Available online at www.sciencedirect.com

SCIENCE DIRECT

Information Processing Letters 90 (2004) 307–313

Information Processing Letters

www.elsevier.com/locate/ipl

A limit characterization for the number of spanning trees of graphs

S.D. Nikolopoulos [a,*], C. Nomikos [a], P. Rondogiannis [b]

[a] Department of Computer Science, University of Ioannina P.O. Box 1186, GR-45110 Ioannina, Greece
[b] Department of Informatics and Telecommunications, University of Athens, Panepistimiopolis, GR-15784 Athens, Greece

Received 16 June 2003; received in revised form 2 December 2003

Available online 9 April 2004

Communicated by K. Iwama

Abstract

In this paper we propose a limit characterization of the behaviour of classes of graphs with respect to their number of spanning trees. Let $\{G_n\}$ be a sequence of graphs $G_0, G_1, G_2, \ldots$ that belong to a particular class. We consider graphs of the form $K_n - G_n$ that result from the complete graph K_n after removing a set of edges that span G_n. We study the spanning tree behaviour of the sequence $\{K_n - G_n\}$ when $n \to \infty$ and the number of edges of G_n scales according to n. More specifically, we define the *spanning tree indicator* $\alpha(\{G_n\})$, a quantity that characterizes the spanning tree behaviour of $\{K_n - G_n\}$. We derive closed formulas for the spanning tree indicators for certain well-known classes of graphs. Finally, we demonstrate that the indicator can be used to compare the spanning tree behaviour of different classes of graphs (even when their members never happen to have the same number of edges).

Keywords: Graphs; Spanning trees; Combinatorial problems

1. Introduction

There exist many interesting and difficult problems in graph theory that are related to counting the number of spanning trees of graphs. The solution of such problems appears to have a direct impact in certain areas of Computer Science. For example, it appears that in order to build a reliable communication network, a good choice would be to select a graph topology that maximizes the number of spanning trees (see [2,4,7]).

Let $K_n - G$ be the graph that results from the complete graph K_n after removing a set of edges that span G. A well-known problem in graph theory is that of calculating the number of spanning trees of the graph $K_n - G$. Many cases have been examined depending on the choice of G. For example, there exist closed formulas for the cases where G is a pairwise disjoint set of edges [11], when it is a chain of edges [6], a cycle [5], a star [10], a multistar [8,12], a complete graph [1], a multi-complete/star graph [3], a quasi-threshold graph [9], and so on (see Berge [1] for an exposition of the main results).

We are seeking alternative approaches that would give a more direct interpretation of the spanning tree behaviour of a class. The basic idea is to compare

* Corresponding author.
E-mail addresses: stavros@cs.uoi.gr (S.D. Nikolopoulos), cnomikos@cs.uoi.gr (C. Nomikos), prondo@di.uoa.gr (P. Rondogiannis).

doi:10.1016/j.ipl.2004.03.001

the number of spanning trees of $K_n - G$ with those of K_n. It would be convenient if the comparison gave a quantity that is independent of n. For this purpose, let $\{G_n\}$ be a sequence of graphs that belong to a particular class of graphs. We study the spanning tree behaviour of the sequences of graphs of the form $\{K_n - G_n\}$, when $n \to \infty$ and the number of edges of G_n scales according to n. More specifically, we define the *spanning tree indicator* $\alpha(\{G_n\})$, a quantity that characterizes the spanning tree behaviour of $\{K_n - G_n\}$. We derive the spanning tree indicators for certain well-known classes of graphs. We show that the indicator can be used to compare different classes of graphs in terms of their number of spanning trees even in the case in which the members of the two classes never happen to have the same number of edges.

The paper is organized as follows. In Section 2 we define the notion of *spanning tree indicator* and examine its properties. Section 3 contains the derivations of the spanning tree indicators for certain well-known classes of graphs. Section 4 presents an example comparison of the spanning tree behaviour of two given classes of graphs. Finally, Section 5 concludes the paper by discussing possible future extensions.

2. The limit characterization

In this section we define the main concepts for characterizing the limit spanning tree behaviour of classes of graphs.

