Recognizing HHDS-Free Graphs

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Abstract. In this paper, we consider the recognition problem on the HHDS-free graphs, a class of homogeneously orderable graphs, and we show that it has polynomial time complexity. In particular, we describe a simple $O(n^2m)$-time algorithm which determines whether a graph G on n vertices and m edges is HHDS-free. To the best of our knowledge, this is the first polynomial-time algorithm for recognizing this class of graphs.

Keywords: HHD-free graphs, HHDS-free graphs, sun, homogeneously orderable graphs, perfectly orderable graphs, recognition.

1 Introduction

In the late 1990s, Brandstädt, Dragan, and Nicolai [2] defined the *homogeneously orderable graphs* as those graphs admitting a homogeneous elimination order (a vertex ordering $v_1, v_2, \ldots, v_n$ is a *homogeneous elimination ordering* if for every i, v_i is h-extremal in the subgraph induced by $v_i, v_{i+1}, \ldots, v_n$; a vertex v is h-extremal in a graph G if the set $D_2(v)$ of vertices at distance at most 2 from v in G contains a proper homogeneous dominating set, i.e., there exists a set $H \subset D_2(v)$ such that H is a homogeneous set in G and $D_2(v) \subseteq N[H]$). They showed that the class of homogeneously orderable graphs contains the class of *homogeneous graphs* introduced by D'Atri, Moscarini, and Sassano [7]. The larger class of homogeneously orderable graphs seems to be more interesting for several reasons; among these are algorithmic reasons, e.g., the (cardinality) Steiner tree problem is solvable in polynomial time on homogeneously orderable graphs [7].

In this paper, we consider a subclass of homogeneously orderable graphs, namely, the HHDS-free graphs. A graph is *HHDS-free* if it contains no induced hole (i.e., a chordless cycle on ≥ 5 vertices), house, domino (see Figure 1), or sun. In [2], Brandstädt, Dragan, and Nicolai proved that a graph G is HHDS-free if and only if G is hereditary homogeneously orderable, i.e., every induced subgraph of G is homogeneously orderable.

The definition of the class of homogeneously orderable graphs implies that this class is a generalization of both the class of dually chordal and the class of distance-hereditary graphs [2,3]. Bandelt and Mulder [1] showed that a graph G is distance-hereditary if and only if it contains no induced house, hole, domino, or gem; then, since every sun contains a gem [2,3], distance-hereditary graphs

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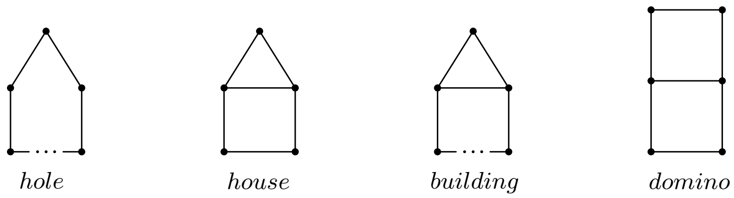

Fig. 1. Some useful graphs

are HHDS-free. Additionally, the HHD-free graphs properly generalize the class of chordal (or triangulated) graphs [9]; a graph is {house,hole,domino}-free or HHD-free if it contains no induced house, hole, or domino. In [11], Hoàng and Khouzam proved that the HHD-free graphs admit a *perfect order*, and thus are *perfectly orderable* [4,13,16]; as a result, the HHDS-free graphs are perfectly orderable as well. A superclass of the HHD-free graphs, which also properly generalizes the class of chordal graphs, is the class of {house,hole}-free or HH-free graphs; Chvátal conjectured [5] and later Hayward [10] proved that the complement $\overline{G}$ of an HH-free graph G is perfectly orderable.

In [3], it is mentioned that the recognition complexity of HHDS-free graphs is open. Yet, several recognition algorithms have been proposed for graph classes that are defined or characterized by forbidden induced holes, houses, or dominos (see [3,9]). Indeed, Hoàng and Khouzam [11], while studying the class of brittle graphs (a well known class of perfectly orderable graphs which contains the HHD-free graphs), showed that the HHD-free graphs can be recognized in $O(n^4)$ time, where n denotes the number of vertices of the input graph. An improved result was obtained by Hoàng and Sritharan [12] who presented an $O(n^3)$-time algorithm for recognizing HH-free graphs and showed that HHD-free graphs can be recognized in $O(n^3)$ time as well; one of the key ingredients in their algorithms is the reduction of a subproblem to the recognition of chordal graphs. Based on the result in [12], recently, Nikolopoulos and Palios [14] presented an $O(\min\{nm\,\alpha(n),\, nm + n^2 \log n\})$-time and $O(n + m)$-space algorithm for recognizing HHD-free graphs, where m is the number of edges of the input graph and $\alpha(n)$ is the very slowly growing inverse of the Ackerman's function.

The main result of this paper is that an HHD-free graph G is also HHDS-free if and only if there is no vertex v of G such that v is the top of a house or a "building" in an auxiliary graph which is a modification of G; a building, which is a generalization of a house, is a cycle on at least 5 vertices with a single chord (i.e., an edge joining two nonconsecutive vertices of the cycle) connecting two vertices of the cycle which are at distance 2 (see Figure 1). This result enables us to describe an $O(n^2 m)$-time algorithm for recognizing whether an input graph on n vertices and m edges is HHDS-free. The space required by the algorithm is $O(n^2)$.

