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# Minimal separators in $P_4$-sparse graphs

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**Abstract**

In this paper, we determine the minimal separators of $P_4$-sparse graphs and establish bounds on their number. Specifically, we show that a $P_4$-sparse graph $G$ on $n$ vertices and $m$ edges has fewer than $2n/3$ minimal separators of total description size at most $4m/3$. The bound on the number of minimal separators is tight and is also tight for the class of cographs, a well known subclass of the $P_4$-sparse graphs. Our results enable us to present a linear-time and linear-space algorithm for computing the number of minimal separators of a given $P_4$-sparse graph; the algorithm can be modified to report the minimal separators of the input graph in linear time and space as well.


*Keywords:* Minimal separation; $P_4$-sparse graphs; Cographs; Perfect graphs

## 1. Introduction

The class of $P_4$-*sparse* graphs was introduced by Hoàng in his doctoral dissertation [13] as the class of graphs for which every set of five vertices induces at most one $P_4$ (chordless path on four vertices). Hoàng gave a number of characterizations for these graphs and showed that the $P_4$-sparse graphs are perfect in the sense of Berge (a graph $G$ is perfect if for every induced subgraph $H$ of $G$, the chromatic number of $H$ equals the clique number of $H$), and in fact perfectly orderable in the sense of Chvátal; see [4,15].

The study of $P_4$-sparse graphs led naturally to constructive characterizations that implied several linear-time recognition algorithms; such an algorithm was given in [18]. Additionally, optimization problems, such as, the coloring, the minimum clique cover, the maximum weighted clique and stable set, and the minimum fill-in, can be solved in linear time as well [20,14]. The class of $P_4$-sparse graphs generalizes both the *complement reducible* graphs, also known as *cographs*, and the $P_4$-*reducible* graphs: the well known class of cographs was introduced in the early 1970s by Lerchs [21] as the class of graphs for which no induced subgraph is isomorphic to a $P_4$; Jamison and Olariu [17] introduced the class of $P_4$-reducible graphs as those in which no vertex belongs to more than one induced $P_4$. Both cographs and $P_4$-reducible graphs can be recognized in linear time [5,7,17].

The class of $P_4$-sparse, and also the classes of cographs and $P_4$-reducible graphs, have been studied extensively in recent years and find applications in many areas of applied mathematics, computer science, and engineering, which

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involve graphs that feature "local density" and also nice algorithmic and structural properties. Indeed, the structure of the $P_4$-sparse graphs incorporates such local density properties since they are graphs with few $P_4$s; we note that researchers tend to equate the notion of local density with the absence of $P_4$s.

We are interested in studying the minimal separators of $P_4$-sparse graphs. A subset $S$ of vertices of a graph $G$ is called a *separator* if the subgraph $G[V(G) - S]$ of $G$ induced by $V(G) - S$ is not connected; note that the definition implies that the empty set ∅ is a separator of every disconnected graph and of no connected graph. A separator $S$ of $G$ is called an *ab-separator* if the vertices $a$ and $b$ of $G$ belong to different connected components in $G[V(G) - S]$. A separator $S$ is called a *minimal ab-separator* if it is an $ab$-separator and no proper subset of $S$ is an $ab$-separator, and $S$ is called a *minimal separator* if it is a minimal $ab$-separator for a pair $\{a, b\}$ of vertices of $G$ [3,2]; again, the way we defined the separators of a graph implies that ∅ is a minimal separator of every disconnected graph. We also mention that, in general, minimal separators of a graph can be generated by computing the neighborhoods of the connected components of subgraphs resulting after the removal of certain vertex sets [3].

The interest in minimal separators comes both from the fact that the minimal separators are used as a structural tool in several algorithmic applications and from their close relationship to the extremities of a graph [1]; detecting and using extremities is a way (although not the only one) of dealing with graph problems such as searching, recognition, and decomposition, since they allow only "local manipulations", which in many cases ensure efficiency. For many graph classes, recognition and several optimization problems were solved by appropriate processing of the graphs' extremities: Berry [1] mentions the work of Roberts who used extremities to characterize proper interval graphs [23], of Golumbic and Gross who defined an edge-elimination scheme for chordal bipartite graphs [16], of Fulkerson and Gross who introduced an elimination scheme which they used to characterize chordal graphs with an associated greedy recognition algorithm [11], of Dahlhaus et al. who extended elimination schemes to domination graphs [10], and of Corneil et al. who showed that strongly chordal graphs have strongly simplicial vertices and that AT-graphs have a domination path with extremities at each end [6].

In this paper, we determine the minimal separators of $P_4$-sparse graphs, and show that such a graph $G$ has fewer than $2n/3$ minimal separators whose total description size does not exceed $4m/3$, where $n$ is the number of vertices and $m$ the number of edges of $G$; in particular, if $G$ is disconnected the number of its minimal separators is at most $(2n - 1)/3$ whereas if it is connected it is at most $(2n - 2)/3$. We also show that these bounds are tight by exhibiting a family of $P_4$-sparse graphs which have that many separators; these graphs are cographs which shows that these lower bounds apply for the class of cographs as well. Based on our results, we also describe a linear-time algorithm which computes the number of minimal separators of a given $P_4$-sparse graph; the algorithm can be easily modified to report the minimal separators of the input graph in linear time.