A *class* $\mathcal{G}$ of graphs is a set of graphs sharing certain common properties. For example, the class of complete graphs is the set $\{K_0, K_1, \ldots, K_i, \ldots\}$ of all complete graphs. Given a class $\mathcal{G}$, we write $\{G_n\}$ to denote a *sequence* $G_0, G_1, \ldots, G_i, \ldots$ of graphs, where each G_i belongs to $\mathcal{G}$. For example, $K_1, K_1, K_2, K_2, \ldots$ is a sequence consisting of complete graphs. Throughout the paper, the number of edges of a given graph G is denoted by $edges(G)$ and the number of vertices by $vrt(G)$. The number of spanning trees of a given graph G is denoted by $N(G)$.

Definition 2.1. Let $\mathcal{G}$ be a class of graphs and $\{G_n\}$ be a sequence of graphs in $\mathcal{G}$. The *spanning tree indicator* $\alpha(\{G_n\})$ of $\{G_n\}$ is defined as $\lim_{n\to\infty} \frac{N(K_n - G_n)}{N(K_n)}$.

Obviously, for every sequence of graphs $\{G_n\}$, we have: $0 \leqslant \alpha(\{G_n\}) \leqslant 1$.

The spanning tree indicator is useful for comparing sequences of graphs whose corresponding elements have “almost” the same number of edges. Such sequences will be called *comparable* and are defined as follows:

Definition 2.2. Let $\mathcal{G}$ and $\mathcal{F}$ be two classes of graphs, and let $\{G_n\}$ and $\{F_n\}$ be two sequences of graphs from the corresponding classes. Then, the sequences $\{G_n\}$ and $\{F_n\}$ will be called *comparable* if

$$\lim_{n\to\infty} \frac{edges(G_n)}{n} = \lim_{n\to\infty} \frac{edges(F_n)}{n} = \beta > 0.$$

Given a sequence $\{G_n\}$, $\lim_{n\to\infty} \frac{edges(G_n)}{n}$ is denoted by $\gamma(\{G_n\})$.

In particular, we are interested in those classes of graphs whose spanning tree indicator is immune to “small changes”. The following definition formalizes this issue:

Definition 2.3. Let $\mathcal{G}$ be a class of graphs. $\mathcal{G}$ is called *β-stable* if for all sequences $\{G_n\}$ and $\{G'_n\}$ from $\mathcal{G}$ with $\gamma(\{G_n\}) = \gamma(\{G'_n\}) = \beta$, it is $\alpha(\{G_n\}) = \alpha(\{G'_n\})$.

For sequences of graphs that are comparable, we use their indicators in order to compare their limit behaviour (“how good they are”) in terms of spanning trees. This is expressed by the following, easily derivable, proposition:

Proposition 2.1. *If $\{G_n\}$ and $\{F_n\}$ are comparable and $\alpha(\{G_n\}) < \alpha(\{F_n\})$ then there exists $n_0 \in N$ such that for every $n > n_0$, $N(K_n - G_n) < N(K_n - F_n)$.*

Proof. Straightforward, using the definition of limits. □

We would like to demonstrate that the above result does not only hold when the elements of $\{G_n\}$ happen to have a larger number of edges than the corresponding elements of $\{F_n\}$. The following proposition resolves this issue.

Theorem 2.1. *Let $\mathcal{G}$ and $\mathcal{F}$ be two β-stable classes of graphs and let $\{G_n\}$ and $\{F_n\}$ be two comparable* *sequences of graphs from the corresponding classes having the properties* $\gamma(\{G_n\}) = \gamma(\{F_n\}) = \beta$ *and* $\alpha(\{G_n\}) < \alpha(\{F_n\})$. *Then there exists a sequence* $\{F'_n\}$ *of the class* $\mathcal{F}$ *that is comparable to* $\{G_n\}$, *such that* $edges(G_n) < edges(F'_n)$ *and* $\alpha(\{G_n\}) < \alpha(\{F'_n\})$.