2 Theoretical Framework

We consider finite undirected graphs with no loops or multiple edges. Let G be such a graph; then, $V(G)$ and $E(G)$ denote the set of vertices and of edges of G, respectively. Let $S \subseteq V(G)$ be a set of vertices of G; the subgraph of G induced by S is denoted by $G[S]$. The *neighborhood* $N(x)$ of a vertex $x \in V(G)$ is the set of all the vertices of G that are adjacent to x. We use $M(x)$ to denote the set $V(G) - \big(N(x) \cup \{x\}\big)$ of non-neighbors of x in G. An *independent* (or *stable*) *set* is a set of vertices no two of which are adjacent.

A path $v_0 v_1 \ldots v_k$ of a graph G is called *simple* if none of its vertices occurs more than once; it is called a *cycle* (*simple cycle*) if $v_0 v_k \in E(G)$. A simple path (cycle) is *chordless* if $v_i v_j \notin E(G)$ for any two non-consecutive vertices v_i, v_j in the path (cycle). A chordless path (chordless cycle, respectively) on n vertices is commonly denoted by P_n (C_n, respectively).

A graph is *chordal* (or *triangulated*) if and only if every cycle of length strictly greater than 3 possesses a chord (i.e., an edge joining two nonconsecutive vertices of the cycle) [3,9,17]. The following definition is taken from [3].

Definition 1. [6,8] *A* sun (*or* trampoline) *is a chordal graph G on $2n$ vertices for some $n \geq 3$ whose vertex set can be partitioned into two sets, $U = \{u_0, u_1, \ldots, u_{n-1}\}$ and $W = \{w_0, w_1, \ldots, w_{n-1}\}$, such that W is an independent set and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j+1 \mod n$.*

A sun on $2k$ vertices is often called a *k-sun*. A sun such that the set U induces a complete graph is called a *complete sun*. It has been shown that every sun contains a complete sun [6,8]; yet, determining whether a graph contains a complete sun does not seem easier than determining whether it contains a sun. We prove the following lemma.

Lemma 1. *Let H be a graph whose vertices can be partitioned into two sets $U = \{u_0, u_1, \ldots, u_{k-1}\}$ and $W = \{w_0, w_1, \ldots, w_{k-1}\}$ of $k \geq 3$ vertices each, such that W is an independent set and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j+1 \mod k$. Then, H is a sun with partition sets U and W if and only if the subgraph $H[U]$ is chordal and the vertices $u_0, u_1, \ldots, u_{k-1}$ form a cycle $u_0 u_1 \cdots u_{k-1}$.*

Proof. ($\Longrightarrow$) Since H is a sun, then H is chordal and thus the subgraph $H[U]$ is chordal as well. Moreover, for all $i = 0, 1, \ldots, k-1$, the vertices u_i and $u_{i+1 \bmod k}$ are adjacent in H since a chordless path from $u_{i+1 \bmod k}$ to u_i in the (connected) graph induced by $\{u_{i+1 \bmod k}, w_{i+1 \bmod k}, \ldots, u_{i-1}, w_{i-1},\ u_i\}$ in H has to be of length 1; otherwise, the vertices of the path along with vertex w_i would induce a chordless cycle on 4 or more vertices, a contradiction to the chordality of H.
($\Longleftarrow$) Since $H[U]$ is chordal, the lemma follows easily from the fact that no w_i ($0 \leq i < k$) participates in a chordless cycle on 4 or more vertices since w_i's only neighbors, u_i and $u_{i+1 \bmod k}$, are adjacent in H. ∎

Let G be a graph and let v be an arbitrary vertex of G. Let us define the following set of non-edges of G

$$E_v \;=\; \{\, xz \mid x, z \in M(v) \text{ and } \exists y \in M(v) \text{ such that } xyz \text{ is a } P_3 \text{ of } G \,\}$$

which we call P_3*-edges*. Then, we construct the graph $\widehat{G}_v$ from G as follows:

$$V(\widehat{G}_v) = V(G) \qquad \text{and} \qquad E(\widehat{G}_v) = E(G) \cup E_v.$$

Note that the definition of P_3-edges implies that $E(G) \cap E_v = \emptyset$. If the graph G has n vertices and m edges, then the graph $\widehat{G}_v$ has n vertices and $O(n^2)$ edges.

Definition 2.

- ▷ *We collectively call a house or a building a* generalized house *or* g-house *for short.*
- ▷ *If vertex v is the top of a house or a building, then v is the* top *of the g-house. If v at the top is adjacent to vertices u, w in the g-house, we say that the* roof *of the g-house is $(v; u, w)$. The vertices of the g-house that do not belong to its roof form a chordless path which we call the* base *of the g-house.*
- ▷ *A g-house is* shorter *than another g-house if it involves fewer vertices.*

Our HHDS-free graph recognition algorithm relies on the following theorem.

