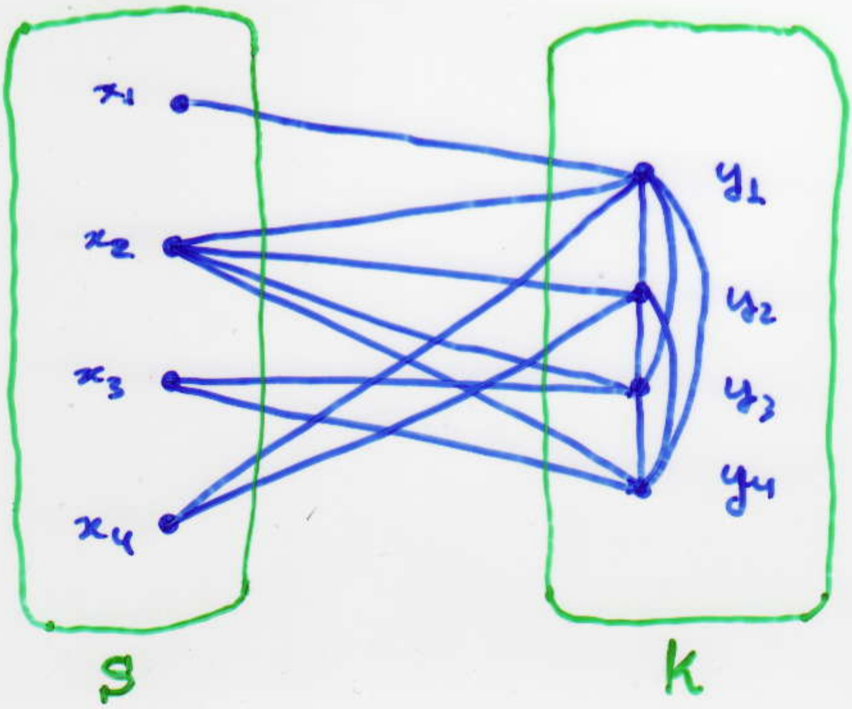
Split Graphs

- An undirected graph $G=(V,E)$ is defined to be split if there is a partition $V = S + K$ of its vertex set:

  S = stable set and K = complete set.

- In general, the partition $V = S + K$ of a split graph will not be unique.

- S will not necessarily be a maximal stable set.
  K " " maximal clique.



- Since a stable set of G is a complete set of $\bar{G}$ and vice versa, we have an immediate result.

- **Theorem:** G is a split graph iff $\bar{G}$ is a split graph.

- The next theorem follows from the work of Hammer and Simeone[1977].

- **Theorem:** Let G be a split graph and $V = S + K$. Exactly one of the following conditions holds:

(i) $|S| = \alpha(G)$ and $|K| = \omega(G)$

(ii) $|S| = \alpha(G)$ and $|K| = \omega(G) - 1$

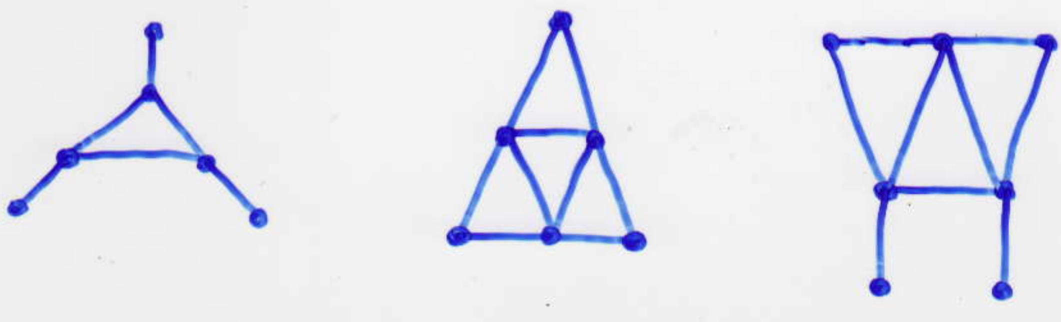
(iii) $|S| = \alpha(G) - 1$ and $|K| = \omega(G)$

- 

- **Theorem** (Földes and Hammer [1977]).
  Let G be an undirected graph. The following conditions are equivalent:
  (i) G is a split graph;
  (ii) G and $\bar{G}$ are triangulated graphs;
  (iii) G contains no induced subgraph $\cong 2K_2, C_4, C_5$

- A characterization of when a split graph is also a comparability graph is given by the following theorem:

- **Theorem**: If G is a split graph, then G is a comparability graph iff G contains no induced subgraph isomorphic to H_1, H_2 or H_3



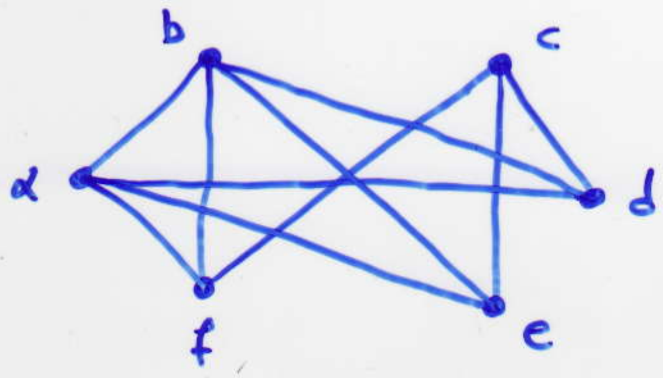
H_1 H_2 H_3

Gographs (or Complement reducible graphs)