Our results enable us to include the class of $P_4$-sparse graphs, and also the classes of cographs and $P_4$-reducible graphs, to the collection of graph classes which have a linear number of minimal separators, as are the chordal graphs and the weakly chordal graphs; we note that the number of minimal separators in a chordal graph is at most $n$ [24], while the corresponding number in a weakly chordal graph is at most $n + m$ [2].

The paper is organized as follows. In Section 2, we present the notation, related terminology, and background results which we need for our algorithm. In Section 3, we present our bounds on the number of minimal separators of $P_4$-sparse graphs and the algorithm. Finally, Section 4 concludes the paper with a summary of our results and possible extensions.

## 2. Theoretical framework

We consider finite undirected graphs with no loops or multiple edges. For a graph $G$, we denote its vertex and edge set by $V(G)$ and $E(G)$, respectively. Let $S$ be a subset of the vertex set of a graph $G$. Then, the subgraph of $G$ induced by $S$ is denoted by $G[S]$.

A *path* in the graph $G$ is a sequence of vertices $v_0 v_1 \cdots v_k$ such that $v_i v_{i+1} \in E(G)$, $0 \leqslant i \leqslant k-1$; we say that this is a path from $v_0$ to $v_k$ and that its *length* is $k$. A path is called *simple* if no vertex occurs more than once; it is called *trivial* if its length is equal to 0. A simple path $v_0 v_1 \cdots v_k$ is *chordless* if $v_i v_j \notin E(G)$ for any two non-consecutive vertices $v_i, v_j$ in the path. Throughout the paper, the chordless path on $k$ vertices is denoted by $P_k$; in particular, a chordless path on four vertices is denoted by $P_4$. If the graph $G$ contains a path from a vertex $x$ to a vertex $y$, we say that $x$ *is connected to* $y$. The graph $G$ is *connected* if $x$ is connected to $y$ for every pair of vertices $x, y \in V(G)$. The *connected components* (or *components*) of $G$ are the equivalence classes of the "is connected to" relation on the vertex set $V(G)$

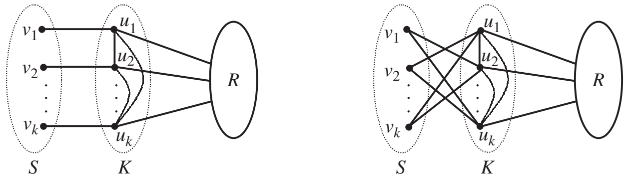


Fig. 1. A thin and a thick spider.

of $G$. The *co-connected components* (or *co-components*) of $G$ are the connected components of the complement $\overline{G}$ of the graph $G$.

The *neighborhood* $N(x)$ of a vertex $x$ of the graph $G$ is the set of all the vertices of $G$ which are adjacent to $x$. The *closed neighborhood* of $x$ is defined as $N[x] := N(x) \cup \{x\}$. The *degree* of a vertex $x$ in a graph $G$, denoted degree$(x)$, is the number of edges incident on $x$; thus, degree$(x) = |N(x)|$. A *clique* is a set of pairwise adjacent vertices; a *stable set* is a set of pairwise non-adjacent vertices.

### 2.1. $P_4$-sparse graphs

Below, we review properties and characterizations of $P_4$-sparse graphs and prove results which we use in our algorithm for the solution of the problem of finding the minimal separators of a $P_4$-sparse graph.

A graph $G$ is called a *spider* if the vertex set $V(G)$ of the graph $G$ admits a partition into sets $S$, $K$, and $R$ such that:

S1: $|S| = |K| \geqslant 2$, the set $S$ is a stable set, and the set $K$ is a clique;
S2: all the vertices in $R$ are adjacent to all the vertices in $K$ and to no vertex in $S$;
S3: there exists a bijection $f : S \to K$ such that exactly one of the following statements holds:
  (i) for each vertex $v \in S$, $N(v) \cap K = \{f(v)\}$;
  (ii) for each vertex $v \in S$, $N(v) \cap K = K - \{f(v)\}$.

The triple $(S, K, R)$ is called the *spider-partition*. A graph $G$ is a *prime spider* if $G$ is a spider with $|R| \leqslant 1$. If the condition of case S3(i) holds then the spider $G$ is called a *thin spider*, whereas if the condition of case S3(ii) holds then $G$ is a *thick spider*; note that the complement of a thin spider is a thick spider and vice versa. Fig. 1 depicts two spiders with partition sets $S = \{v_1, v_2, \ldots, v_k\}$, $K = \{u_1, u_2, \ldots, u_k\}$, and $R$: the spider on the left is a thin spider with $N(v_i) \cap K = \{u_i\}$, whereas the one on the right is a thick spider with $N(v_i) \cap K = K - \{u_i\}$, for every $v_i \in S$.

**Observation 2.1** (*Jamison and Olariu [19], Observation 2.8*). *If a graph G is a spider, then exactly one of the following statements holds*:

(i) *for every* $v \in S$ *and* $u \in K$, degree$(v) = 1$ *and* degree$(u) = |V(G)| - |S|$;
(ii) *for every* $v \in S$ *and* $u \in K$, degree$(v) = |K| - 1$ *and* degree$(u) = |V(G)| - 2$.