Theorem 1. *Let G be an HHD-free graph. The graph G contains a sun if and only if there exists a vertex v such that the graph $\widehat{G}_v$ defined above with respect to v contains a house or a building with v at its top.*

Proof. ($\Longrightarrow$) Suppose that the graph G contains a sun induced by the sets of vertices $U = \{u_0, u_1, \ldots, u_{k-1}\}$ and $W = \{w_0, w_1, \ldots, w_{k-1}\}$, where $k \geq 3$ (see Definition 1). Then, in the graph $\widehat{G}_{w_0}$, the vertices $w_0, u_0, u_1, w_1, w_2, \ldots, w_{k-1}$ induce a house or a building with vertex w_0 at its top (see Figure 2 for an example where $k = 5$; dashed edges indicate P_3-edges); note that $u_0u_1 \in E(G)$ (see Lemma 1), that the vertices u_0 and u_1 are not adjacent to any of the vertices $w_1, w_2, \ldots, w_{k-2}$ and $w_2, w_3, \ldots, w_{k-1}$, respectively, and that, for all $i = 1, 2, \ldots, k-2$, the vertices w_i and w_{i+1} induce a P_3-edge.
($\Longleftarrow$) Suppose that there exists a vertex v which is the top of a house or a building in $\widehat{G}_v$, i.e., v is the top of a g-house. Then, the following holds:

> **Fact 1.** If the vertex v is the top of a g-house in the graph $\widehat{G}_v$, with roof $(v; u, w)$, then every edge in the base of a *shortest* g-house with roof $(v; u, w)$ is a P_3-edge.

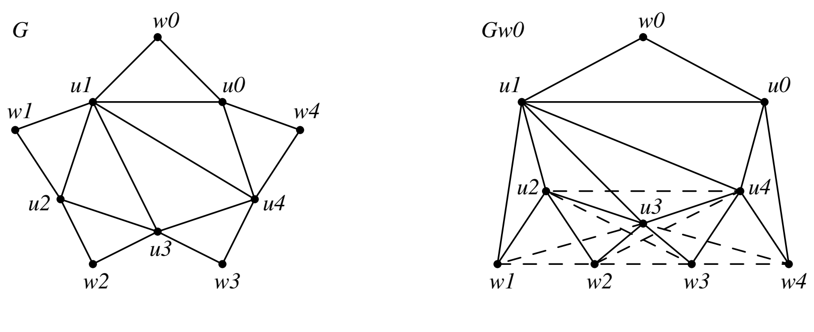

Fig. 2

Fact 1 is established in Lemma 2. Thus, if a shortest g-house with roof $(v; u, w)$ has base $p_1 p_2 \cdots p_k$, then each $p_i p_{i+1}$ $(1 \le i \le k-1)$ is a P_3-edge; let us replace each such edge with a corresponding P_3 $p_i q_i p_{i+1}$ in G. Then, from the fact that we are considering a shortest g-house, we conclude that for $i = 1, 2, \ldots, k-1$, the vertex q_i is not adjacent to any of the vertices in $\{p_1, p_2, \ldots, p_{i-1}, p_{i+2}, \ldots, p_k\}$ (as in the proof of Lemma 2), which implies that the q_is are all distinct (note that the q_is may be arbitrarily adjacent to one other); the situation is depicted in Figure 3 where dashed lines indicate potential edges.

Additionally, vertex u is adjacent to at least one of the vertices $q_1, q_2, \ldots, q_{k-1}$. If u were not adjacent to any of them, then if x is the leftmost neighbor of w among $q_1, q_2, \ldots, q_{k-1}, p_k$ and if ρ is a chordless path from p_1 to x in the (connected) graph induced by the vertices $\{p_1, q_1, p_2, q_2, \ldots, x\}$ in G, the vertices v, u, w, and the vertices of the path ρ induce a house or a building in G (with v at its top), which contradicts the fact that the graph G is HHD-free. Thus, u is adjacent to at least one q_i. In fact, we can show the following:

> **Fact 2.** There exists an integer r, where $1 \le r \le k-1$, such that the vertex u is adjacent to precisely $q_1, q_2, \ldots, q_r$ among the q_is, otherwise the graph G contains a sun.

Fact 2 is established in Lemma 6 (case (b)) with the aid of Lemma 4: since u is adjacent to both p_1 and a vertex q_i, then Lemma 4 implies that it is also adjacent to q_1; then, for $r = \max\{\, j \mid uq_j \in E(G) \,\}$, Lemma 6 (case (b)) implies that if there exists a vertex q_i $(2 \le i \le r-1)$ which is not adjacent to u, then the graph G contains a sun, as desired.

So, let us consider the case where the vertex u is adjacent to each of the vertices $q_1, q_2, \ldots, q_r$, where $1 \le r \le k-1$. Similarly, we assume that there exists an integer ℓ, where $1 \le \ell \le k-1$, such that the vertex w is adjacent to each of the vertices $q_\ell, q_{\ell+1}, \ldots, q_{k-1}$. Then, it has to be that $r \ge \ell$; if $r < \ell$, then the vertices v, u, w, and the vertices of a chordless path from q_r to q_ℓ in the (connected) graph induced by $\{q_r, p_{r+1}, q_{r+1}, \ldots, p_\ell, q_\ell\}$ induce a house or a building in G, a contradiction. In fact, $r = k-1$ and $\ell = 1$, i.e., the vertices u, w are adjacent to each of the vertices $q_1, q_2, \ldots, q_{k-1}$. Suppose for contradiction that $r \le k-2$ which implies that $k \ge 3$ since $r \ge 1$; then, because $r \ge \ell$, the vertex w is

