# Online Social Networks and Media

Community detection

# Notes on Homework 1

1. You should write your *own code for generating the graphs.*

You may use SNAP graph primitives (e.g., add node/edge)

2. For the *degree distribution*:
you should produce **5** *plots* (simple distribution, bins of equal size, bins of exponential size, cumulative, zipf)

3. For all your measurements, generate a number, say 100, different graphs and report *average values* (this refers to the graphs in cases (a) and (c))

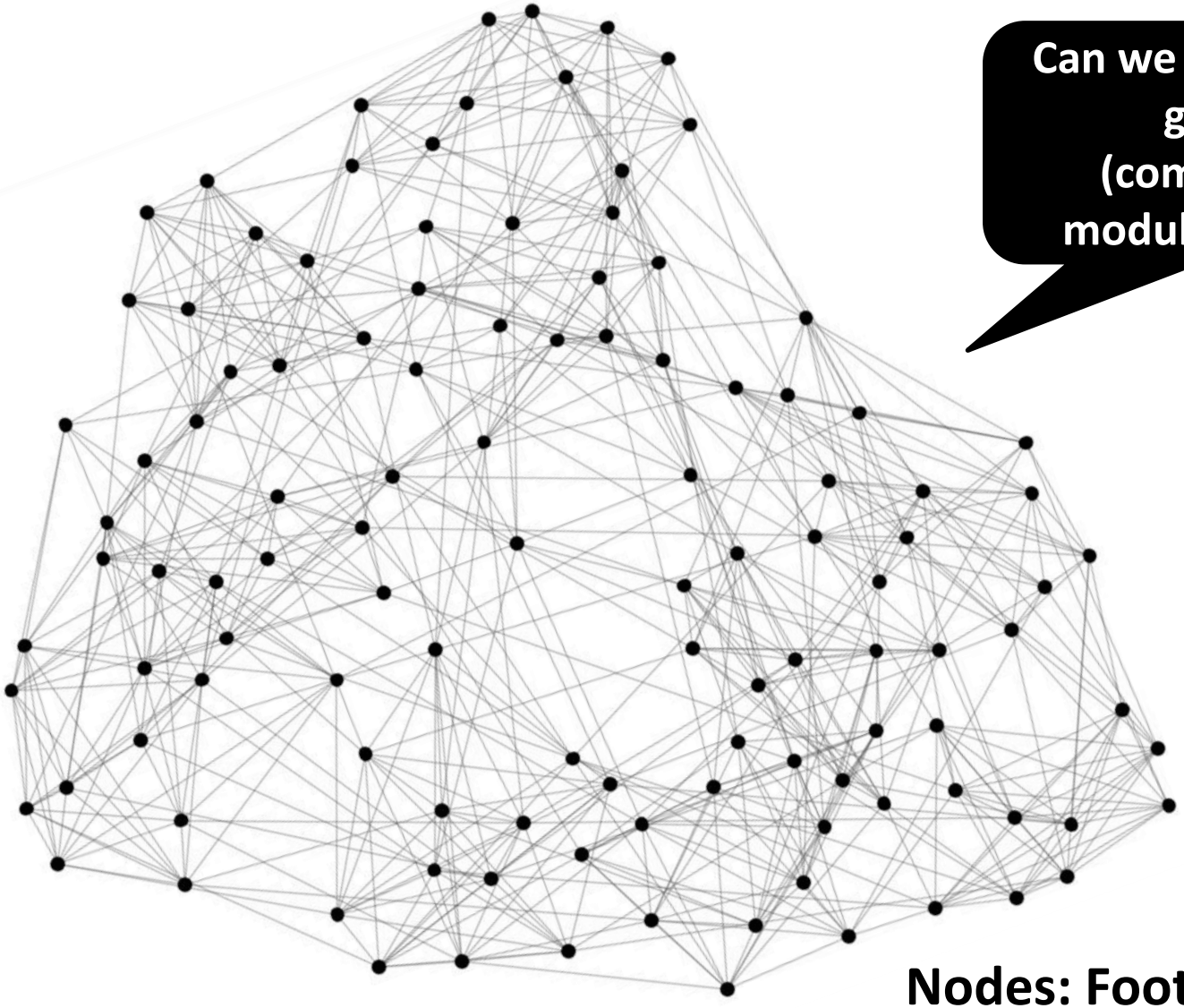
# Introduction

Real networks are *not random graphs*

Communities
aka: groups, clusters, cohesive subgroups, modules

*(informal) Definition: groups of vertices* which probably share *common properties* and/or play *similar roles* within the graph

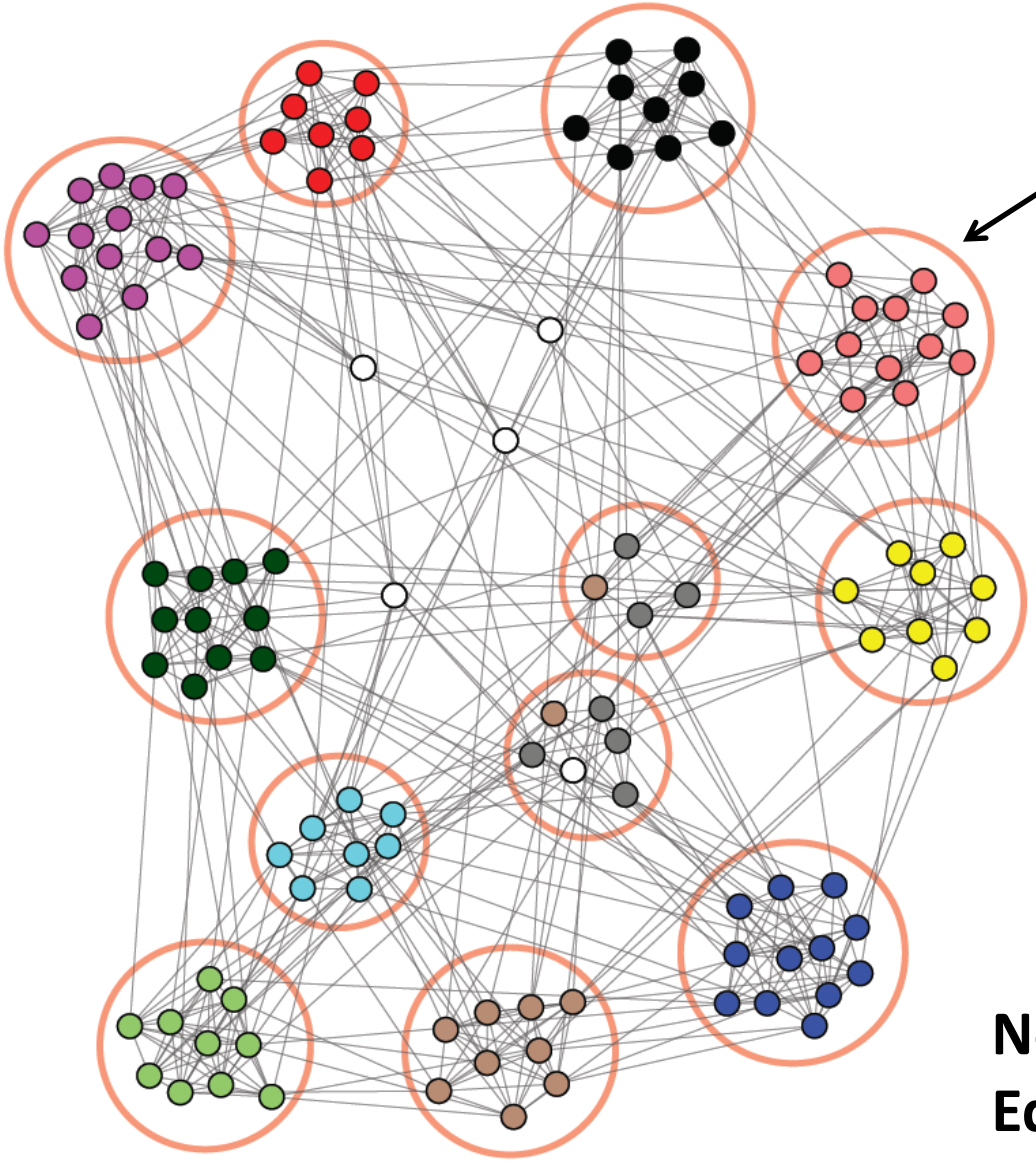
Some are *explicit (emic)* (e.g., Facebook (groups), LinkedIn (groups, associations), etc), we are interested in *implicit (etic)* ones



Can we identify node groups? (communities, modules, clusters)

**Nodes: Football Teams**
**Edges: Games played**

# NCAA Football Network

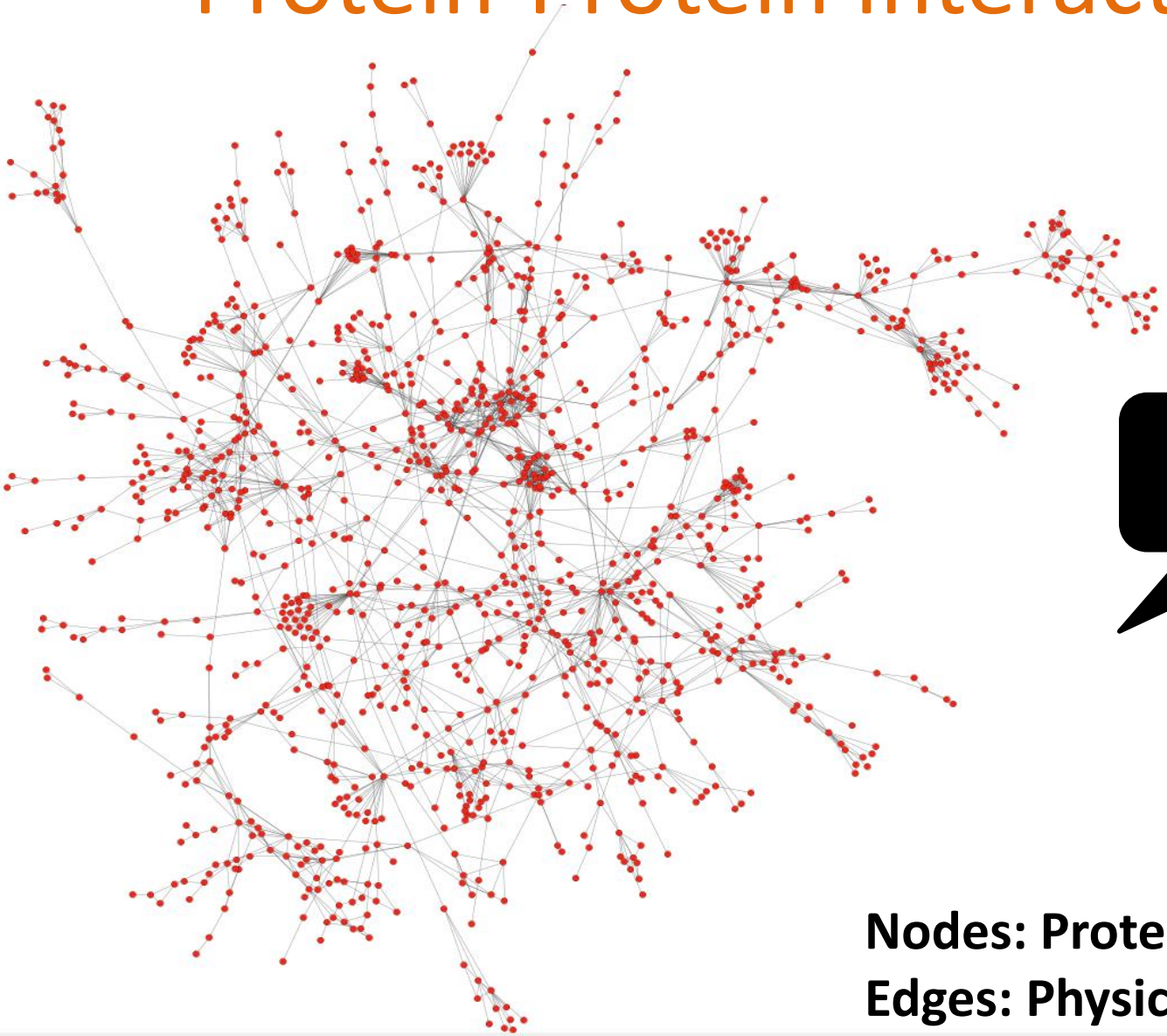


**NCAA conferences**

- Mid American
- Big East
- Atlantic Coast
- SEC
- Conference USA
- Big 12
- Western Athletic
- Pacific 10
- Mountain West
- Big 10
- Sun Belt
- Independents

**Nodes: Football Teams**
**Edges: Games played**

# Protein-Protein Interactions

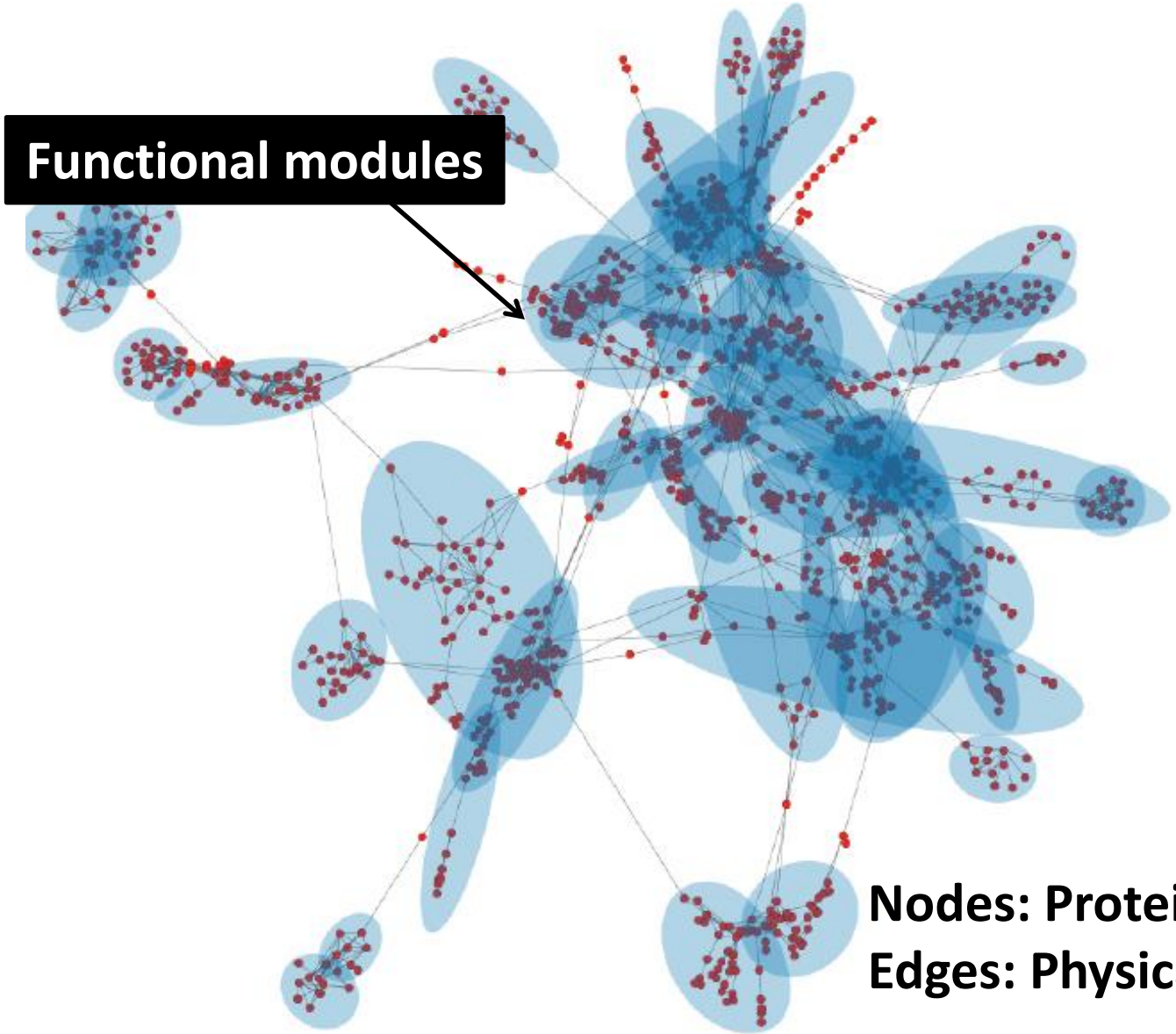


Can we identify functional modules?

