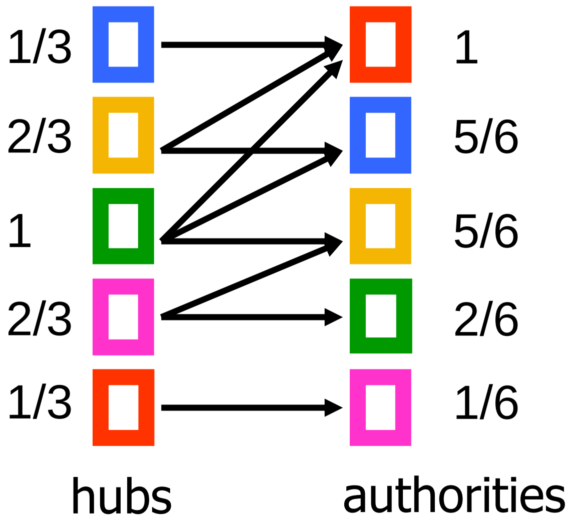
Example

Step 1: I operation



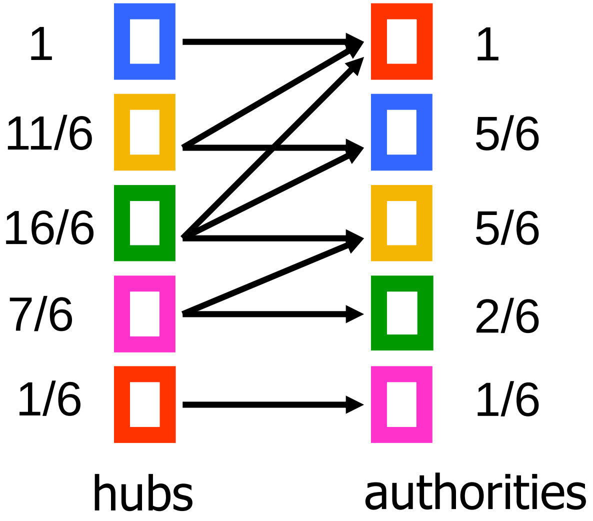
Example

Step 1: Normalization (Max norm)