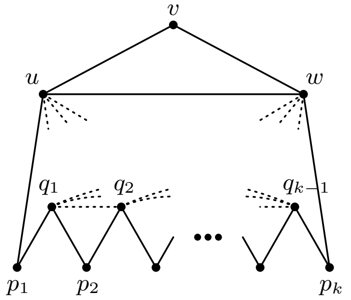

Fig. 3

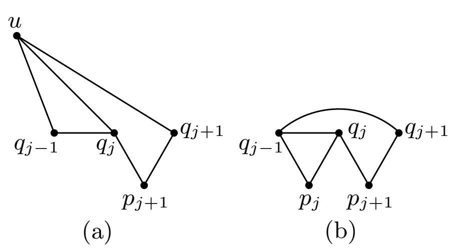

Fig. 4

adjacent to both q_{k-2} and q_{k-1}. Moreover, $q_{k-2}q_{k-1} \in E(G)$ (for otherwise the vertices $w, q_{k-2}, p_{k-1}, q_{k-1}, p_k$ would induce a house in G with vertex p_k at its top, a contradiction); then, the vertices $p_{k-2}, q_{k-2}, q_{k-1} \in M(v)$ induce a P_3 in G, that is, $p_{k-2}q_{k-1}$ would be a P_3-edge in $\widehat{G}_v$, which implies that the vertices $v, u, p_1, p_2, \ldots, p_{k-2}, q_{k-1}, w$ induce a g-house in $\widehat{G}_v$ with roof (v, u, w); note that q_{k-1} is not adjacent to $p_1, p_2, \ldots, p_{k-3}$ nor to u. This, however, contradicts the minimality of the g-house induced by $v, u, p_1, p_2, \ldots, p_k, w$. Thus, the assumption that $r \leq k-2$ led us to a contradiction. Hence, $r = k-1$ (i.e., vertex u is adjacent to each of the vertices $q_1, q_2, \ldots, q_{k-1}$); similarly, vertex w is adjacent to each of these vertices as well.

If there exists a vertex q_i that is adjacent to a vertex q_j but is not adjacent to a vertex $q_{j'}$, where $1 \leq i < j' < j \leq k-1$, then clearly $k \geq 4$ and Lemma 6 along with Lemma 4 imply that the graph G contains a sun: since q_i is adjacent to both p_{i+1} and q_j, then Lemma 4 implies that it is also adjacent to q_{i+1} (note that the graph G is HHD-free and contains the path $p_{i+1}q_{i+1}p_{i+2}q_{i+2} \cdots p_jq_j$, with chords only between q_is, and the vertex q_i is not adjacent to any of $p_{i+2}, p_{i+3}, \ldots, p_j$); then, Lemma 6 (case (b)) implies that since vertex q_i is not adjacent to vertex $q_{j'}$, where $i+2 \leq j' \leq j-1$, the graph G contains a sun.

Suppose now that no vertex q_i as in the previous paragraph exists; that is, for all $i = 1, 2, \ldots, k-2$, if q_i is adjacent to a vertex q_j, where $1 \leq i < j \leq k-1$, then q_i is adjacent to each of $q_{i+1}, q_{i+2}, \ldots, q_j$. Then Lemma 5 implies that the subgraph of G induced by the vertices $w, u, q_1, q_2, \ldots, q_{k-1}$ is chordal; recall that $uw \in E(G)$ and both u and w are adjacent to each of the vertices $q_1, q_2, \ldots, q_{k-1}$. Additionally, we take advantage of the fact that u is adjacent to each of the vertices $q_1, q_2, \ldots, q_{k-1}$ in order to show by induction on i that $q_iq_{i+1} \in E(G)$ for all $i = 1, 2, \ldots, k-2$. For the basis step, we observe that if $q_1q_2 \notin E(G)$ then the vertices u, p_1, q_1, p_2, q_2 induce a house in G (with vertex p_1 at its top), a contradiction. For the inductive step, we assume that $q_{j-1}q_j \in E(G)$ where $j \geq 2$, and suppose for contradiction that $q_jq_{j+1} \notin E(G)$; if $q_{j-1}q_{j+1} \notin E(G)$, then the vertices $u, q_{j-1}, q_j, p_{j+1}, q_{j+1}$ induce a house in G with vertex q_{j-1} at its top (Figure 4(a)), which leads to a contradiction, whereas if $q_{j-1}q_{j+1} \in E(G)$, then the vertices $q_{j-1}, p_j, q_j, p_{j+1}, q_{j+1}$ induce a house in G with vertex p_j at its top (Figure 4(b)), a contradiction again. Therefore, $q_jq_{j+1} \in E(G)$, and from the induction, $q_iq_{i+1} \in E(G)$ for all $i = 1, 2, \ldots, k-2$. This result, the chordality of the subgraph $G[\{w, u, q_1, q_2, \ldots, q_{k-1}\}]$, the fact that $uw \in E(G)$, $uq_1 \in E(G)$, and $wq_{k-1} \in E(G)$, and Lemma 1 imply that the subgraph of G induced by the vertices $v, u, p_1, q_1, p_2, q_2, \ldots, p_{k-1}, q_{k-1}, p_k, w$ is a sun with partition sets $U = \{u, q_1, q_2, \ldots, q_{k-1}, w\}$ and $W = \{v, p_1, p_2, \ldots, p_k\}$. ∎