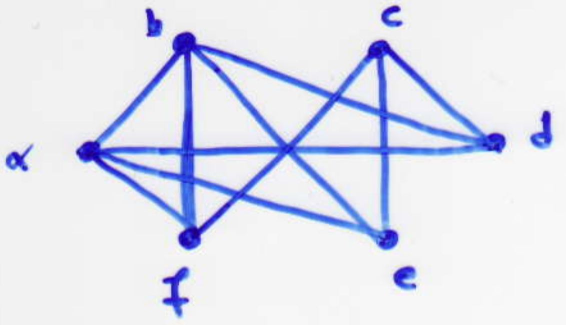
- Cographs are defined as the class of graphs formed from a single vertex under the closure of the operations of union and complement.

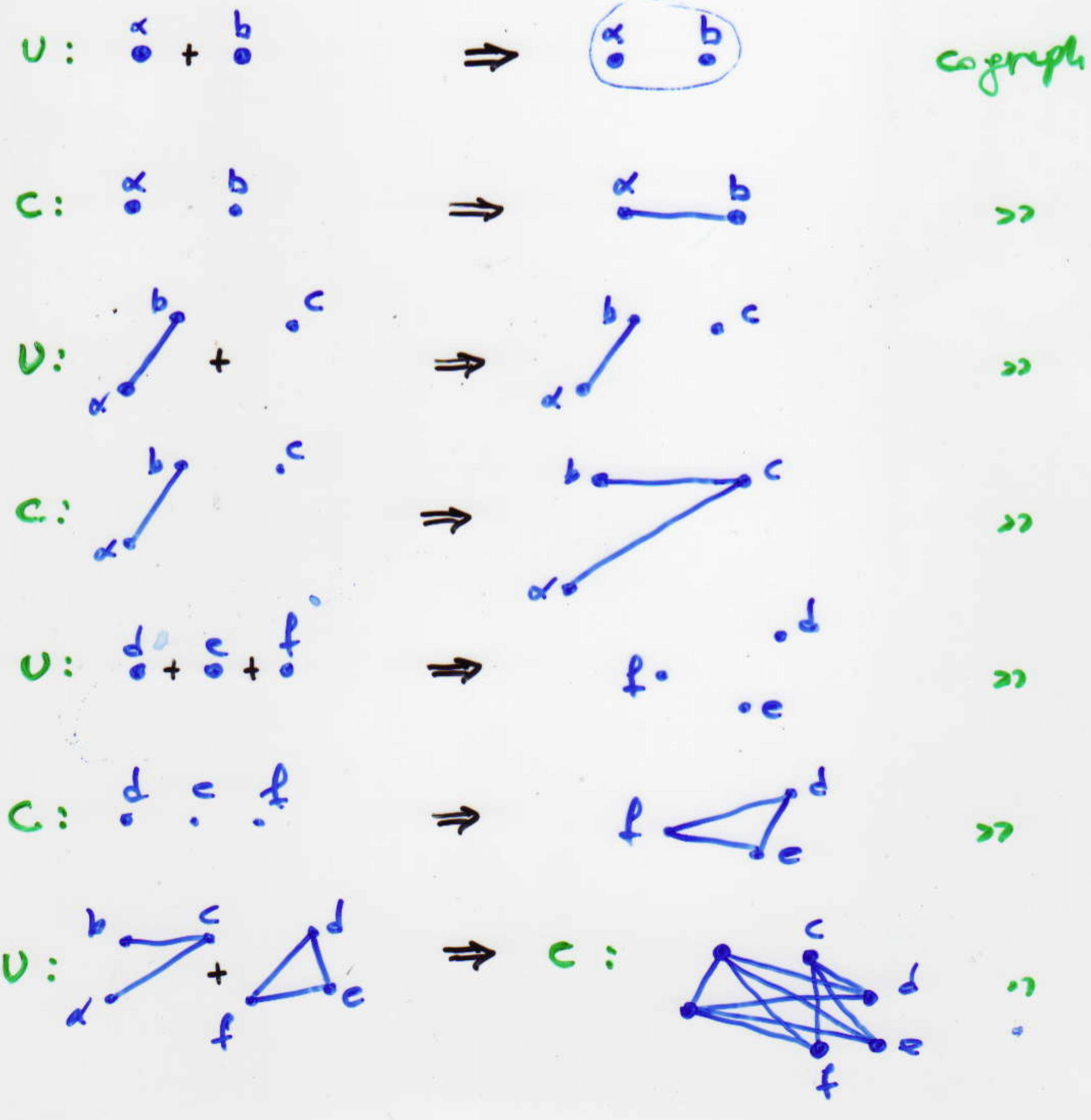
- More precisely, the class of cographs can be defined recursivly as follows:
  - (i) a single-vertex graph is a cograph;
  - (ii) the disjoint union of a cograph is a cograph;
  - (iii) the complement of a cograph is a cograph;

- Example:



- Construction of the following cograph:



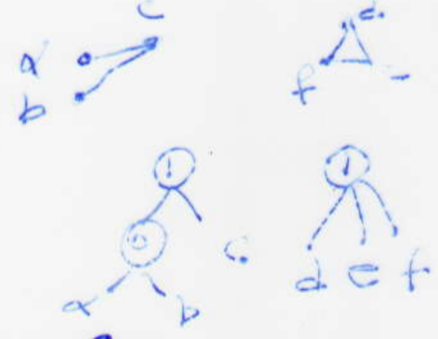


- Cograph were introduced in the early 1970s by Lerchs.

