Information Processing Letters 75 (2000) 265–273

Information Processing Letters

www.elsevier.com/locate/ipl

On the performance of the first-fit coloring algorithm on permutation graphs

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Received 6 July 1999; received in revised form 19 April 2000
Communicated by A. Tarlecki

Abstract

In this paper we study the performance of a particular on-line coloring algorithm, the First-Fit or Greedy algorithm, on a class of perfect graphs namely the permutation graphs. We prove that the largest number of colors $\chi_{FF}(G)$ that the First-Fit coloring algorithm (FF) needs on permutation graphs of chromatic number $\chi(G) = \chi$ when taken over all possible vertex orderings is not linearly bounded in terms of the off-line optimum, if χ is a fixed positive integer. Specifically, we prove that for any integers $\chi > 0$ and $k \geqslant 0$, there exists a permutation graph G on n vertices such that $\chi(G) = \chi$ and $\chi_{FF}(G) \geqslant \frac{1}{2}((\chi^2 + \chi) + k(\chi^2 - \chi))$, for sufficiently large n. Our result shows that the class of permutation graphs $\mathcal{P}$ is not First-Fit χ-bounded; that is, there exists no function f such that for all graphs $G \in \mathcal{P}$, $\chi_{FF}(G) \leqslant f(\omega(G))$. Recall that for perfect graphs $\omega(G) = \chi(G)$, where $\omega(G)$ denotes the clique number of G.

Keywords: On-line coloring; First-Fit algorithm; Algorithms; Permutation graphs; Perfect graphs; Combinatorial problems

1. Introduction

A *coloring* (or proper coloring) of a graph G is an assignment of positive integers called "colors" to its vertices so that no two adjacent vertices have the same color. The *coloring problem* is to color a graph with as few colors as possible; that is, to minimize the number of colors (see Jensen and Toft [4]). An *on-line coloring* of a graph G is a procedure that immediately colors the vertices of G taken from a list without looking ahead or changing the colors already assigned. More precisely, an on-line coloring of G is an algorithm that properly colors G by receiving its vertices in some order $v_1, v_2, \ldots, v_n$. The color of v_i is assigned by only looking at the subgraph of G induced by the set $\{v_1, v_2, \ldots, v_i\}$, and the color of v_i never changes thereafter.

Let G be a graph with an ordering $v_1 < v_2 < \cdots < v_n$ of its vertices and let A be an on-line coloring algorithm with input $(G, <)$. Over all such possible orderings $<$, let $\chi_A(G)$ denote the maximum number of colors used by A to color G. Clearly, $\chi_A(G)$ measures the worst-case behaviour of A on G. The minimum number of colors required to color G off-line is called chromatic number of G, and is denoted by $\chi(G)$.

The simplest on-line coloring is the *First-Fit algorithm* (also sometimes called "the Greedy algorithm"); we will refer to it by the abbreviation FF throughout the paper. Given $(G, <)$ as input, FF works by receiving the vertices of the graph G one vertex at time

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0020-0190/00/$ – see front matter
PII: S0020-0190(00)00109-5

in the given order $v_1 < v_2 < \cdots < v_n$ and assigning the smallest possible integer from $\mathbb{Z}^+$ as the color to vertex v_i ($1 \leqslant i \leqslant n$); that is, the smallest color not yet assigned to any vertex adjacent to v_i among the previously colored vertices. We note that if the vertices of G are considered in an ideal sequence then $\chi_{\mathrm{FF}}(G) = \chi(G)$; to construct such a sequence first find an optimal coloring of G and then put all vertices with the same color in consecutive positions in the sequence.

Our objective is to study the performance of the coloring algorithm FF on permutation graphs, a well-known class of perfect graphs. A graph $G = (V, E)$ is a *permutation graph* if and only if there exists a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ on vertex set $V = \{1, 2, \ldots, n\}$ such that $(i, j) \in E$ if and only if $(i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$, for all $i, j \in V$, where $\pi^{-1}(i)$ is the index of the element i in π [1,8,9].

Many researchers have extensively studied on-line coloring algorithms [2–5]. Most of their work is devoted to the proof of upper bounds for the $\chi_{\mathrm{FF}}(G)$; that is, the worst-case behaviour of the coloring algorithm FF [2,4]. We mention here some of them in the case of subfamilies of perfect graphs: $\chi_{\mathrm{FF}}(G) \leqslant \omega(G) + 1$ if G is a split graph; $\chi_{\mathrm{FF}}(G) \leqslant \frac{3}{2}\omega(G)$ if G is the complement of a bipartite graph; $\chi_{\mathrm{FF}}(G) \leqslant 2\omega(G) - 1$ if G is the complement of a chordal graph; $\chi_{\mathrm{FF}}(G) \leqslant 40\omega(G)$ if G is an interval graph [5], where $\omega(G)$ denotes the clique number of G. (Kierstead and Trotter [7] presented an on-line algorithm for coloring an interval graph G with at most $3\omega(G) - 2$ colors and showed that no on-line algorithm could do better; their algorithm was almost, but not quite, the FF algorithm.) These results say that the on-line coloring algorithm FF can color all these subfamilies of perfect graphs by a number of colors that is linearly bounded in respect to the off-line optimum. It is well known that for perfect graphs $\chi(G) = \omega(G)$; hereafter $\chi(G) = \chi$.