Lemma 2. *Let G be an HHD-free graph, v a vertex of G, and $\widehat{G}_v$ be the auxiliary graph defined above with respect to v. If the vertex v is the top of a g-house in the graph $\widehat{G}_v$ and if u and w are the neighbors of v in the g-house, then every edge in the base of a shortest g-house with roof $(v; u, w)$ is a P_3-edge.*

Proof. Let a shortest g-house with roof $(v; u, w)$ have base $p_1p_2 \cdots p_k$, where $k \geq 2$ (Figure 5(a)). Since G does not contain a house or a hole, the path $p_1 \cdots p_k$

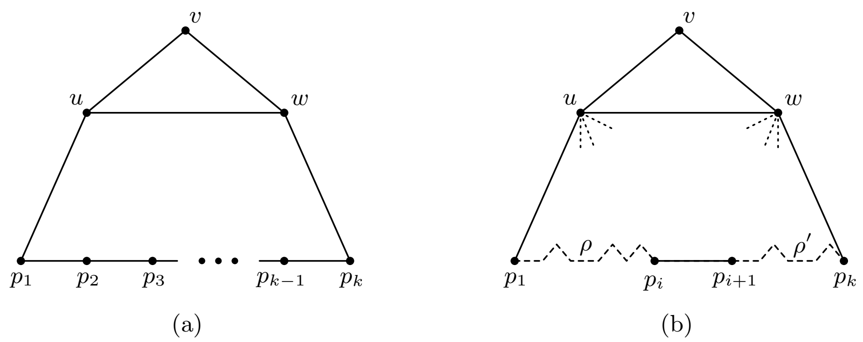

Fig. 5

contains P_3-edges; let us replace each P_3-edge $p_i p_{i+1}$ $(1 \leq i < k)$ by a corresponding P_3 $p_i q_i p_{i+1}$ of G. Then, each such vertex q_i is not adjacent to any vertex in $\{p_1, \ldots, p_{i-1}, p_{i+2}, \ldots, p_k\}$: if q_i were adjacent to p_j, for some $j \in \{1, 2, \ldots, i-1\}$ then the vertices p_j, q_i, p_{i+1} would induce a P_3 in G, and thus $p_j p_{i+1}$ would be a P_3-edge, which would imply that the vertices $v, u, p_1, \ldots, p_j, p_{i+1}, \ldots, p_k, w$ would induce a g-house with roof $(v; u, w)$ in $\widehat{G}_v$, in contradiction to the minimality of the g-house induced by $v, u, p_1, p_2, \ldots, p_k, w$; a similar argument leads to a contradiction if q_i were adjacent to p_j, for some $j \in \{i+2, i+3, \ldots, k\}$. The fact that q_i is not adjacent to any vertex in $\{p_1, \ldots, p_{i-1}, p_{i+2}, \ldots, p_k\}$ also implies that the vertices q_i are all different.

We will show next that every edge $p_i p_{i+1}$ is a P_3-edge. Suppose for contradiction that $p_i p_{i+1}$ is not a P_3-edge; hence, it is an edge of G instead. Consider a chordless path ρ in G from p_1 to p_i in the (connected) graph induced by $\{p_1, q_1, p_2, \ldots, q_{i-1}, p_i\}$ and a chordless path ρ' from p_{i+1} to p_k in the (connected) graph induced by $\{p_{i+1}, q_{i+1}, p_{i+2}, \ldots, q_{k-1}, p_k\}$. We show that the concatenation of the path ρ, the edge $p_i p_{i+1}$, and the path ρ' forms a chordless path in G (see Figure 5(b)). If there were a chord, this would have been an edge $q_\ell q_r$, where $\ell < i$ and $r \geq i+1$. Let us consider an edge $q_\ell q_r$ that minimizes the difference $r - \ell$; then, the vertices of the path ρ from q_ℓ to p_i, and the vertices of the path ρ' from p_{i+1} to q_r induce a cycle in G. In fact, they induce a chordless cycle due to the minimality of $q_\ell q_r$, the chordlessness of ρ and ρ', and the fact that p_i sees none of the vertices of ρ' except for p_{i+1}, and that p_{i+1} sees none of the vertices of ρ except for p_i. Additionally, because G contains no hole, it must be the case that $\ell = i-1$ and $r = i+1$, i.e., the vertices $q_\ell, p_i, p_{i+1}, q_r$ form a C_4. Then, the vertices q_ℓ, q_r, p_{r+1} induce a P_3 in G and thus the edge $q_\ell p_{r+1}$ is a P_3-edge in $\widehat{G}_v$. If neither u nor w see q_ℓ then the vertices $v, u, p_1, p_2, \ldots, p_\ell, q_\ell, p_{r+1}, p_{r+2}, \ldots, p_k, w$ would form a g-house in $\widehat{G}_v$ with roof $(v; u, w)$ which is shorter than the g-house induced by $v, u, p_1, \ldots, p_k, w$, in contradiction to the minimality of the latter g-house; hence, at least one of u, w sees q_ℓ, and similarly at least one of u, w sees q_r. On the other hand, neither u nor w see both q_ℓ and q_r, since G does not contain a house. Therefore, either u sees q_ℓ and w sees q_r or u sees q_r and w sees q_ℓ; in either case, the vertices

v, u, q_ℓ, q_r, w induce a house (recall that $uw \in E(G)$); a contradiction. Thus, no chord exists, and the concatenation of the path ρ, the edge $p_i p_{i+1}$, and the path ρ' forms a chordless path π in G (Figure 5(b)).