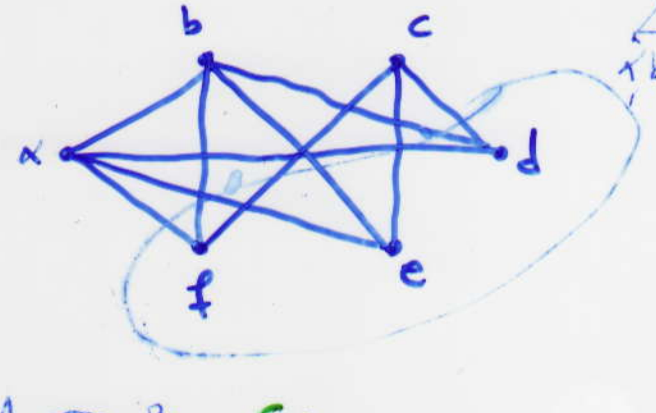
- Lerchs has shown, among other properties, the following two very nice algorithmic properties:

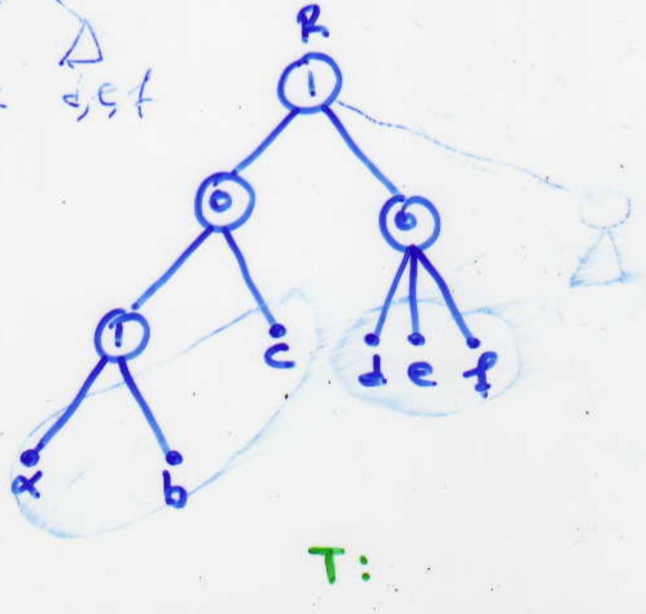
  (P1) cographs are exactly the P_4 restricted graphs

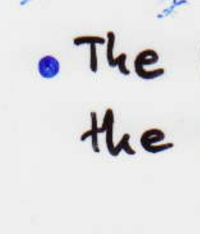
  (P2) cographs have a unique tree representation called cotree.



- A cograph and its cotree:







G:

T:

- The root R will have only one (0) node child iff the represented cograph is disconnected.

Permutation Representation of a Cograph

- Let G be a cograph with cotree T.

- We plot the vertices of G on an $n \times n$ grid.

- For each node $\alpha \in T$, we denote:
  $G(\alpha)$ the subgraph $G[L(T(\alpha))]$.
  $L(T(\alpha))$ the set of the leaf nodes in $T(\alpha)$.
  $T(\alpha)$ the subtree of T rooted at α.

- The vertices of $G(\alpha)$ will be plotted on a $n(\alpha) \times n(\alpha)$ square region $R(\alpha)$ on the grid, where $n(\alpha)$ is the number of vertices in $G(\alpha)$.

- We represent

$$R(\alpha) = [x_1(\alpha), y_1(\alpha), x_2(\alpha), y_2(\alpha)]$$

where

$(x_1(\alpha), y_1(\alpha))$ is the lower-left corner, and
$(x_2(\alpha), y_2(\alpha))$ is the upper-right corner.

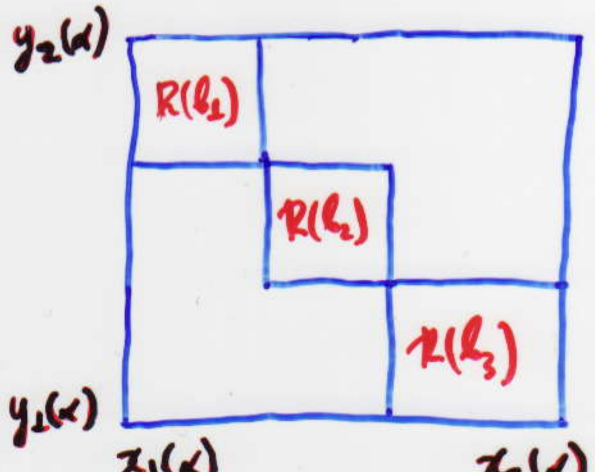
- Our algorithm is a top-down computation on T.