Proof. Let $\hat{\imath}(n)$ be the minimum index $m \geqslant n$ such that $edges(G_n) < edges(F_m)$; $\hat{\imath}(n)$ always exists since $\lim_{n\to\infty} edges(F_n) = \infty$. We define $F'_n = F_{\hat{\imath}(n)}$. We show that $\{F'_n\}$ is comparable to $\{G_n\}$, that is $\gamma(\{F'_n\}) = \gamma(\{G_n\}) = \beta$, and that $\alpha(\{F'_n\}) = \alpha(\{F_n\})$.

Since $\gamma(\{G_n\}) = \gamma(\{F_n\}) = \beta$, for every $\varepsilon > 0$ there exists k_ε such that for every $n \geqslant k_\varepsilon$ and $m \geqslant k_\varepsilon$, the following are true:

$$(\beta - \varepsilon) \cdot n \leqslant edges(G_n) \leqslant (\beta + \varepsilon) \cdot n$$

and

$$(\beta - \varepsilon) \cdot m \leqslant edges(F_m) \leqslant (\beta + \varepsilon) \cdot m.$$

Now for every $\varepsilon > 0$, we define the following function $\tilde{\imath}_\varepsilon$:

$$\tilde{\imath}_\varepsilon(n) = \left\lfloor \frac{\beta+\varepsilon}{\beta-\varepsilon} \cdot n \right\rfloor + 1 > \frac{\beta+\varepsilon}{\beta-\varepsilon} \cdot n.$$

For $n \geqslant k_\varepsilon$, we have

$$edges(F_{\tilde{\imath}_\varepsilon(n)}) \geqslant (\beta - \varepsilon) \cdot \tilde{\imath}_\varepsilon(n) > (\beta + \varepsilon) \cdot n \geqslant edges(G_n).$$

Since $\hat{\imath}(n)$ is the minimum index $m \geqslant n$ such that $edges(G_n) < edges(F_m)$ we have that $\hat{\imath}(n) \leqslant \tilde{\imath}_\varepsilon(n)$ for $n \geqslant k_\varepsilon$. Consequently we have:

$$\lim_{n\to\infty} \frac{\hat{\imath}(n)}{n} \leqslant \lim_{n\to\infty} \frac{\tilde{\imath}_\varepsilon(n)}{n} \leqslant \lim_{n\to\infty} \frac{\frac{\beta+\varepsilon}{\beta-\varepsilon} \cdot n + 1}{n} = \frac{\beta+\varepsilon}{\beta-\varepsilon}.$$

This should hold for arbitrary positive ε, which implies

$$\lim_{n\to\infty} \frac{\hat{\imath}(n)}{n} = 1.$$

Then,

$$\frac{edges(F'_n)}{n} = \frac{edges(F_{\hat{\imath}(n)})}{n} = \frac{edges(F_{\hat{\imath}(n)})}{\hat{\imath}(n)} \cdot \frac{\hat{\imath}(n)}{n}.$$

Thus,

$$\gamma\big(\{F'_n\}\big) = \lim_{n\to\infty} \frac{edges(F_{\hat{\imath}(n)})}{\hat{\imath}(n)} \cdot \lim_{n\to\infty} \frac{\hat{\imath}(n)}{n} = \beta,$$

which implies that the sequences $\{F'_n\}$ and $\{G_n\}$ are comparable.

Since $\mathcal{F}$ is a β-stable class and

$$\gamma\big(\{F_n\}\big) = \gamma\big(\{F'_n\}\big) = \beta,$$

we have $\alpha(\{F_n\}) = \alpha(\{F'_n\})$. Thus,

$$\alpha\big(\{G_n\}\big) < \alpha\big(\{F'_n\}\big). \qquad \square$$

3. Indicators for well-known classes of graphs

In the following, we derive the spanning tree indicators for sequences whose members belong to certain well-known classes of graphs. We give a complete proof for the case of the complete graphs; the other cases follow in a similar way.