**Nodes: Proteins**

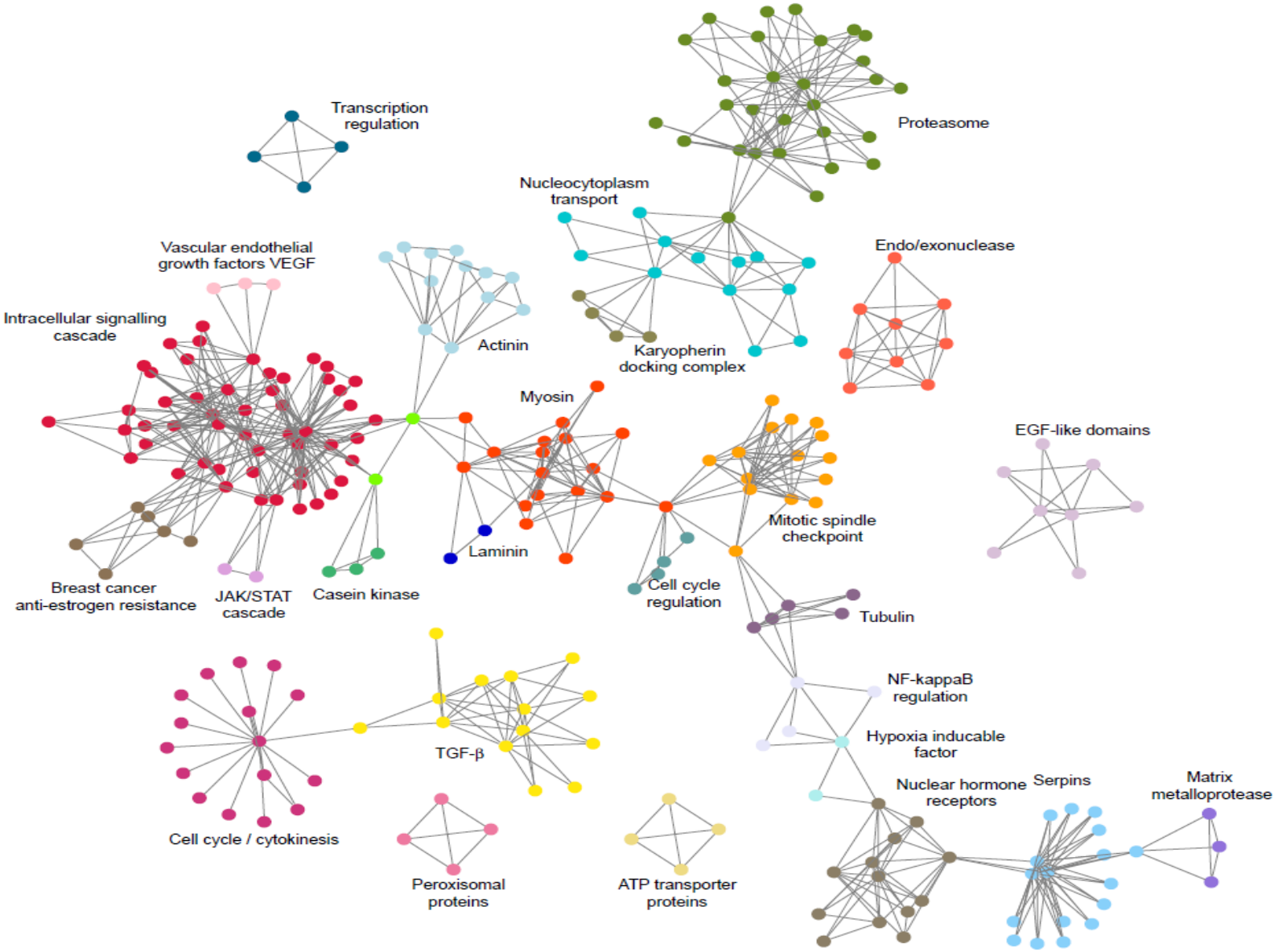
**Edges: Physical interactions**

# Protein-Protein Interactions

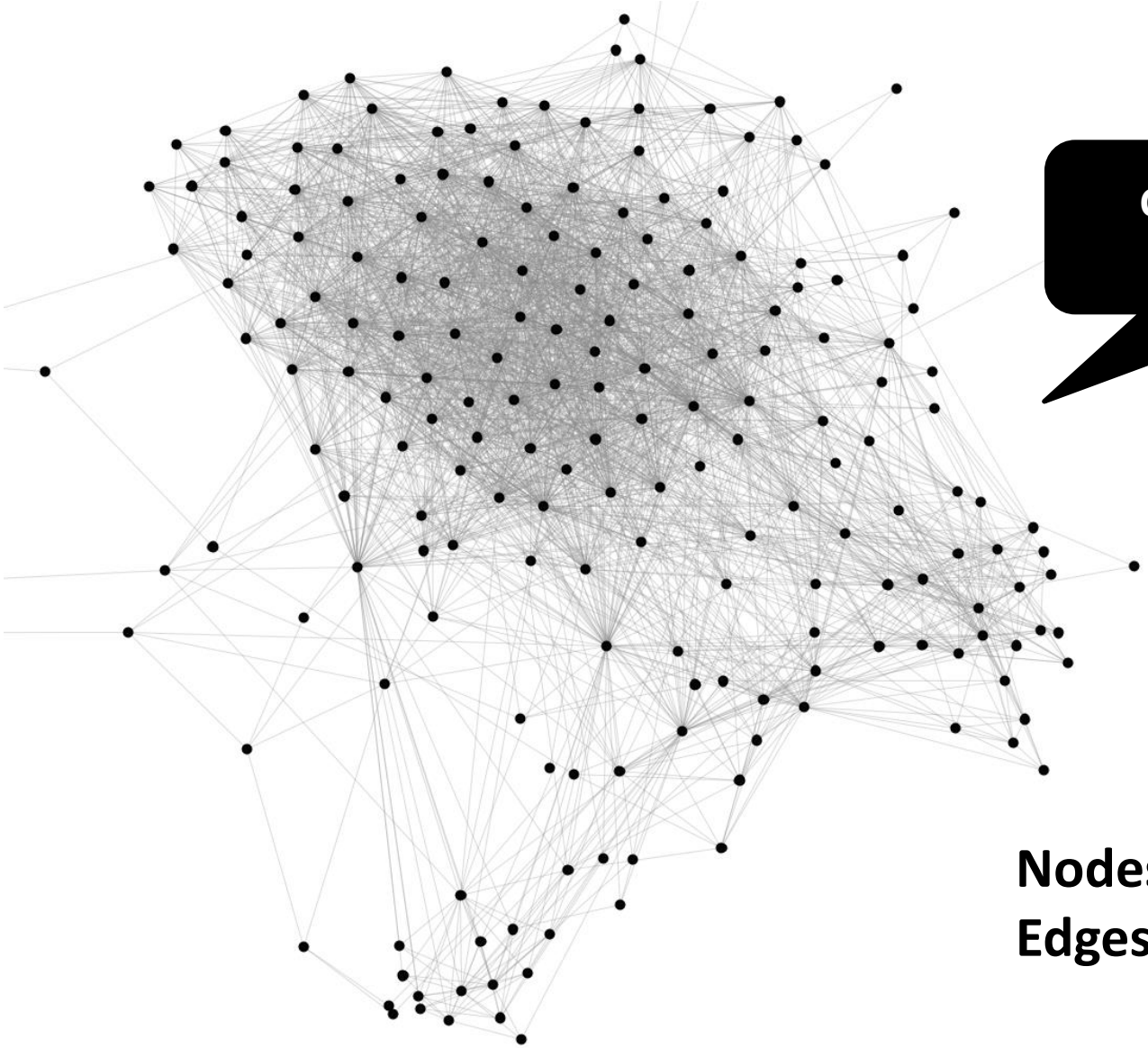



**Nodes: Proteins**
**Edges: Physical interactions**

# Protein-Protein Interactions



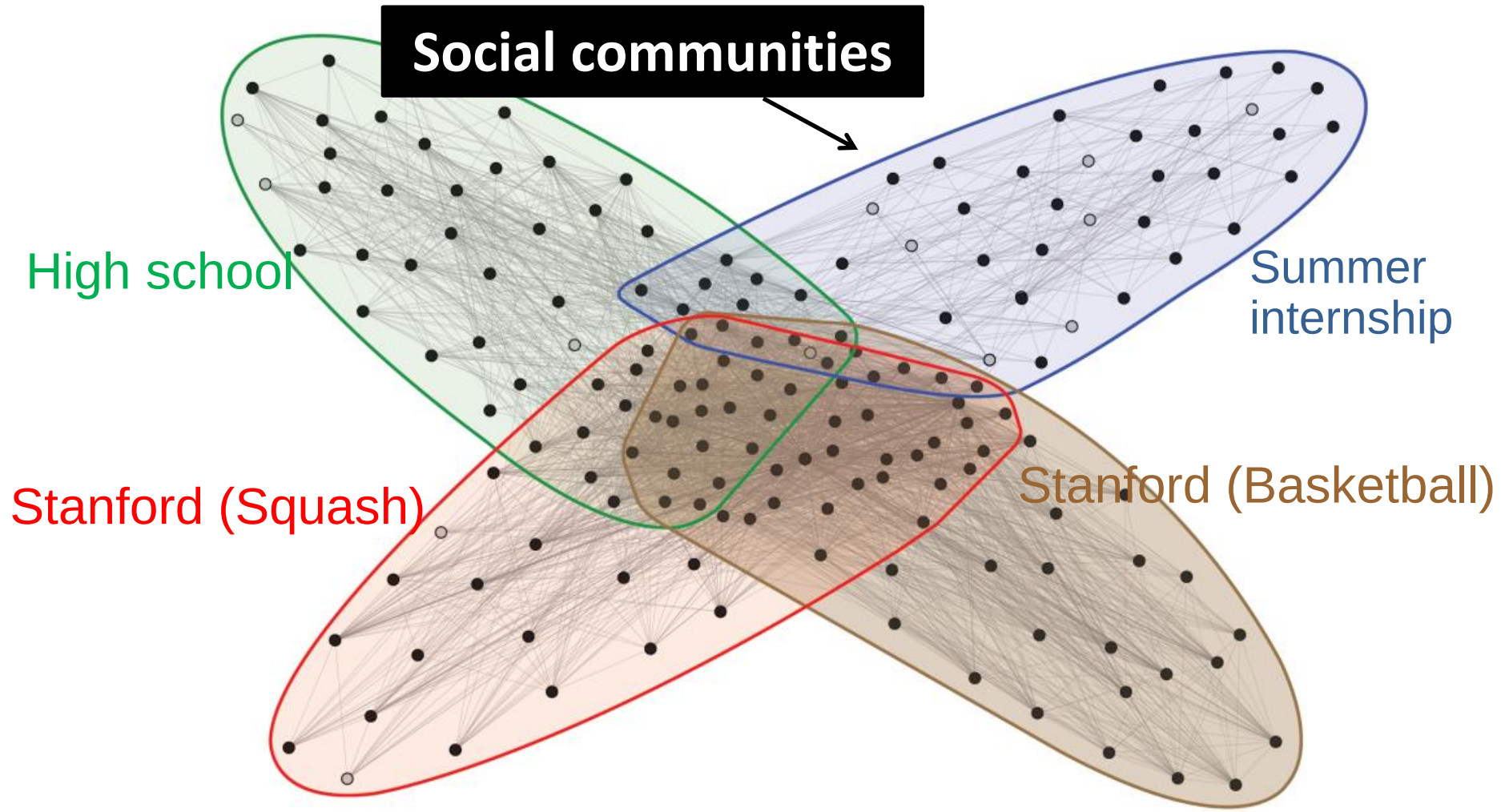
# Facebook Network



**Can we identify social communities?**

**Nodes: Facebook Users**
**Edges: Friendships**

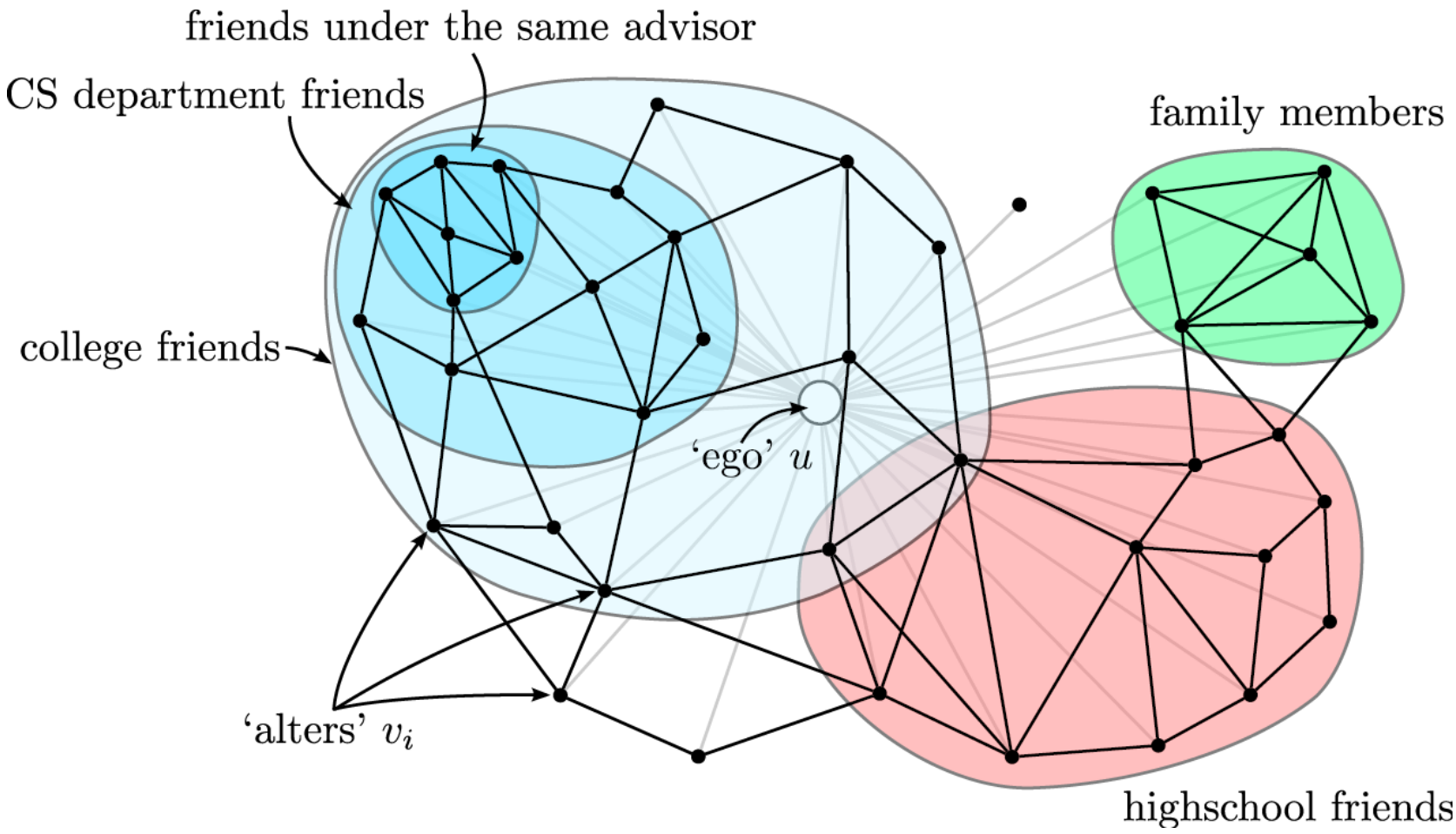
# Facebook Network




**Nodes: Facebook Users**

**Edges: Friendships**

# Twitter & Facebook




social circles, circles of trust

# Outline

## PART I

1. Introduction: what, why, types?
2. Cliques and vertex similarity
3. Background: How it relates to “cluster analysis”
4. Hierarchical clustering (betweenness)
5. Modularity
6. *How to evaluate (if time allows)*

## PART II

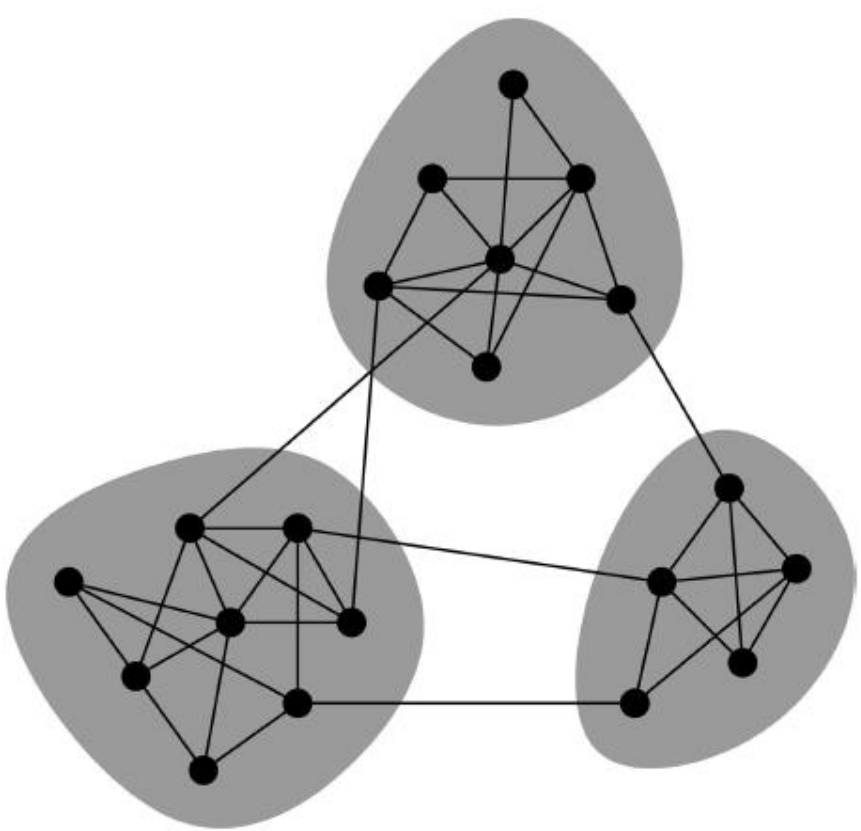
Cuts, Spectral clustering, Denser subgraphs, community evolution

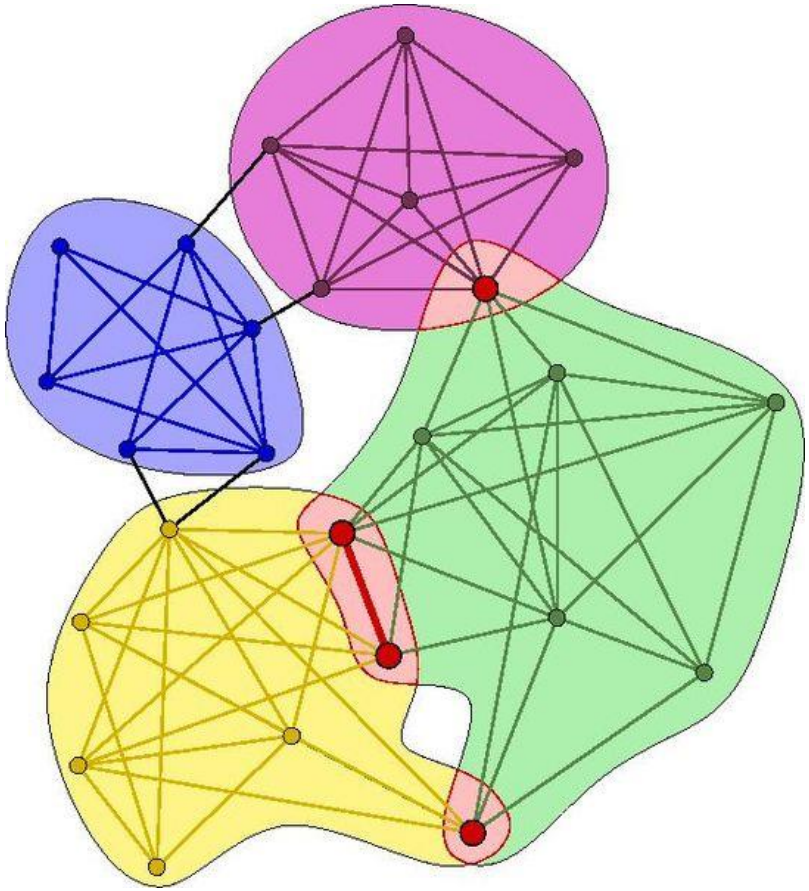
# Why? (some applications)

- *Knowledge discovery*
- Groups based on common interests, behavior, etc (e.g., Canadians who call USA, readings tastes, etc)
  - *Recommendations, marketing*
- *Collective behavior* (observable at the group, not the individual level, local view is noisy and ad hoc)
- *Performance-wise* (partition a large graph into many machines, assigning web clients to web servers, routing in ad hoc networks, etc)
- *Classification of the nodes* (by identifying modules and their boundaries)
- Summary, visual *representation* of the graph

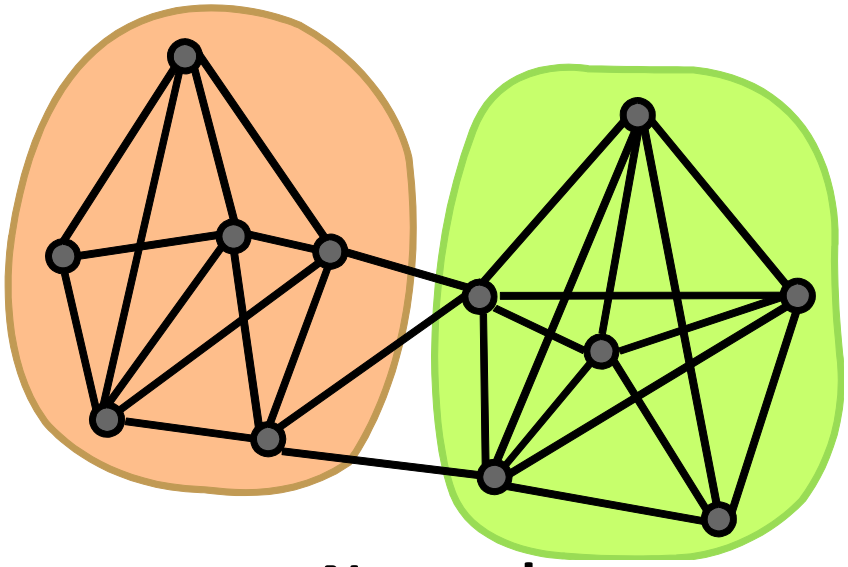
# Community Types

## Non-overlapping vs. overlapping communities

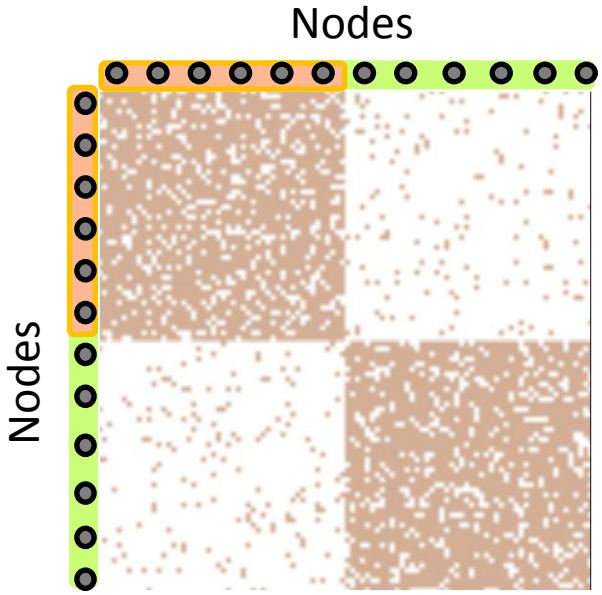




# Non-overlapping Communities



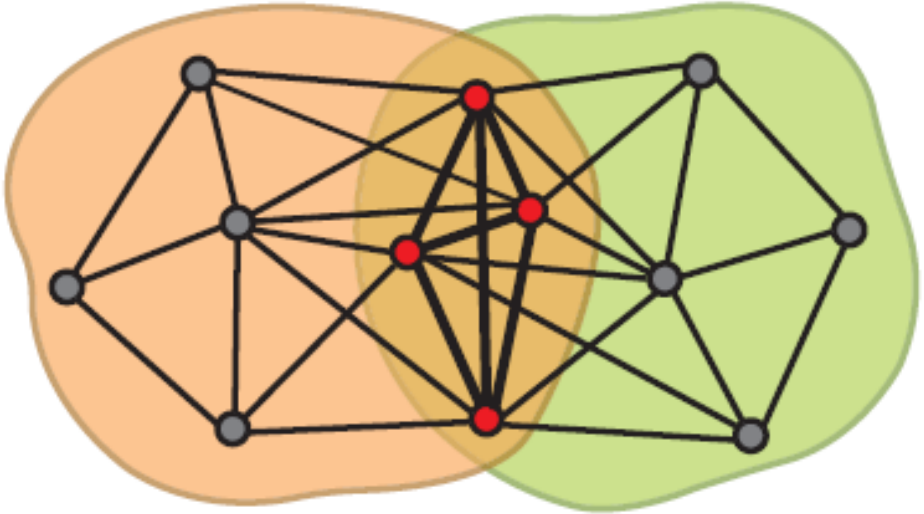
**Network**

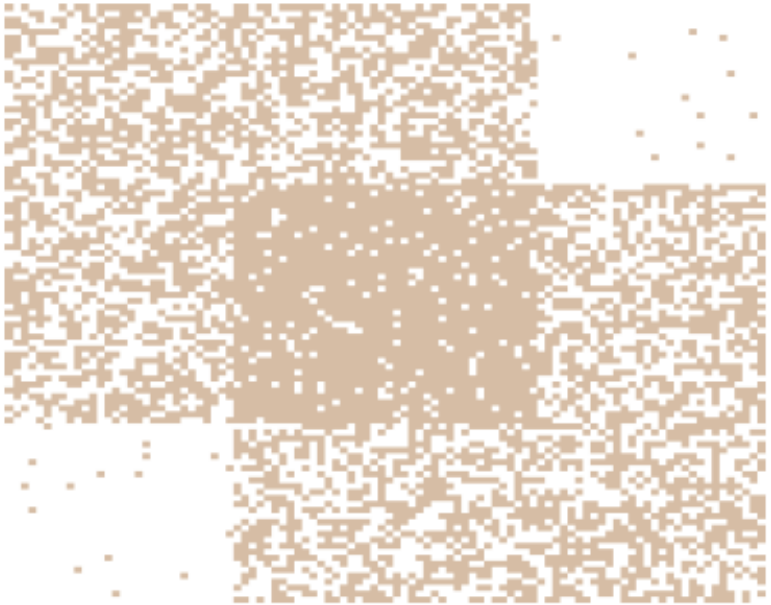



**Adjacency matrix**

# Overlapping Communities

What is the structure of community overlaps:
*Edge density in the overlaps is higher!*

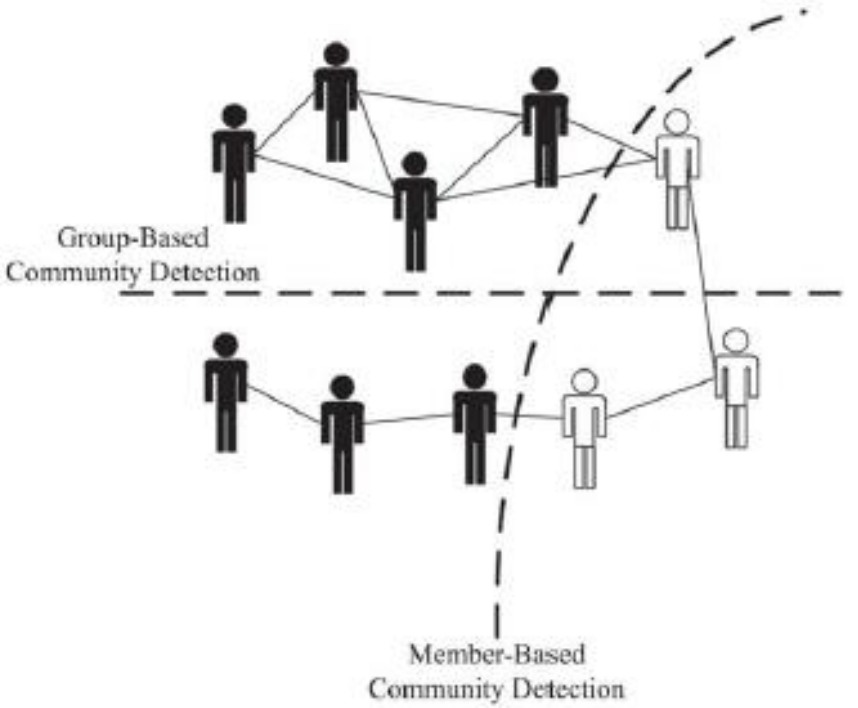




Communities as “tiles”

# Community Types

## Member-based (local) vs. group-based



# Community Detection

Given a graph G(V, E), find subsets $C_i$ of V,

such that $\bigcup_i C_i \subseteq V$

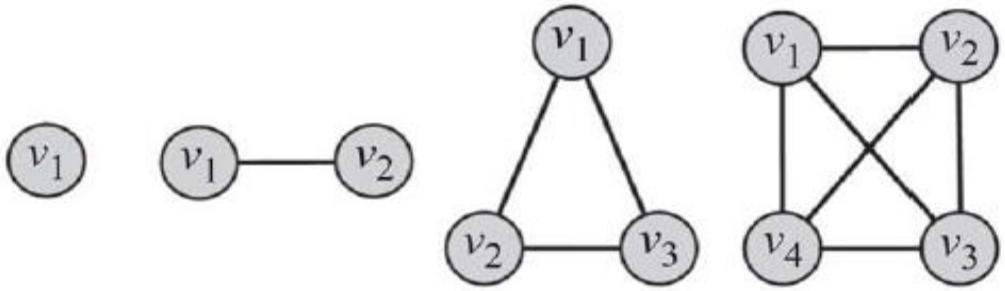
- Edges can also represent content or attributes shared by individuals (in the same location, of the same gender, etc)
- Undirected graphs
- Unweighted (easily extended)
- Attributed, or labeled graphs

*Multipartite graphs* – e.g., affiliation networks, citation networks, customers-products: reduced to unipartited projections of each vertex class

# Cliques (degree similarity)

Clique: a maximum *complete subgraph* in which all pairs of vertices are connected by an edge.