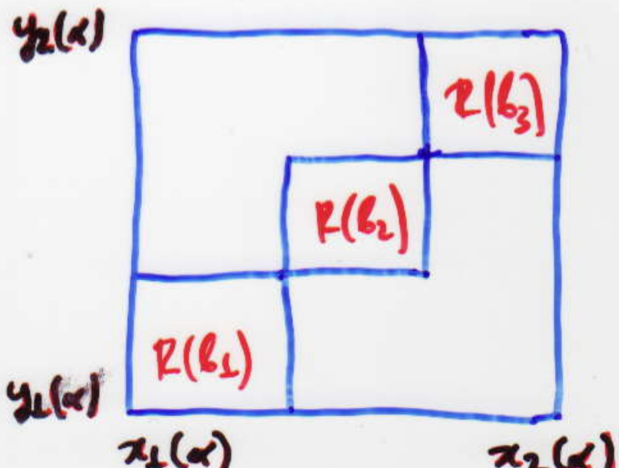
- For the root r of T, let $R(r) = [1, 1, n, n]$.

- Consider an internal node α with children $\beta_1, \beta_2, \ldots, \beta_k$.

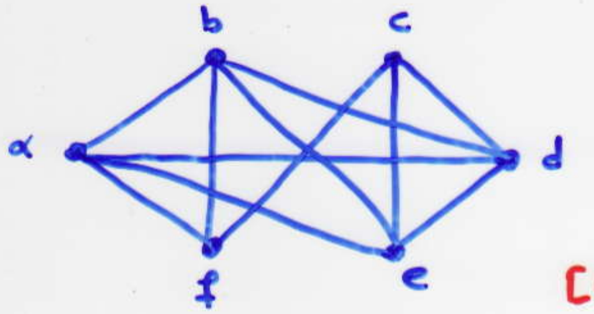
- Suppose $R(\alpha) = [x_1(\alpha), y_1(\alpha), x_2(\alpha), y_2(\alpha)]$ has been computed, we describe how to comput
$$R(\beta_i) = [x_1(\beta_i), y_1(\beta_i), x_2(\beta_i), y_2(\beta_i)]$$
for each $1 \le i \le k$.

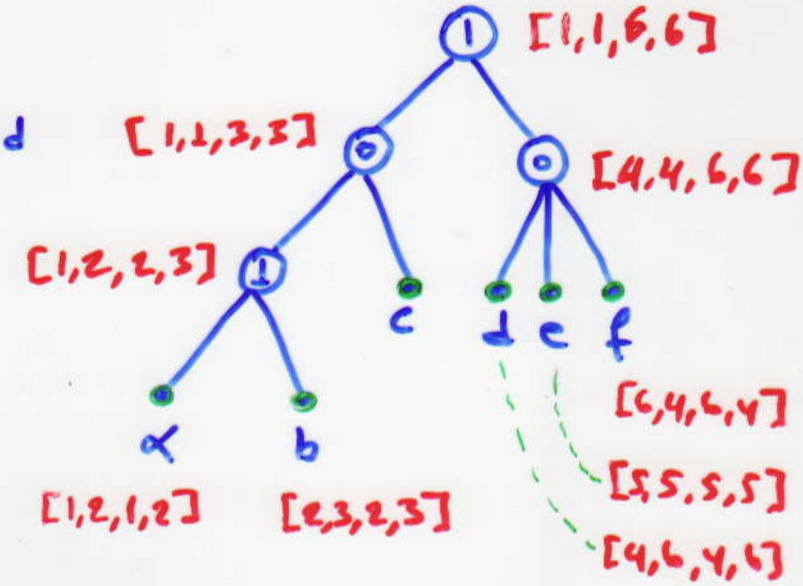
- Case 1: α is a 0-node
- Case 2: α is a 1-node

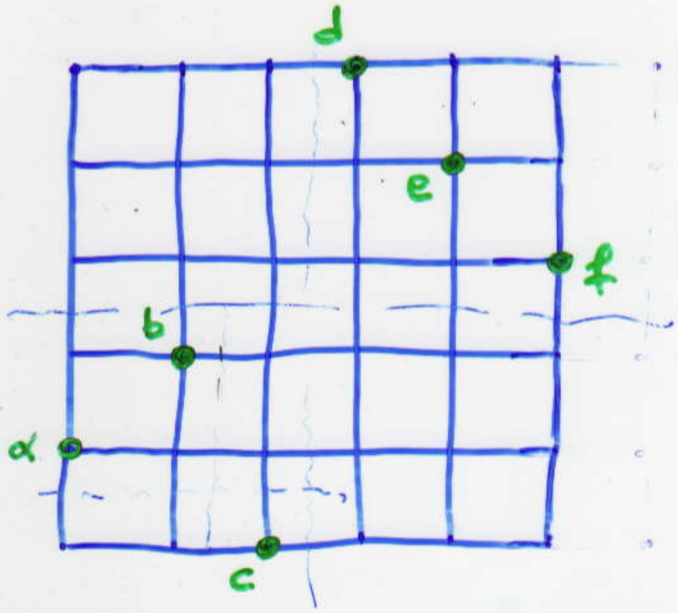




• Example:





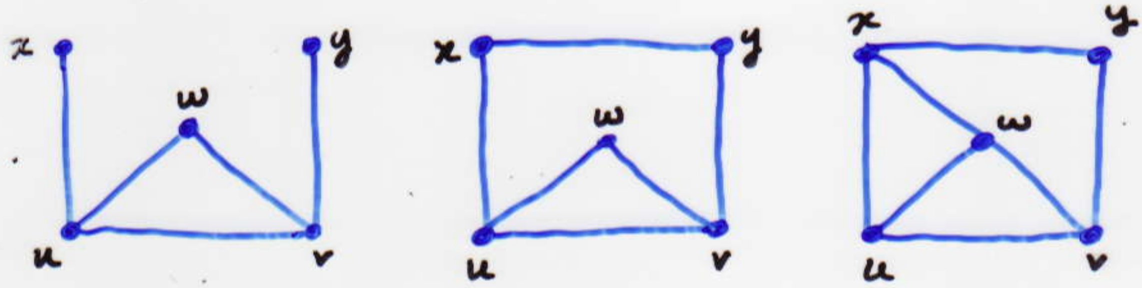


$$\pi = [2,3,1,6,5,4]$$

Recognition Properties of Cograph

- **Theorem**: Let G be a graph. The following statments are equivalent:
  (i) G is a cograph;
  (ii) G does not contain P_4 as a subgraph;

- **Example**:



- For every actual edge $(x,y) \in AE$ we define:

$$AV^{x}_{(x,y)} = \{ z \in V \mid (x,z) \in E \text{ and } (y,z) \notin E \}$$

$$AV^{y}_{(x,y)} = \{ z \in V \mid (x,z) \notin E \text{ and } (y,z) \in E \}$$

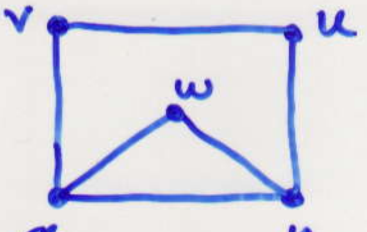
- For every pair $x, y \in V$: $(x,y) \notin E$ we define:

$$FV_{\{x,y\}} = \{ z \in V \mid (x,z) \in E \text{ and } (y,z) \in E \}$$

- **Theorem**: Let $G=(V,E)$ be an A-free graph. Then, G is a cograph.

- **Theorem**: Let $G=(V,E)$ be a graph. The following statments are equivalent:
  (i) G is a cograph;
  (ii) There exists no actual edge $(x,y) \in AE$ and vertices $v \in AV^{x}_{(x,y)}$, $u \in AV^{y}_{(x,y)}$ such that either $N(v) \cup N(y) \neq FV_{\{v,y\}}$ or $N(u) \cup N(x) \neq FV_{\{u,x\}}$.

- **Example**:



1) $(x,y) \in AE$

$N(v) \cup N(y) = \{x,u\}$

$FV_{\{v,y\}} = \{x,u\}$

$N(u) \cup N(x) = \{v,y\}$

$FV_{\{v,y\}} = \{v,y\}$

2) $(v,x) \in AE$

$$\left.\begin{array}{l} N(u) \cup N(x) = \{v,w,y\} \\ F_{\{u,x\}} = \{v,y\} \end{array}\right\} \Rightarrow \text{G is not a cograph.}$$

- Algorithm Recognition - Cographs

1. For every $(x,y) \in AE$ do
   1.1. comput $AV^{x}_{(x,y)}$ and $AV^{y}_{(x,y)}$;

2. For every pair $x, y \in V$: $(x,y) \notin E$ do
   2.1. compute $FV_{\{x,y\}}$;

3. For every actual edge $(x,y) \in AE$ do
   3.1. for every vertex $v \in AV^{x}_{(x,y)}$ do
        if $N(v) \cup N(y) \neq FV_{\{x,y\}}$ then P_4; stop;
   3.2. for every vertex $u \in AV^{y}_{(x,y)}$ do
        if $N(u) \cup N(x) \neq FV_{\{x,y\}}$ then P_4; stop;

end.

- The algorithm runs in $O(\delta \cdot m)$ where δ is the degree of the input graph.

- The algorithm can be implemented in $O(L)$ time with $O(\delta \cdot m)$ processors on a CRCW-PRAM.

Threshold Graphs

- Chvátal and Hammer have proved the following result:

- **Theorem**: Let $G=(V,E)$ be a graph. The following statements are equivalent:

  (i) G is a threshold graph; 

  (ii) G has no induced subgraphs $\cong$ to P_4, C_4 or $2K_2$

- There exist an $O(n)$-time sequential algorithm for recognizing threshold graphs.

- The algorithm is based on the degree partition.

- Let $\delta(v)$ be the degree of the vertex v of an undirected graph.

- We define the degree partition of a graph $G=(V,E)$ in which we associate vertices having the same degree.

- Let $0 < \delta_1 < \delta_2 < \dots < \delta_k < |V|$ be the degrees of the nonisolated vertices.

- δ_i are distinct; there may be many vertices of degree δ_i.

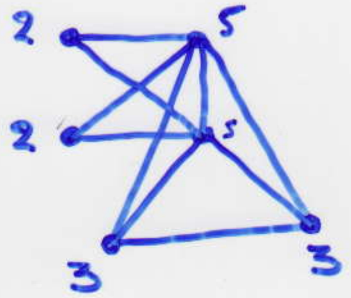
- Define: $\delta_0 = 0$ and $\delta_{k+1} = |V| - 1$

- The degree partition of V is given by

$$V = D_0 + D_1 + \dots + D_k$$

where D_i = set of all vertices of degree δ_i.