The main result of this paper is summarized in the following theorem:

Theorem 1. *For any integers $\chi > 0$ and $k \geqslant 0$, there exists a permutation graph G such that the chromatic number of G is equal to χ and the on-line First-Fit coloring algorithm uses*

$$c_{\mathrm{FF}}(G) = \frac{\chi(\chi + 1)}{2} + k\frac{\chi(\chi - 1)}{2}$$

colors to color G.

A class of graphs $\mathcal{G}$ is First-Fit χ-bounded (or, FF χ-bounded) if there exists a function f such that for all graphs $G \in \mathcal{G}$, $\chi_{\mathrm{FF}}(G) \leqslant f(\omega(G))$ [4,6]. In this paper we show that, contrary to known results for other graph classes, the class of permutation graphs is not FF χ-bounded. In Theorem 1, k may be any function of χ. Thus, we obtain:

Corollary 1. *The class of permutation graphs is not FF χ-bounded; that is, there exists no function f such that for all permutation graphs G, $\chi_{\mathrm{FF}}(G) \leqslant f(\chi(G))$.*

2. $A(n)$ and $B(n)$ permutations

In this section we define two types of permutations $A(m)$ and $B(n)$ of lengths m and n, respectively, which we shall use as tools for constructing a permutation graph G on which $\chi_{\mathrm{FF}}(G)$ is greater than or equal to the values given in Theorem 1. We represent a permutation of length n as a rearrangement of $N_n = (1, 2, \ldots, n)$.

Moreover, we define two operations on permutations which we call *x-insertion* and *y-insertion*. Each of these operations is applied on two permutations, say, A and B of lengths m and n, respectively, and produces a permutation of length $m + n$, by inserting the permutation B into A in a specific manner.

2.1. Construction of $A(n)$ and $B(n)$

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_m)$ be two sequences of lengths n and m, respectively, whose elements are drawn from a linearly ordered set S. We shall use the notation $C = [A, B]$ to denote the sequence $C = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$.

We construct n sequences $A_1, A_2, \ldots, A_n$ of lengths $n, 2(n - 1), 3(n - 2), \ldots, n$, respectively. Let

$$A_1 = [A_{11}, A_{12}, \ldots, A_{1(n-1)}, A_{1n}],$$
$$A_2 = [A_{21}, A_{22}, \ldots, A_{2(n-1)}],$$
$$\vdots$$
$$A_n = [A_{n1}]$$

be these sequences, where A_{ij} is a sequence of length i, $1 \leqslant i \leqslant n$. The elements of A_{ij} are denoted by a_{ij}^k, where $k = 1, 2, \ldots, i$; that is, $A_{ij} = (a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^i)$.

First, we compute the sequence $A_1 = [A_{11}, A_{12}, \ldots, A_{1(n-1)}, A_{1n}]$, whose elements are sequences of length 1 each; that is $A_1 = (a_{11}^1, a_{12}^1, \ldots, a_{1n}^1)$, where $A_{1j} = (a_{1j}^1)$. The elements of the sequence A_1 are defined as follows:

$$a_{11}^1 = n$$

and

$$a_{1j}^1 = (n - j + 1) + \frac{1}{2} \sum_{i=0..j-2} (n-i)(n-i+1), \quad j = 2, 3, \ldots, n.$$

Next we compute the sequence $A_i = [A_{i1}, A_{i2}, \ldots, A_{(n-i+1)}]$, for $i = 2, 3, \ldots, n$. The elements of the sequence $A_{ij} = (a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^i)$, $1 \leqslant j \leqslant n - i + 1$, are defined as follows:

$$a_{ij}^1 = a_{1j}^1 - i + 1$$

and

$$a_{ij}^k = a_{ij}^{k-1} + (n - j + 1) - (k - 2), \quad k = 2, 3, \ldots, i.$$

Having computed the sequences $A_1, A_2, \ldots, A_n$, let us now define the following three sequences:

$$A(n) = \left[A_{11}, A_{12}, \ldots, A_{1(n-1)}, A_{1n}, A_{21}, A_{22}, \ldots, A_{2(n-1)}, \ldots, A_{n1}\right],$$
$$A^*(n) = \left[A_{1n}, A_{2(n-1)}, \ldots, A_{n1}\right],$$
$$B(n) = (1, 2, \ldots, n).$$