The vertex u is not adjacent to any vertex in the path ρ'. If it were, let t' be the leftmost such vertex; clearly, $t' \neq p_{i+1}$. Moreover, let t be the rightmost vertex of ρ which is adjacent to u; t is well defined since $up_1 \in E(G)$ and $t \neq p_i$. But then, the vertex u and the vertices in the part of the path π from t to t' induce a hole in G, which leads to a contradiction; thus, u is not adjacent to any vertex in ρ'. Similarly, w is not adjacent to any vertex in ρ. But then G contains a hole: it is induced by the vertices u, w, and the vertices of the path π from the rightmost neighbor of u in ρ (which is to the left of p_i) to the leftmost neighbor of w in ρ' (which is to the right of p_{i+1}). This however contradicts the fact that G is HHD-free, and therefore we conclude that the base of the g-house induced by $u, v, w, p_1, p_2, \ldots, p_k$ consists entirely of P_3-edges. ∎

Lemma 3. *Let G be a graph which contains a C_4 $abcd$ and a path ρ from c to d (different from the path cd) whose vertices other than its endpoints c and d are adjacent neither to a nor to b. Then, the graph G contains a hole, a house, or a domino.*

Lemma 4. *Let G be an HHD-free graph that contains a path $p_s q_s p_{s+1} q_{s+1} \cdots p_t q_t$, where $t \geq s+1$, with chords only between q_is, and let x be a vertex of G that is adjacent to p_s and is not adjacent to any of $p_{s+1}, p_{s+2}, \ldots, p_t$. If the vertex x is adjacent to q_t, then it is also adjacent to q_s.*

Proof. Suppose for contradiction that $xq_s \notin E(G)$. Let $t' = \min\{\, i \mid s+1 \leq i \leq t \text{ and } xq_i \in E(G)\,\}$; the vertex $q_{t'}$ is well defined since x is adjacent to q_t. Then, $q_s q_{t'} \in E(G)$, otherwise the length of a chordless path from q_s to $q_{t'}$ in the (connected) graph induced by $\{q_s, p_{s+1}, q_{s+1}, \ldots, p_{t'}, q_{t'}\}$ in G would be of length at least 2 and the vertices of the path along with x and p_s would induce a hole in G, a contradiction. But then, the vertices $x, p_s, q_s, q_{t'}$ induce a C_4 in G and G contains the path $q_s p_{s+1} q_{s+1} \cdots p_{t'} q_{t'}$ whose vertices other than its endpoints are adjacent neither to x nor to p_s. Thus, Lemma 3 applies, implying that the graph G contains a hole, a house, or a domino, in contradiction to the fact that G is HHD-free. Therefore, the vertex x is adjacent to q_s. ∎

Lemma 5. *Let H be a graph that does not contain holes, and $v_1, v_2, \ldots, v_k$ ($k \geq 3$) be an ordering of a subset of vertices of H such that, for all $i = 1, 2, \ldots, k-1$, if v_i is adjacent to v_j, where $i < j \leq k$, then v_i is adjacent to each of the vertices $v_{i+1}, v_{i+2}, \ldots, v_j$. Then, the subgraph of H induced by the vertices $v_1, v_2, \ldots, v_k$ is chordal.*

Proof. Since the graph H does not contain holes, we only need to show that the subgraph induced by the vertices $v_1, v_2, \ldots, v_k$ does not contain a C_4. Suppose for contradiction that it contained a C_4, say, $v_a v_b v_c v_d$, and suppose without loss of generality that $a = \min\{a, b, c, d\}$. Then, we distinguish the following cases:

(i) $b = \max\{a, b, c, d\}$: then, v_a is adjacent to v_b but is not adjacent to v_c and yet $c < b$ (see Figure 6(a)), a contradiction;

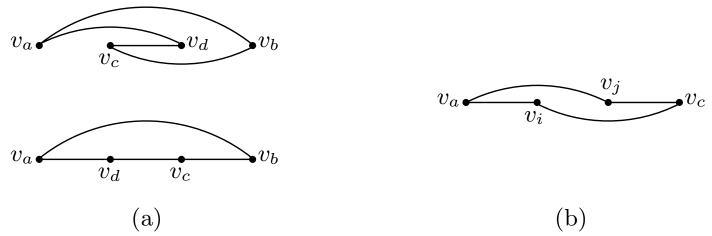

Fig. 6. Different cases for the C_4 $v_a v_b v_c v_d$

(ii) $c = \max\{a, b, c, d\}$: then, if $i = \min\{b, d\}$ and $j = \max\{b, d\}$, v_i is adjacent to v_c but is not adjacent to v_j and yet $i < j < c$ (see Figure 6(b)), a contradiction;

(iii) $d = \max\{a, b, c, d\}$: the case is similar to case (i) and leads to a contradiction.