- Example:



$\delta_1 = 2$

$\delta_2 = 3$

$\delta_3 = 5$

$0 < \delta_1 < \delta_2 < \delta_3 < 6$

- The main idea of the recognition Algorithm:

(a) First, it brings together all vertices with the same degree.

(b) The vertex set V is partitioned into $k+1$ disjoint vertex sets

$$V = D_0 + D_1 + \cdots + D_k$$

satisfying the property

$$u \in D_i \Leftrightarrow \delta(u) = i$$

$0 \le i \le k$.

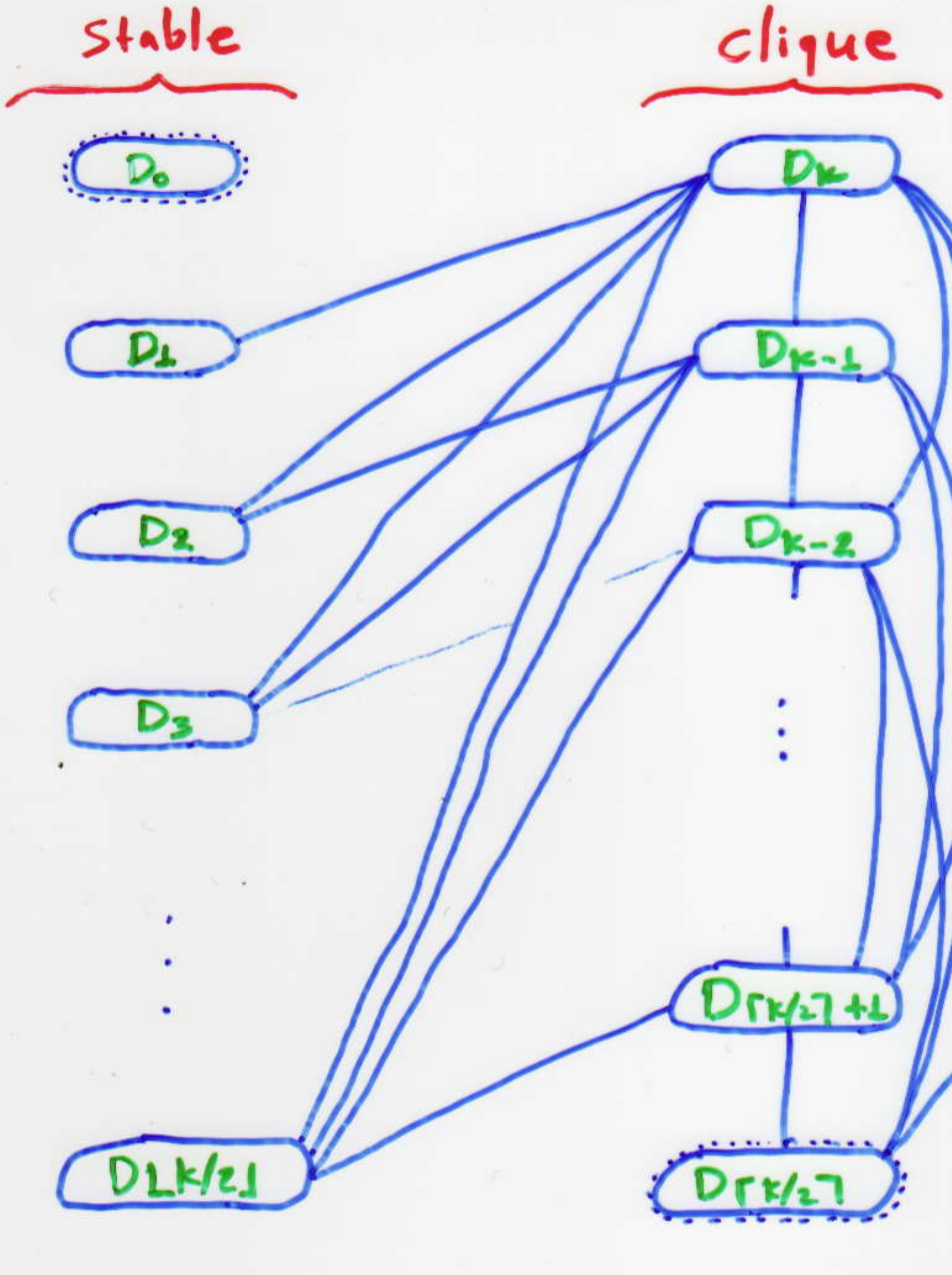
(c) Then, the algorithm uses the following result:

G is a threshold graph iff the recursions below are satisfied:

$$\delta_{i+1} = \delta_i + |D_{k-i}| \quad (i = 0, 1, \ldots, \lfloor k/2 \rfloor - 1)$$

$$\delta_i = \delta_{i+1} + |D_{k-i}| \quad (i = k, k-1, \ldots, \lfloor k/2 \rfloor + 1)$$

- The typical structure of a threshold graph:



- D_0 contains all isolated vertices (may be empty).

- $D_{\lceil k/2 \rceil}$ only exists if k is odd

- $\delta_0 = 0$ and $\delta_{k+1} = |V| - 1$

A-free Graphs

- A graph $G=(V,E)$ is called an A-free graph if every edge of G is either free or semi-free.