A *clique of size k* is a subgraph of *k* vertices where the degree of all vertices in the induced subgraph is *k -1* .



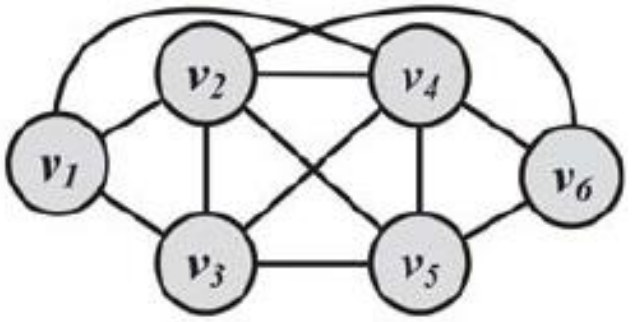

✓ Cliques vs complete graphs

# Cliques (degree similarity)

Search for

- the maximum clique (the one with the largest number of vertices) or
- for all *maximal cliques* (cliques that are not subgraphs of a larger clique; i.e., cannot be expanded further).

Both problems are NP-hard, as is verifying whether a graph contains a clique larger than size *k*.



# Cliques

**Algorithm 6.1** Brute-Force Clique Identification

**Require:** Adjacency Matrix $A$, Vertex $v_x$

1: **return** Maximal Clique $C$ containing $v_x$
2: CliqueStack = $\{\{v_x\}\}$, Processed = $\{\}$;
3: **while** CliqueStack not empty **do**
4: C=pop(CliqueStack); push(Processed,C);
5: $v_{last}$ = Last node added to C;
6: $N(v_{last}) = \{v_i | A_{v_{last},v_i} = 1\}$.
7: **for all** $v_{temp} \in N(v_{last})$ **do**
8: **if** $C \bigcup \{v_{temp}\}$ is a clique **then**
9: push(CliqueStack, $C \bigcup \{v_{temp}\}$);
10: **end if**
11: **end for**
12: **end while**
13: Return the largest clique from Processed

Enumerate all cliques.

Checks all permutations!

For 100 vertices, $2^{99}$- 1 different cliques

# Cliques

## Pruning

- Prune all vertices (and incident edges) with degrees less than k - 1.
- Effective due to the power-law distribution of vertex degrees

"Exact cliques" are *rarely observed* in real networks.

E.g., a clique of 1,000 vertices has (999x1000)/2 = 499,500 edges.

- A single edge removal results in a subgraph that is no longer a clique.
- That represents less than 0.0002% of the edges

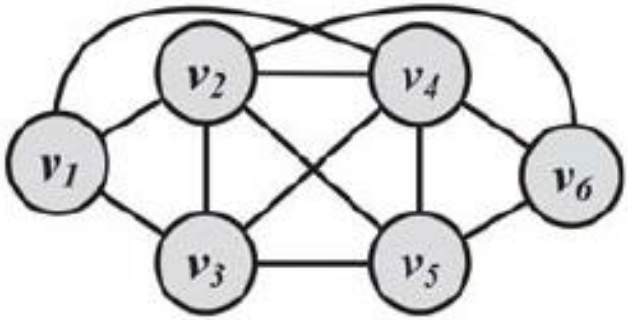
# Relaxing Cliques

All vertices have *a minimum degree* but not necessarily *k* -1

## *k-plex*

For a set of vertices V, for all u, $d_u \geq |V| - k$
where $d_u$ is the degree of v in the induced subgraph

*What is k for a clique?*




Maximal

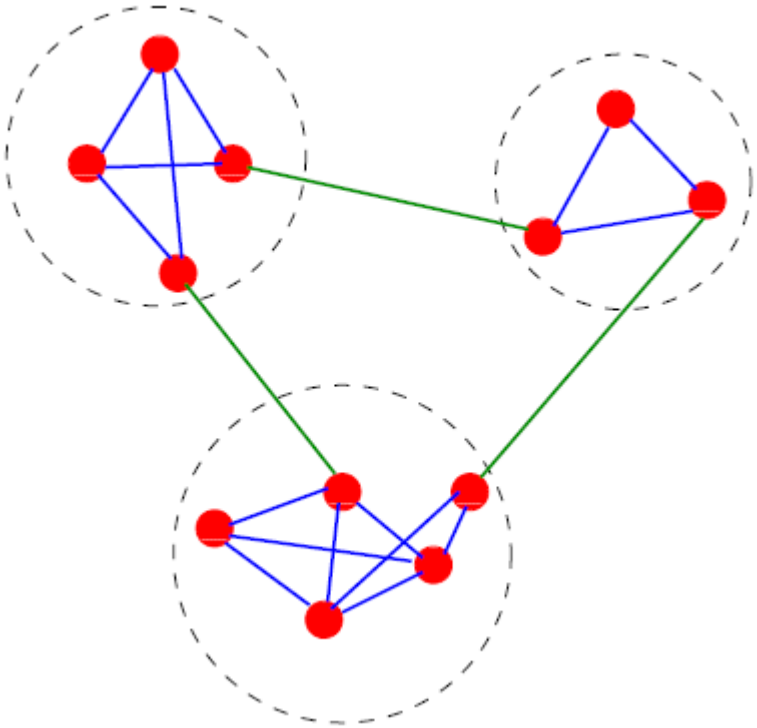
1-plex : $\{v_2, v_3, v_4, v_5\}$

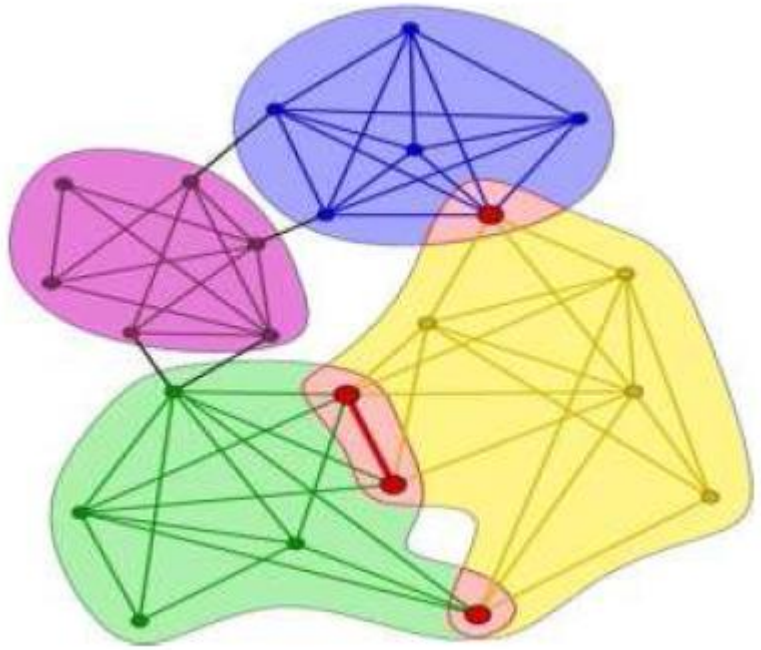
2-plex : $\{v_1, v_2, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5, v_6\}$

3-plex : $\{v_1, v_2, v_3, v_4, v_5, v_6\}$

# Clique Percolation Method (CPM): Using cliques as seeds

Assumption: communities are formed from a set of cliques and edges that connect these cliques.





# Clique Percolation Method (CPM): Using cliques as seeds

Two *k*-cliques are adjacent if they share *k* - 1 vertices.

The union of adjacent *k*-cliques is called *k*-clique chain.

Two *k*-cliques are *connected* if they are part of a *k*-clique chain.

A ***k-clique community*** is the largest connected subgraph obtained by the union of a *k*-clique and of all *k*-cliques which are connected to it.

# Clique Percolation Method (CPM): Using cliques as seeds

1. Given *k*, find all cliques of size *k*.
2. Create graph (clique graph) where all cliques are vertices, and two cliques that share *k* - 1 vertices are connected via an edge.
3. Communities are the connected components of this graph.

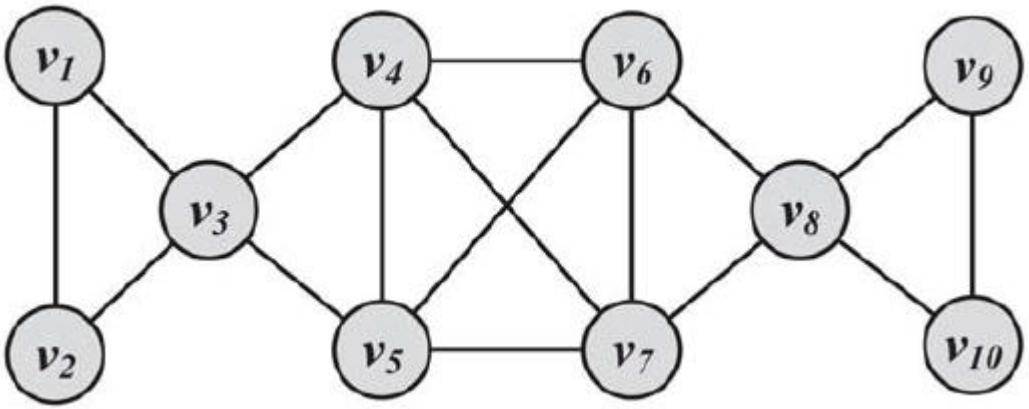
**Algorithm 6.2** Clique Percolation Method (CPM)

**Require:** parameter $k$

1: **return** Overlapping Communities
2: $Cliques_k$ = find all cliques of size $k$
3: Construct clique graph $G(V, E)$, where $|V| = |Cliques_k|$
4: $E = \{e_{ij} \mid$ clique $i$ and clique $j$ share $k - 1$ nodes$\}$
5: Return all connected components of $G$

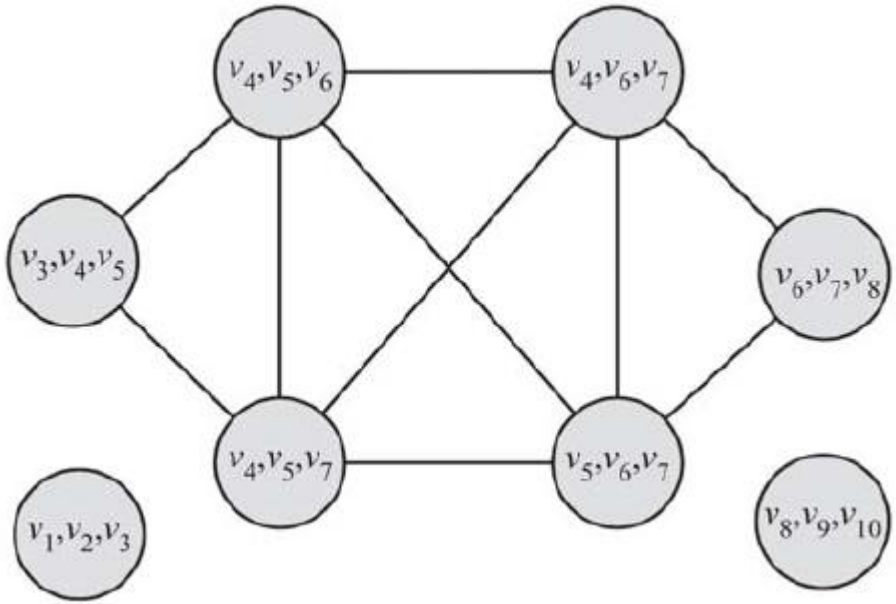
# Clique Percolation Method (CPM): Using cliques as seeds

Input graph, let $k = 3$



# Clique Percolation Method (CPM): Using cliques as seeds

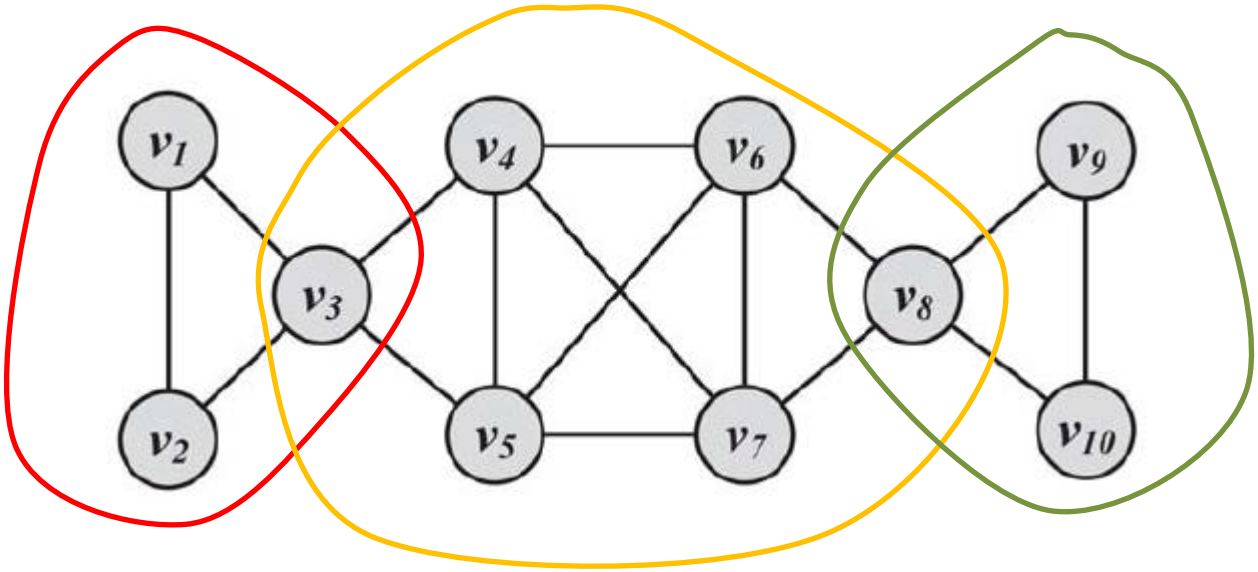
Clique graph for $k = 3$




(v1, v2, ,v3), (v8, v9, v10), and (v3, v4, v5, v6, v7, v8)

# Clique Percolation Method (CPM): Using cliques as seeds

Result




(v1, v2, ,v3), (v8, v9, v10), and (v3, v4, v5, v6, v7, v8)

*Note: the example protein network was detected using a CPM algorithm*

# Clique Percolation Method (CPM): Using cliques as seeds

- A k-clique community is identified by making a k-clique "roll" over adjacent k-cliques, where rolling means rotating a k-clique about the k-1 vertices it shares with any adjacent k-clique.
- By construction, *overlapping*.
- There may be vertices belonging to nonadjacent k-cliques, which could be reached by different paths and end up in different clusters.
- There are also vertices that cannot be reached by any *k*-clique
- In the example, if instead of $k = 3$, for the maximal cliques?
- Theoretical complexity grows exponential with size, but efficient on sparse graphs

# Vertex similarity

- Define similarity between two vertices
- Place similar vertices in the same cluster

- Use traditional *cluster analysis*

# Vertex similarity

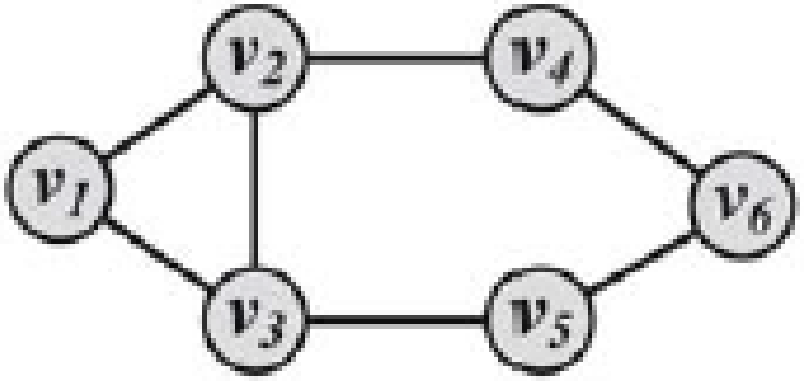
- Structural equivalence: based on the overlap between their neighborhoods

$$\sigma(v_i, v_j) = |N(v_i) \cap N(v_j)|$$

- Normalized to [0, 1], e.g.,

$$\sigma_{\text{Jaccard}}(v_i, v_j) = \frac{|N(v_i) \cap N(v_j)|}{|N(v_i) \cup N(v_j)|}$$

# Vertex similarity




$$\sigma_{\text{Jaccard}}(v_2, v_5) = \frac{|\{v_1, v_3, v_4\} \cap \{v_3, v_6\}|}{|\{v_1, v_3, v_4, v_6\}|} = 0.25$$

# Other definitions of vertex similarity

Use the adjacency matrix A,

$$d_{ij} = \sqrt{\sum_{k \neq i,j} (A_{ik} - A_{jk})^2}$$

# Other definitions of vertex similarity

If we map vertices u, v to n-dimensional points A, B in the Euclidean space,

$$d_{AB}^{E} = \sum_{k=1}^{n} \sqrt{(a_k - b_k)^2}$$

$$d_{AB}^{M} = \sum_{k=1}^{n} |a_k - b_k|$$

$$d_{AB}^{\infty} = \max_{k \in [1,n]} |a_k - b_k|$$

$$\rho_{AB} = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\sum_{k=1}^{n} a_k^2} \sqrt{\sum_{k=1}^{n} b_k^2}}$$

# Other definitions of vertex similarity

Many more – we shall revisit this issue when we talk about *link prediction*

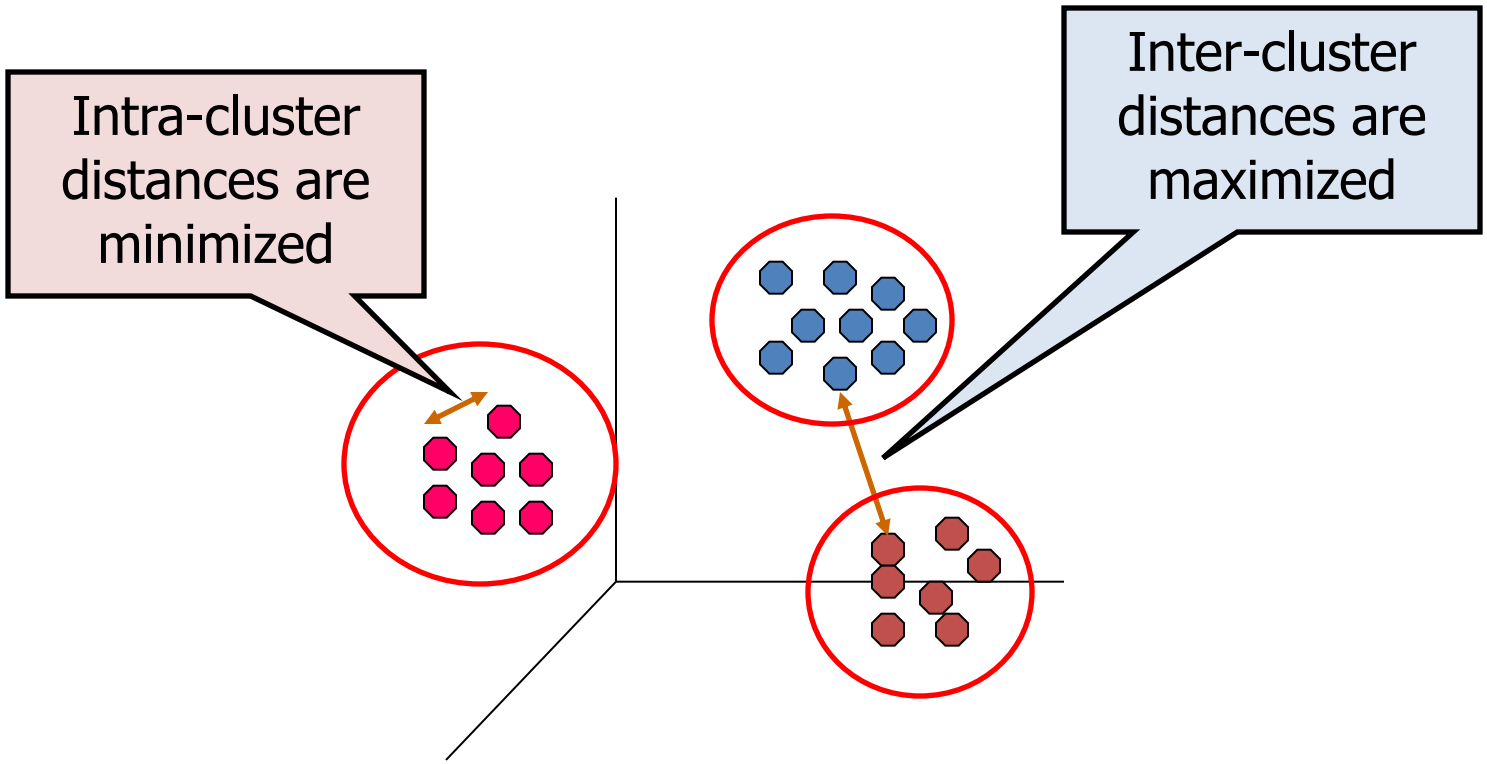
# Outline

## PART I

1. Introduction: what, why, types?
2. Cliques and vertex similarity
3. Background: cluster analysis
4. Hierarchical clustering (betweenness)
5. Modularity
6. How to evaluate

# What is Cluster Analysis?

Finding groups of objects such that the objects in a group will be similar (or related) to one another and different from (or unrelated to) the objects in other groups



# Notion of a cluster can be ambiguous

How many clusters?

