Θεωρία Γραφημάτων

Θεμελιώσεις-Αλγόριθμοι-Εφαρμογές

Ενότητα 8

ΤΕΛΕΙΑ ΓΡΑΦΗΜΑΤΑ

Εισαγωγή

Βασικοί Αλγόριθμοι Γραφημάτων

Πολυπλοκότητα χώρου και χρόνου: *O και Ω*

Τέλεια Γραφήματα

- Κλάσεις
- Ιδιότητες
- Προβλήματα

Αλγοριθμικές Τεχνικές …

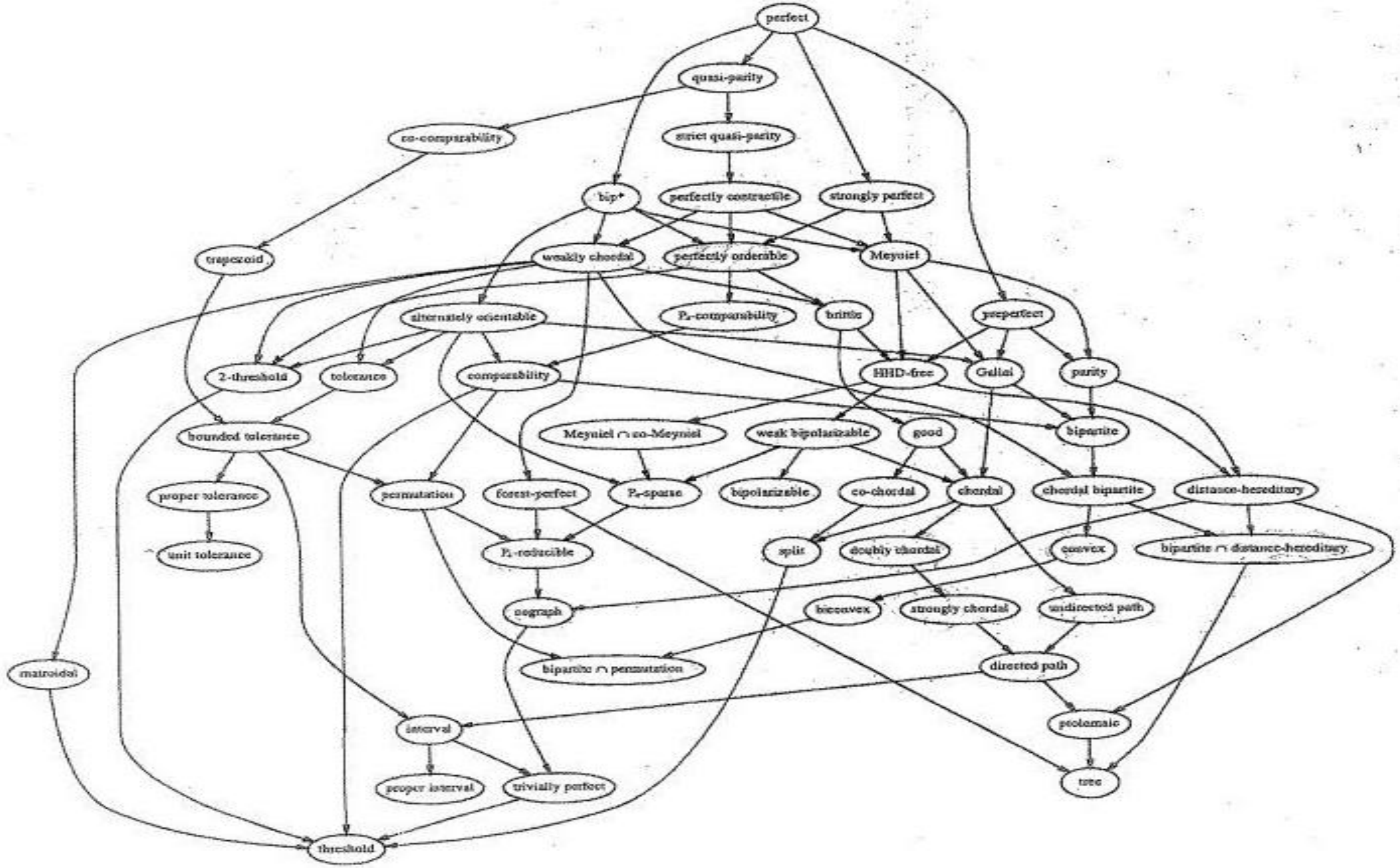
Αλγόριθμοι Προβλημάτων Αναγνώρισης και Βελτιστοποίησης