It follows from the definitions that the sequences $A(n)$ and $A^*(n)$ contain $m = n(n+1)(n+2)/6$ and $m^* = n(n+1)/2$ elements, respectively. Moreover, by construction the sequence $A(n)$ is a permutation on N_m. For example, let us consider the sequences $A(3)$ and $A^*(3)$. By definition $A(3) = [A_1, A_2, A_3]$ and $A^*(3) = [A_{13}, A_{22}, A_{31}]$, where $A_1 = [A_{11}, A_{12}, A_{13}]$, $A_2 = [A_{21}, A_{22}]$ and $A_3 = [A_{31}]$. It is easy to see that, $A_{11} = (3)$, $A_{12} = (8)$, $A_{13} = (10)$, $A_{21} = (2, 5)$, $A_{22} = (7, 9)$, $A_{31} = (1, 4, 6)$, and therefore $A_1 = (3, 8, 10)$, $A_2 = (2, 5, 7, 9)$, $A_3 = (1, 4, 6)$. Thus, $A(3) = (3, 8, 10, 2, 5, 7, 9, 1, 4, 6)$ and $A^*(3) = (10, 7, 9, 1, 4, 6)$.

2.2. Insertion operations

Let $A = (a_1, a_2, \ldots, a^*, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_m)$ be two permutations on N_n and N_m, respectively. We define an operation on A and B which produces a permutation A_x on N_{n+m} as follows:

$$A_x = (a_1', a_2', \ldots, a_n', b_1', b_2', \ldots, b_m'),$$

where

(i) $a_i' = a_i$ for all $a_i \leqslant a^*$,
(ii) $a_i' = a_i + m$ for all $a_i > a^*$,
(iii) $b_i' = a^* + i, \; 1 \leqslant i \leqslant m$.

The above operation is called *x-insertion* and denoted by x-insert$(A; a^*, B)$. The element a^* is called a *pivot*. Additionally, we define the *y-insertion* operation on A and B, denoted by y-insert$(A; a^*, B)$, which produces a permutation A_y on N_{n+m} as follows:

$$A_y = \left(a_1, a_2, \ldots, a_i, b_1', b_2', \ldots, b_m', a_{i+1}, a_{i+2}, \ldots, a_n\right),$$

where

(i) $a_i = a^*$,
(ii) $b_i' = n + i, \; 1 \leqslant i \leqslant m$.

Let A be a permutation on N_n, and let $A^* = (a_1^*, a_2^*, \ldots, a_m^*)$ and $B = (b_1, b_2, \ldots, b_m)$ be two sequences such that $A^* \subseteq A$, and $\|A^*\| = \|B\|$. In such a case, we shall use the notation x-insert$(A; A^*, B)$ to denote the sequence of operations x-insert$(A; a_i^*, (1))$, for $i = 1, 2, \ldots, m$; recall that, (1) is a permutation on N_1. In a similar manner, we shall use the notation y-insert$(A; A^*, B)$.

3. The input $(G, <)$ of the FF algorithm

In this section we construct a permutation graph G and an ordering $<$ of its vertices such that the algorithm FF with input $(G, <)$ uses $c_{\mathrm{FF}}(G)$ colors to color G, where $c_{\mathrm{FF}}(G)$ equals the values given in Theorem 1. We first describe a strategy which transforms a permutation π of length n into a geometric scheme, which is a set of n planar points with specific x- and y-coordinates, and then we show how a permutation graph is defined by such a scheme.

3.1. Permutations and schemes

A set P of n points $\{p_1, p_2, \ldots, p_n\}$ in the plane such that $x(p_i) \neq x(p_j)$ and $y(p_i) \neq y(p_j)$ for every

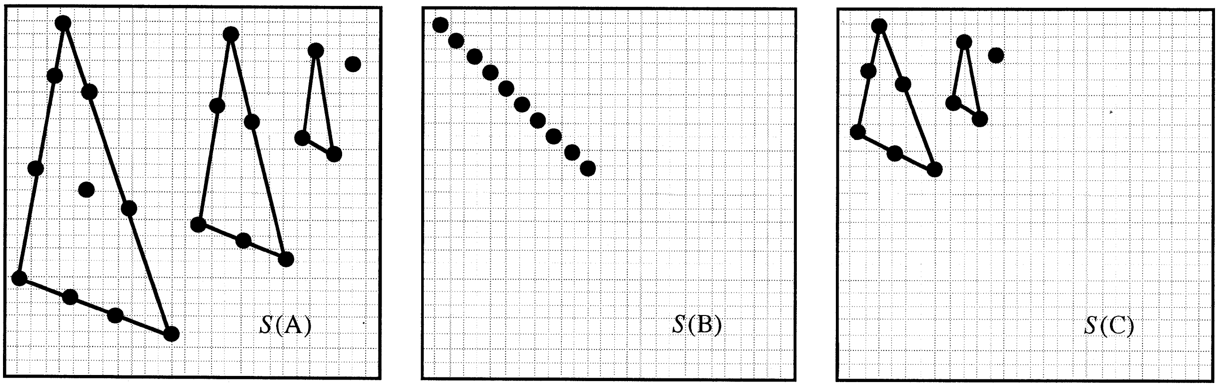

Fig. 1. The three basic schemes $A(4)$-scheme, $B(10)$-scheme and $A(3)$-scheme.