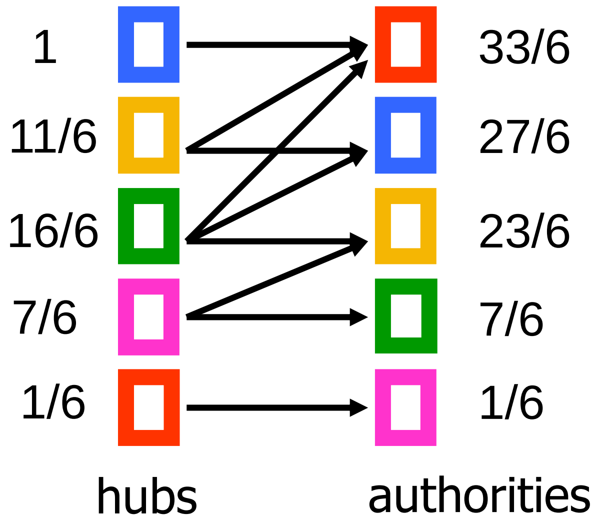
Example

Step 2: O step



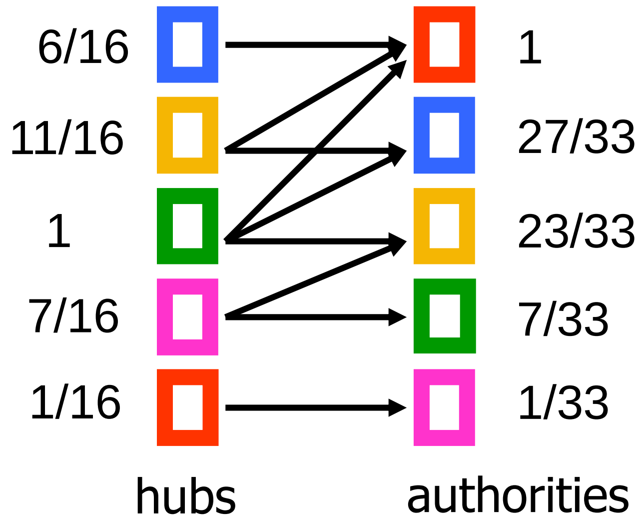
Example

Step 2: I step



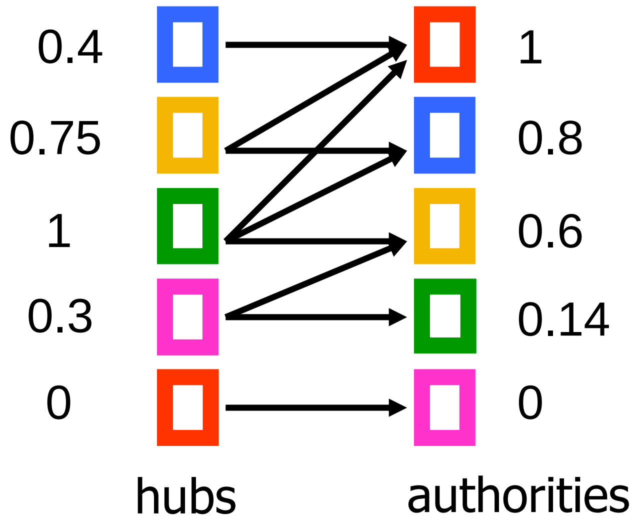
Example

Step 2: Normalization



Example

Convergence



HITS and eigenvectors

- The HITS algorithm is a power-method eigenvector computation
- In vector terms
  - $a^t = A^T h^{t-1}$ and $h^t = A a^{t-1}$
  - $a^t = A^T A a^{t-1}$ and $h^t = A A^T h^{t-1}$
  - Repeated iterations will converge to the eigenvectors

- The authority weight vector a is the eigenvector of $A^T A$
- The hub weight vector h is the eigenvector of $A A^T$

- The vectors a and h are called the singular vectors of the matrix A

Singular Value Decomposition

$$A = U \quad \Sigma \quad V^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_r \end{bmatrix}$$

$[n \times r] \; [r \times r] \; [r \times n]$

- **r** : rank of matrix A
- $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$: singular values (square roots of eig-vals AA^T, A^TA)
- $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r$: left singular vectors (eig-vectors of AA^T)
- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$: right singular vectors (eig-vectors of A^TA)
- $$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

Why does the Power Method work?

- If a matrix R is real and symmetric, it has real eigenvalues and eigenvectors: (λ_1, w_1), (λ_2, w_2), ... , (λ_r, w_r)
  - r is the rank of the matrix
  - $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|$
- For any matrix R, the eigenvectors $w_1, w_2, \ldots, w_r$ of R define a basis of the vector space
  - For any vector x, $Rx = \alpha_1 w_1 + a_2 w_2 + \cdots + a_r w_r$
- After t multiplications we have:

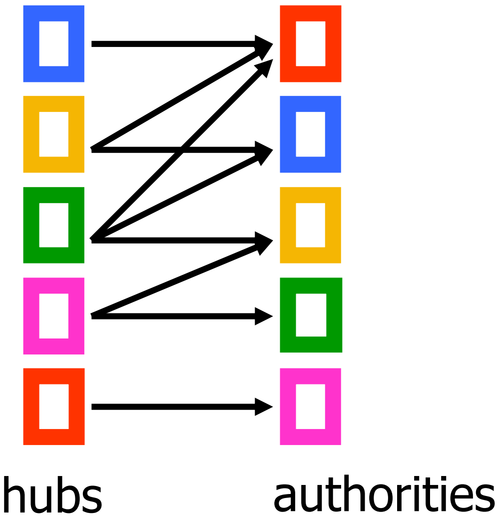
$$R^t x = \lambda_1^{t-1} \alpha_1 w_1 + \lambda_2^{t-1} a_2 w_2 + \cdots + \lambda_2^{t-1} a_r w_r$$

- Normalizing leaves only the term w_1.

OTHER ALGORITHMS

The SALSA algorithm [LM00]

- Perform a random walk alternating between hubs and authorities



- What does this random walk converge to?

- The graph is essentially undirected, so it will be proportional to the degree.

Social network analysis

- Evaluate the centrality of individuals in social networks
  - degree centrality
    - the (weighted) degree of a node
  - distance centrality
    - the average (weighted) distance of a node to the rest in the graph

$$D_c(v) = \frac{1}{\sum_{u \neq v} d(v,u)}$$

  - betweenness centrality
    - the average number of (weighted) shortest paths that use node v

$$B_c(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

Counting paths – Katz 53

- The importance of a node is measured by the weighted sum of paths that lead to this node
- $A^m[i,j]$ = number of paths of length m from i to j
- Compute

$$P = bA + b^2A^2 + \cdots + b^mA^m + \cdots = (I - bA)^{-1} - I$$

- converges when $b < \lambda_1(A)$
- Rank nodes according to the column sums of the matrix P

Bibliometrics

- Impact factor (E. Garfield 72)
  - counts the number of citations received for papers of the journal in the previous two years
- Pinsky-Narin 76
  - perform a random walk on the set of journals
  - P_{ij} = the fraction of citations from journal i that are directed to journal j