Again, items (i) and (ii) correspond to the case of a thin spider and a thick spider, respectively.

**Observation 2.2** (*Jamison and Olariu [19], Observation 2.9*). *If a graph G is a spider with partition* $(S, K, R)$ *and R is non-empty*, *then for every* $v \in S$, $u \in K$, *and* $r \in R$, degree$(v) <$ degree$(r) <$ degree$(u)$.

It is not difficult to see that a spider with $|K| = |S| = k$ contains exactly $k(k-1)/2 + k'$ $P_4$s, where $k'$ is the number of $P_4$s in the subgraph $G[R]$. From the definition of the spider and Observations 2.1 and 2.2, it follows that if $G$ is a spider, then the sets $S$, $K$, and $R$ are unique (see also [19]). Finally, from the properties of a spider $G$, and also from the definition of the $P_4$-sparse graphs, it easily follows that $G$ is $P_4$-sparse iff the graph $G[R]$ is $P_4$-sparse.

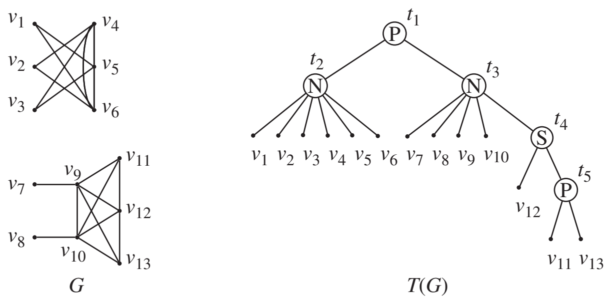


Fig. 2. A disconnected $P_4$-sparse graph $G$ on 13 vertices and the corresponding modular decomposition tree $T(G)$.

Let us now return to general $P_4$-sparse graphs. Then, the following result holds:

**Lemma 2.1** (*Jamison and Olariu [19], Theorem 2.13*). *For a graph G, the following conditions are equivalent*:

(i) *G is a $P_4$-sparse graph*;
(ii) *for every induced subgraph H of G with at least two vertices, exactly one of the following statements is satisfied*:
  (ii.1) *H is disconnected*;
  (ii.2) *$\overline{H}$ is disconnected*;
  (ii.3) *H is a spider*.

### 2.2. Modular decomposition

A subset $M$ of vertices of a graph $G$ is said to be a *module* of $G$, if every vertex outside $M$ is either adjacent to all vertices in $M$ or to none of them. The empty set, the singletons, and the vertex set $V(G)$ are *trivial* modules and whenever $G$ has only trivial modules it is called a *prime* (or *indecomposable*) *graph*. A non-trivial module is also called *homogeneous set*. A module $M$ of the graph $G$ is called a *strong module*, if for any module $M'$ of $G$, either $M' \cap M = \emptyset$ or one module is included into the other. Furthermore, a module in $G$ is also a module in $\overline{G}$.

The *modular decomposition* of a graph $G$ is a linear-space representation of all the partitions of $V(G)$ where each partition class is a module. The *modular decomposition tree* $T(G)$ of the graph $G$ (or *md-tree* for short) is a unique labeled tree associated with the modular decomposition of $G$ in which the leaves of $T(G)$ are the vertices of $G$ and the set of vertices of $G$ which are the leaves of the subtree rooted at an internal node induces a strong module of $G$. Thus, the md-tree $T(G)$ represents all the strong modules of $G$. An internal node is labeled by either $P$ (for *parallel* module), $S$ (for *series* module), or $N$ (for *neighborhood* module). The module corresponding to a $P$-node induces a disconnected subgraph of $G$, that of an $S$-node induces a connected subgraph of $G$ whose complement is a disconnected subgraph, and that of an $N$-node induces a connected subgraph of $G$ whose complement is also a connected subgraph. Note that if the md-tree $T(G)$ does not contain any $N$-node then it is precisely the special case of the cotree and $G$ is a cograph, since it has only parallel and series modules.

The process of constructing the md-tree $T(G)$ of the graph $G$ is defined as follows: parallel modules are decomposed into their connected components, series modules into their co-connected components, and neighborhood modules into their maximal strong submodules. The efficient construction of the md-tree $T(G)$ has received a great deal of attention. It can be shown that for every graph $G$ the tree $T(G)$ is unique up to isomorphism and it can be constructed in linear time; the first linear-time algorithms for the construction of the md-tree are described in [8,22], while more recent and more practical ones can be found in [9,12]. In Fig. 2 we show an example of a disconnected $P_4$-sparse graph and its md-tree.

Let $t$ be an internal node of the md-tree $T(G)$ of a graph $G$. We denote by $M(t)$ the module corresponding to $t$ which consists of the set of vertices of $G$ associated with the subtree of $T(G)$ rooted at node $t$; note that $M(t)$ is a strong module for every (internal or leaf) node $t$ of $T(G)$. Let $\{a_1, a_2, \ldots, a_p\}$ be the set of children of node $t$ in $T(G)$. We denote by $G(t)$ the *representative graph* of the module $M(t)$ defined as follows:

- $V(G(t)) = \{a_1, a_2, \ldots, a_p\}$ and
- $E(G(t)) = \{a_i a_j \mid v_k v_\ell \in E(G),\ v_k \in M(a_i)$ and $v_\ell \in M(a_j)\}$.