$p_i, p_j \in P$ ($1 \leqslant i, j \leqslant n$ and $i \neq j$), is called *scheme* and denoted by $S(P)$. Let $S(P)$ and $S(Q)$ be two schemes of n and m points, respectively, such that $x(p_i) \neq x(q_j)$ and $y(p_i) \neq y(q_j)$ for every $p_i \in P$ and $q_j \in Q$ ($1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$). The *union* of the schemes $S(P)$ and $S(Q)$ is defined to be the scheme $S(P \cup Q)$ of $n + m$ points. The number of points in a scheme, say, $S(P)$, is denoted by $|S(P)|$. A point $p_i \in S(P)$ is said to be *dominated* by $p_j \in S(P)$ (or p_j *dominates* p_i) if $x(p_i) < x(p_j)$ and $y(p_i) < y(p_j)$.

Let π be a permutation on N_n. A π*-scheme* (or *permutation scheme*) is defined to be a scheme of n points $\{p_1, p_2, \ldots, p_n\}$ such that $(x(p_i), y(p_i)) = (i, -\pi^{-1}(i))$, $1 \leqslant i \leqslant n$.

The $A(n)$*-scheme* and the $B(n)$*-scheme* are called *basic schemes*, where $A(n)$ and $B(n)$ are the two permutations which we defined in Section 2. Recall that, $A(n)$ and $B(n)$ are permutations of lengths $n(n + 1)(n + 2)/6$ and $n(n + 1)/2$, respectively. The parameter n of the $A(n)$-scheme (respectively $B(n)$-scheme) is called *degree* of the $A(n)$-scheme (respectively $B(n)$-scheme). For notation convenience we shall omit the parameter n of the basic scheme $A(n)$-scheme (respectively $B(n)$-scheme) and we shall denote it by $S(A)$ (respectively $S(B)$).

In Fig. 1 there are three basic schemes: an $S(A)$ scheme of degree 4, an $S(B)$ scheme of degree 10 and an $S(A)$ scheme of degree 3; that is, an $A(4)$-scheme, a $B(10)$-scheme and an $A(3)$-scheme.

We next show how a permutation graph is defined by a π-scheme. Let π be a permutation on N_n and let G be a graph with $V(G) = \{1, 2, \ldots, n\}$ and $(i, j) \in E(G)$ if and only if $(i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$. Let $\mathrm{S}(\pi) = \{p_1, p_2, \ldots, p_n\}$ be the π-scheme of the permutation π. Then, we define the graph $G[\pi]$ as follows:

$V(G[\pi]) = \{p_1, p_2, \ldots, p_n\}$, and

$(p_i, p_j) \in E(G[\pi])$ if and only if p_i dominates p_j.

By definition G is a permutation graph and $G[\pi] = G$. Thus, given a permutation π on N_n, the combinatorial object $G[\pi]$ and the geometric object $S(\pi)$ are in one-to-one correspondence; by definition π and $G[\pi]$ are also in one-to-one correspondence.

3.2. Construction of $G[\pi_{\mathrm{FF}}]$

Let us now construct a permutation scheme, say, $S_{\mathrm{FF}} := S(\pi_{\mathrm{FF}})$, and an ordering $<$ of its points (we shall define it in Section 3.3) such that the algorithm FF with input $(G, <)$ uses $c_{\mathrm{FF}}(G)$ colors to color G, where $G = G[\pi_{\mathrm{FF}}]$. Recall that the graph $G[\pi_{\mathrm{FF}}]$ and the permutation scheme $S(\pi_{\mathrm{FF}})$ are in one-to-one correspondence.

Given an integer $\chi > 0$, we first construct the basic schemes $S(A)$, $S(B)$ and $S(C)$ by using the permutations $A(\chi)$, $B(\chi(\chi + 1)/2)$ and $A(\chi - 1)$, respectively. Then we construct the scheme $S(A \cup B \cup C)$ of Fig. 2. This construction can be done by first y-inserting the scheme $S(B)$ into $S(A)$ using $A^*(\chi)$ as pivot; that is, y-insert$(A; A^*(\chi), B)$, and then x-inserting the scheme $S(C)$ into $S(A \cup B)$ with pivot