Six Clusters

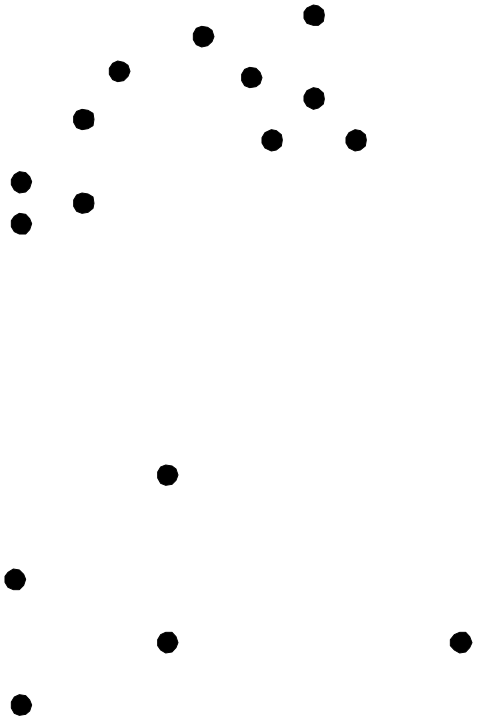
Two Clusters

Four Clusters

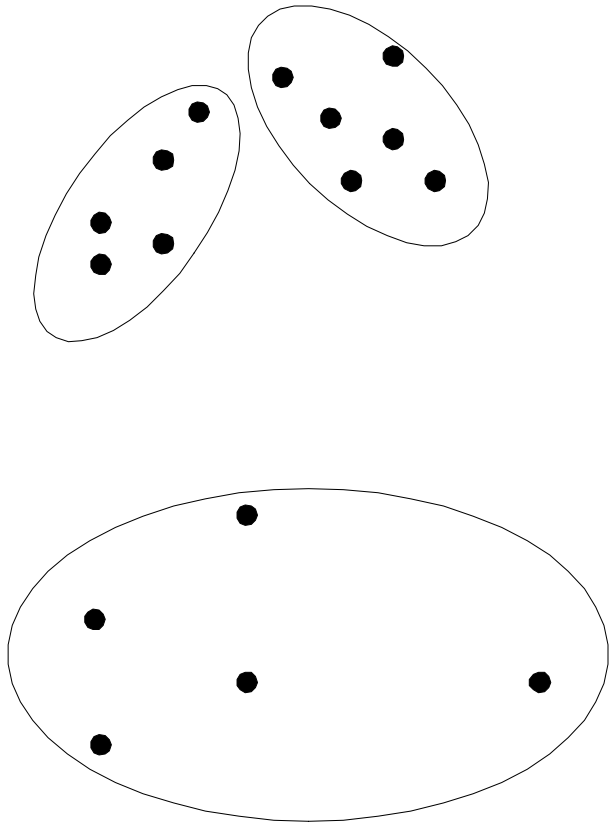
# Types of Clustering

- A clustering is a set of clusters
- Important distinction between hierarchical and partitional sets of clusters
- Partitional Clustering
  - Division of data objects into non-overlapping subsets (clusters) such that each data object is in exactly one subset
  - Assumes that the number of clusters is given
- Hierarchical clustering
  - A set of nested clusters organized as a hierarchical tree

# Partitional Clustering



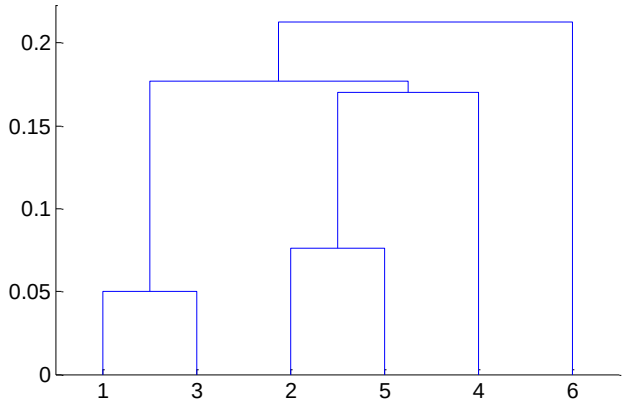
Original Points

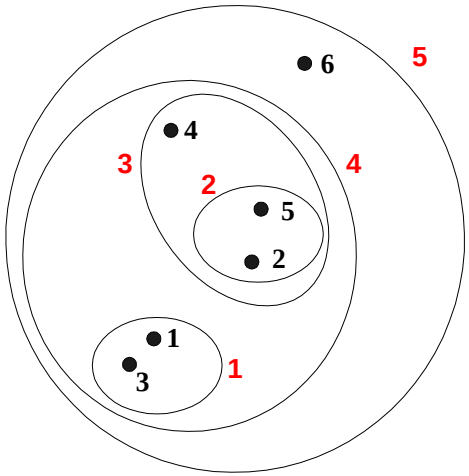


A Partitional Clustering

# Hierarchical Clustering

- Produces a set of nested clusters organized as a hierarchical tree
- Can be visualized as a dendrogram
  - A tree like diagram that records the sequences of merges or splits



# Other Distinctions Between Sets of Clusters

- Exclusive versus non-exclusive
  - In non-exclusive clustering, points may belong to multiple clusters.
  - Can represent multiple classes or ‘border’ points
- Fuzzy versus non-fuzzy
  - In fuzzy clustering, a point belongs to every cluster with some weight between 0 and 1
  - Weights must sum to 1
  - Probabilistic clustering has similar characteristics
- Partial versus complete
  - In some cases, we only want to cluster some of the data
- Heterogeneous versus homogeneous
  - Cluster of widely different sizes, shapes, and densities

# Clusters defined by an objective function

Finds clusters that minimize or maximize an objective function.

- *Enumerate all possible ways of dividing the points* into clusters and evaluate the `goodness' of each potential set of clusters by using the given objective function. (NP Hard)
- Can have *global* or *local* objectives.
  - Hierarchical clustering algorithms typically have local objectives
  - Partitional algorithms typically have global objectives
- A variation of the global objective function approach is to *fit the data to a parameterized model*.
  - Parameters for the model are determined from the data.
  - Mixture models assume that the data is a ‘mixture' of a number of statistical distributions.

# Clustering Algorithms

- K-means
- Hierarchical clustering
- Density clustering

# K-means Clustering

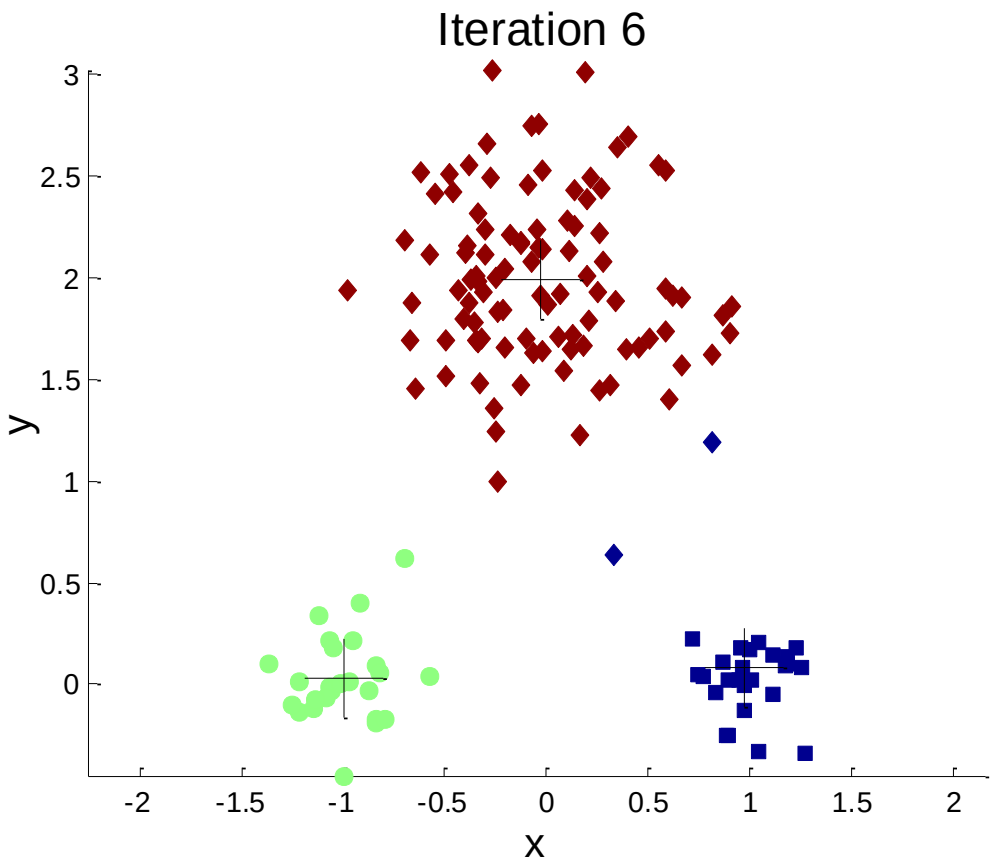
1: Select $K$ points as the initial centroids.
2: **repeat**
3: Form $K$ clusters by assigning all points to the closest centroid.
4: Recompute the centroid of each cluster.
5: **until** The centroids don't change

- Partitional clustering approach
- Each cluster is associated with a *centroid* (center point)
- Each point is assigned to the cluster with the closest centroid
- Number of clusters, K, must be specified
- The basic algorithm is very simple

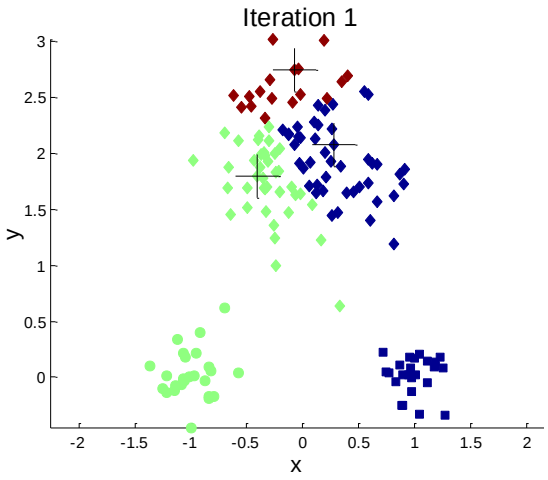
# K-means Clustering

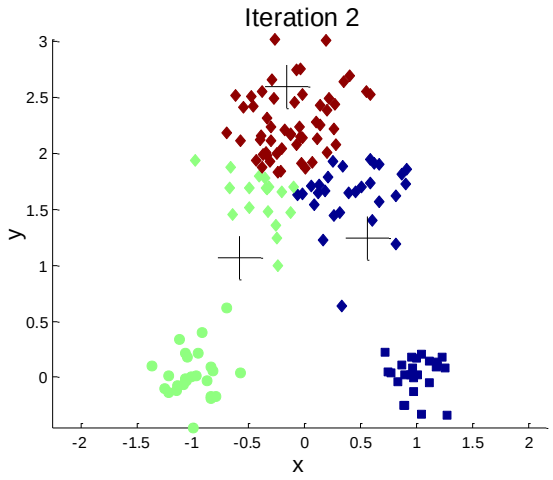
- Initial centroids are often chosen randomly.
  - Clusters produced vary from one run to another.
- The centroid is (typically) the mean of the points in the cluster.
- 'Closeness' is measured by Euclidean distance, cosine similarity, correlation, etc.
- K-means *will converge* for common similarity measures mentioned above.
- Most of the convergence happens in the first few iterations.
  - Often the stopping condition is changed to 'Until relatively few points change clusters'
- Complexity is O( n * K * I * d )
  - n = number of points, K = number of clusters, I = number of iterations, d = number of attributes

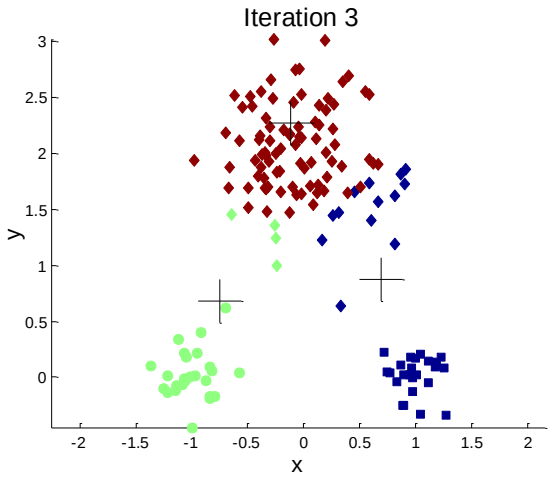
# Example

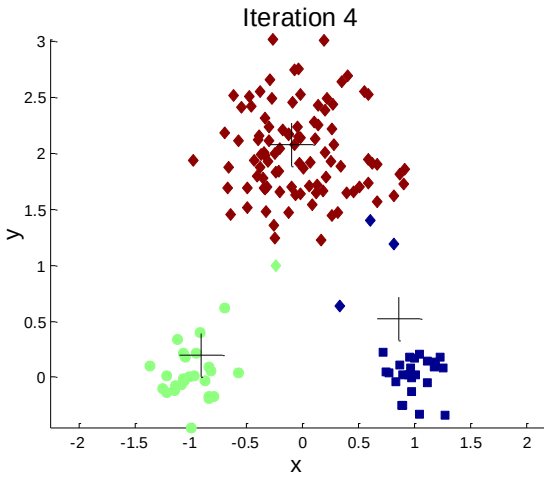


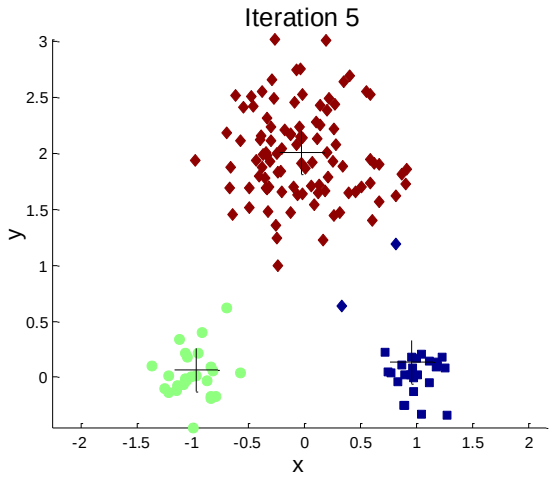
# Example

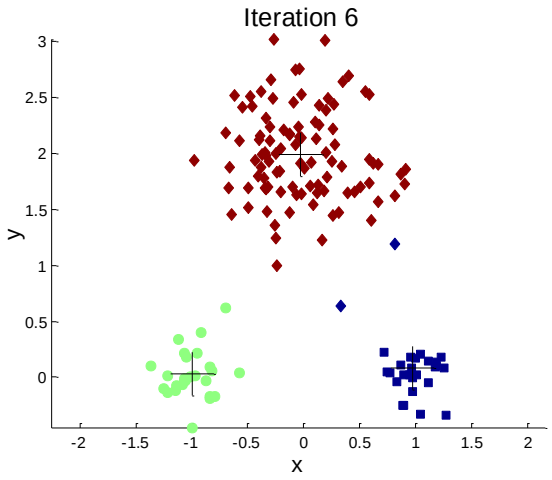

Iteration 1
x
y


Iteration 2
x
y

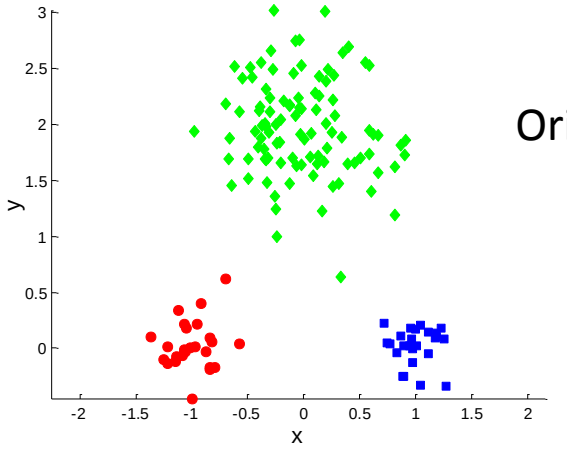

Iteration 3
x
y


Iteration 4
x
y

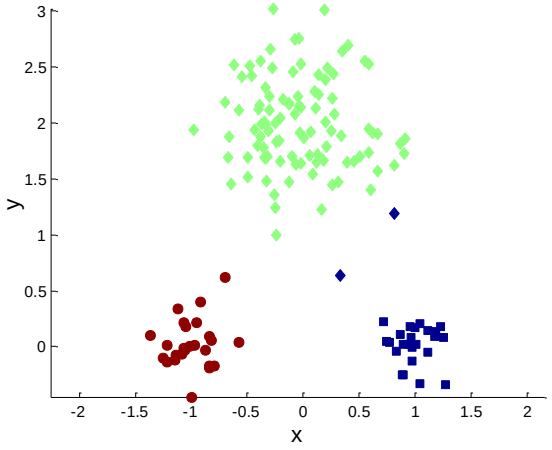

Iteration 5
x
y


Iteration 6
x
y

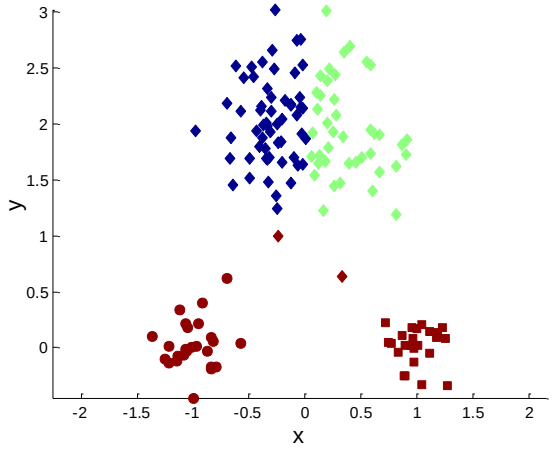
# Two different K-means clusterings




Original Points




Optimal Clustering




Sub-optimal Clustering

Importance of choosing initial points

# K-means Clusters

- Most common measure is Sum of Squared Error (SSE)
  - For each point, the error is the distance to the nearest cluster
  - To get SSE, we square these errors and sum them.

$$SSE = \sum_{i=1}^{K} \sum_{x \in C_i} dist^2(m_i, x)$$

  - $x$ is a data point in cluster $C_i$ and $m_i$ is the representative point for cluster $C_i$
    - can show that $m_i$ corresponds to the center (mean) of the cluster
  - Given two clusters, we can choose the one with the smallest error
  - One easy way to reduce SSE is to increase K, the number of clusters
    - A good clustering with smaller K can have a lower SSE than a poor clustering with higher K

# Limitations of K-means

- K-means has problems when clusters are of differing
  - Sizes
  - Densities
  - Non-globular shapes

- K-means has problems when the data contains outliers.

# Pre-processing and Post-processing

- Pre-processing
  - Normalize the data
  - Eliminate outliers
- Post-processing
  - Eliminate small clusters that may represent outliers
  - Split ‘loose’ clusters, i.e., clusters with relatively high SSE
  - Merge clusters that are ‘close’ and that have relatively low SSE
  - Can use these steps during the clustering process

# Hierarchical Clustering

- Two main types of hierarchical clustering
  - Agglomerative:
    - Start with the points (vertices) as individual clusters
    - At each step, merge the closest pair of clusters until only one cluster (or k clusters) left
  - Divisive:
    - Start with one, all-inclusive cluster (the whole graph)
    - At each step, split a cluster until each cluster contains a point (vertex) (or there are k clusters)
- Traditional hierarchical algorithms use a similarity or distance matrix
  - Merge or split one cluster at a time

# Strengths of Hierarchical Clustering

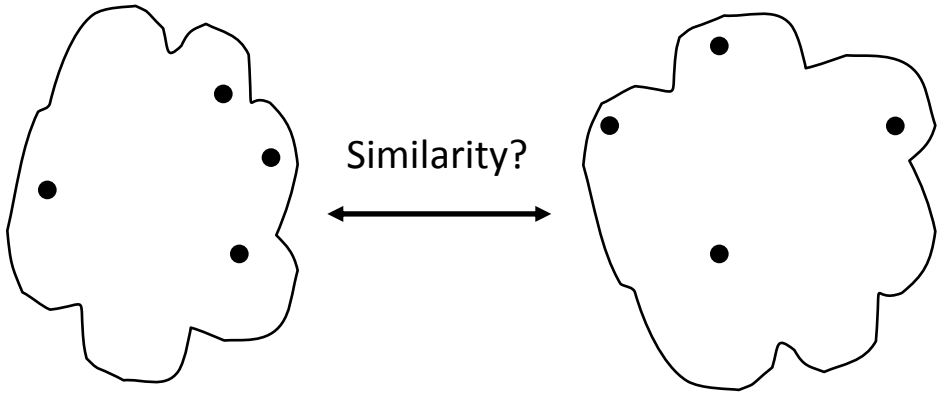
- Do not have to assume any particular number of clusters
  - Any desired number of clusters can be obtained by ‘cutting’ the dendogram at the proper level

- They may correspond to meaningful taxonomies
  - Example in biological sciences (e.g., animal kingdom, phylogeny reconstruction, …)

# Agglomerative Clustering Algorithm

- Popular hierarchical clustering technique
- Basic algorithm is straightforward
  1. [Compute the proximity matrix]
  2. Let each data point be a cluster
  3. **Repeat**
  4. Merge the *two closest clusters*
  5. [Update the proximity matrix]
  6. **Until** only a single cluster remains
- Key operation is the *computation of the proximity of two clusters*
  - Different approaches to defining the distance between clusters distinguish the different algorithms

# How to Define Inter-Cluster Similarity

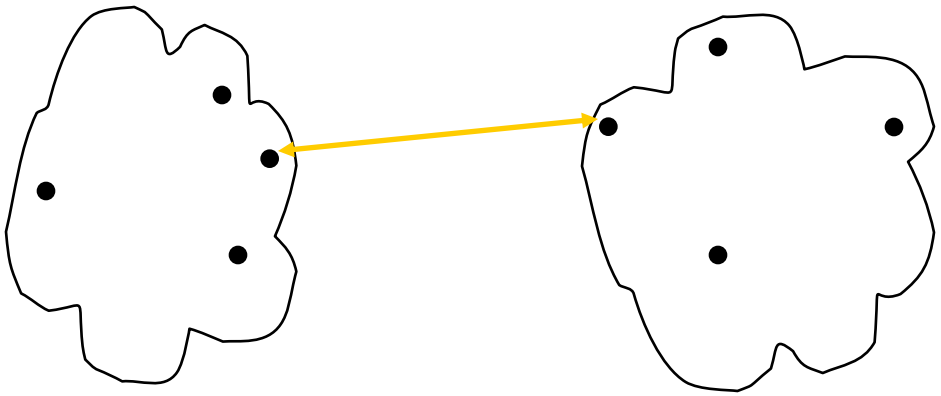



| | p1 | p2 | p3 | p4 | p5 | . . . |
|---|---|---|---|---|---|---|
| p1 | | | | | | |
| p2 | | | | | | |
| p3 | | | | | | |
| p4 | | | | | | |
| p5 | | | | | | |
| . | | | | | | |

. .