Αλγόριθμοι Θεωρίας Γραφημάτων

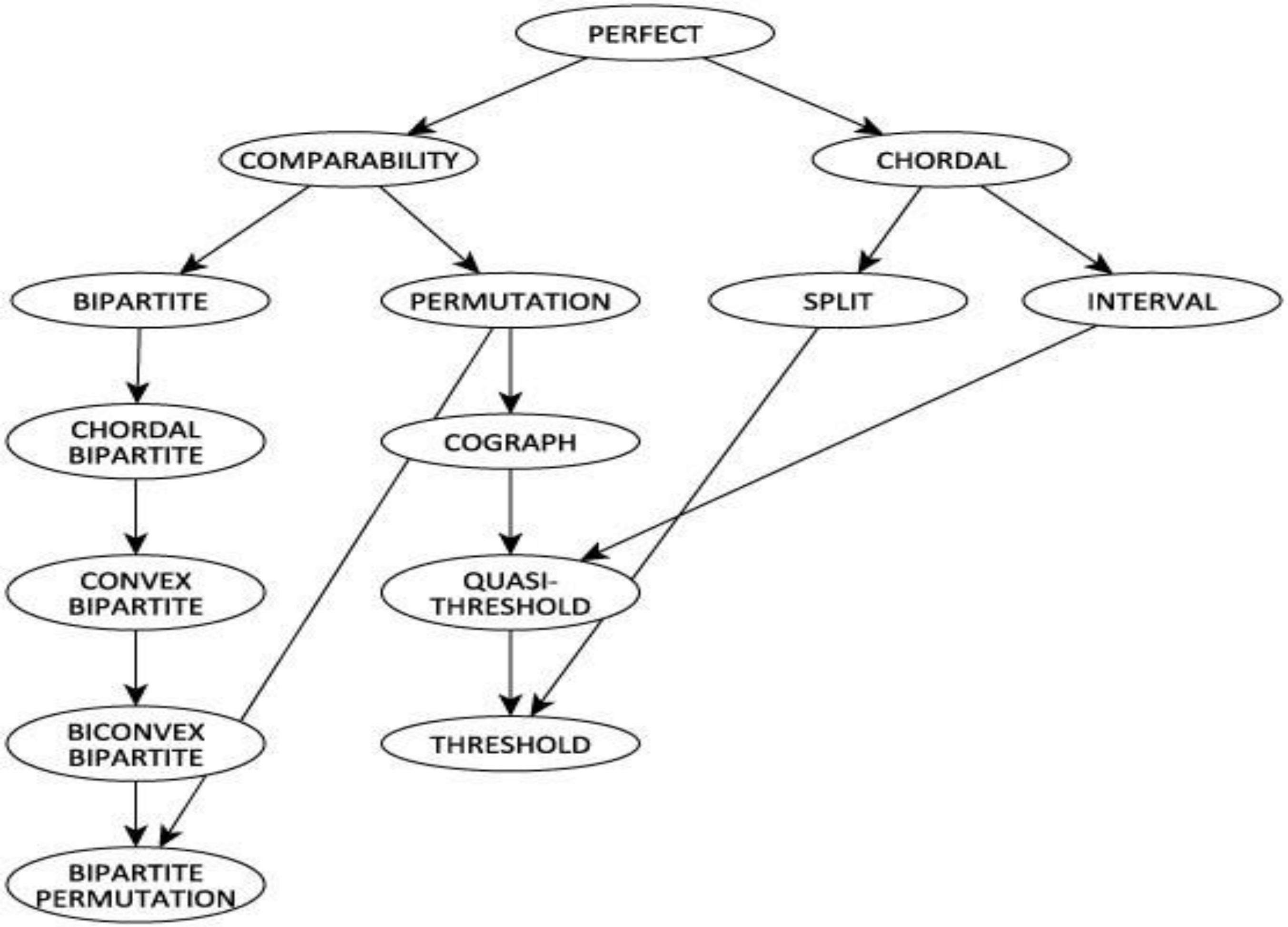
- Πολυωνυμικοί Αλγόριθμοι... (Γραμμικοί)
- Προβλήματα: NP-Πλήρη
- Επιλογές →
  - Προσέγγιση Λύσης
  - Περιορισμοί Ιδιοτήτων

Τέλεια Γραφήματα, ...