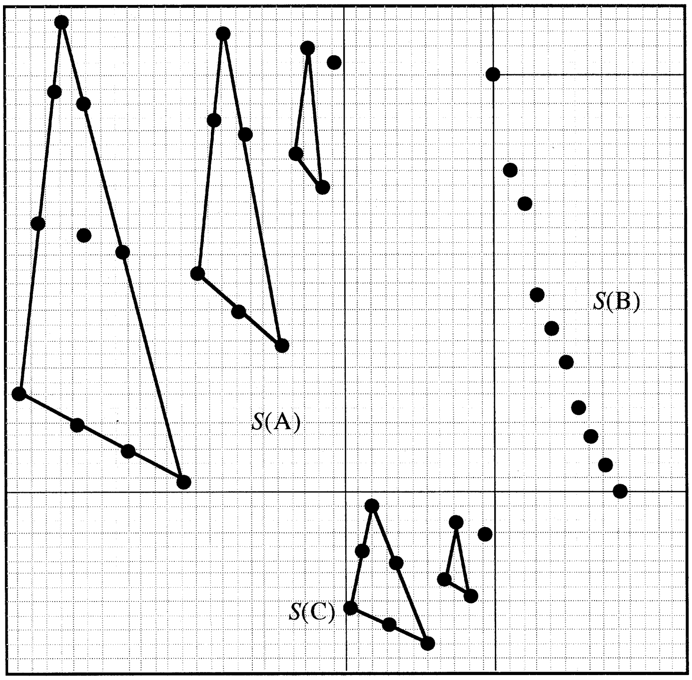

Fig. 2. The permutation scheme $S(A \cup B \cup C)$.

the first element a_1^* of $A^*(\chi)$; that is, x-insert$(A \cup B; a_1^*, C)$.

We next show the way we can extend the scheme $S(A \cup B \cup C)$, by creating and inserting various basic schemes into $S(A \cup B \cup C)$, so that the resulting scheme is S_{FF}. In order to do that we first set $S_{FF} := S(A \cup B_0 \cup C_0)$, where $B_0 = B$ and $C_0 = C$. Then, we construct the scheme $S(B_i')$ by using the permutation $B(b)$ of length $b = |B_i|$ and x-insert it into the scheme S_{FF} using B_i as pivot, $i \geqslant 0$. The result of the x-insertion operation is an updated scheme S_{FF} which is the union of S_{FF} with $S(B_i')$. Next, we construct the scheme $S(B_{i+1})$ by using the permutations $B(b)$, where $b = |C_i^* \cup B_i'|$, and y-insert it into the scheme S_{FF} using $C_i^* \cup B_i'$ as pivot. Now, the result of the y-insertion operation is an updated scheme S_{FF} which is the union of S_{FF} with $S(B_{i+1})$. Finally, we construct the scheme $S(C_{i+1})$ by using the permutation $A(\chi - 1)$ and x-insert this scheme into the scheme S_{FF} with pivot the point b_i', where b_i' is a point of the scheme $S(B_i')$ such that $x(b_i') = |S_{FF}| + |S(B_{i+1})|$ and $y(b_i') = |S_{FF}|$. The resulting permutation scheme S_{FF} is the union of S_{FF} with $S(C_{i+1})$.

Clearly, we can extend the permutation scheme S_{FF} by repeatedly applying the above construction process for $i = 1, 2, \ldots, k-1$ (see Fig. 3). Again, the resulting scheme S_{FF} and the graph $G[\pi_{FF}]$ are in one-to-one correspondence.

We are now in a position to give a formal description of the way we can construct a permutation scheme S_{FF} for which we shall define an ordering $<$ such that the algorithm FF with input $(G, <)$ uses $c_{FF}(G)$ colors to color G, where $G = G[\pi_{FF}]$. In the proposed algorithm we shall use the notation "x-insert$(A; B, C) \Rightarrow S(Q)$" to denote that the scheme $S(Q)$ is produced by x-inserting the permutation C into A using B as pivot. The construction algorithm is formally presented (see Algorithm `Scheme_SFF`).

By construction, the geometric object S_{FF} consists of the three basic schemes $S(A)$, $S(B)$ and $S(C)$ of degrees χ, $\chi(\chi+1)/2$ and $\chi - 1$, respectively, and some number of basic schemes $S(B_i')$, $S(B_i)$ and $S(C_i)$ of various degrees, where χ is a fixed positive

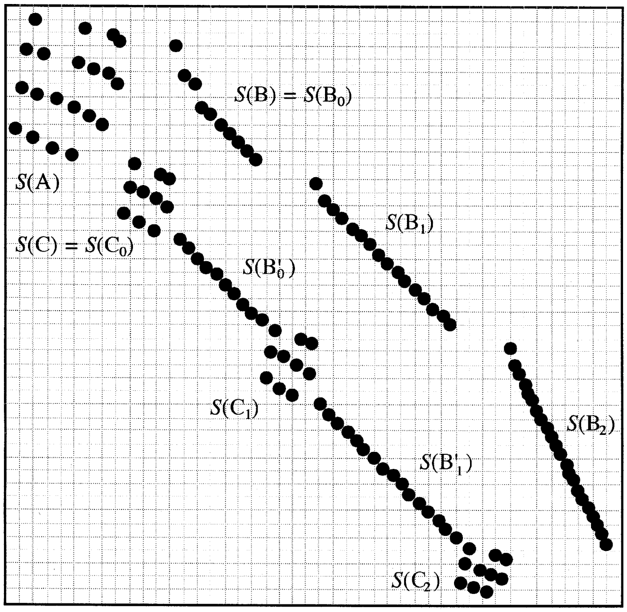

Fig. 3. The geometric object S_{FF}, where π is a permutation on N_{131}.