- We define

$$\operatorname{cent}(G)=\{x\in V \mid N[x]=V\}$$

- **Theorem:** Let G be a graph. Then the following statements are equivalent:
  - (i) G is a A-free graph;
  - (ii) G has no induced subgraph $\cong$ to P_4 or C_4;
  - (iii) Every connected induced subgraph $G[S]$, $S\subseteq V$, satisfies $\operatorname{cent}(G[S])\neq\emptyset$.

- **Lemma:** The following two statements hold:
  - (i) G is an A-free iff $G-\operatorname{cent}(G)$ is an A-free;
  - (ii) If $G-\operatorname{cent}(G)\neq\emptyset$, then $G-\operatorname{cent}(G)$ contains at least two components.

- Let G be an A-free graph. Then,

$$V_1 = \mathrm{cent}(G).$$

- Put $G_1 = G$

$$G - V_1 = G_2 \cup G_3 \cup \dots \cup G_r$$

where

G_i is a component of $G - V_1$, $r \geq 2$.

- Then, G_i is an A-free graph, and so

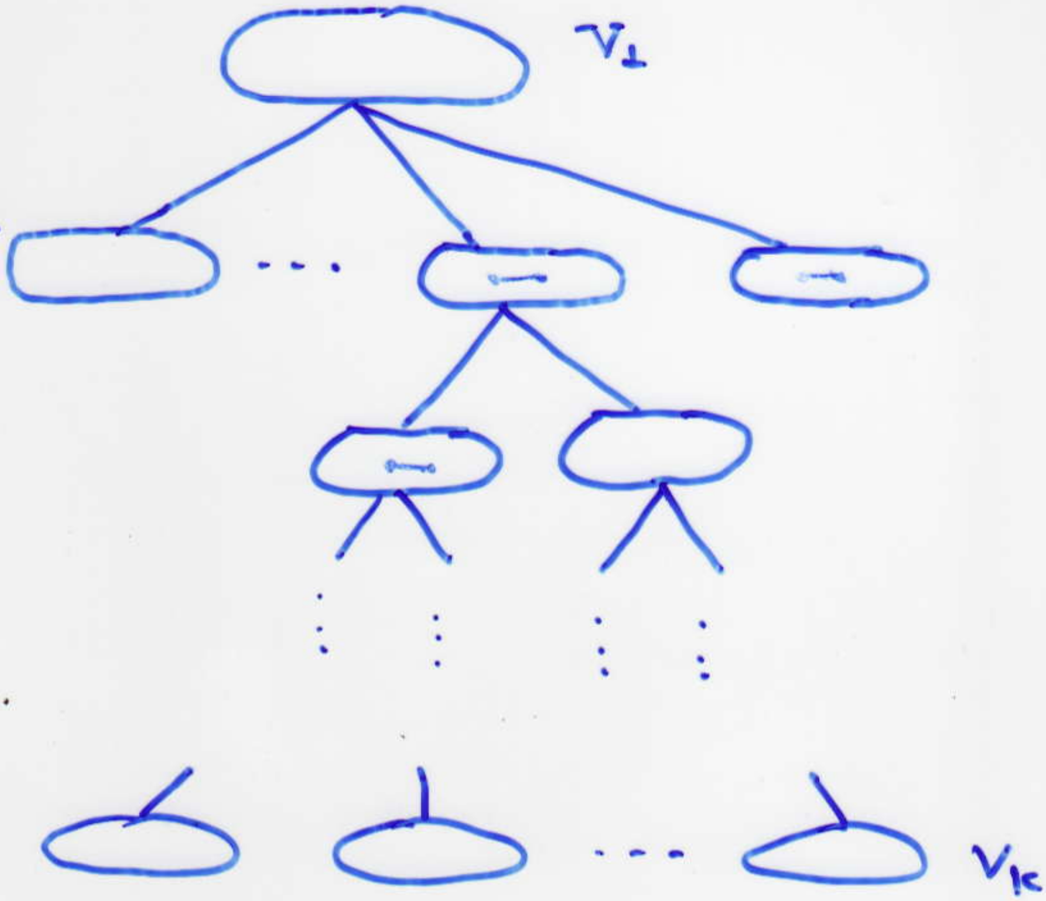
$$V_i = \mathrm{cent}(G_i) \neq \emptyset$$

- We finally obtain the following partition of V:

$$V = V_1 + V_2 + \dots + V_k$$

where $V_i = \mathrm{cent}(G_i)$.

- The typical structure of an Λ-free graph:



- Moreover we can define a partial order $\leq$ on $\{V_1, V_2, \dots, V_k\}$ as follows:

$$V_i \leq V_j \text{ if } V_i = \mathrm{cent}(G_i) \text{ and } V_j \subseteq V(G_i).$$

- Let G be a connected A-free graph, and let $V(G) = V_1 + V_2 + \cdots + V_k$ be the partition defined above.

Then this partition and the partially ordered set $(\{V_i\}, \leq)$ have the following properties:

(P1) If $V_i \leq V_j$ then every vertex of V_i and every vertex of V_j are joined by an edge.

(P2) For every V_i, $\mathrm{cent}(G[\{\cup V_i \mid V_i \leq V_j\}]) = V_i$.