3.1. Complete graphs

Let K_n be the complete graph on n vertices. Consider now the class $\mathcal{K} = \{K_1, K_2, \ldots, K_i, \ldots\}$ and let $\{G_n\}$ be a sequence of graphs belonging to $\mathcal{K}$ such that[1]:

$$\lim_{n\to\infty} \frac{edges(G_n)}{n} = \beta \in (0, 1).$$

The number of spanning trees of $K_n - K_r$ is given by [1]:

$$N(K_n - K_r) = n^{n-2} \cdot \left(1 - \frac{r}{n}\right)^{r-1}.$$

Using the above equation, we get that:

$$\begin{aligned} \alpha\big(\{G_n\}\big) &= \lim_{n\to\infty} \left(1 - \frac{vrt(G_n)}{n}\right)^{vrt(G_n)-1} \\ &= \lim_{n\to\infty} \left(1 - \frac{vrt(G_n)\cdot(vrt(G_n)-1)}{2n} \times \frac{2}{vrt(G_n)-1}\right)^{vrt(G_n)-1}. \end{aligned}$$

Since G_n is a complete graph, we have that:

$$\lim_{n\to\infty} \frac{vrt(G_n)\cdot(vrt(G_n)-1)}{2n} = \lim_{n\to\infty} \frac{edges(G_n)}{n} = \beta.$$

[1] As it will be shown in Section 4, such a sequence exists for class $\mathcal{K}$ (this can also be easily established for all other classes considered in this paper).

This implies that for any $\varepsilon > 0$ there exists an n_0 such that for every $n \geqslant n_0$:

$$\beta - \varepsilon \leqslant \frac{vrt(G_n) \cdot (vrt(G_n) - 1)}{2n} \leqslant \beta + \varepsilon.$$

Using the above inequalities we can obtain lower and upper bounds for the indicator of $\{G_n\}$. For arbitrarily small ε:

$$\lim_{n\to\infty}\left(1 - \frac{2\cdot(\beta+\varepsilon)}{vrt(G_n)-1}\right)^{vrt(G_n)-1} \leqslant \alpha(\{G_n\}) \leqslant \lim_{n\to\infty}\left(1 - \frac{2\cdot(\beta-\varepsilon)}{vrt(G_n)-1}\right)^{vrt(G_n)-1},$$

which implies:

$$e^{-2\cdot(\beta+\varepsilon)} \leqslant \alpha(\{G_n\}) \leqslant e^{-2\cdot(\beta-\varepsilon)}.$$

Since the above inequalities hold for arbitrarily small ε, we conclude that:

$$\alpha(\{G_n\}) = e^{-2\cdot\beta}.$$

Notice that the above limit only depends on β and not on any other characteristic of $\{G_n\}$ and therefore the corresponding class is β-stable.

3.2. *Star graphs*

Let $\mathcal{S} = \{S_0, S_1, \ldots, S_i, \ldots\}$ be the class of star graphs, where S_r is the star graph on $r + 1$ vertices. Let $\{G_n\}$ be a sequence on $\mathcal{S}$ such that:

$$\lim_{n\to\infty} \frac{edges(G_n)}{n} = \beta \in (0, 1).$$

It is well known [1] that the number of spanning trees of $K_n - S_r$ is given by the following formula:

$$N(K_n - S_r) = n^{n-2} \cdot \left(1 - \frac{1}{n}\right)^{r-2} \cdot \left(1 - \frac{r}{n}\right).$$

The spanning tree indicator for the sequence $\{G_n\}$ can be derived by the following limit:

$$\alpha(\{G_n\}) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{edges(G_n)-1} \times \left(1 - \frac{edges(G_n)+1}{n}\right).$$

It can be easily shown that the above limit exists and is equal to:

$$\alpha(\{G_n\}) = e^{-\beta} \cdot (1 - \beta).$$

Notice that for every β, the indicator for this class of graphs is less than that of the class of complete graphs.