Algorithm `Scheme_SFF:`

Step 1. Construct the scheme $S(A)$ by using the permutations $A(\chi)$;

Step 2. Construct the scheme $S(B) = S(B_0)$ by using the permutations $B(b)$ where $b = \chi(\chi+1)/2$, and apply the operation y-insert$(A;\, A^*(\chi),\, B_0) \Rightarrow S(A \cup B_0)$;

Step 3. Construct the scheme $S(C) = S(C_0)$ by using the permutation $A(\chi - 1)$, and apply the operation x-insert$(A \cup B;\, a_1^*,\, C_0) \Rightarrow S(A \cup B_0 \cup C_0)$;
Set $S_{\mathrm{FF}} := S(A \cup B_0 \cup C_0)$;

Step 4. for $i = 0, 1, \ldots, k-2$

4.1 Construct the scheme $S(B_i')$ by using the permutation $B(b)$ where $b = |B_i|$, and apply x-insert$(S_{\mathrm{FF}};\, B_i,\, B_i') \Rightarrow S_{\mathrm{FF}}^1$;

4.2 Construct the scheme $S(B_{i+1})$ by using the permutations $B(b)$ where $b = |C_i^* \cup B_i'|$, and apply y-insert$(S_{\mathrm{FF}}^1;\, C_i^* \cup B_i',\, B_{i+1}) \Rightarrow S_{\mathrm{FF}}^2$;

4.3 Construct the scheme $S(C_{i+1})$ by using the permutation $A(\chi - 1)$, select the point b_i' from B_i' such that $x(b_i') = |S_{\mathrm{FF}}| + |S(B_{i+1})|$ and $y(b_i') = |S_{\mathrm{FF}}|$, and apply x-insert$(S_{\mathrm{FF}}^2;\, b_i',\, C_{i+1}) \Rightarrow S_{\mathrm{FF}}^3$;

4.4 Set $S_{\mathrm{FF}} := S_{\mathrm{FF}}^3$;

end;

end

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integer and $i \geqslant 0$. The schemes $S(A)$ and $S(C_i)$ are constructed by using the permutation A, while the schemes $S(B_i')$ and $S(B_i)$ are constructed by using the permutation B (see Section 2). We say that the schemes $S(A)$, $S(C_i)$, $S(B_i')$ and $S(B_i)$ are of *A-type*, *C-type*, *B'-type* and *B-type*, respectively. The geometric object of Fig. 3 is produced by Algorithm `Scheme_SFF` after two iterations of Step 4; that is, for $k = 3$.

3.3. An ordering of $V(G[\pi_{\mathrm{FF}}])$

We are interested in finding an ordering $<$ of the n points of the scheme S_{FF}; that is, $p_1 < p_2 < \cdots < p_n$