(P3) For every two V_s and V_t : $V_s \leq V_t$,
$G[\{\cup V_i \mid V_s \leq V_i \leq V_t\}]$ is a complet graph.
Morever, for every maximal element V_t of $(\{V_i\}, \leq)$, $G[\{\cup V_i \mid V_1 \leq V_i \leq V_t\}]$ is a maximal complete subgraph of G.

(P4) Every edge (x,y) : $x, y \in V_i$, $(x,y) \in FE$;
" " : $x \in V_i$, $y \in V_j$, $V_i \neq V_j$, $(x,y) \in SE$;

- G is a threshold graph iff G has no induced subgraphs isomorphic to C_4, P_4 or $2K_2$.

- **Theorem**: The threshold graphs are precisely those A-free graphs containing no induced subgraph isomorphic to $2K_2$.

- **Theorem**: Let G be an A-free graph. The following statements are equivalent:

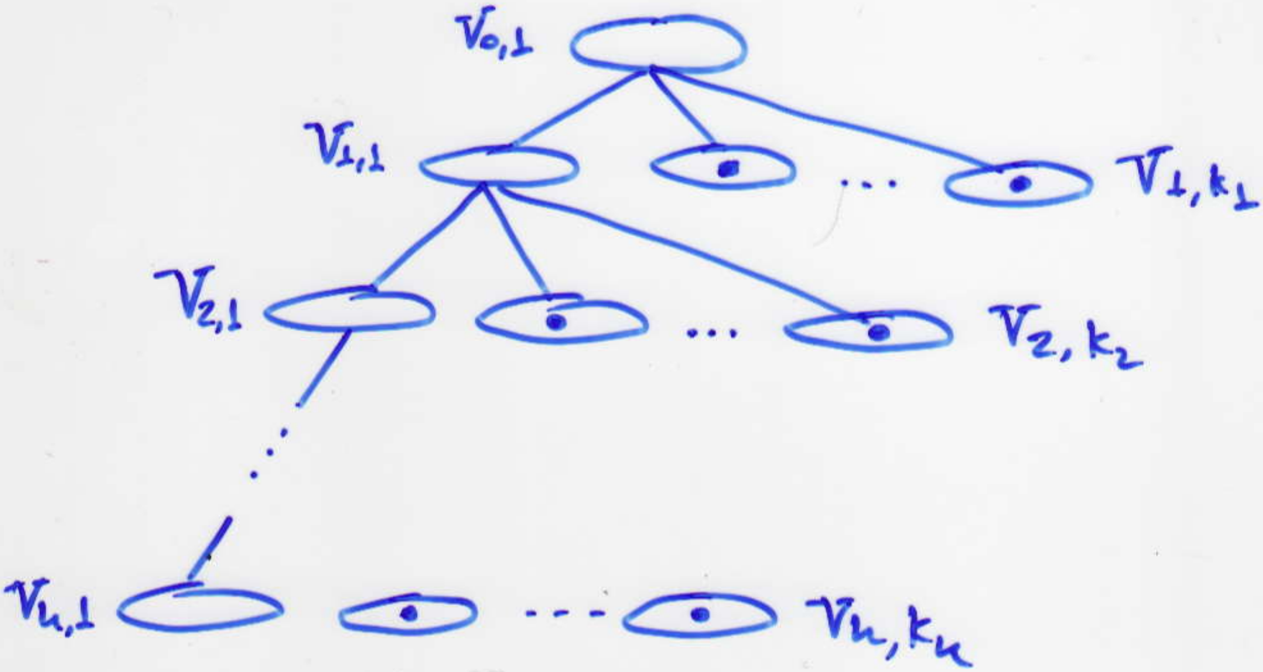
  (i) G contains no $2K_2$;

  (ii) $\overline{G}$ is an A-free graph;

Proof. (i) $\Rightarrow$ (ii): Let $T_c(G)$ be the cent-tree of G and let $V_{1,1}, V_{1,2}, \ldots, V_{1,k_1}$ be the children of the node $V_{0,1} = cent(G)$;

Then, $\exists$ at ~~least~~ most one child of $V_{0,1}$: $|V_{1,1}| \geq 1$.

We can easily prove that the typical structure of an A-free graph which contains no induced subgraph ≅ to $2K_2$ is the following:



The cent-tree has the following properties:

(P1) $K = V_{0,1} \cup V_{1,1} \cup \dots \cup V_{h,1}$ is a clique

(P2) $S = V - K$ is an indipendent set.

(P3) For every $x, y \in S$: $level(x) < level(y)$, $N(x) \subseteq N(y)$.

We can easily prove that $\bar{G}$ is an A-free graph.

(ii) $\Rightarrow$ (i): Suppose G contains $2K_2$. Then, $\bar{G}$ contains C_4, a contradiction.

- **Theorem:** Let $G=(V,E)$ be an undirected graph. The following statements are equivalent:
  (i) G is a threshold graph;
  (ii) G and $\overline{G}$ are A-free graphs;

- The free, semi-free and actual vertices of a connected graph G with n vertices and m edges can be computed in $O(\delta \cdot m)$ time.

- Thus, threshold graphs can be recognized in $O(\delta \cdot m + n^2)$ time.

- A-free, cographs and threshold graphs can be recognized in $O(1)$ time by using $O(n \cdot m)$ processors on a CRCW-PRAM
  or
  in $O(\log n)$ time by using $O(n \cdot m / \log n)$ processors.