Proximity Matrix

# How to Define Inter-Cluster Similarity



| | p1 | p2 | p3 | p4 | p5 | . . . |
|---|---|---|---|---|---|---|
| p1 | | | | | | |
| p2 | | | | | | |
| p3 | | | | | | |
| p4 | | | | | | |
| p5 | | | | | | |
| . | | | | | | |

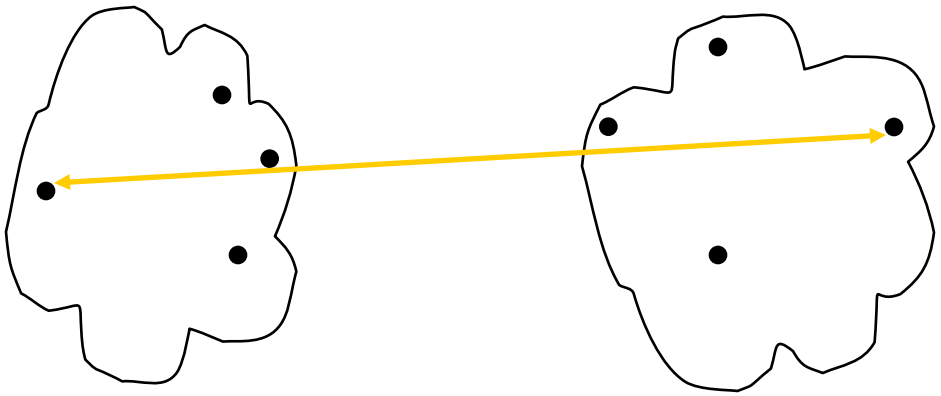
Proximity Matrix

MIN or single link

based on the two most similar (closest) points in the different clusters

(sensitive to outliers)

# How to Define Inter-Cluster Similarity



| | p1 | p2 | p3 | p4 | p5 | . . . |
|---|---|---|---|---|---|---|
| p1 | | | | | | |
| p2 | | | | | | |
| p3 | | | | | | |
| p4 | | | | | | |
| p5 | | | | | | |
| . | | | | | | |

. .

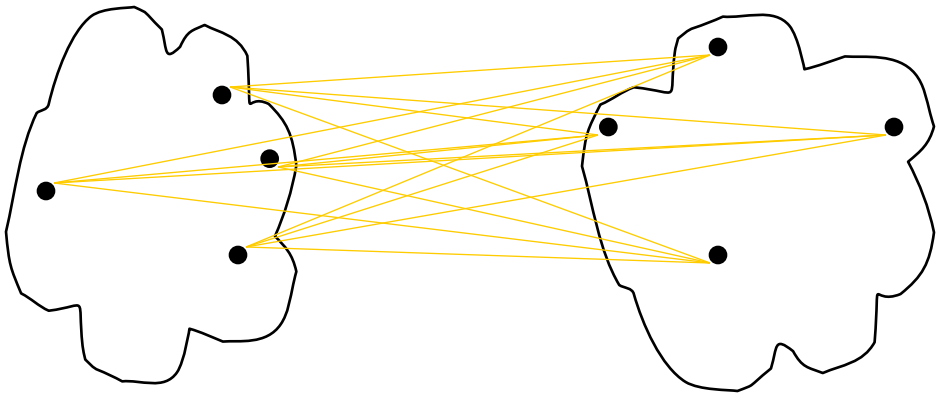
Proximity Matrix

## MAX or complete linkage

Similarity of two clusters is based on the two least similar (most distant) points in the different clusters

(Tends to break large clusters
Biased towards globular clusters)

# How to Define Inter-Cluster Similarity



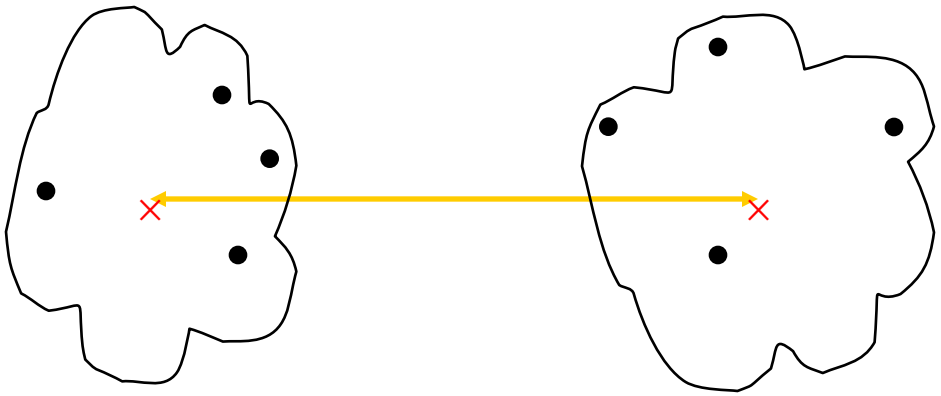
| | p1 | p2 | p3 | p4 | p5 | . . . |
|---|---|---|---|---|---|---|
| p1 | | | | | | |
| p2 | | | | | | |
| p3 | | | | | | |
| p4 | | | | | | |
| p5 | | | | | | |
| . | | | | | | |
| . | | | | | | |
| . | | | | | | |

Proximity Matrix

## Group Average

Proximity of two clusters is the average of pairwise proximity between points in the two clusters.

# How to Define Inter-Cluster Similarity



Distance Between Centroids

| | p1 | p2 | p3 | p4 | p5 | . . . |
|---|---|---|---|---|---|---|
| p1 | | | | | | |
| p2 | | | | | | |
| p3 | | | | | | |
| p4 | | | | | | |
| p5 | | | | | | |
| . | | | | | | |

.

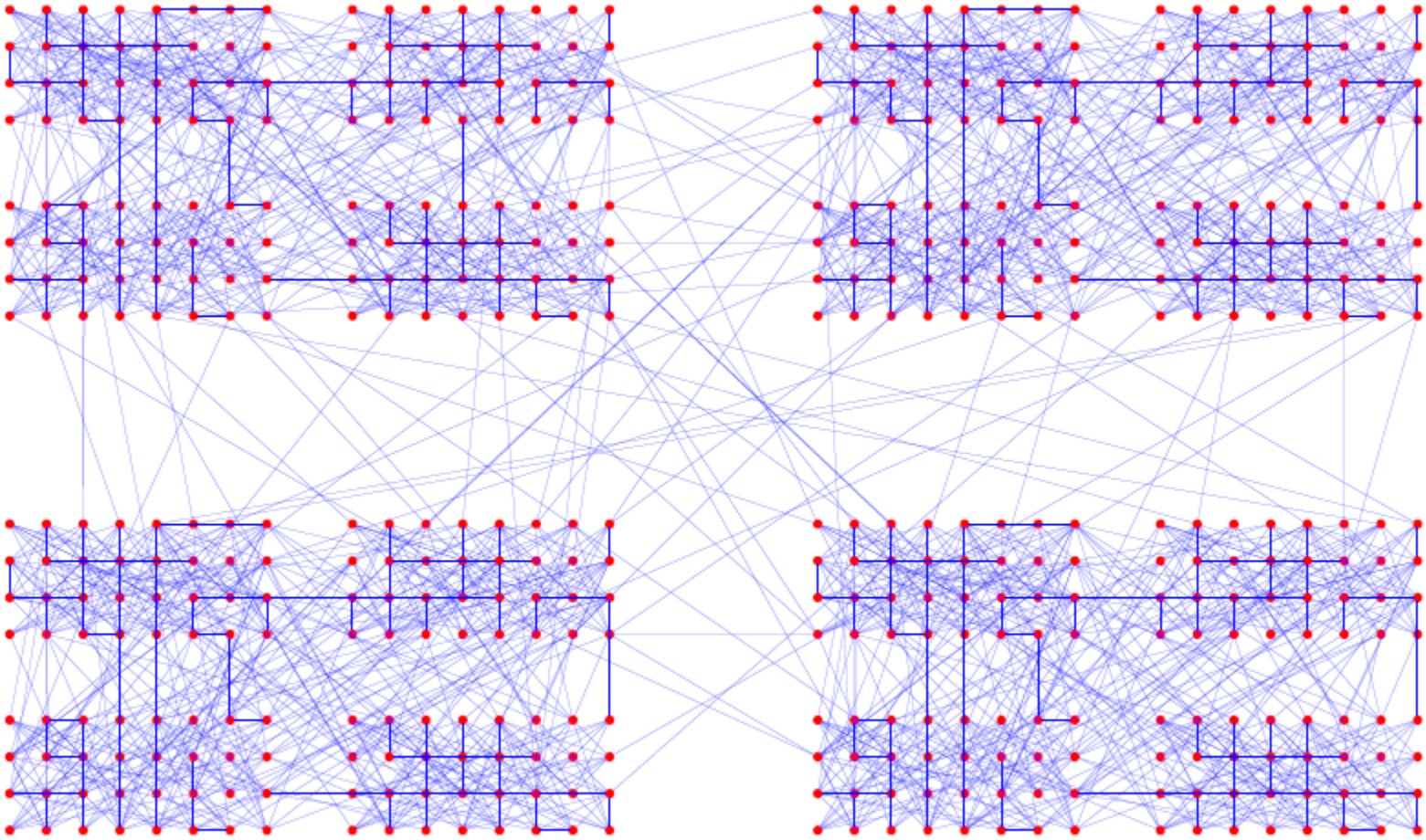
.

Proximity Matrix

# Cluster Similarity: Ward's Method

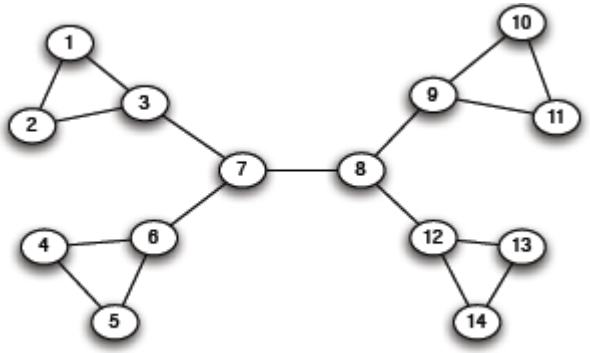
- Similarity of two clusters is based on the increase in squared error when two clusters are merged
  - Similar to group average if distance between points is distance squared
- Less susceptible to noise and outliers
- Biased towards globular clusters
- Hierarchical analogue of K-means
  - Can be used to initialize K-means

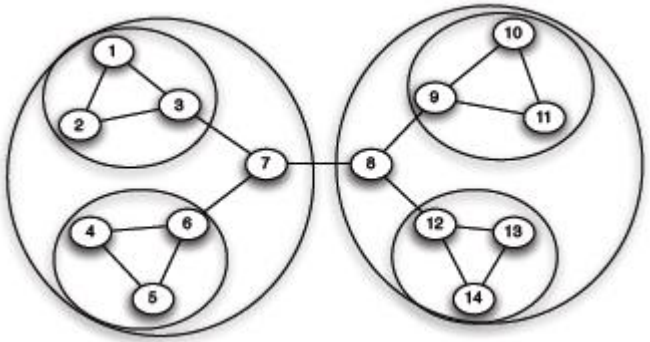
# Example of a Hierarchically Structured Graph



# Graph Partitioning

- Divisive methods: try to identify and remove the "spanning links" between densely-connected regions
- Agglomerative methods: Find nodes that are likely to belong to the same region and merge them together (bottom-up)



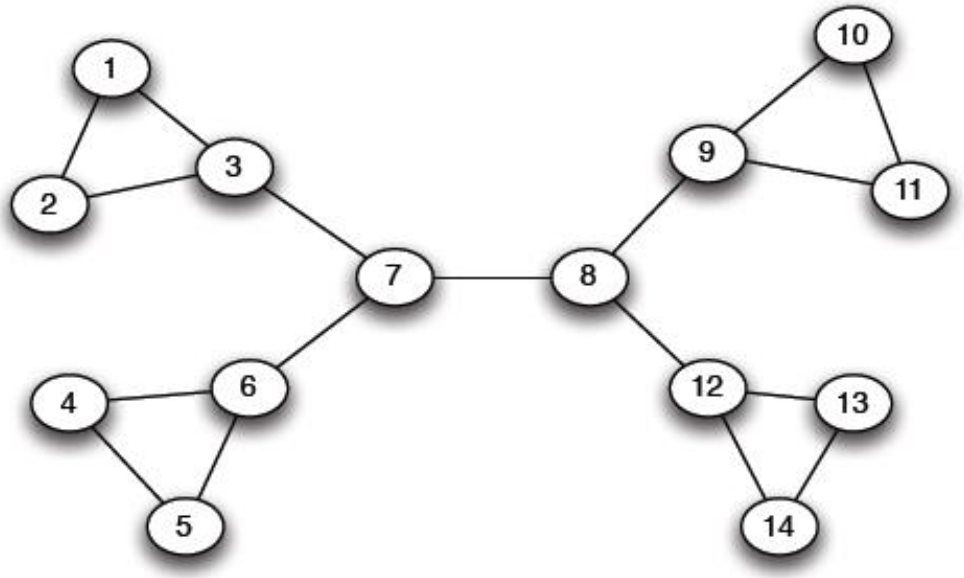



# The Girvan Newman method

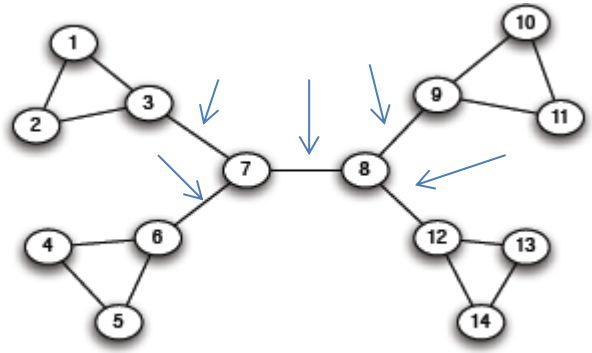
## Hierarchical divisive method

- Start with the whole graph
- Find edges whose removal "partitions" the graph
- Repeat with each subgraph until single vertices

*Which edge?*



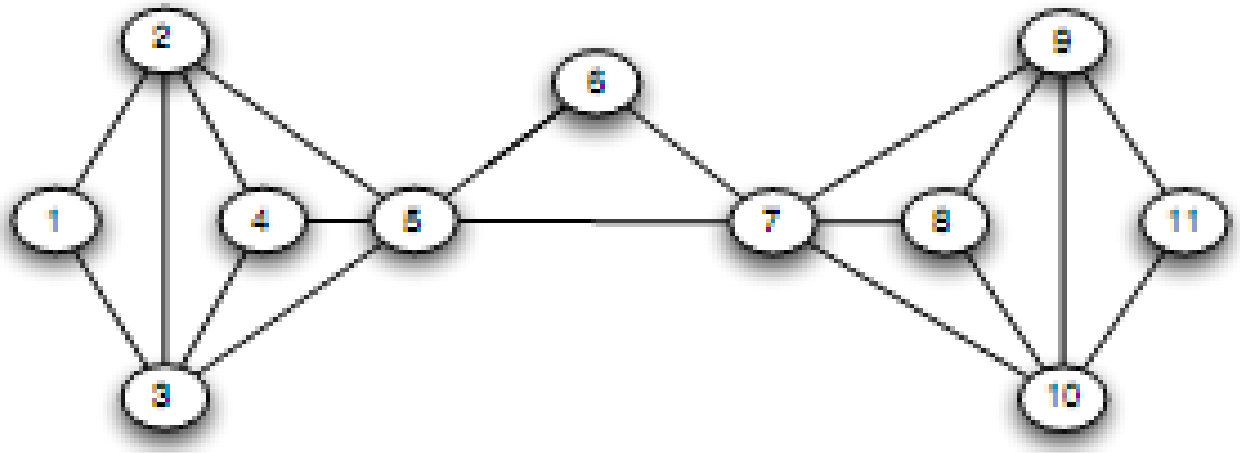
# The Girvan Newman method

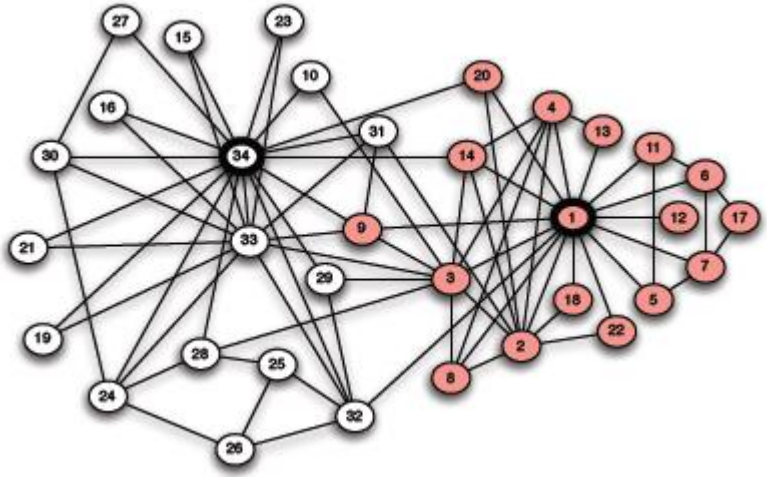



Use bridges or cut-edge (if removed, the nodes become disconnected)

Which one to choose?

# The Girvan Newman method

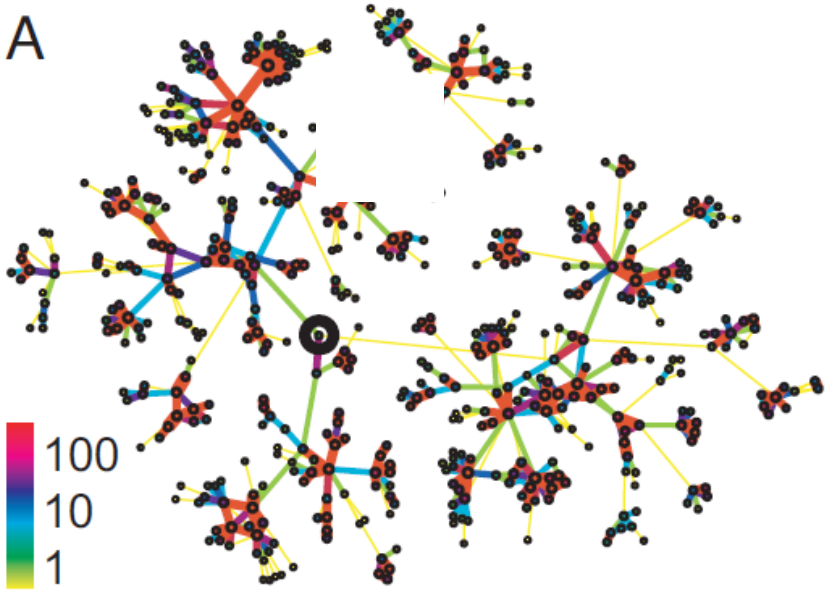


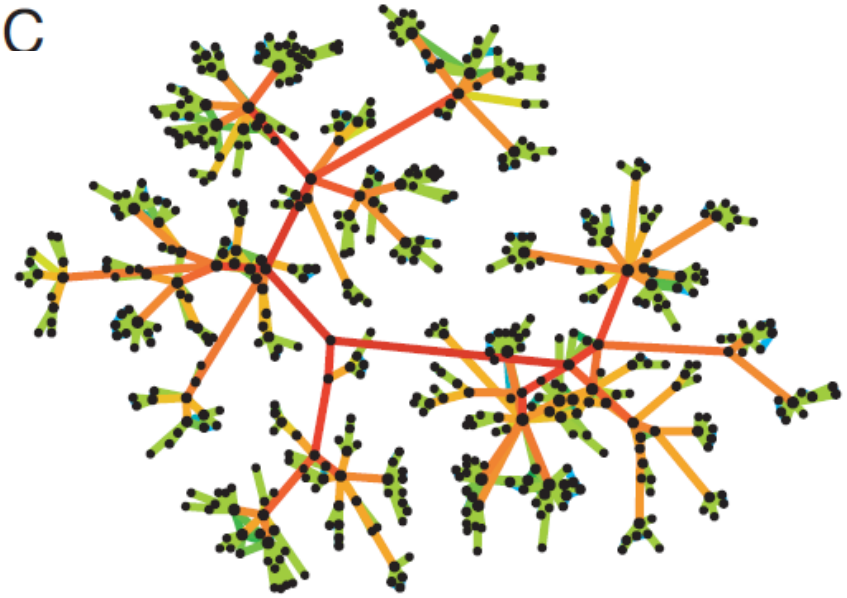

There may be none!

# Strength of Weak Ties

- Edge betweenness: Number of shortest paths passing over the edge
- Intuition:




**Edge strengths (call volume) in a real network**

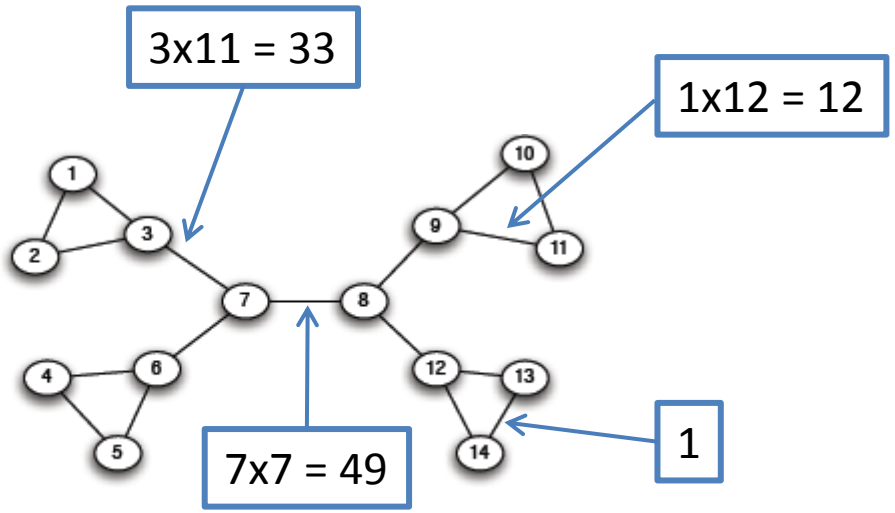


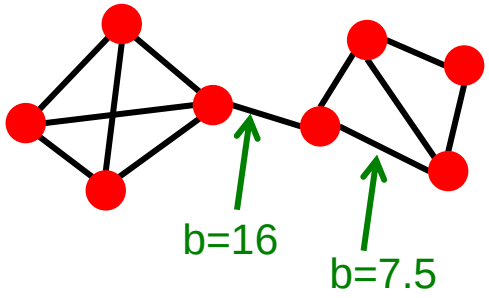

**Edge betweenness in a real network**

# Edge Betweenness

Betweenness of an edge (a, b): number of pairs of nodes x and y such that the edge (a, b) lies on the shortest path between x and y - since there can be several such shortest paths edge (a, b) is credited with the fraction of those shortest paths that include (a, b).

$$\mathrm{bt}(a,b) = \sum_{x,y} \frac{\#shortest\_paths(x,y)through(a,b)}{\#shortest\_paths(x,y)}$$




Edges that have a high probability to occur on a randomly chosen shortest path between two randomly chosen nodes have a high betweenness.