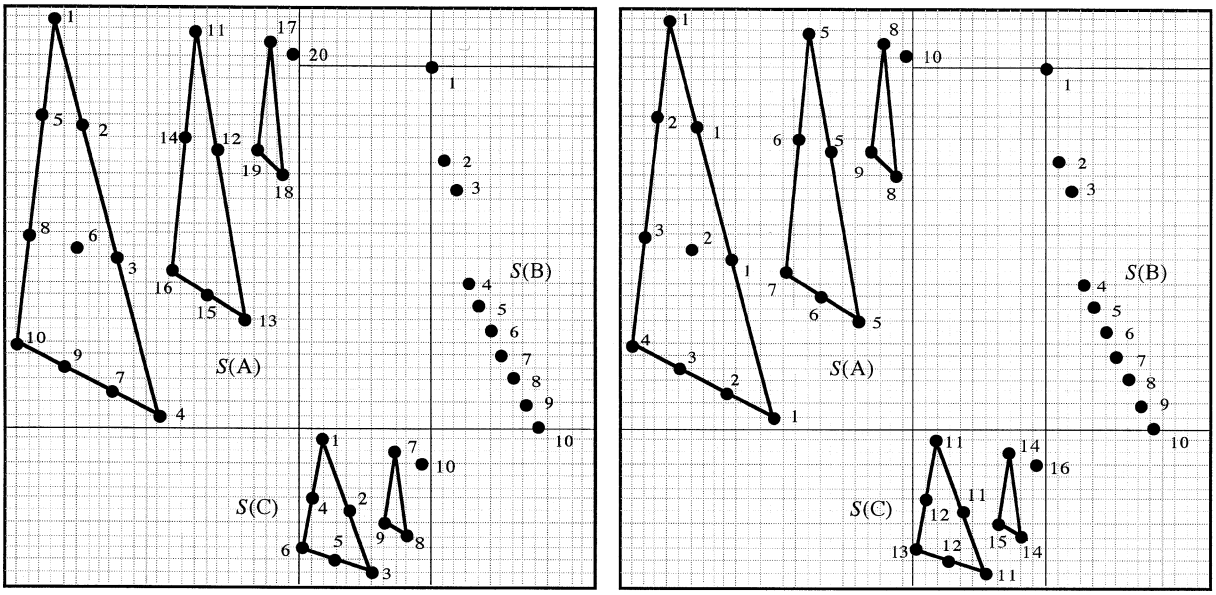

Fig. 4. The orderings of the points of the three basic schemes $S(A)$, $S(B)$ and $S(C)$, and the FF coloring of the scheme $S(A \cup B \cup C)$.

such that the coloring algorithm FF with input $(G, <)$ uses $c_{FF}(G)$ colors to color G, where $G = G[\pi_{FF}]$ and $c_{FF}(G) = \frac{1}{2}((\chi^2 + \chi) + k(\chi^2 - \chi))$. To this end, we order the points of the two basic schemes as shown in Fig. 4; that is, we order the points of the scheme $S(C_i)$ in the same way as $S(C)$, since $S(C_i)$ and $S(C)$ have the same structure, $i > 0$. Finally, the points of the scheme $S(B'_i)$ (respectively $S(B_i)$) are ordered such that $p_i < p_j$ if and only if $x(p_i) < x(p_j)$ for every $p_i, p_j \in S(B'_i)$ (respectively $S(B_i)$).

Having defined an ordering of the points of each individual scheme of the geometric object S_{FF} (see Fig. 4), let us now define an ordering $<_s$ on its schemes. Suppose that S_{FF} consists of an A-type scheme, k C-type schemes, k B-type schemes and $k-1$ B'-type schemes. The ordering $<_s$ on the components of S_{FF} (i.e., $S(A), S(B_i), S(B'_i)$ and $S(C_i)$, $0 \leqslant i \leqslant k-1$) is defined as follows:

(i) $S(A) <_s S(B_0) <_s S(C_0)$;

(ii) $S(C_i) <_s S(B'_i) <_s S(B_{i+1}) <_s S(C_{i+1})$, $i = 0, 1, \ldots, k-1$.

Let $S(P), S(Q)$ be two schemes of S_{FF} and let p, q be two points such that $p \in S(P)$ and $q \in S(Q)$. Then, $p < q$ if and only if $S(P) <_s S(Q)$. Thus, we have defined an ordering $<$ on the points of S_{FF}. In Fig. 4 we show the orderings of the points of the basic schemes $S(A)$, $S(B)$ and $S(C)$; left figure, and the FF coloring of the scheme $S(A \cup B \cup C)$; right figure. We note that, $S(A) <_s S(B) <_s S(C)$.

4. The performance of the FF algorithm

Let S_{FF} be a permutation scheme of degree n constructed by Algorithm `Scheme_SFF`. Let χ be the degree of the basic scheme $S(A)$ of S_{FF} and let k be the number of schemes $S(C_0), S(C_1), \ldots, S(C_{k-1})$ in S_{FF}. Consider the permutation graph $G[\pi_{FF}]$ which corresponds to the permutation scheme S_{FF} and let $(G[\pi_{FF}], <)$ be the input of the algorithm FF, where $<$ is the ordering constructed in Section 2. Then, the following statements hold:

(i) $\chi(\chi+1)/2$ colors are assigned to scheme $S(A)$;

(ii) zero new colors are assigned to scheme $S(B)$; the scheme $S(B)$ is colored with the $\chi(\chi+1)/2$ colors of $S(A)$;

(iii) $\chi(\chi-1)/2$ new colors are assigned to scheme $S(C_i)$, $i = 0, 1, \ldots, k-1$;

(iv) zero new colors are assigned to scheme $S(B_i')$; the scheme $S(B_i')$ is colored with the colors of $S(B_i)$, $i = 0, 1, \ldots, k-2$;

(v) zero new colors are assigned to scheme $S(B_i)$; the scheme $S(B_i)$ is colored with the colors of $S(C_{i-1} \cup B_{i-1}')$, $i = 0, 1, \ldots, k-1$;

Thus, $c_{\mathrm{FF}}(G[\pi_{\mathrm{FF}}]) = \chi(\chi+1)/2 + k\chi(\chi-1)/2$, where $\chi = \chi(G[\pi_{\mathrm{FF}}])$. Thus, Theorem 1 is proved.