In all cases, we reached a contradiction, which implies that the subgraph of H induced by the vertices $v_1, v_2, \ldots, v_k$ is chordal. ∎

Lemma 6. *Let G be an HHD-free graph that contains a path $q_s p_{s+1} q_{s+1} \cdots p_t q_t$, where $t \geq s + 2$, with chords only between q_is, and let x be a vertex of G that is adjacent to q_s and q_t, and is not adjacent to any of $p_{s+1}, p_{s+2}, \ldots, p_t$.*

(a) Suppose that the vertex x is not adjacent to the vertices $q_{s+1}, q_{s+2}, \ldots, q_{t-1}$, and that for $i = s, s+1, \ldots, t-1$, if the vertex q_i is adjacent to q_j (where $i < j \leq t$) then it is adjacent to each of the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$. Then, the vertices $x, q_s, p_{s+1}, q_{s+1}, \ldots, p_t, q_t$ induce a sun in G.

(b) If there exists a vertex q_i ($s + 1 \leq i \leq t - 1$) that is not adjacent to x, then the graph G contains a sun.

Proof. (a) First, the set $\{q_s, q_{s+1}, \ldots, q_t\}$ contains at least 3 vertices. Next, due to the property of the q_is, Lemma 5 implies that the subgraph of G induced by the vertices $q_s, q_{s+1}, \ldots, q_t$ is chordal. In light of Lemma 1 and of the fact that the vertex x is adjacent to q_s and q_t only, and each vertex p_i ($s + 1 \leq i \leq t$) is adjacent to q_{i-1} and q_i only, we need only prove that the vertices $q_s, q_{s+1}, \ldots, q_t$ induce a cycle $q_s q_{s+1} \cdots q_t$ in G.

We begin by showing that the vertex q_s is adjacent to at least one vertex in $\{q_s, q_{s+1}, \ldots, q_t\}$; if it were not, then the vertices x, q_s, p_{s+1}, and the vertices of a chordless path from q_{s+1} to q_t in the (connected) graph induced by $\{q_{s+1}, p_{s+2}, q_{s+2}, \ldots, p_t, q_t\}$ would induce a hole in G, a contradiction. If q_ℓ is that vertex, i.e., $q_s q_\ell \in E(G)$, then $q_s q_t \in E(G)$: this is trivially true if $q_\ell = q_t$; if $q_\ell \neq q_t$, then because the graph G contains the path $x q_t p_t q_{t-1} \cdots p_{\ell+1} q_\ell$, where $\ell \leq t - 1$, with chords only between q_is, and the vertex q_s is adjacent to x and q_ℓ but is not adjacent to any of $p_t, p_{t-1}, \ldots, p_{\ell+1}$, Lemma 4 applies, implying that q_s is adjacent to q_t in G. From this fact and from the property of the vertices q_i ($s \leq i < t$) that if q_i is adjacent to q_j, where $i < j \leq t$, then q_i is adjacent to each of the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$, we conclude that q_s is adjacent to each

of the vertices $q_{s+1}, q_{s+2}, \ldots, q_t$; this in turn enables us to additionally show (by induction on i) that $q_i q_{i+1} \in E(G)$ for all $i = s+1, s+2, \ldots, t-1$. For the basis step, we note that if $q_{s+1} q_{s+2} \notin E(G)$, then the vertices $q_s, p_{s+1}, q_{s+1}, p_{s+2}, q_{s+2}$ induce a house in G with vertex p_{s+1} at its top, a contradiction. For the inductive step, assume that $q_{j-1} q_j \in E(G)$ where $j \geq s+2$. We show that $q_j q_{j+1} \in E(G)$; if not, then the vertices $q_s, q_{j-1}, q_j, p_{j+1}, q_{j+1}$ induce a house in G with vertex q_{j-1} at its top, a contradiction. Our inductive proof is complete implying that $q_i q_{i+1} \in E(G)$ for all $i = s+1, s+2, \ldots, t-1$; then, because $q_s q_{s+1} \in E(G)$ and $q_s q_t \in E(G)$, we have that the vertices $q_s, q_{s+1}, \ldots, q_t$ indeed induce a cycle $q_s q_{s+1} \cdots q_t$ in G.
(b) Since the vertex x is adjacent to q_s and q_t, and is not adjacent to a vertex in $\{q_{s+1}, q_{s+1}, \ldots, q_{t-1}\}$, we can find vertices q_ℓ, q_r, where $s \leq \ell < r \leq t$, such that x is adjacent to q_ℓ and q_r but is not adjacent to any of $q_{\ell+1}, q_{\ell+2}, \ldots, q_{r-1}$. Then, if for each vertex q_i ($\ell \leq i \leq r-1$), the fact that q_i is adjacent to a vertex q_j, where $i < j \leq r$, implies that q_i is adjacent to each of the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$, Lemma 6 (case (a)) applies, implying that the vertices $x, q_\ell, p_{\ell+1}, q_{\ell+1}, \ldots, p_r, q_r$ induce a sun in G. Suppose now that there exists a vertex q_i ($\ell \leq i \leq r-1$) that is adjacent to a vertex q_j and is not adjacent to a vertex $q_{j'}$, where $i < j' < j \leq r$. Let us collect all such vertices in a (non-empty) set S.