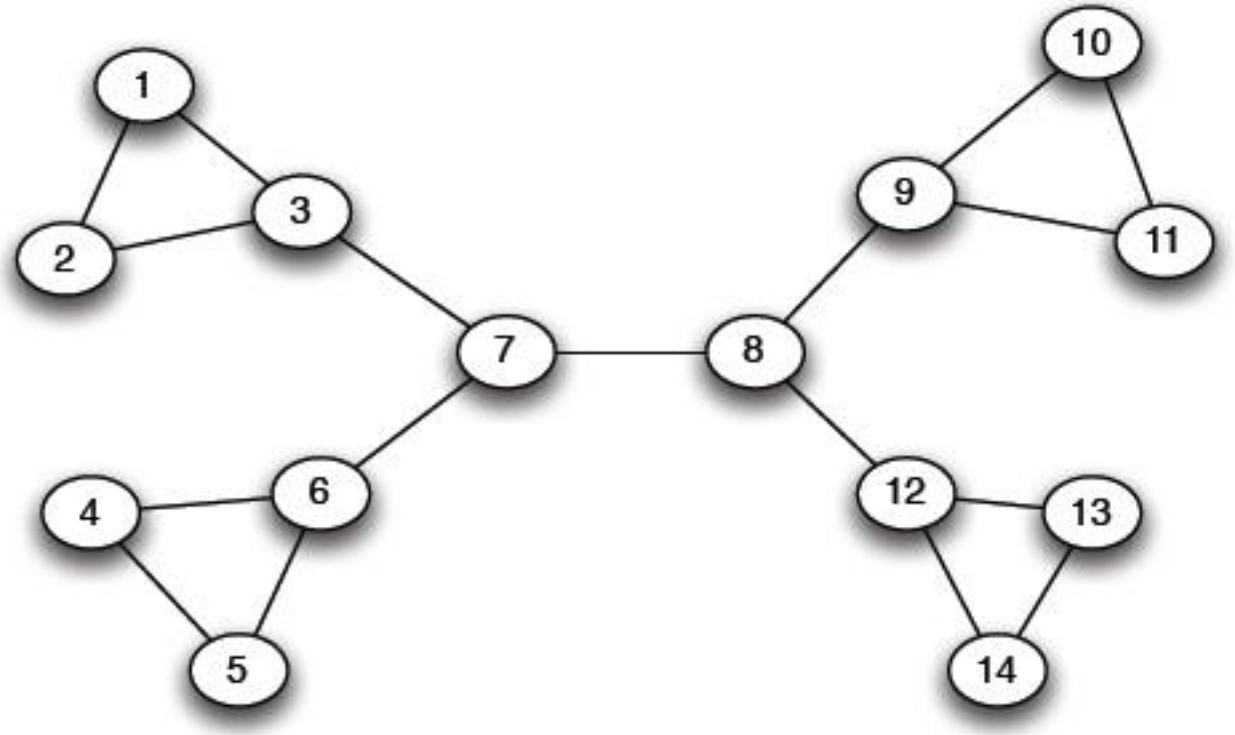
*Traffic (unit of flow)*

# The Girvan Newman method

» **Undirected unweighted networks**

– Repeat until no edges are left:
  - Calculate betweenness of edges
  - Remove edges with highest betweenness
– Connected components are communities
– Gives a hierarchical decomposition of the network

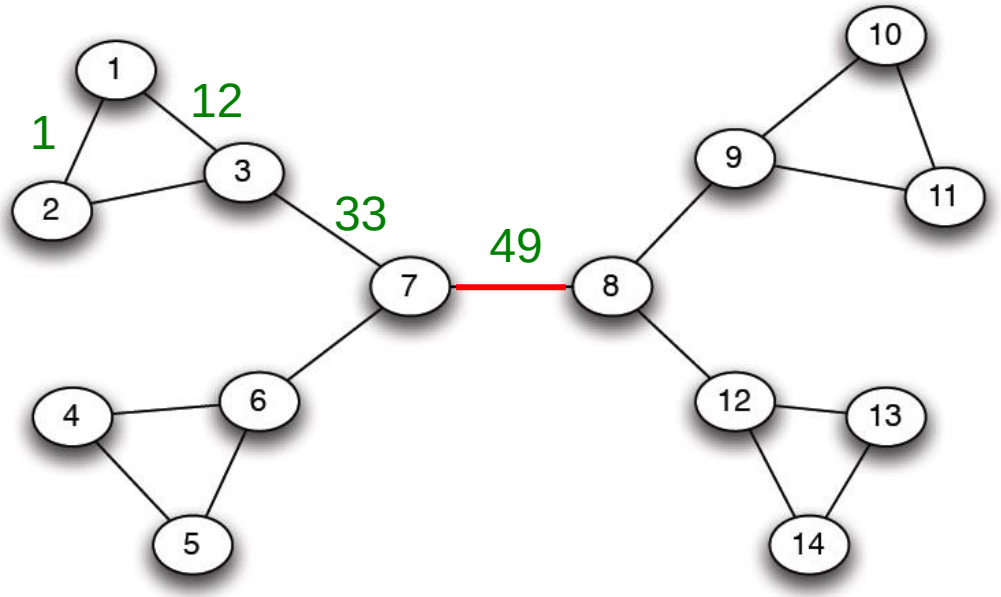
# Girvan Newman method: An example




Betweenness(7, 8)= 7x7 = 49 Betweenness(1, 3) = 1X12=12

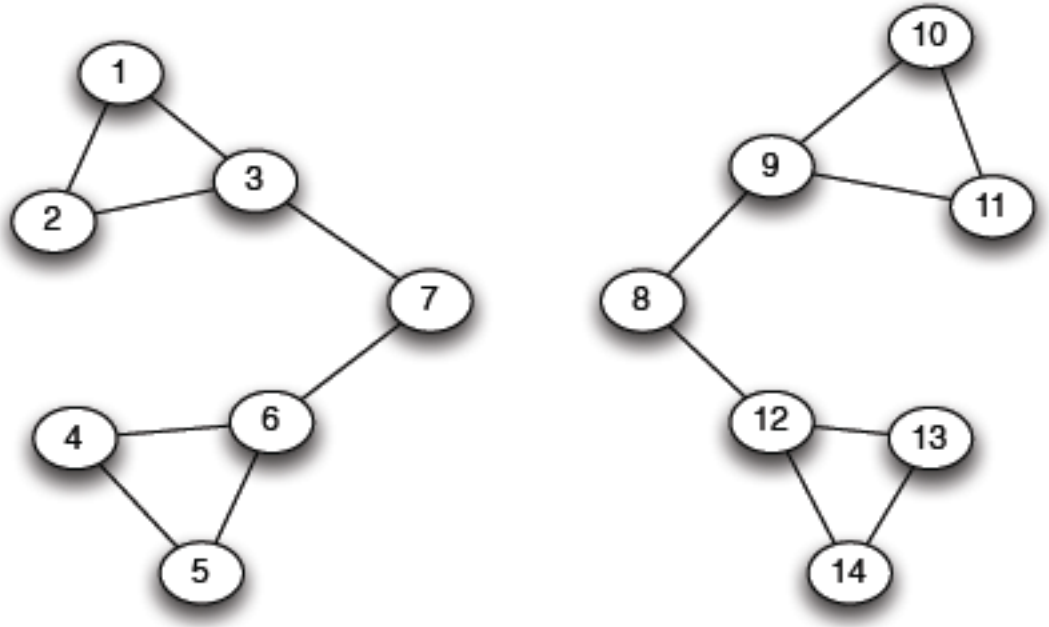
Betweenness(3, 7)=Betweenness(6, 7)=Betweenness(8, 9) = Betweenness(8, 12)= 3X11=33