We now compute the number $n = n(\chi, k)$ of vertices of the graph $G[\pi_{\mathrm{FF}}]$ as a function of χ and k, where χ is the chromatic number of the graph $G[\pi_{\mathrm{FF}}]$ (or, equivalently, the degree of the scheme $S(A)$ of S_{FF}) and k is the number of schemes of C-type in the permutation scheme S_{FF}.

Let

$$\chi_{\mathrm{FF}}^i = \frac{\chi(\chi+1)}{2} + \frac{i\chi(\chi-1)}{2}, \quad 0 \leqslant i \leqslant k.$$

Notice that χ_{FF}^i is the number of colors of the scheme $S(A \cup B_0 \cup C_1 \cup \cdots \cup C_{i-1})$, $1 \leqslant i \leqslant k$. Recall that n_a, n_b and n_c denote the number of points in the schemes $S(A)$, $S(B)$ and $S(C)$, respectively. Then, it is easy to see that the minimum number n_0 of vertices of a graph $G[\pi_{\mathrm{FF}}]$ on which FF uses χ_{FF}^0 colors is $n_0 = n_a$ (we note that the algorithm FF with input $(S_{\mathrm{FF}}, <)$ also uses χ_{FF}^0 colors to color the scheme $S(C \cup B)$ which consists of $n_a + n_b > n_0$ points); the minimum number n_1 of vertices of $G[\pi_{\mathrm{FF}}]$ on which FF uses χ_{FF}^1 colors is $n_1 = n_0 + n_b + n_c$; the minimum number n_2 of vertices of $G[\pi_{\mathrm{FF}}]$ on which FF uses χ_{FF}^2 colors is $n_2 = n_1 + \chi_{\mathrm{FF}}^0 + \chi_{\mathrm{FF}}^1 + n_c$; and so on. Thus,

$$\begin{aligned}
n_1 &= n_a + n_b + n_c, \\
n_2 &= n_1 + \chi_{\mathrm{FF}}^0 + \chi_{\mathrm{FF}}^1 + n_c, \\
&\vdots \\
n_k &= n_{k-1} + \chi_{\mathrm{FF}}^{k-2} + \chi_{\mathrm{FF}}^{k-1} + n_c.
\end{aligned}$$

Then we have,

$$\begin{aligned}
n_k &= n_a + n_b + n_c \\
&\quad + (\chi_{\mathrm{FF}}^0 + \chi_{\mathrm{FF}}^1 + \cdots + \chi_{\mathrm{FF}}^{k-2}) \\
&\quad + (\chi_{\mathrm{FF}}^1 + \chi_{\mathrm{FF}}^2 + \cdots + \chi_{\mathrm{FF}}^{k-1}) \\
&\quad + (k-1)n_c \\
&= n_a + n_b + kn_c + 2(k-1)\chi(\chi+1)/2 \\
&\quad + (k-1)^2\chi(\chi-1)/2.
\end{aligned}$$

We have shown that the scheme $S(A)$ consists of $n_a = \chi(\chi+1)(\chi+2)/6$ points, the scheme $S(B)$ consists of $n_b = \chi(\chi+1)/2$ points and the scheme $S(C)$ consists of $n_c = n_a - \chi(\chi+1)/2$ points. Thus,

$$n(\chi, k) = \begin{cases} (k+1)n_a + (k-1)(k\chi - k + 2)\chi/2, \\ \qquad \text{for } k \geqslant 1, \\ n_a, \quad \text{for } k = 0, \end{cases}$$

where $n_a = \chi(\chi+1)(\chi+2)/6$.

Thus, we have proved that the largest number of colors $\chi_{\mathrm{FF}}(G)$ that the on-line coloring algorithm FF needs on permutation graphs G with n vertices and chromatic number χ when taken over all possible vertex orderings is no less than $\frac{1}{2}((\chi^2 + \chi) + k(\chi^2 - \chi))$, where k is a nonnegative integer. The graph we constructed for which the algorithm FF uses that many colors has $n = n(\chi, k) \geqslant \chi(\chi+1)(\chi+1)/6$ vertices.

5. Conclusions

In this paper we studied the behaviour of the on-line coloring algorithm FF on the class of permutation graphs. We used a simple graphical representation of such graphs in the plane which makes possible intuitive description of the construction of the "bad" permutation graph $G[\pi_{\mathrm{FF}}]$. Based on this graph, we showed that the class of permutation graphs is not FF χ-bounded: for any integers $\chi > 0$ and $k \geqslant 0$, there exists a permutation graph G on n vertices such that $\chi(G) = \chi$ and $\chi_{\mathrm{FF}}(G) \geqslant \frac{1}{2}((\chi^2 + \chi) + k(\chi^2 - \chi))$, for sufficiently large n. Recall that, a class of perfect graphs $\mathcal{P}$ is FF χ-bounded if there exists a function f such that for all graphs $G \in \mathcal{P}$, $\chi_{\mathrm{FF}}(G) \leqslant f(\chi(G))$.

Acknowledgement

The authors thank the referees for comments which improved the presentation of this paper.

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