Note that by the definition of a module, if a vertex of $M(a_i)$ is adjacent to a vertex of $M(a_j)$ then every vertex of $M(a_i)$ is adjacent to every vertex of $M(a_j)$. Thus $G(t)$ is isomorphic to the graph induced by a subset of $M(t)$ consisting of a single vertex from each maximal strong submodule of $M(t)$ in the modular decomposition of $G$. It is easy to show that the following lemma holds (see also [14]):

**Lemma 2.2.** *Let $G$ be a graph, $T(G)$ its modular decomposition tree, and $t$ an internal node of $T(G)$. Then, $G(t)$ is an edgeless graph if $t$ is a P-node, $G(t)$ is a complete graph if $t$ is an S-node, and $G(t)$ is a prime graph if $t$ is an N-node.*

In particular, for the case of $P_4$-sparse graphs, Giakoumakis and Vanherpe [14] showed the following result:

**Lemma 2.3.** *Let $G$ be a graph and let $T(G)$ be its modular decomposition tree. The graph $G$ is $P_4$-sparse iff for every N-node $t$ of $T(G)$, $G(t)$ is a prime spider with a spider-partition $(S, K, R)$ and no vertex in $S \cup K$ is an internal node of $T(G)$.*

(Recall that the graph $G(t)$ has as vertices the children of the node $t$ of $T(G)$.) The above lemma implies that every $N$-node $t$ of the md-tree $T(G)$ of a $P_4$-sparse graph $G$ has either $2k$ or $2k+1$ children, where $|S| = |K| = k \geqslant 2$ and $|R| \leqslant 1$; the sets $S$, $K$, and $R$ form the spider-partition of the graph $G(t)$. More precisely, the $N$-node $t$ has $k$ children which correspond to $S$, $k$ children which correspond to $K$, and either no other child if $R = \emptyset$ or one more child if $R \neq \emptyset$ (in this case, $|R| = 1$ and this child is the root of a subtree of $T(G)$). The children which correspond to $S$ and $K$ are leaves in $T(G)$ and, thus, they are vertices of $G$, while the child which corresponds to $R$, if $R \neq \emptyset$, is either a leaf (i.e., a vertex of $G$) or an internal node labeled by either $P$, $S$, or $N$. The md-tree $T(G)$ depicted in Fig. 2 contains two $N$-nodes, the nodes $t_2$ and $t_3$. The graph $G(t_2)$ is a prime spider on six vertices with spider-partition $S = \{v_1, v_2, v_3\}$, $K = \{v_4, v_5, v_6\}$, and $R = \emptyset$, while $G(t_3)$ is a prime spider on five vertices with spider-partition $S = \{v_7, v_8\}$, $K = \{v_9, v_{10}\}$, and $R = \{t_4\}$; the internal node $t_4$ is an $S$-node. The graph $G[M(t_2)]$ is a spider (prime spider) with $R = \emptyset$, and the graph $G[M(t_3)]$ is also a spider (non-prime spider)with $R = \{v_{11}, v_{12}, v_{13}\}$.

Lemma 2.3 also implies a linear-time algorithm for the recognition problem, and linear-time algorithms for classical optimization problems on $P_4$-sparse graphs [14].

## 3. Minimal separators

In this section, we prove theoretical results on minimal separators for the class of $P_4$-sparse graphs, provide tight bounds on their number, and describe a linear-time algorithm for counting the number of minimal separators of a given $P_4$-sparse graph. Recall that we consider that the empty set is a minimal separator of every disconnected graph (this works well for our purposes; see Remark 3.1); if this is not desired, then for a disconnected graph, ignore the empty set from the set of minimal separators given below and subtract 1 from their number.

**Lemma 3.1.** *Let $G$ be a $P_4$-sparse graph. Then,*

(i) *$G$ is disconnected: if $C_1, C_2, \ldots, C_p$, $p \geqslant 2$, are the connected components of $G$ and $\{\sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,\ell_i}\}$ are the sets of minimal separators of the subgraph $G[C_i]$, $1 \leqslant i \leqslant p$, respectively, then the minimal separators of $G$ are*
  - *the empty set $\emptyset$ and*
  - *the separators $\sigma_{i,j}$ for all $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant \ell_i$.*

(ii) *$\overline{G}$ is disconnected: if $C'_1, C'_2, \ldots, C'_q$, $q \geqslant 2$, are the co-components of $G$ and $\{\sigma'_{i,1}, \sigma'_{i,2}, \ldots, \sigma'_{i,\ell'_i}\}$ are the sets of minimal separators of the subgraph $G[C'_i]$, $1 \leqslant i \leqslant q$, respectively, then the minimal separators of $G$ are*
  - *the sets $\sigma'_{i,j} \cup (V(G) - C'_i)$ for all $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant \ell'_i$.*

(iii) *G is a spider*: *if* $(S, K, R)$ *is the spider partition of G and* $\{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$ *is the set of minimal separators of the subgraph* $G[R]$, *then the minimal separators of G are*

- *the sets* $\sigma_i \cup K$, *for each separator* $\sigma_i$, $1 \leqslant i \leqslant \ell$, *of* $G[R]$ *and*
- *the subsets of K of size* 1 *if G is a thin spider or the subsets of K of size* $|K| - 1$ *if G is a thick spider*.