For each vertex q_i in S (which is adjacent, say, to q_{j_i} where $i+1 < j_i$), Lemma 4 implies that q_i is adjacent to q_{i+1}; note that G is HHD-free and contains the path $p_{i+1} q_{i+1} \cdots p_{j_i} q_{j_i}$, and q_i is adjacent to p_{i+1} and q_{j_i}. Then, for each vertex $q_i \in S$, we can find indices ℓ_i and r_i where $i < \ell_i < r_i \leq r$, such that q_i is adjacent to q_{ℓ_i} and q_{r_i} but is not adjacent to any of the vertices $q_{\ell_i+1}, q_{\ell_i+2}, \ldots, q_{r_i-1}$, and the difference $r_i - \ell_i$ is minimized. Let $q_{\hat{\imath}}$ be a vertex in S such that $r_{\hat{\imath}} - \ell_{\hat{\imath}} = \min_{q_i \in S}\{r_i - \ell_i\}$; the minimality of $q_{\hat{\imath}}$ implies that for $i = \ell_{\hat{\imath}}, \ell_{\hat{\imath}}+1, \ldots, r_{\hat{\imath}}-1$, if the vertex q_i is adjacent to q_j (where $i < j \leq r_{\hat{\imath}}$) then it is adjacent to each of the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$. This, the fact that the graph G contains the path $q_{\ell_{\hat{\imath}}} p_{\ell_{\hat{\imath}}+1} q_{\ell_{\hat{\imath}}+1} \cdots p_{r_{\hat{\imath}}} q_{r_{\hat{\imath}}}$, where $r_{\hat{\imath}} \geq \ell_{\hat{\imath}}+2$, with chords only between q_is, and the fact that vertex $q_{\hat{\imath}}$ is adjacent to $q_{\ell_{\hat{\imath}}}$ and $q_{r_{\hat{\imath}}}$ but is not adjacent to any of $q_{\ell_{\hat{\imath}}+1}, q_{\ell_{\hat{\imath}}+2}, \ldots, q_{r_{\hat{\imath}}-1}$ imply that Lemma 6 (case (a)) applies, and therefore, the vertices $q_{\hat{\imath}}, q_{\ell_{\hat{\imath}}}, p_{\ell_{\hat{\imath}}+1}, q_{\ell_{\hat{\imath}}+1}, \ldots, p_{r_{\hat{\imath}}}, q_{r_{\hat{\imath}}}$ induce a sun in G. ∎

3 The Algorithm

The recognition algorithm takes advantage of Theorem 1. We start by checking whether the input graph G is HHD-free. If it is not, then clearly G is not HHDS-free. Otherwise, for each vertex v of G, we construct the auxiliary graph $\widehat{G}_v$ and check whether v is the top of a house or a building in $\widehat{G}_v$; if this is so for any vertex v, then G is not HHDS-free. We note that in order to check whether v is the top of a house or a building in $\widehat{G}_v$, we can use the algorithms in [12] (Algorithm High) and [14] (Algorithm Not-in-HHB) which for a graph H and a vertex x return true if and only if the vertex x belongs to a hole or is the top of a house or a building in H; Lemma 7 establishes that v does not belong to a hole in $\widehat{G}_v$ if G is HHD-free.

Lemma 7. *Let G be an HHD-free graph, v a vertex of G, and $\widehat{G}_v$ be the auxiliary graph defined in Section 2 with respect to v. Then, the vertex v does not belong to a hole in the graph $\widehat{G}_v$.*

Formally, the recognition algorithm works as follows:

Algorithm Rec-HHDS-free

1. **if** G is not HHD-free
   then return "G is not HHDS-free";
2. **for** each vertex v of G **do**
   2.1 construct the auxiliary graph $\widehat{G}_v$;
   2.2 **if** v is the top of a house or a building in $\widehat{G}_v$
   then return "G is not HHDS-free"; $\{G$ *contains a sun*$\}$
3. **return** "G is HHDS-free".

The correctness of the algorithm follows from Theorem 1.

Time and Space Complexity. Let n and m be the number of vertices and edges of the input graph G. Step 1 can be executed in $O(\min\{nm\alpha(n),\, nm + n^2 \log n\})$ time and $O(n+m)$ space [14]. In Step 2, the construction of the auxiliary graph $\widehat{G}_v$ can be done in $O(nm)$ time and requires $O(n^2)$ space. Then, we check whether vertex v is the top of a house or a building by means of the Algorithm Not-in-HHB [14], which for a graph on N vertices and M edges takes $O(N + \min\{M\alpha(N),\, M + N \log N\})$ time and $O(N+M)$ space; since $\widehat{G}_v$ has n vertices and $O(n^2)$ edges, Substep 2.2 takes $O(n^2)$ time and space. Thus, the entire execution of Step 2 for all the vertices of G takes $O(n^2 m)$ time and $O(n^2)$ space. Step 3 takes constant time and space.

Therefore, we obtain the following theorem.

Theorem 2. *Let G be an undirected graph on n vertices and m edges. Then, there exists an algorithm for determining whether G is an HHDS-free graph in $O(n^2 m)$ time and $O(n^2)$ space.*

4 Concluding Remarks

We have presented a recognition algorithm for the class of HHDS-free graphs running in $O(n^2 m)$ time with $O(n^2)$ space. To the best of our knowledge, it is the first polynomial-time algorithm for recognizing the class of HHDS-free graphs. The proposed recognition algorithm can be augmented to provide a certificate (an induced house, hole, domino, or sun) in linear additional time and space whenever it decides that the input graph is not HHDS-free: for a house, hole, or domino, see [15]; for a sun, we take advantage of the proof of Theorem 1, which is constructive. Finally, the use of P_3-edges enables us to recognize {house, hole, domino, 3-sun}-free graphs in $O(n^2 m)$ time and $O(n)$ space.

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