# Girvan-Newman: Example




Need to re-compute betweenness at every step

# Girvan Newman method: An example

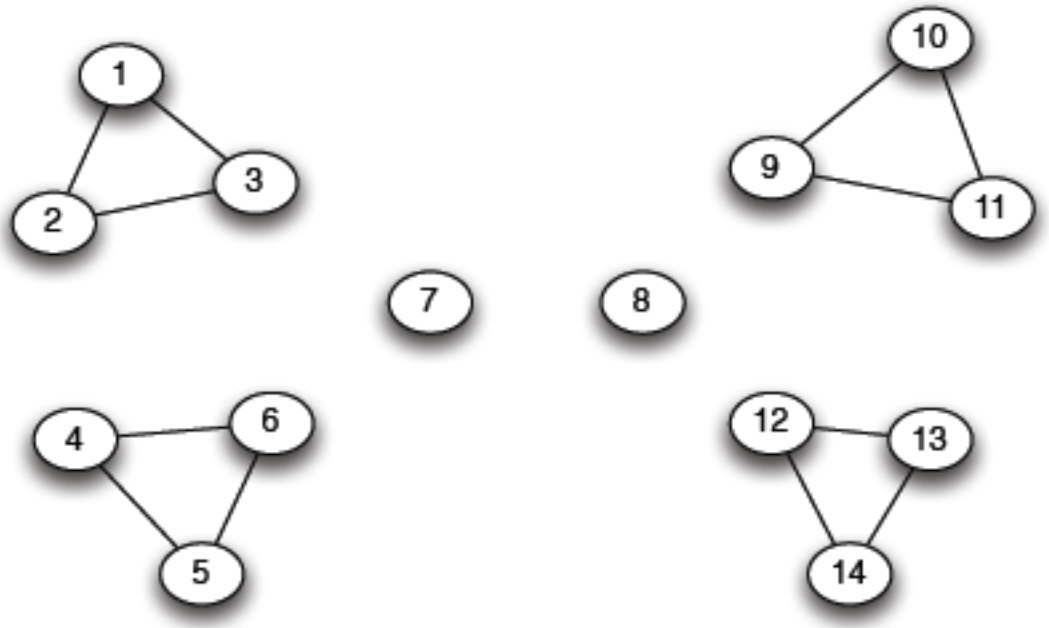



(a) *Step 1*

Betweenness(1, 3) = 1X5=5

Betweenness(3,7)=Betweenness(6,7)=Betweenness(8,9) = Betweenness(8,12)= 3X4=12

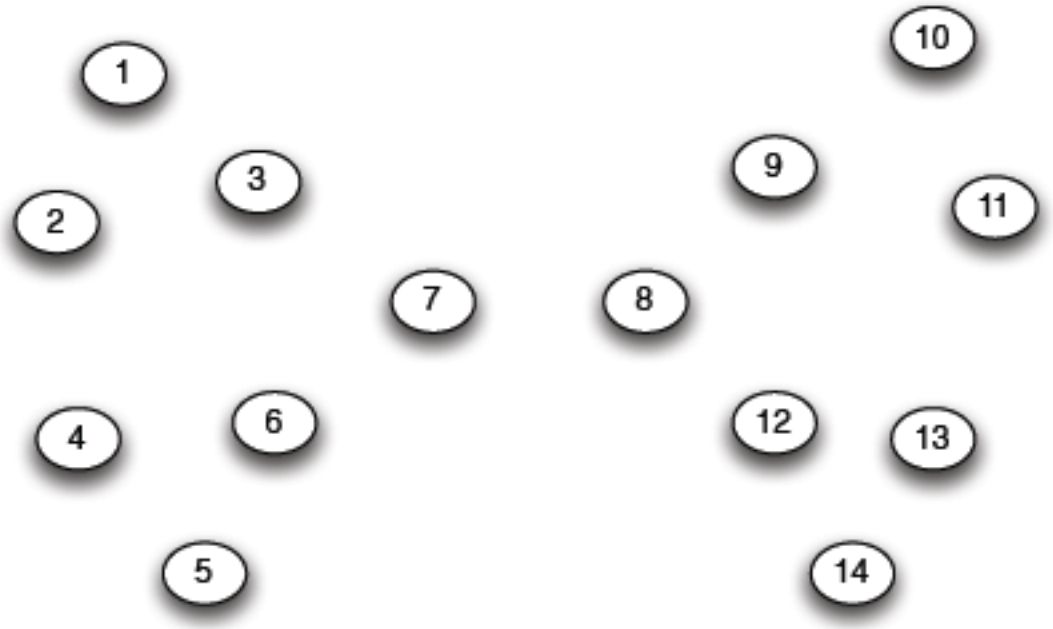
# Girvan Newman method: An example




(b) *Step 2*

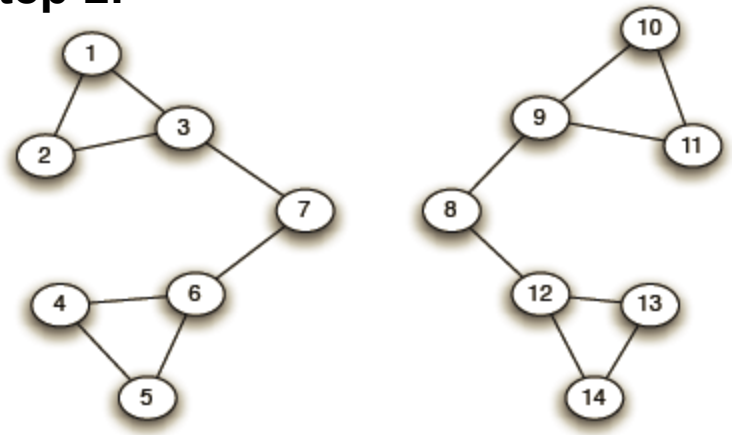
Betweenness of every edge = 1

# Girvan Newman method: An example

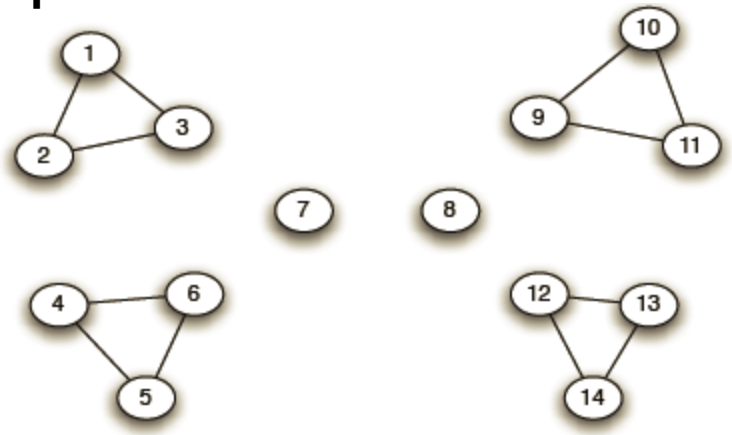


# Girvan-Newman: Example

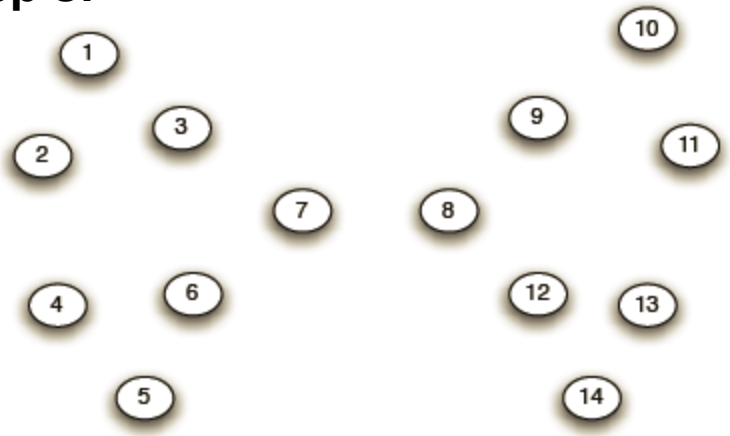
**Step 1:**



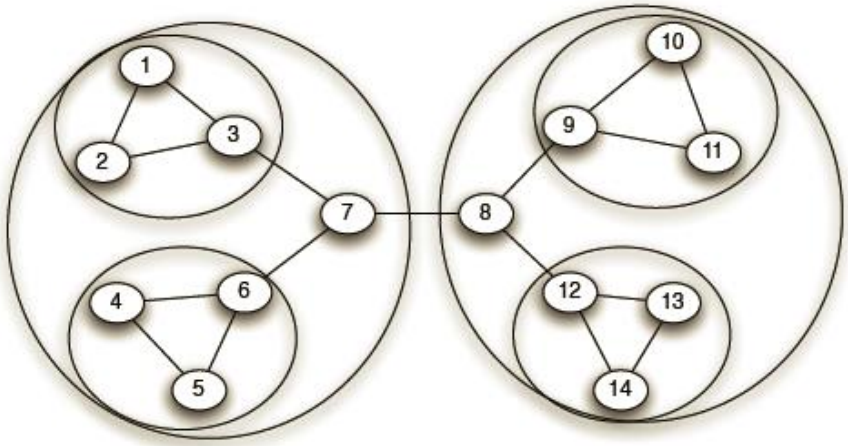

**Step 2:**



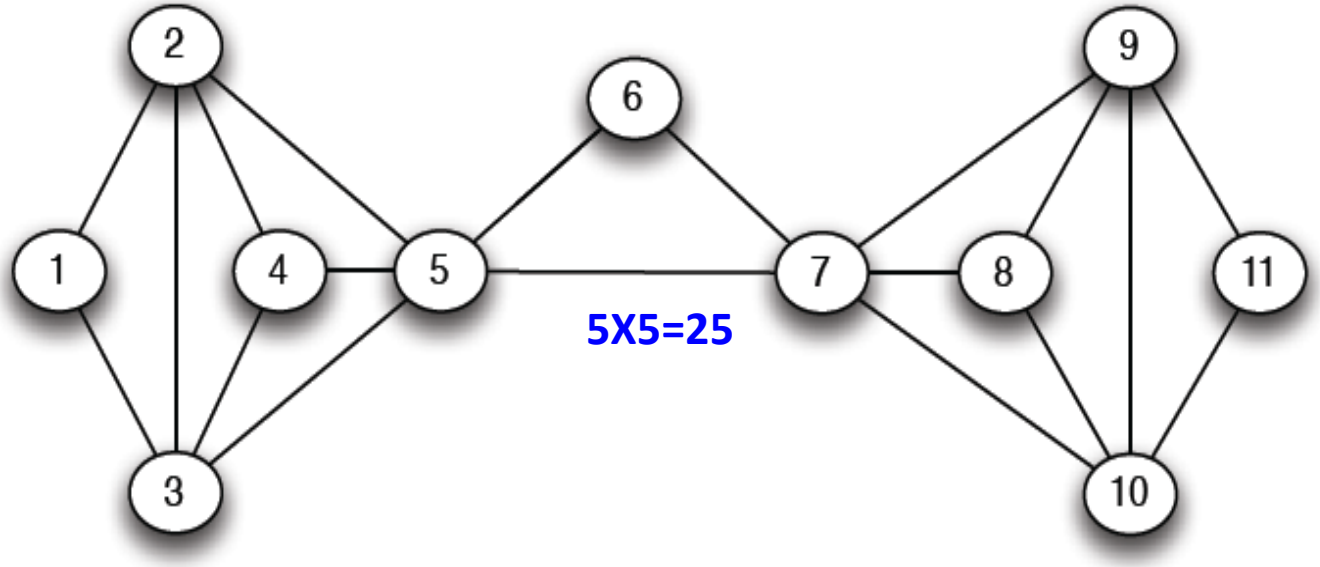

**Step 3:**



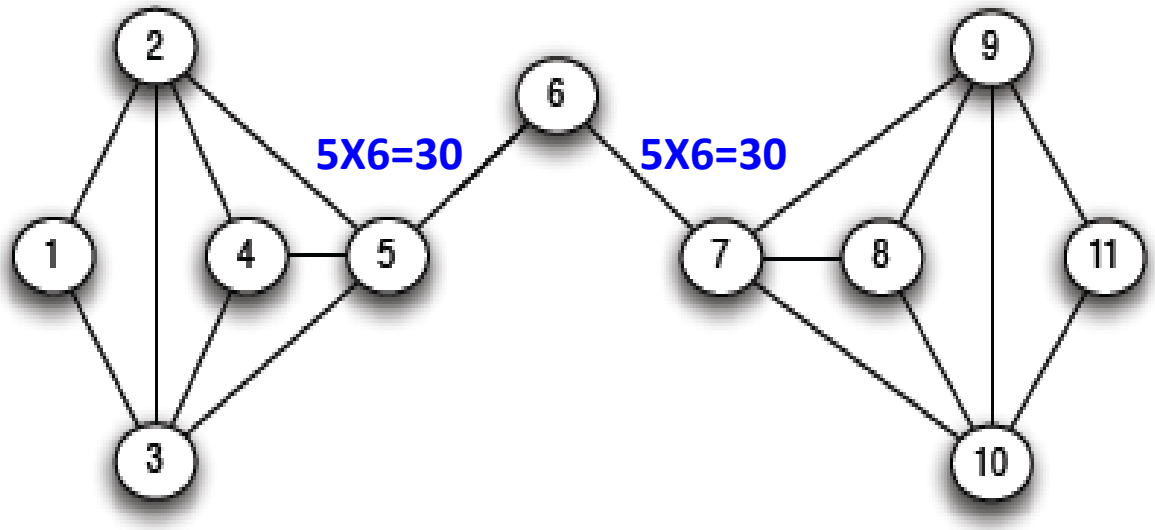

**Hierarchical network decomposition:**



# Another example

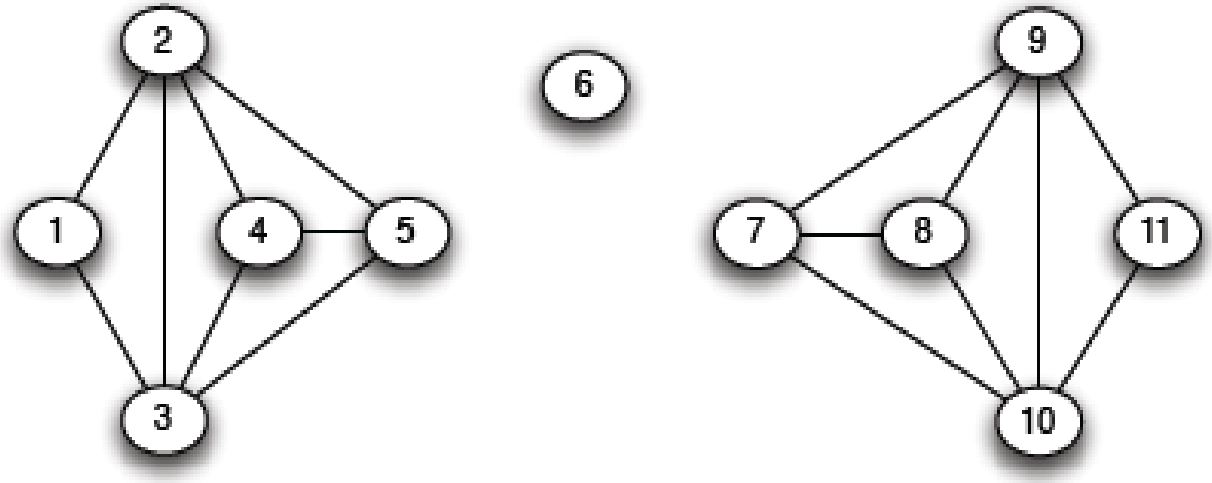


# Another example




(a) *Step 1*

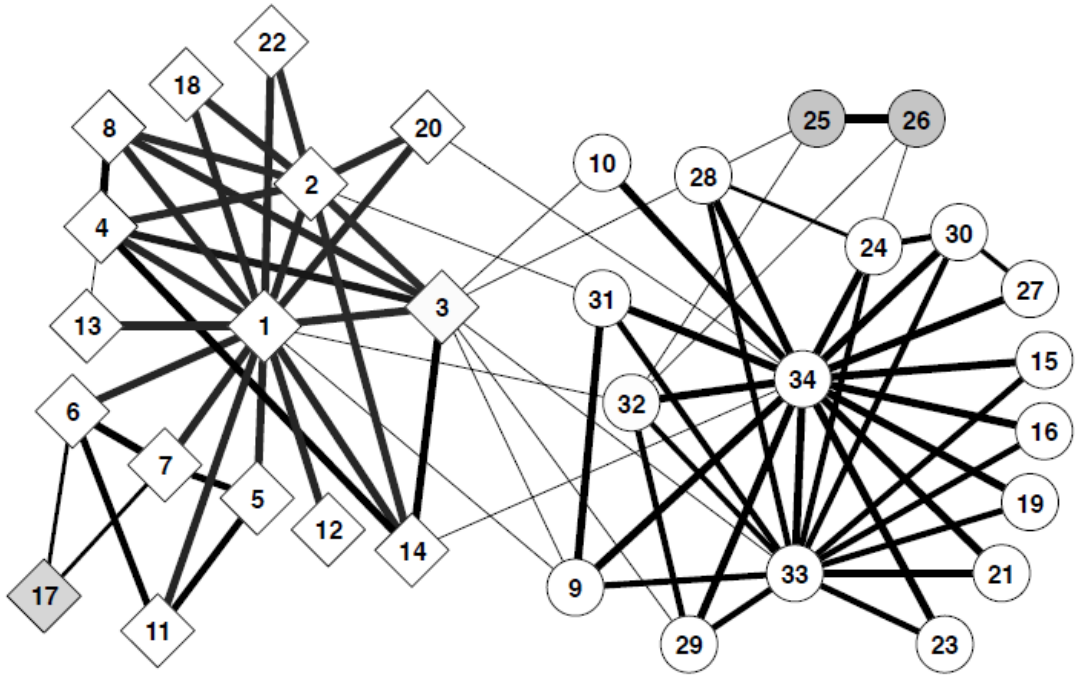
# Another example

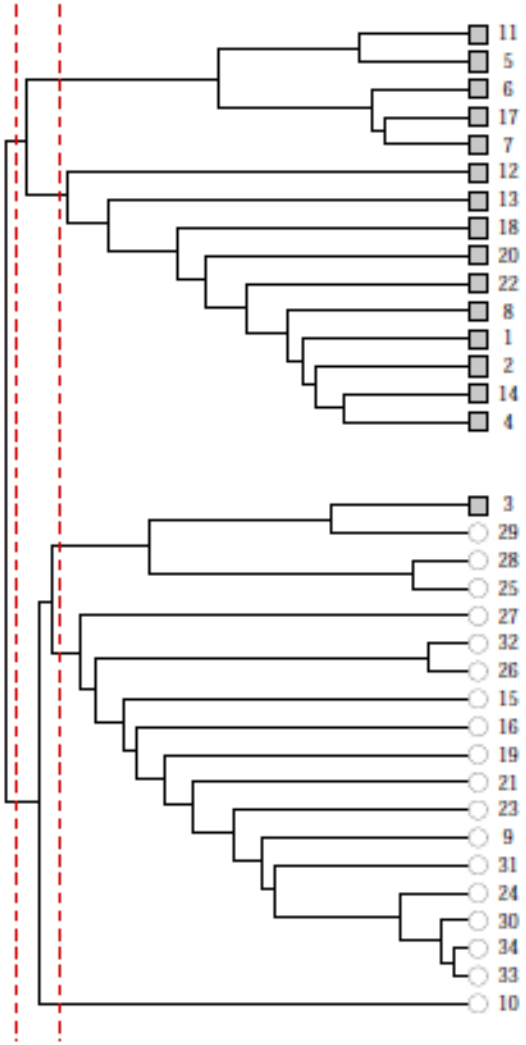



(b) *Step 2*

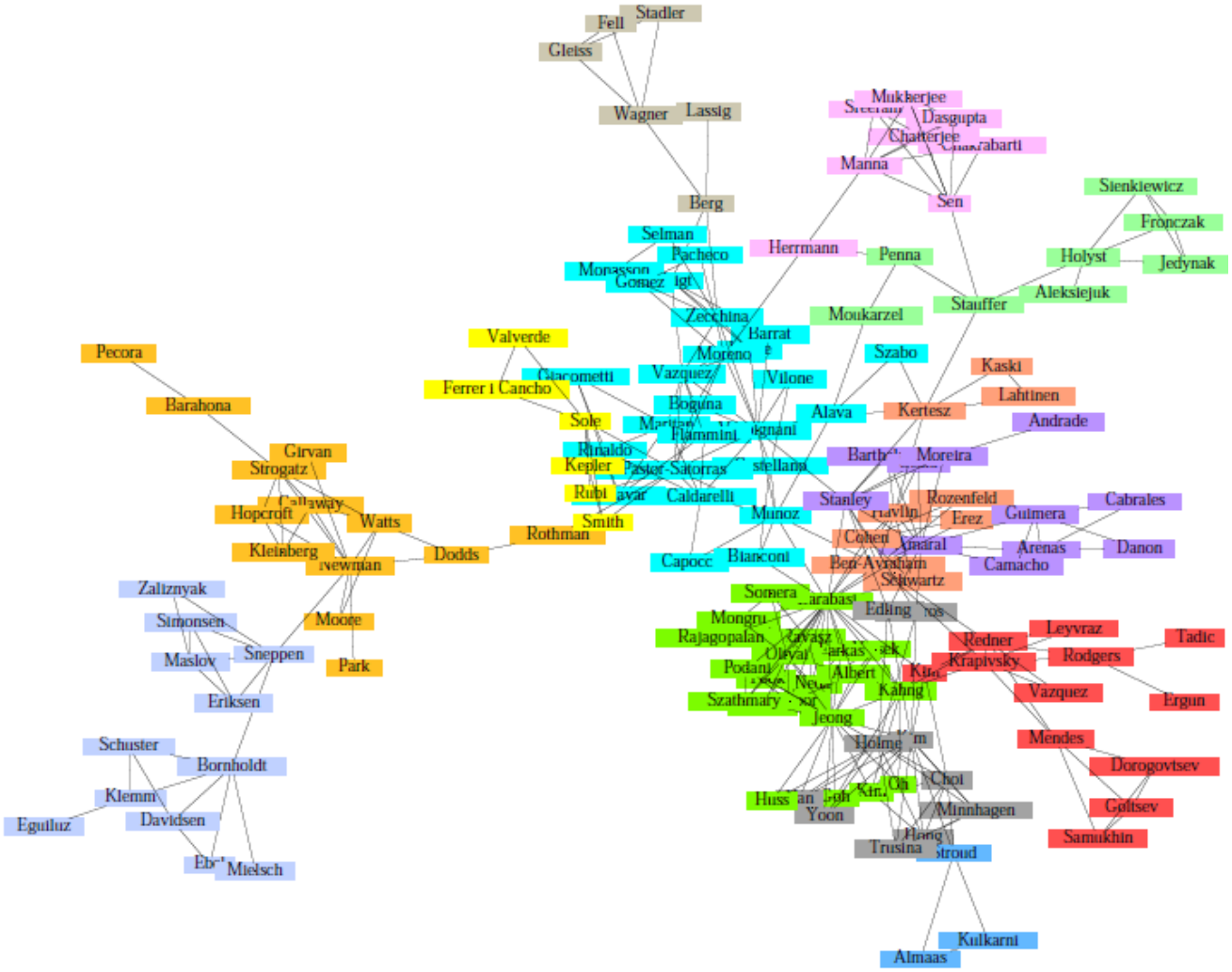
# Girvan-Newman: Results

- **Zachary's Karate club:**
  Hierarchical decomposition



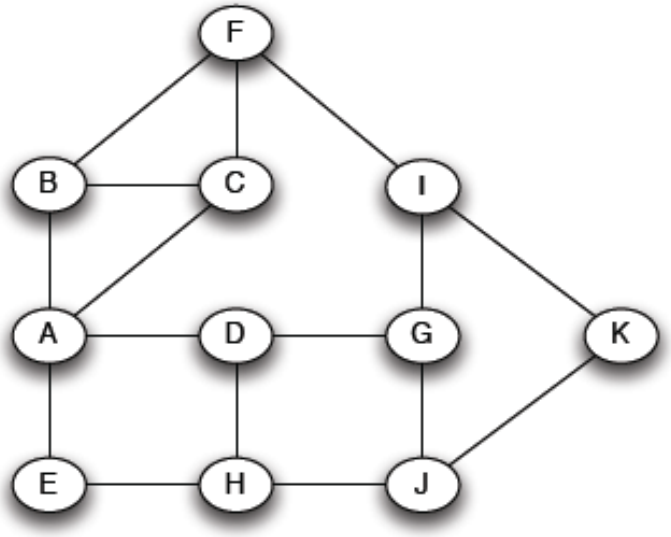
# Girvan-Newman: Results




Communities in physics collaborations

# How to Compute Betweenness?

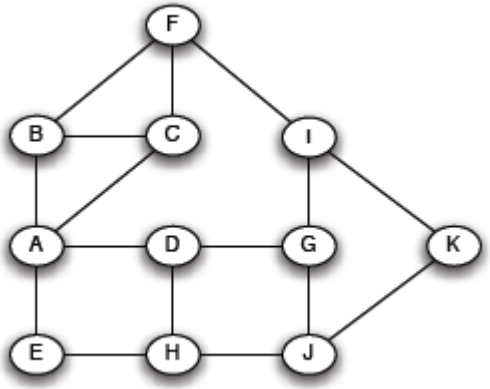
- Want to compute betweenness of paths starting at node $A$



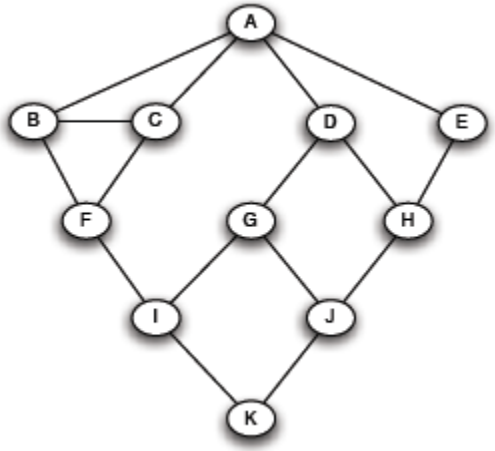
# Computing Betweenness

1. Perform a *BFS* starting from A
2. Determine the number of shortest path from A to each other node
3. Based on these numbers, determine the amount of flow from A to all other nodes that uses each edge

# Computing Betweenness: step 1



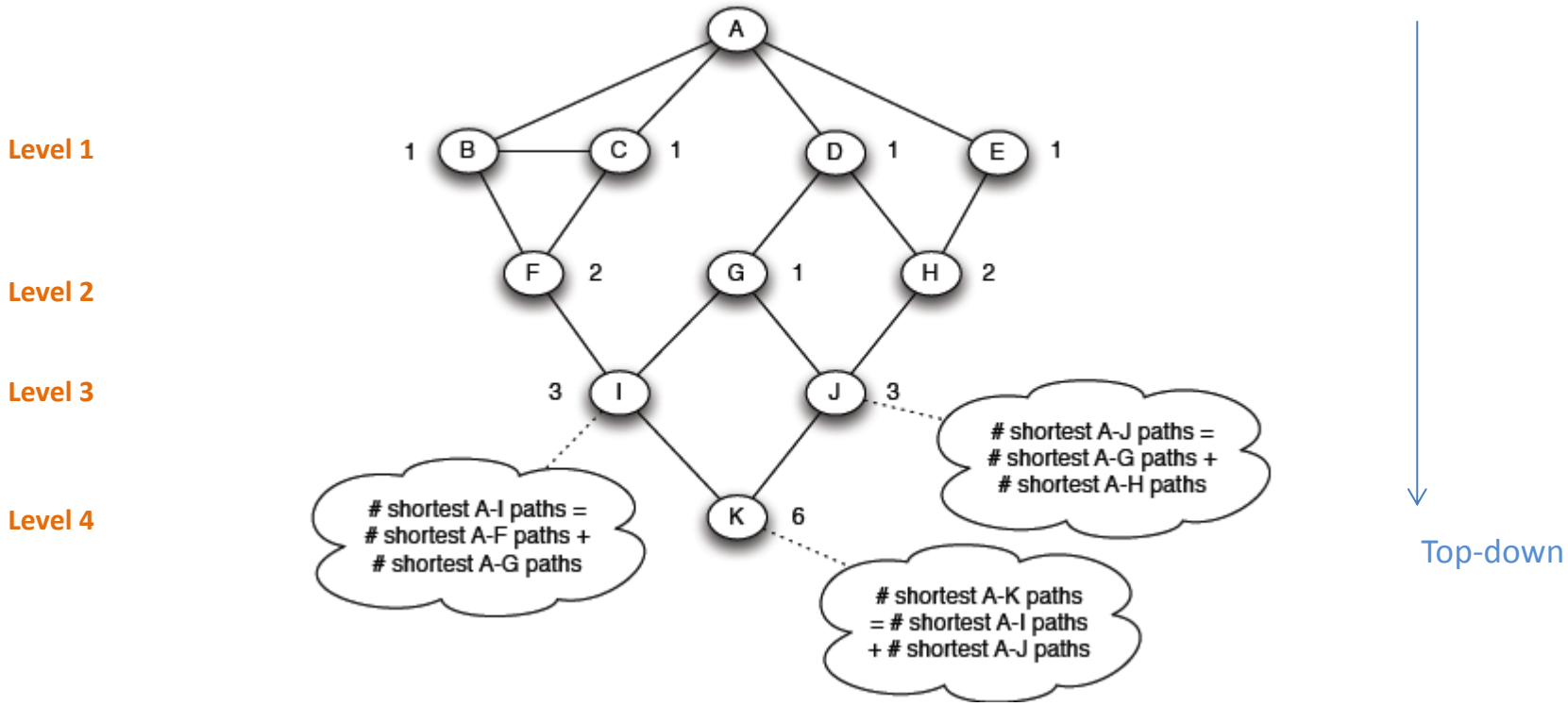

Initial network




BFS on A

# Computing Betweenness: step 2

Count how many shortest paths from A to a specific node

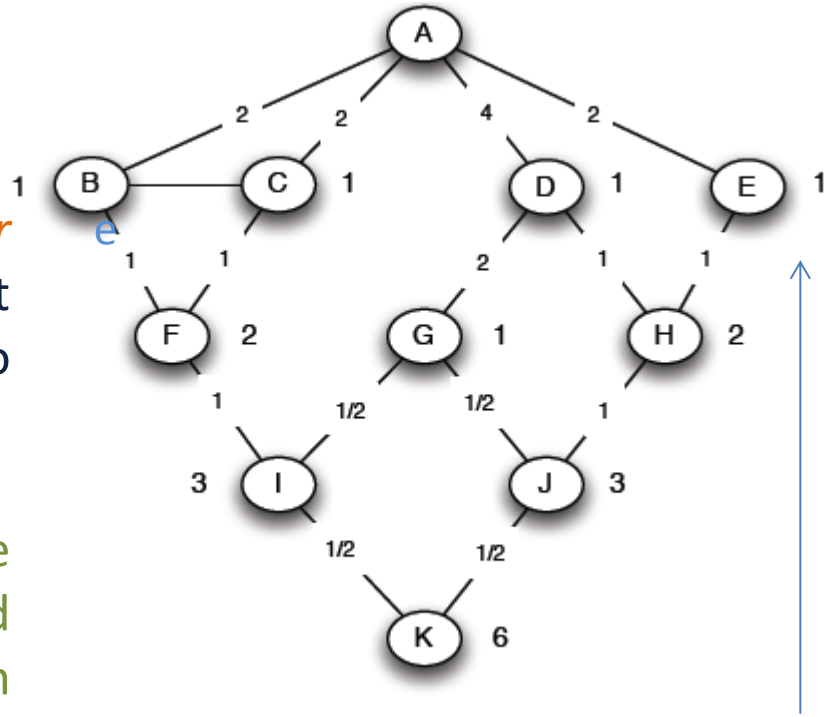


# Computing Betweenness: step 3

Compute betweenness by working up the tree: If there are multiple paths count them fractionally

For *each edge e*: calculate the sum *over all nodes Y* of the fraction of shortest paths *from the root A to Y* that go through e.

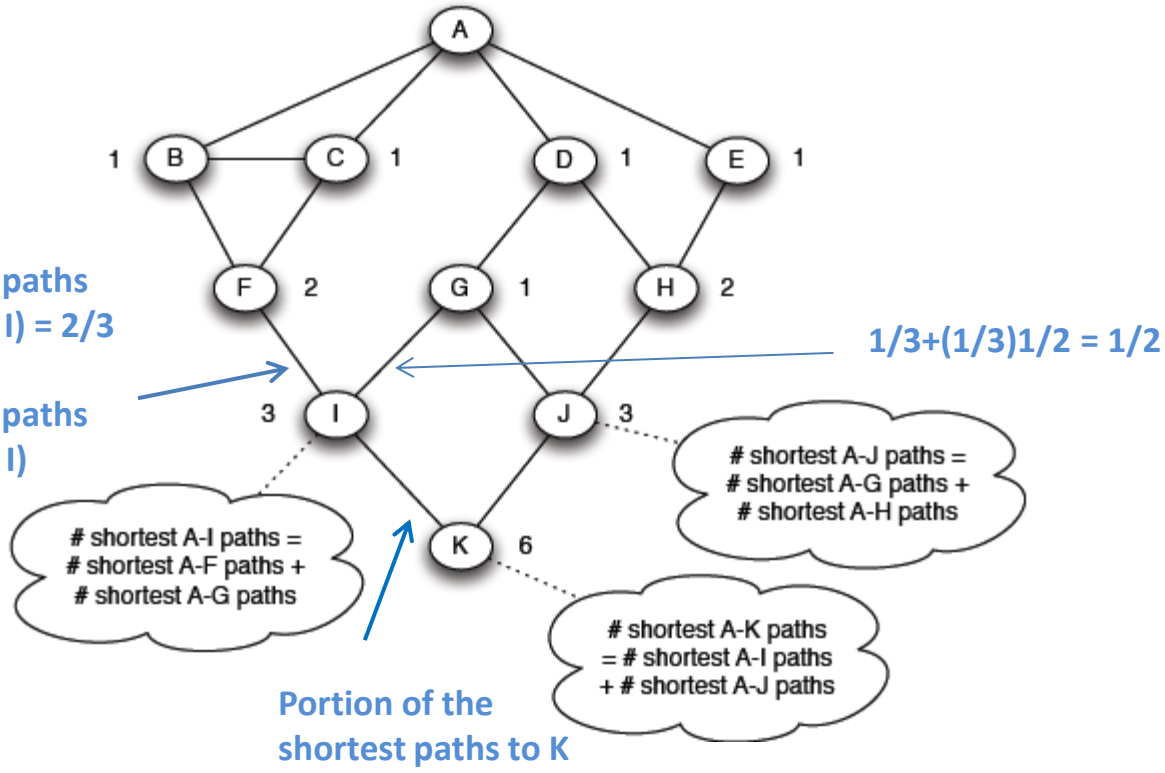
Each edge (X, Y) participates in the shortest-paths from the root to Y and to nodes (at levels) below Y -> Bottom up calculation



# Computing Betweenness: step 3

Count the flow through each edge

$$credit(e) = \sum_{X,Y} \frac{|shortest - path(X,Y) \text{through } e|}{|shortest\_path(X,Y)\}|}$$




**Portion of the shortest paths to I that go through (F, I) = 2/3**
**+**
**Portion of the shortest paths to K that go through (F, I) (1/2)(2/3) = 1/3**
**= 1**

**1/3+(1/3)1/2 = 1/2**

**Portion of the shortest paths to K that go through (I, K) = 3/6 = 1/2**

# Computing Betweenness: step 3

**The algorithm:**

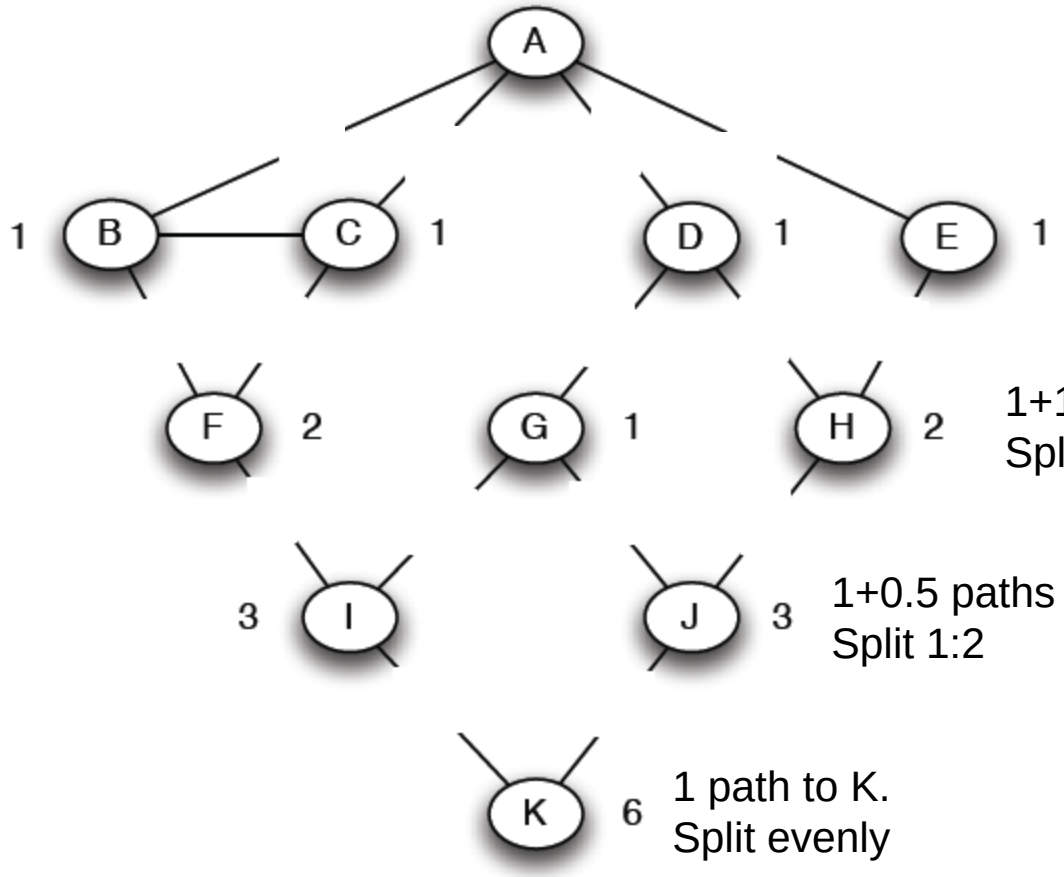
•Add edge **flows**:

-- node flow =

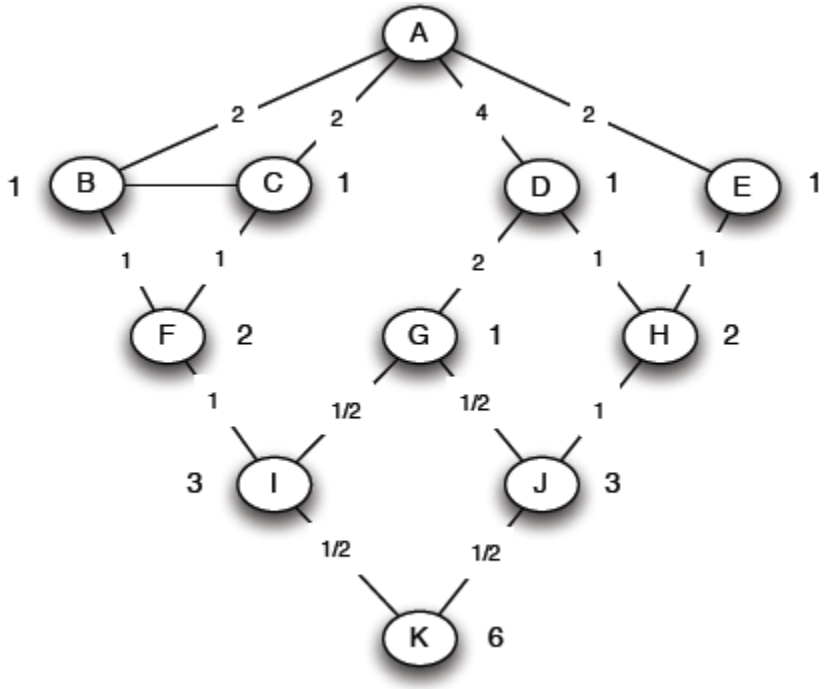
1+∑child edges

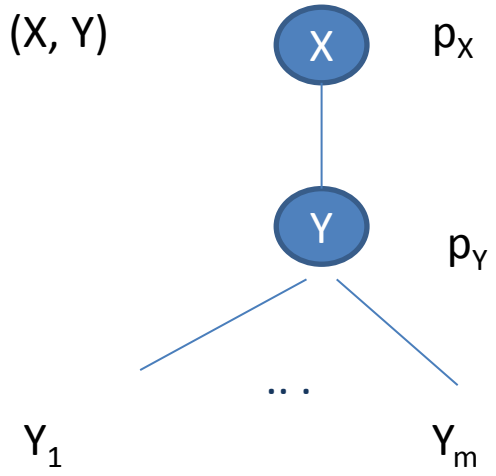
-- split the flow up based on the parent value

• Repeat the BFS procedure for each starting node $U$



# Computing Betweenness: step 3



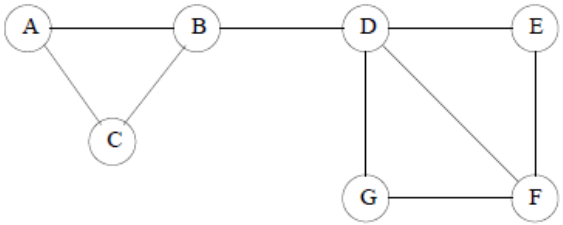

$$flow(X,Y) = p_X / p_Y \sum_{Y_i childof Y} (p_X / p_Y)\, flow(Y, Y_i)$$

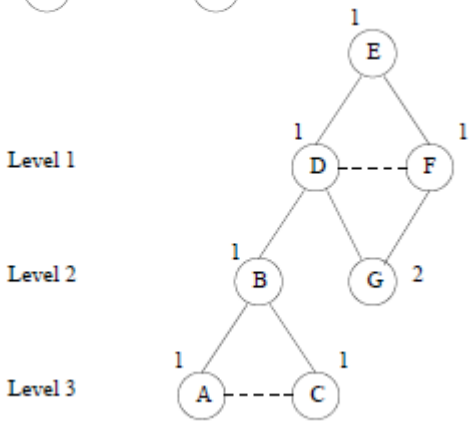
# Computing Betweenness

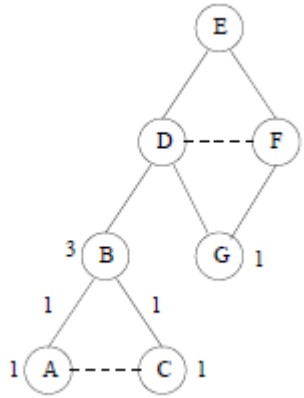
Repeat the process for all nodes

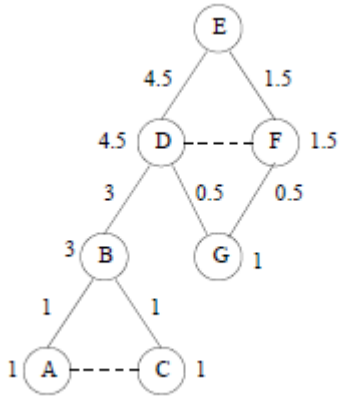
Sum over all BFSs

# Example

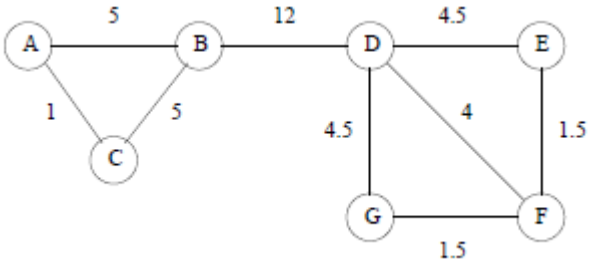

A
B
D
E
C
G
F

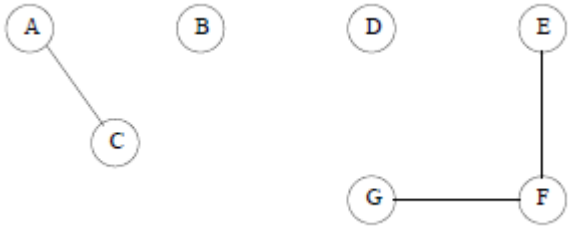

Level 1
Level 2
Level 3
1
E
1
1
D
F
1
1
B
G
2
1
1
A
C


E
D
F
3
B
G
1
1
1
1
A
C
1


E
4.5
1.5
4.5
D
F
1.5
3
0.5
0.5
3
B
G
1
1
1
1
A
C
1

# Example



# Computing Betweenness

Issues

- Test for connectivity?
- Re-compute all paths, or only those affected
- Parallel computation
- Sampling

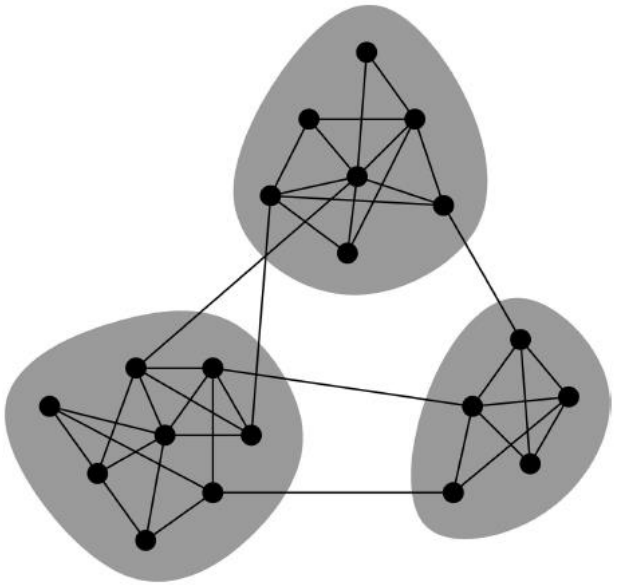
# Outline

## PART I

1. Introduction: what, why, types?
2. Cliques and vertex similarity
3. Background: Cluster analysis
4. Hierarchical clustering (betweenness)
5. Modularity
6. How to evaluate

# Modularity

- Communities: sets of **tightly connected nodes**
- Define: **Modularity $Q$**
  - A measure of how well a network is partitioned into communities
  - Given a partitioning of the network into groups $s \in S$:



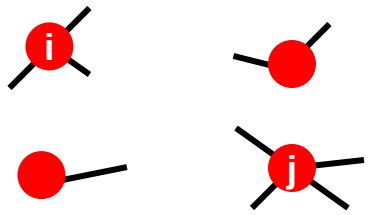
$$Q \propto \sum_{s \in S} [\ (\#\text{ edges within group } s) - (\text{expected } \#\text{ edges within group } s)\ ]$$

**Need a null model!**

*a copy of the original graph keeping some of its structural properties but without community structure*

# Null Model: Configuration Model

- Given real $G$ on $n$ nodes and $m$ edges, construct rewired network $G$'
  - Same degree distribution but random connections
  - Consider $\boldsymbol{G}$' as a multigraph
  - The expected number of edges between nodes $i$ and $j$ of degrees $\boldsymbol{d_i}$ and $\boldsymbol{d_j}$ equals to: $\boldsymbol{d_i} \cdot \frac{\boldsymbol{d_j}}{\boldsymbol{2m}} = \frac{\boldsymbol{d_i d_j}}{\boldsymbol{2m}}$



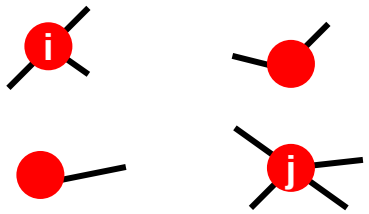

*For any edge going out of i randomly, the probability of this edge getting connected to node j is* $\frac{\boldsymbol{d_j}}{\boldsymbol{2m}}$

*Because the degree for i is $d_i$, we have $d_i$ number of such edges*

Note:

$$\sum_{u \in N} d_u = 2m$$

# Null Model: Configuration Model




- The expected number of edges in (multigraph) **G'**:
  - $= \frac{1}{2}\sum_{i \in N}\sum_{j \in N}\frac{d_i d_j}{2m} = \frac{1}{2}\cdot\frac{1}{2m}\sum_{i \in N} d_i\left(\sum_{j \in N} d_j\right) =$
  - $= \frac{1}{4m} 2m \cdot 2m = m$

Note:

$$\sum_{u \in N} d_u = 2m$$

# Modularity

- Modularity of partitioning S of graph G:
  - $Q \propto \sum_{s \in S}$ [ (# edges within group $s$) – (expected # edges within group $s$) ]
  - $Q(G,S) = \underbrace{\frac{1}{2m}} \sum_{s \in S} \sum_{i \in s} \sum_{j \in s} \left( A_{ij} - \frac{d_i d_j}{2m} \right)$

Normalizing cost.: -1<Q<1

$A_{ij}$ = 1 if i→j, 0 else

- Modularity values take range [−1, 1]
  - It is positive if the number of edges within groups exceeds the expected number
  - 0.3-0.7 < Q means significant community structure

# Modularity

Greedy method of Newman (one of the many ways to use modularity)

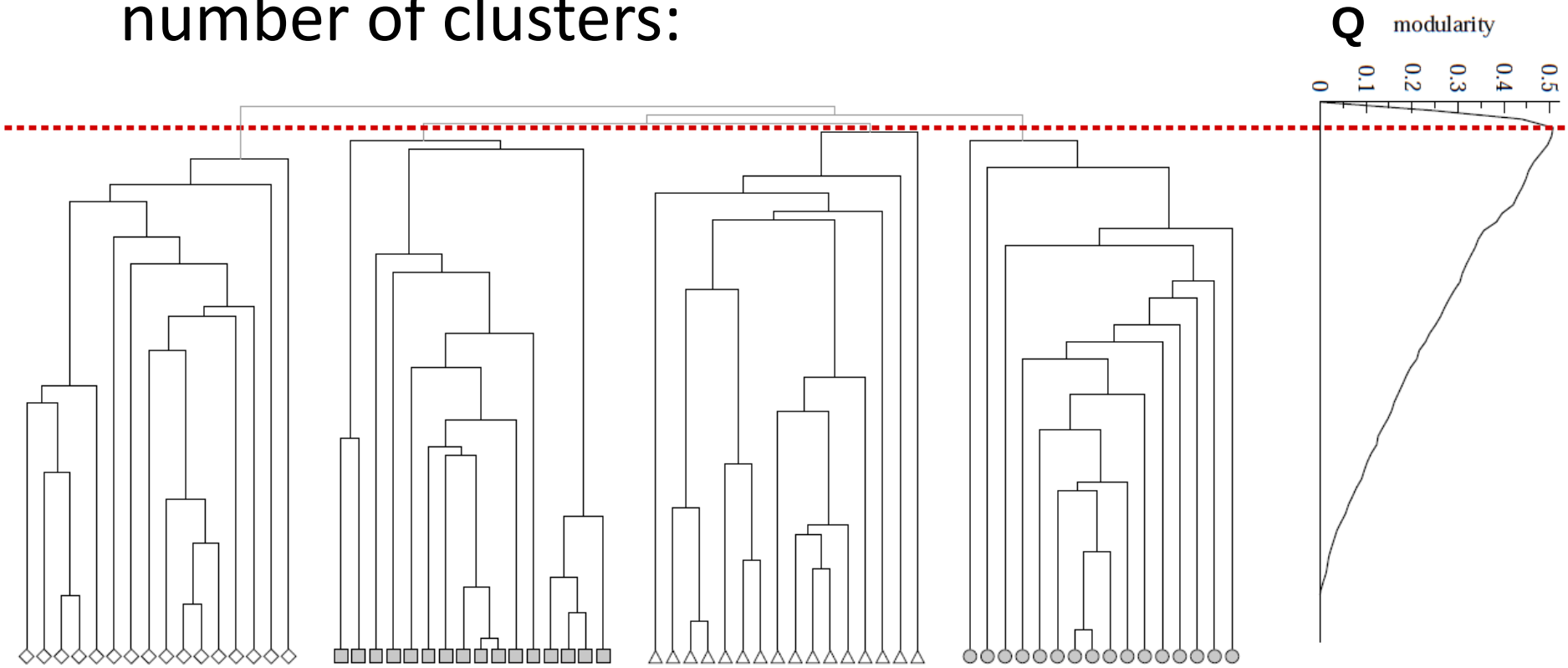
*Agglomerative hierarchical clustering method*

1. Start with a state in which each vertex is the sole member of one of n communities
2. Repeatedly join communities together in pairs, choosing at each step the join that results in the greatest increase (or smallest decrease) in Q.

Since the joining of a pair of communities between which there are no edges can never result in an increase in modularity, we ***need only consider those pairs between which there are edges***, of which there will at any time be at most m

# Modularity: Number of clusters

- Modularity is useful for selecting the number of clusters:



# Modularity: Cluster quality

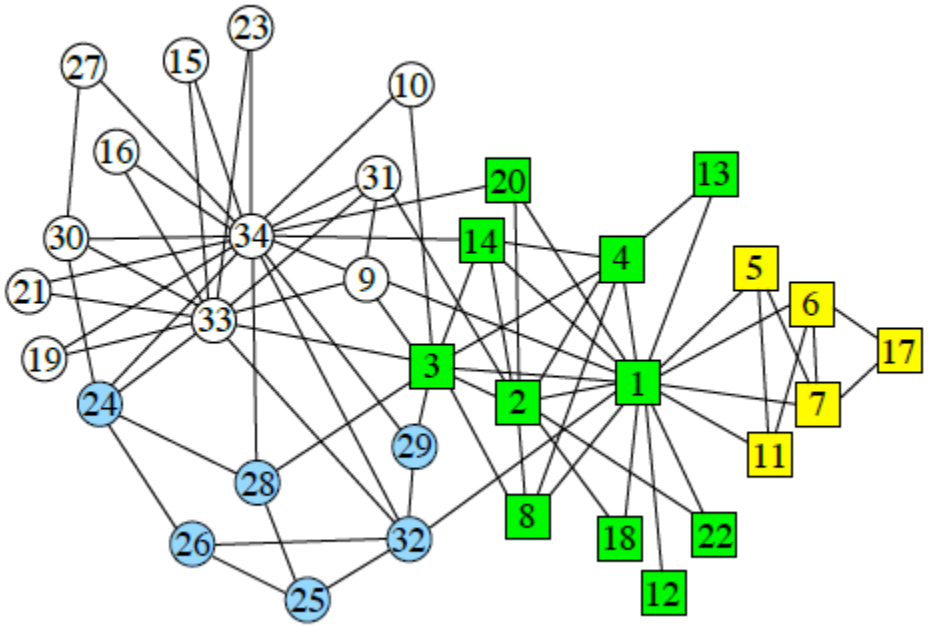
When a given clustering is “good”?

Also, it is both a local (per individual cluster) and global measure

# Community Evaluation

- With ground truth
- Without ground truth

# Evaluation with ground truth




Zachary's Karate Club

Club president (34) (circles) and instructor (1) (rectangles)

# Metrics: purity

the fraction of instances that have labels equal to the label of the community's majority

$$Purity = \frac{1}{N}\sum_{i=1}^{k}\max_{j}|C_i \cap L_j|$$

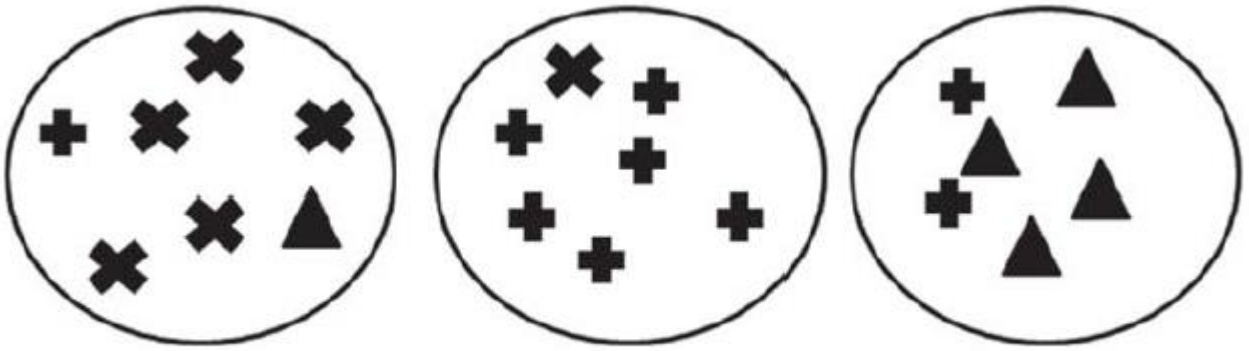

(5+6+4)/20 = 0.75

# Metrics

Based on pair counting: the number of pairs of vertices which are classified in the same (different) clusters in the two partitions.

- True Positive (**TP**) Assignment: when similar members are assigned to the same community. This is a *correct* decision.
- True Negative (**TN**) Assignment: when dissimilar members are assigned to different communities. This is a *correct* decision.
- False Negative (**FN**) Assignment: when similar members are assigned to different communities. This is an *incorrect* decision.
- False Positive (**FP**) Assignment: when dissimilar members are assigned to the same community. This is an *incorrect* decision.

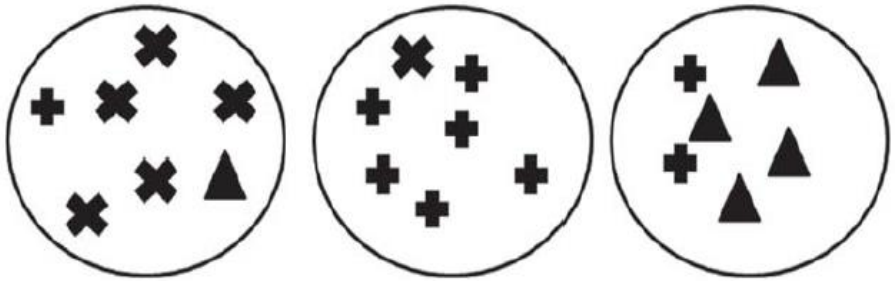
# Metrics: pairs




For TP, we need to compute the number of pairs with the same label that are in the same community

$$TP = \underbrace{\binom{5}{2}}_{\text{Community 1}} + \underbrace{\binom{6}{2}}_{\text{Community 2}} + \underbrace{\left(\binom{4}{2} + \binom{2}{2}\right)}_{\text{Community 3}} = 32$$

# Metrics: pairs




For TN: compute the number of dissimilar pairs in dissimilar communities

$$TN = \underbrace{(5\times 6 + 1\times 1 + 1\times 6 + 1\times 1)}_{\text{Communities 1 and 2}}$$

$$+ \underbrace{(5\times 4 + 5\times 2 + 1\times 4 + 1\times 2)}_{\text{Communities 1 and 3}}$$

$$+ \underbrace{(6\times 4 + 1\times 2 + 1\times 4}_{\text{Communities 2 and 3}} = 104.$$

# Metrics: pairs


For FP, compute dissimilar pairs that are in the same community.

$$FP = \underbrace{(5\times 1 + 5\times 1 + 1\times 1)}_{\text{Community 1}} + \underbrace{(6\times 1)}_{\text{Community 2}} + \underbrace{(4\times 2)}_{\text{Community 3}} = 25$$

For FN, compute similar members that are in different communities.

$$FN = \underbrace{(5\times 1)}_{\times} + \underbrace{(6\times 1 + 6\times 2 + 2\times 1)}_{+} + \underbrace{(4\times 1)}_{\triangle} = 29$$

# Metrics: pairs

Precision (P): the fraction of pairs that have been correctly assigned to the same community.

$$TP/(TP+FP)$$

Recall (R): the fraction of pairs assigned to the same community of all the pairs that should have been in the same community.

$$TP/(TP+FN)$$

F-measure

$$2PR/(P+R)$$

# Evaluation without ground truth

- Cluster Cohesion: Measures how closely related are objects in a cluster
- Cluster Separation: Measure how distinct or well-separated a cluster is from other clusters
- Example: Squared Error
  - Cohesion is measured by the within cluster sum of squares (SSE)

$$WSS = \sum_{i} \sum_{x \in C_i} (x - m_i)^2$$

  - Separation is measured by the between cluster sum of squares

$$BSS = \sum_{i} |C_i| (m - m_i)^2$$

    - Where $|C_i|$ is the size of cluster i

# Evaluation without ground truth

$$\delta_{int}(\mathcal{C}) = \frac{\#\text{ internal edges of } \mathcal{C}}{n_c(n_c-1)/2}$$

$$\delta_{ext}(\mathcal{C}) = \frac{\#\text{ inter-cluster edges of } \mathcal{C}}{n_c(n-n_c)}$$

# Evaluation without ground truth

With semantics:

- (ad hoc) analyze other attributes (e.g., profile, content generated) for coherence
- human subjects (user study) Mechanical Turk

Visual representation (similarity/adjacency matrix, word clouds, etc)


(a) U.S . Constitution


(b) Sports

# Basic References


- Jure Leskovec, Anand Rajaraman, Jeff Ullman, Mining of Massive Datasets, Chapter 10, http://www.mmds.org/
- Reza Zafarani, Mohammad Ali Abbasi, Huan Liu, Social Media Mining: An Introduction, Chapter 6, http://dmml.asu.edu/smm/
- Santo Fortunato: Community detection in graphs. CoRR abs/0906.0612v2 (2010)
- Pang-Ning Tan, Michael Steinbach, Vipin Kumar, Introduction to Data Mining, Chapter 8, http://www.users.cs.umn.edu/~kumar/dmbook/index.php

# Questions?