**Proof.** First, observe that cases (i)–(iii) cover all possibilities (Lemma 2.1).

(i) Since the graph $G$ is disconnected, the empty set is a separator; it is important to note that $\sigma_{i,j} \neq \emptyset$ since the subgraphs $G[C_1], G[C_2], \ldots, G[C_p]$ are connected. In order to determine the minimal separators of $G$, we need to examine pairs of non-adjacent vertices of $G$. Clearly, we need only consider pairs of vertices that both belong to the same connected component $C_i$ of $G$. But this implies that the sought separators are precisely the separators $\sigma_{i,j}$.

(ii) We examine pairs of vertices of $G$ to determine their minimal separators. Since any two vertices belonging to different co-components of $G$ are adjacent in $G$, we need only consider pairs of vertices that both belong to the same co-component $C'_i$; any two such vertices are adjacent in $G$ iff they are adjacent in the induced subgraph $G[C'_i]$. Then, it is not difficult to see that the set $\sigma'_{i,j} \cup (V(G) - C'_i)$ is an $xy$-separator in $G$ for two vertices $x, y \in C'_i$ iff $\sigma'_{i,j}$ is an $xy$-separator in $G[C'_i]$. Finally, the minimality of $\sigma'_{i,j} \cup (V(G) - C'_i)$ follows from the minimality of the separator $\sigma'_{i,j}$ in $G[C'_i]$ and the fact that any vertex in $V(G) - C'_i$ is adjacent to all vertices in $C'_i$.

(iii) In order to determine the minimal separators, we need to consider pairs of vertices $x, y$ such that either $x \in S$ and $y \in R \cup (K - N(x)) \cup (S - \{x\})$, or $x, y \in R$. In the latter case, the set $\sigma_i \cup K$ is an $xy$-separator of $G$ iff $\sigma_i$ is an $xy$-separator of $G[R]$; additionally, the minimality of $\sigma_i \cup K$ follows from the minimality of $\sigma_i$ and the fact that all vertices in $K$ are adjacent to all vertices in $R$. For the former case, we distinguish cases depending on whether the graph $G$ is a thin or a thick spider. Suppose that $G$ is a thin spider and let $f(v) \in K$ be the only neighbor of vertex $v \in S$. Consider a vertex $x \in S$. Then, for any vertex $y \in R \cup (K - \{f(x)\})$, the minimal $xy$-separator is $\{f(x)\}$. If $y \in S - \{x\}$, then the removal of $f(x)$ or $f(y)$ suffices to disconnect $x$ from $y$ so that the minimal separators are $\{f(x)\}$ and $\{f(y)\}$. Thus, by taking into account all vertices in $S$, we obtain as minimal separators all the subsets of the set $K$ of size 1. In a similar fashion, we can show that if $G$ is a thick spider, the minimal $xy$-separators of $G$ when $x \in S$ are all the subsets of the set $K$ of size $|K| - 1$. □

**Remark 3.1.** It is important to note that in case (ii) of Lemma 3.1 if a subgraph $G[C'_i]$ is disconnected, the set $V(G) - C'_i$ is a minimal separator; this separator is contained in the separators described in Lemma 3.1(ii) because $G[C'_i]$ contributes the minimal separator $\emptyset$. Additionally, in case (iii), if $G[R]$ is disconnected, its minimal separator $\emptyset$ enables us to correctly include the set $K$ in the set of minimal separators of $G$. On the other hand, no separator of a connected $P_4$-sparse graph is equal to $\emptyset$; in case (ii), $V(G) - C'_i \neq \emptyset$, while in case (iii), $K \neq \emptyset$.

Lemma 3.1 implies the following theorem (note that the total description size of a set of separators is equal to the sum of the sizes of the separators in the set).

**Theorem 3.1.** *Let G be a* $P_4$*-sparse graph on n vertices and m edges. Then, if G is disconnected it has at most* $(2n-1)/3$ *minimal separators*, *whereas if it is connected it has at most* $(2n-2)/3$ *minimal separators*. *In either case*, *the total description size of the minimal separators does not exceed* $4m/3$.

**Proof.** We use induction on the number $n$ of vertices of the graph $G$ (note that there exist $P_4$-sparse graphs on any number of vertices). For the basis step, we consider a graph on $n = 1$ vertex and $m = 0$ edges: the graph is $P_4$-sparse, is connected, and has no separators; since $0 \leqslant (2 \cdot 1 - 2)/3$ and $0 \leqslant 4 \cdot 0/3$, the basis case holds. For the inductive hypothesis, we assume that the theorem holds for all $P_4$-sparse graphs on at most $n - 1 \geqslant 1$ vertices. The inductive step requires us to show that the theorem holds for all $P_4$-sparse graphs on $n$ vertices. Let us consider such a graph $G$ and let $m$ be the number of its edges. Then, by Lemma 2.1, exactly one of the following three cases holds:

(i) *G is disconnected*: Let $C_1, C_2, \ldots, C_p$, $p \geqslant 2$, be the connected components of $G$. If $n_1, n_2, \ldots, n_p$ and $m_1, m_2, \ldots, m_p$ are the numbers of vertices and edges of the subgraphs $G[C_1], G[C_2], \ldots, G[C_p]$, respectively, then

$$n = n_1 + n_2 + \cdots + n_p, \tag{1}$$

$$m = m_1 + m_2 + \cdots + m_p, \tag{2}$$

while in accordance with Lemma 3.1(i) we have

$$\text{number of min-separators of } G = 1 + \sum_{i=1}^{p} (\text{number of min-separators of } G[C_i]), \tag{3}$$

$$\text{size of min-separators of } G = \sum_{i=1}^{p} (\text{size of min-separators of } G[C_i]). \tag{4}$$

Additionally, the subgraphs $G[C_1], G[C_2], \ldots, G[C_p]$ are $P_4$-sparse graphs on at most $n-1$ vertices each, and thus the inductive hypothesis applies on each of them; since they are connected, it holds that

$$\text{number of min-separators of } G[C_i] \leqslant \frac{2 \cdot n_i - 2}{3}, \tag{5}$$

$$\text{size of min-separators of } G[C_i] \leqslant \frac{4 \cdot m_i}{3}. \tag{6}$$

Combining the inequality (5) with Eqs. (3) and (1), we have

$$\text{number of min-separators of } G \leqslant 1 + \sum_{i=1}^{p} \frac{2 \cdot n_i - 2}{3} = 1 + \frac{2 \cdot n}{3} - \frac{2p}{3}.$$

Since $p \geqslant 2$, we have $1 - 2p/3 \leqslant -\frac{1}{3}$, which implies that

$$\text{number of min-separators of } G \leqslant \frac{2 \cdot n - 1}{3}. \tag{7}$$

Moreover, Eq. (4) with Eqs. (6) and (2) give

$$\text{size of min-separators of } G \leqslant \sum_{i=1}^{p} \frac{4 \cdot m_i}{3} = \frac{4 \cdot m}{3}. \tag{8}$$

Inequalities (7) and (8) establish the desired bounds for a disconnected $P_4$-sparse graph on $n$ vertices and $m$ edges.

(ii) $\overline{G}$ *is disconnected*: Let $C'_1, C'_2, \ldots, C'_q$, $q \geqslant 2$, be the co-components of the graph $G$ and let $n'_1, n'_2, \ldots, n'_q$ and $m'_1, m'_2, \ldots, m'_q$ be the numbers of vertices and edges of the subgraphs $G[C'_1], G[C'_2], \ldots, G[C'_q]$, respectively. We work as in the previous case and we have

$$n = n'_1 + n'_2 + \cdots + n'_q, \tag{9}$$

$$m = m'_1 + m'_2 + \cdots + m'_q + \frac{1}{2} \sum_{i=1}^{q} n'_i \cdot (n - n'_i). \tag{10}$$

Eq. (10) comes from the fact that, in addition to the edges in the subgraphs $G[C'_1], \ldots, G[C'_q]$, $G$ contains edges between any pair of vertices belonging to different co-components of $G$. Additionally, Lemma 3.1(ii) implies

$$\text{number of min-separators of } G = \sum_{i=1}^{q} (\text{number of min-separators of } G[C'_i]), \tag{11}$$

$$\text{size of min-separators of } G = \sum_{i=1}^{q} \sum_{j=1}^{\ell'_i} (|\sigma'_{i,j}| + (n - n'_i)), \tag{12}$$

where $\sigma'_{i,j}$ $(1 \leqslant j \leqslant \ell'_i)$ are the minimal separators of $G[C'_i]$; since $\sum_{j=1}^{\ell'_i} |\sigma'_{i,j}|$ is precisely the total size of the minimal separators of $G[C'_i]$, Eq. (12) is rewritten

$$\text{size of min-separators of } G = \sum_{i=1}^{q} ((\text{size of min-separators of } G[C'_i]) + \ell'_i \cdot (n - n'_i)). \tag{13}$$

The subgraphs $G[C'_1], G[C'_2], \ldots, G[C'_q]$ are $P_4$-sparse graphs on at most $n-1$ vertices each, and thus the inductive hypothesis applies on each of them; they may be connected or disconnected, and hence

$$\text{number } \ell'_i \text{ of min-separators of } G[C'_i] \leqslant \frac{2 \cdot n'_i - 1}{3}, \tag{14}$$

since $(2 \cdot n'_i - 2)/3 \leqslant (2 \cdot n'_i - 1)/3$, and

$$\text{size of min-separators of } G[C'_i] \leqslant \frac{4 \cdot m'_i}{3}. \tag{15}$$

Then, from Eq. (11) in conjunction with Eqs. (14) and (9) and the fact that $q \geqslant 2$, we have

$$\text{number of min-separators of } G \leqslant \sum_{i=1}^{q} \frac{2 \cdot n'_i - 1}{3} = \frac{2 \cdot n}{3} - \frac{q}{3} \leqslant \frac{2 \cdot n - 2}{3}. \tag{16}$$

Additionally, Eqs. (13) and (15) imply

$$\text{size of min-separators of } G \leqslant \sum_{i=1}^{q} \left( \frac{4 \cdot m'_i}{3} + \ell'_i \cdot (n - n'_i) \right),$$

which in light of Eqs. (14) and (10) gives

$$\text{size of min-separators of } G \leqslant \sum_{i=1}^{q} \left( \frac{4 \cdot m'_i}{3} + \frac{2 \cdot n'_i - 1}{3} \cdot (n - n'_i) \right) \leqslant \frac{4 \cdot m}{3}. \tag{17}$$

Inequalities (16) and (17) establish the desired bounds for a connected $P_4$-sparse graph on $n$ vertices and $m$ edges.