3.3. *Disjoint copies of K_2*

Let D_i be the graph consisting of i disjoint copies of K_2 (that is, i pairwise disjoint edges), and let $\mathcal{D} = \{D_0, D_1, \ldots, D_i, \ldots\}$ be the class of such graphs. Let $\{G_n\}$ be a sequence on $\mathcal{D}$ such that:

$$\lim_{n\to\infty} \frac{edges(G_n)}{n} = \beta \in \left(0, \frac{1}{2}\right).$$

It is also well known [1] that the number of spanning trees of $K_n - D_r$ is given by the following formula:

$$N(K_n - D_r) = n^{n-2} \cdot \left(1 - \frac{2}{n}\right)^r.$$

The spanning tree indicator for the sequence $\{G_n\}$ can be derived by the following limit:

$$\alpha(\{G_n\}) = \lim_{n\to\infty}\left(1 - \frac{2}{n}\right)^{edges(G_n)}.$$

This can be easily shown to be equal to:

$$\alpha(\{G_n\}) = e^{-2\cdot\beta}.$$

Observe that the indicator in this case is identical to the one corresponding to the class of complete graphs.

3.4. *Complete bipartite graphs*

Let $K_{m,r}$ be the complete bipartite graph on $m + r$ vertices. We consider now the class $\mathcal{B} = \{K_{m,1}, K_{m,2}, \ldots, K_{m,i}, \ldots\}$. Let $\{G_n\}$ be a sequence of graphs from $\mathcal{B}$ such that:

$$\lim_{n\to\infty} \frac{edges(G_n)}{n} = \beta \in (0, 1).$$

In order to calculate the spanning tree indicator for $\{G_n\}$ we need to know the formula giving the number of spanning trees of the graph $K_n - K_{m,r}$. We prove the following theorem:

Theorem 3.1. *The number of spanning trees of $K_n - K_{m,r}$ is equal to*

$$n^{n-2} \cdot \left(1 - \frac{r}{n}\right)^{m-1} \cdot \left(1 - \frac{m}{n}\right)^{r-1} \cdot \left(1 - \frac{m+r}{n}\right).$$

Proof. Using the Complement Spanning-Tree Matrix Theorem [1] and a calculation technique very similar to the one presented in [8]. □

The spanning tree indicator for $\{G_n\}$ can be calculated if we take into consideration the fact that $edges(G_n) = m \cdot r$. Therefore, it is:

$$\alpha\big(\{G_n\}\big) = \lim_{n\to\infty}\left(1 - \frac{edges(G_n)}{m \cdot n}\right)^{m-1} \times \left(1 - \frac{m}{n}\right)^{\frac{edges(G_n)}{m}-1} \times \left(1 - \frac{m + \frac{edges(G_n)}{m}}{n}\right).$$

The above limit is easily shown to be equal to:

$$\alpha\big(\{G_n\}\big) = \mathrm{e}^{-\beta} \cdot \left(1 - \frac{\beta}{m}\right)^{m}.$$

Notice that, for $m = 1$ we get the spanning tree indicator for the star graph case.

4. Comparing classes of graphs through indicators

In this section we provide an example of using the proposed approach in order to compare two different classes of graphs. The two classes we consider have been appropriately chosen so as that their members never happen to have the same size.

Assume now that we want to compare these two classes with respect to their number of spanning trees. One possible solution would be to select specific members from the two classes and calculate the number of spanning trees analytically. However, this cannot be performed in a straightforward way since there are no members of the two classes that have the same number of edges. In other words, we can not directly compare the two classes because we do not know which elements from the two classes to select in order to perform the comparison. On the other hand, using the proposed technique, we can construct pairs of comparable sequences from the two classes and then compare the classes through the indicators. Consequently, the limit approach is shown to be effective even when the graphs being considered are not equal in size.

Consider the class $\mathcal{K}$ of all complete graphs and the class $\mathcal{B}$ which is a subclass of the complete bipartite graphs; let $\mathcal{K} = \{K_1, K_2, \ldots, K_i, \ldots\}$ and $\mathcal{B} = \{K_{2,1}, K_{2,4}, \ldots, K_{2,i^2}, \ldots\}$.