(iii) *G is a spider*: Let $(S, K, R)$ be the spider partition of $G$ and let $|S| = |K| = k \geqslant 2$, which implies that $|R| = n - 2k$. Lemma 3.1(iii) implies that

$$\text{number of min-separators of } G = k + (\text{number of min-separators of } G[R]), \tag{18}$$

$$\text{size of min-separators of } G = \sum_{i=1}^{\ell} (|\sigma_i| + k) + \begin{cases} k & \text{if } G \text{ is a thin spider,} \\ k \cdot (k-1) & \text{if } G \text{ is a thick spider,} \end{cases} \tag{19}$$

where $\sigma_i$ $(1 \leqslant i \leqslant \ell)$ are the minimal separators of $G[R]$. Since the subgraph $G[R]$ is $P_4$-sparse on at most $n-1$ vertices, the inductive hypothesis applies on it; as it may be connected or disconnected, we have

$$\text{number } \ell \text{ of min-separators of } G[R] \leqslant \frac{2 \cdot |R| - 1}{3} = \frac{2 \cdot (n - 2k) - 1}{3}, \tag{20}$$

$$\text{size of min-separators of } G[R] = \sum_{i=1}^{\ell} |\sigma_i| \leqslant \frac{4 \cdot m_R}{3}, \tag{21}$$

where $m_R$ is the number of edges of $G[R]$. Eqs. (18) and (20) and the fact that $k \geqslant 2$ imply that

$$\text{number of min-separators of } G \leqslant k + \frac{2 \cdot (n - 2k) - 1}{3} = \frac{2 \cdot n}{3} - \frac{k+1}{3} \leqslant \frac{2 \cdot n - 2}{3} \tag{22}$$

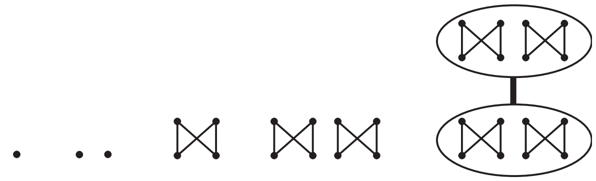


Fig. 3. The graphs $F_0$, $H_1$, $F_1$, $H_2$, and $F_2$.

as desired for a connected $P_4$-sparse graph on $n$ vertices. Since for the number $m$ of edges of the graph $G$ it holds that

$$m = m_R + k \cdot |R| + k \cdot (k-1)/2 + \begin{cases} k & \text{if } G \text{ is a thin spider,} \\ k \cdot (k-1) & \text{if } G \text{ is a thick spider,} \end{cases} \tag{23}$$

and $\sum_{i=1}^{\ell} k = \ell \cdot k \leqslant 2 \cdot k \cdot |R|/3$ (due to Eq. (20)), Eq. (19) in conjunction with Eqs. (21) and (23) gives

$$\text{size of min-separators of } G \leqslant \frac{4 \cdot m}{3}.$$

The inductive proof is complete since, in each case, the statement of the theorem holds for the graph on $n$ vertices. Therefore, the theorem holds for all $P_4$-sparse graphs. □

In fact, we can show that the above bounds on the number of minimal separators of a $P_4$-sparse graph are tight. Consider the following two families of graphs:

$F_0$ is a graph on a single vertex and no edges

and for each $i \geqslant 1$:

$H_i$ is a disconnected graph with two components, each being a copy of the graph $F_{i-1}$,

$F_i$ is a connected graph with two co-components, each being a copy of the graph $H_i$.

(Fig. 3 depicts the graphs $F_i$ and $H_i$ for $i \leqslant 2$; the heavy line connecting a pair of "blobs" in $F_2$ indicates that every vertex in one of these blobs is adjacent to every vertex in the other blob.) It is not difficult to see that the graphs $F_i$ and $H_i$ have the following properties:

P1: each of the graphs $F_i$ $(i \geqslant 0)$ and $H_i$ $(i \geqslant 1)$ is a cograph and hence a $P_4$-sparse graph;
P2: the number of vertices of the graph $F_i$ is $|V(F_i)| = 4^i$, $i \geqslant 0$;
P3: the number of vertices of the graph $H_i$ is $|V(H_i)| = 2 \cdot 4^{i-1}$, $i \geqslant 1$.

Then, we can show the following result:

**Lemma 3.2.** *The number of minimal separators of the graphs* $F_i$ $(i \geqslant 0)$ *and* $H_i$ $(i \geqslant 1)$ *are* $(2 \cdot |V(F_i)| - 2)/3$ *and* $(2 \cdot |V(H_i)| - 1)/3$, *respectively.*

**Proof.** In light of the definition of the graphs and Lemma 3.1(i) and (ii), we observe that:

$$\text{number of min-separators of } H_i = 1 + 2 \cdot (\text{number of min-separators of } F_{i-1}); \tag{24}$$

$$\text{number of min-separators of } F_i = 2 \cdot (\text{number of min-separators of } H_i). \tag{25}$$

Eqs. (24) and (25) imply that

$$\text{number of min-separators of } F_i = 2 + 4 \cdot (\text{number of min-separators of } F_{i-1}).$$

This equality and the fact that $|V(F_i)| = 4^i$ (see Property P2) make it easy to show by induction on $i$ ($i \geqslant 0$) that

$$\text{number of min-separators of } F_i = \frac{2 \cdot |V(F_i)| - 2}{3}.$$

Then, from Eq. (24) and because $|V(H_i)| = 2 \cdot |V(F_{i-1})|$ (see Properties P2 and P3), it follows that

$$\text{number of min-separators of } H_i = 1 + 2 \cdot \frac{2 \cdot |V(F_{i-1})| - 2}{3} = \frac{2 \cdot |V(H_i)| - 1}{3}. \qquad \square$$

Since the graphs $F_i$ and $H_i$ are cographs (Property P1), the lower bounds established in Lemma 3.2 for the number of minimal separators also apply for the class of cographs.

Finally, we describe an algorithm which computes the number of minimal separators of a given $P_4$-sparse graph $G$. The algorithm relies on Lemma 3.1.

*Algorithm MinSepNum*
*Input*: a $P_4$-sparse graph $G$.
*Output*: the number of minimal separators of $G$.

1. Construct the modular decomposition tree $T(G)$ of the input graph $G$;
2. Execute the subroutine *process*(*root*), where *root* is the root node of the tree $T(G)$; the number of minimal separators of $G$ is the number returned by the subroutine.

where the description of the subroutine *process*( ) is as follows:

*process*(node $t$)
*Input*: node $t$ of the modular decomposition tree $T(G)$ of the input graph $G$.
*Output*: the number of minimal separators of the subgraph $H$ of $G$ corresponding to the subtree of $T(G)$ rooted at node $t$.

```
1. if t is a leaf
   then return(0);
2. if t is a P-node {H disconnected}
   then sum ← 0;
        for each child t' of t in T(G) do
            sum ← sum + process(t');
        return(sum + 1);
3. if t is an S-node {H̄ disconnected}
   then sum ← 0;
        for each child t' of t in T(G) do
            sum ← sum + process(t');
        return(sum);
4. if t is an N-node {H spider}
   then k ← the size of the partition sets S and K of the spider H;
          if the partition set R is empty
          then ℓ ← 0;
          else ℓ ← process(t'), where t' is the node of T(G) which corresponds to R;
          return(k + ℓ);
```

The correctness of the algorithm follows from Lemma 3.1. We next compute its time and space complexity. Let $n$ and $m$ be the number of vertices and edges of the input graph $G$. Step 1 of the algorithm can be executed in $O(n+m)$ time and space [8,9,12,22]; we also note that the modular decomposition tree $T(G)$ can be easily obtained from the so-called ps-tree of $G$ [19], which can be computed in $O(n+m)$ time and space using the algorithm in [18] which is simpler than the available modular decomposition algorithms. The subroutine process( ) is called exactly once for each of the O($n$) nodes of the tree $T(G)$, and in addition to the time required by any recursive calls, the processing of such a node $t$ takes $O(d_{T(G)}(t) + 1)$ time and O(1) space, where $d_{T(G)}(t)$ is the number of children of node $t$ in $T(G)$. Thus, the total time for the processing of all the nodes of the tree $T(G)$ is O($n$); the space needed is O($n$). In summary, we have:

**Theorem 3.2.** *The number of minimal separators of a $P_4$-sparse graph $G$ on $n$ vertices and $m$ edges can be computed in* $O(n + m)$ *time using* $O(n + m)$ *space*.

Algorithm MinSepNum can be easily modified to report all the minimal separators of $G$ based on their characterizations given in Lemma 3.1; in light of Theorem 3.1, the resulting algorithm runs in $O(n+m)$ time and requires $O(n+m)$ space.

## 4. Concluding remarks

In this paper, we determined the minimal separators of the $P_4$-sparse graphs and showed that the number of minimal separators of such a graph (or cograph by inclusion) on $n$ vertices and $m$ edges, does not exceed $2n/3$ and that their description size is at most $4m/3$. This result enables us to include these two classes of graphs to the collection of graph classes, such as chordal graphs and weakly chordal graphs, which have a linear number of minimal separators. We also presented a linear-time algorithm for computing the number of minimal separators of a $P_4$-sparse graph, which can be easily modified to report the minimal separators within the same time and space complexity.

The obvious open problem is to compute the number of minimal separators in other classes of graphs; as a first step, one could try to work on classes that properly contain the $P_4$-sparse graphs, e.g., the classes of the $P_4$-laden, $P_4$-lite, $P_4$-extendible, and $P_4$-tidy graphs (see [4] for an exposition of such classes).

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