Notice that $edges(K_{2,i^2}) \neq edges(K_j)$ for every $i, j \geqslant 0$, since:

$$edges(K_{2i}) = \frac{2i \cdot (2i-1)}{2} = 2i^2 - i < 2i^2 = edges(K_{2,i^2})$$

and

$$edges(K_{2i+1}) = \frac{(2i+1) \cdot 2i}{2} = 2i^2 + i > 2i^2 = edges(K_{2,i^2})$$

(that is, $edges(K_{2,i^2})$ is a number between the number of edges of two consecutive complete graphs).

We next show that for every β we can construct sequences $\{F_n\}$ of graphs in $\mathcal{K}$ and $\{G_n\}$ of graphs in $\mathcal{B}$ such that $\gamma(\{F_n\}) = \gamma(\{G_n\}) = \beta$.

Let $F_n = K_m$, where m satisfies the following inequalities:

$$\frac{(m-1)\cdot(m-2)}{2} < \beta \cdot n \leqslant \frac{m \cdot (m-1)}{2}$$

(that is, F_n is the minimum complete graph with no less than $\beta \cdot n$ edges).

Now, for all $n \geqslant \frac{4}{\beta}$, it is $vrt(F_n) \geqslant 4$. In that case:

$$edges(F_n) = \frac{vrt(F_n)\cdot(vrt(F_n)-1)}{2} \leqslant \frac{2\cdot(vrt(F_n)-2)\cdot(vrt(F_n)-1)}{2} < 2\cdot\beta\cdot n.$$

Moreover, since

$$edges(F_n) = \frac{vrt(F_n)\cdot(vrt(F_n)-1)}{2},$$

we get for $n \geqslant \frac{4}{\beta}$:

$$vrt(F_n) = \frac{1 + \sqrt{1 + 8 \cdot edges(F_n)}}{2} < \frac{1 + \sqrt{1 + 16\cdot\beta\cdot n}}{2}.$$

Furthermore,

$$edges(F_n) - \beta\cdot n = \frac{vrt(F_n)\cdot(vrt(F_n)-1)}{2} - \beta\cdot n < \frac{vrt(F_n)\cdot(vrt(F_n)-1)}{2} - \frac{(vrt(F_n)-1)\cdot(vrt(F_n)-2)}{2} = vrt(F_n) - 1 < vrt(F_n).$$

Consequently, for $n \geqslant \frac{4}{\beta}$, we have

$$0 \leqslant \frac{edges(F_n) - \beta \cdot n}{n} < \frac{vrt(F_n)}{n} < \frac{1 + \sqrt{1 + 16 \cdot \beta \cdot n}}{2n}.$$

Since

$$\lim_{n\to\infty} \frac{1 + \sqrt{1 + 16 \cdot \beta \cdot n}}{2n} = 0,$$

we conclude that

$$\lim_{n\to\infty} \frac{edges(F_n) - \beta \cdot n}{n} = 0$$

which is equivalent to $\gamma(\{F_n\}) = \beta$.

Similarly let $G_n = K_{2,m^2}$, where m satisfies the following inequalities:

$$2 \cdot m^2 \leqslant \beta \cdot n < 2 \cdot (m + 1)^2$$

(that is, G_n is the maximum graph in $\mathcal{B}$ with no more than $\beta \cdot n$ edges). Then,

$$\begin{aligned} &\beta \cdot n - edges(G_n) \\ &\quad = \beta \cdot n - 2 \cdot m^2 < 2 \cdot (m+1)^2 - 2 \cdot m^2 \\ &\quad = 4 \cdot m + 2 \leqslant 6m \end{aligned}$$

and

$$m = \sqrt{\frac{edges(G_n)}{2}} \leqslant \sqrt{\frac{\beta \cdot n}{2}}.$$

Consequently, we have

$$0 \leqslant \frac{\beta \cdot n - edges(G_n)}{n} < \frac{6m}{n} \leqslant \frac{6 \cdot \sqrt{\beta \cdot n/2}}{n}.$$

Since

$$\lim_{n\to\infty} \frac{6 \cdot \sqrt{\beta \cdot n/2}}{n} = 0,$$

we conclude that

$$\lim_{n\to\infty} \frac{\beta \cdot n - edges(G_n)}{n} = 0$$

which is equivalent to $\gamma(\{G_n\}) = \beta$.

We finally observe that although $edges(F_n) \geqslant edges(G_n)$ the following holds:

$$\alpha\bigl(\{F_n\}\bigr) = \mathrm{e}^{-2\cdot\beta} > \mathrm{e}^{-\beta} \cdot (1 - \beta/2)^2 = \alpha\bigl(\{G_n\}\bigr).$$

Thus, we conclude that the class $\mathcal{K}$ behaves better that the class $\mathcal{B}$, with respect to the number of spanning trees.

The above example illustrates that classes that do not contain graphs of the same size, can be proved to have different behaviour with respect to number of spanning trees by a simple comparison of their indicators. Similarly, comparison of classes through indicators can be used in cases in which two classes of graphs intuitively appear to have a common behaviour with respect to their number of spanning trees, but this fact cannot be established by considering specific elements of the two classes.

For example consider the classes $\mathcal{K}_e$ and $\mathcal{K}_o$ of complete graphs with even and odd number of vertices, respectively. Intuitively, these two classes have the same behaviour with respect to the number of spanning trees, that is one expects that the number of the spanning trees of $K_n - K_m$ depends on m because of its size and not because of its parity. But comparing specific elements of the two classes fails to separate size from parity.

On the other hand, comparison through indicator provides a mathematical formalism that can be used to prove that the above two classes have actually the same behaviour (since they have the same indicator).

5. Conclusions and future work

In this paper we have introduced a characterization of the limit behaviour of graphs with respect to their number of spanning trees. More specifically, given two classes of graphs whose members have the same growing rate, one can use their indicators in order to deduce which one of the classes has (asymptotically) the best behaviour.

There are certain aspects of this work which we believe that deserve further investigation. The questions we have faced are the following:

- Does there exist a structural description of those classes of graphs that are β-stable?
- Certain families of graphs appear to have the same indicator. Does there exist a simple and general criterion that one could use in order to deduce that two classes of graphs have the same indicator without actually computing analytically the indicators themselves?
- Does there exist a more accurate criterion that could be used to compare the limit spanning-tree

behaviour of classes of graphs that have the same spanning-tree indicator?

We believe that answers to such questions can offer a significant insight on the spanning-tree behaviour of graphs.

References

[1] C. Berge, Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.
[2] C. Cheng, Maximizing the total number of spanning trees in a graph; two related problems in graph theory and optimal design theory, J. Combin. Theory Ser. B (1981) 240–248.
[3] K.-L. Chung, W.-M. Yan, On the number of spanning trees of a multi-complete/star related graph, Inform. Process. Lett. 76 (2000) 113–119.
[4] C.J. Colbourn, The Combinatorics of Network Reliability, Oxford University Press, New York, 1987.
[5] B. Gilbert, W. Myrvold, Maximizing spanning trees in almost complete graphs, Networks 30 (1997) 23–30.
[6] W. Moon, Enumerating labelled trees, in: F. Harary (Ed.), Graph Theory and Theoretical Physics, Academic Press, London, 1967, pp. 261–271.
[7] W. Myrvold, K.H. Cheung, L.B. Page, J.E. Perry, Uniformly-most reliable networks do not always exist, Networks (1991) 417–419.
[8] S.D. Nikolopoulos, P. Rondogiannis, On the number of spanning trees of multi-star related graphs, Inform. Process. Lett. 65 (1998) 183–188.
[9] S.D. Nikolopoulos, C. Papadopoulos, The number of spanning trees in K_n-complements of quasi-threshold graphs, Graphs Combin., in press.
[10] P.V. O'Neil, The number of trees in a certain network, Notices Amer. Math. Soc. 10 (1963) 569.
[11] L. Weinberg, Number of trees in a graph, Proc. IRE 46 (1958) 1954–1955.
[12] W.-M. Yan, W. Myrvold, K.-L. Chung, A formula for the number of spanning trees of a multi-star related graph, Inform. Process. Lett. 68 (1998) 295–298.