Κλάσεις Τέλειων Γραφημάτων



Κλάσεις Τέλειων Γραφημάτων



Γραφήματα Τομής

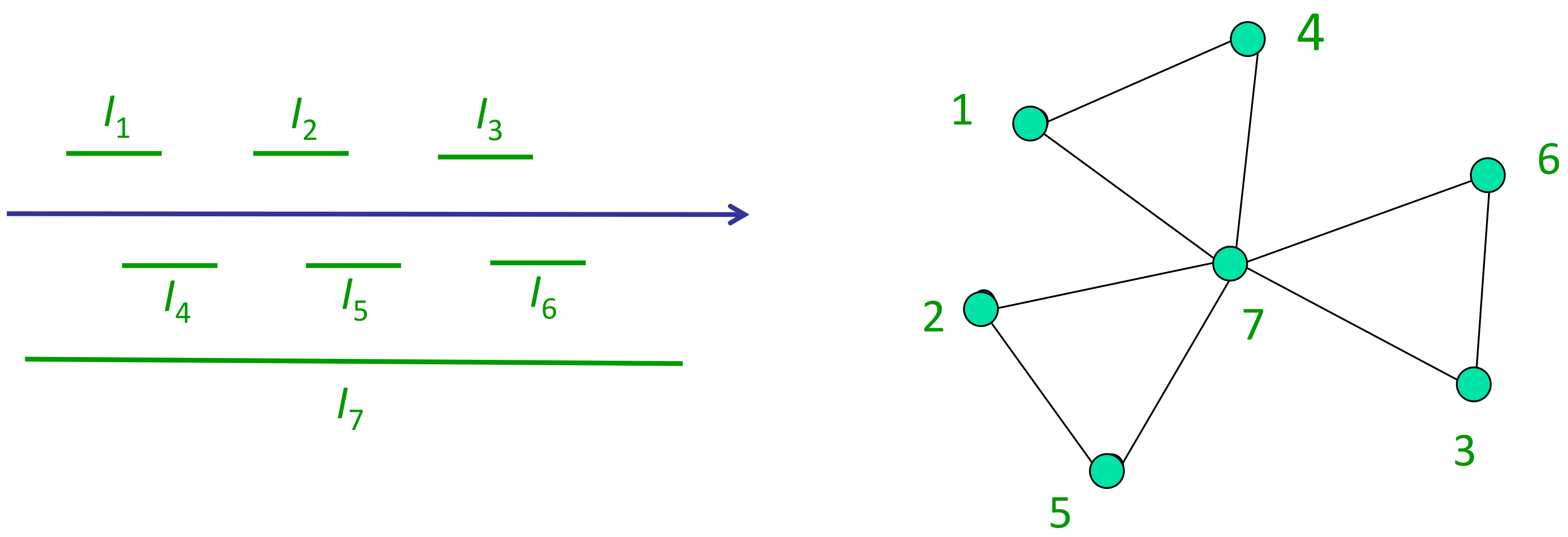
- Let *F* be a family of nonempty sets.

- The intersection graph of *F* is obtained be representing each set in F by a vertex:

$$x \rightarrow y \iff S_X \cap S_Y \neq \varnothing$$

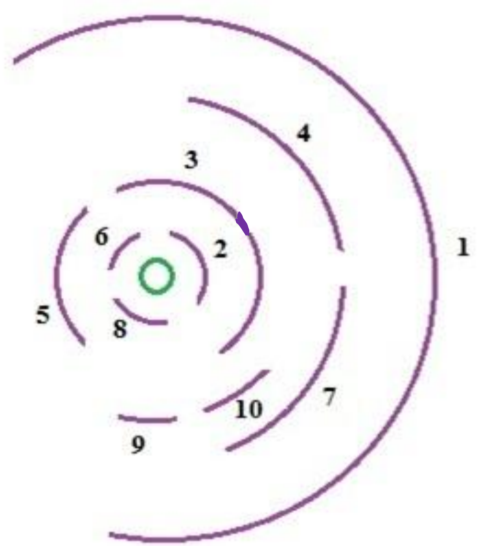
Γραφήματα Τομής (Διαστημάτων)

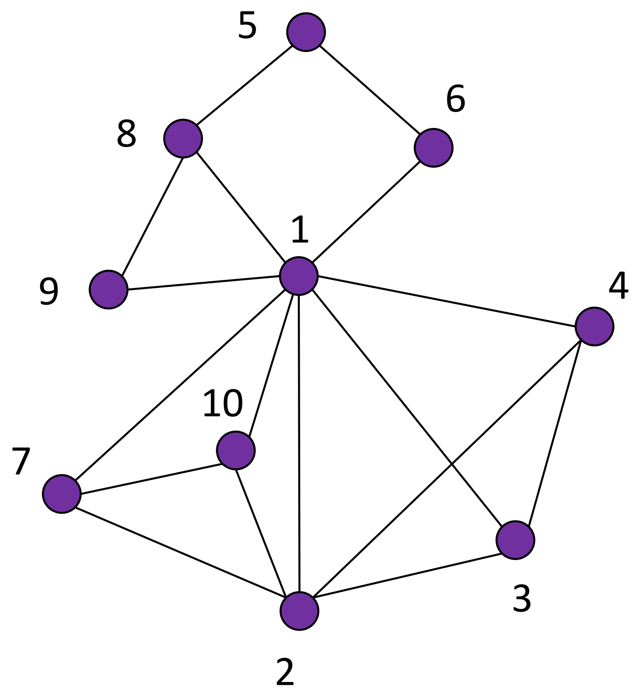
- The intersection graph of a family of intervals on a linearly ordered set (like the real line) is called an interval graph



Γραφήματα Τομής (Κυκλικών-τόξων)

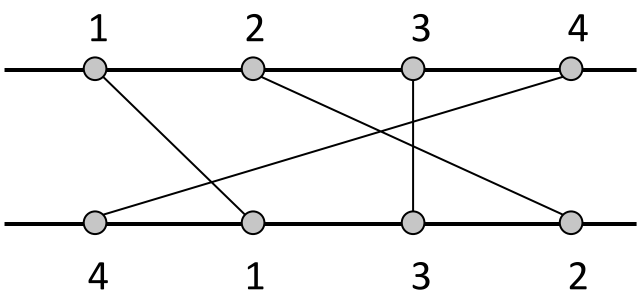
- Circular-arc graphs properly contain the internal graphs.



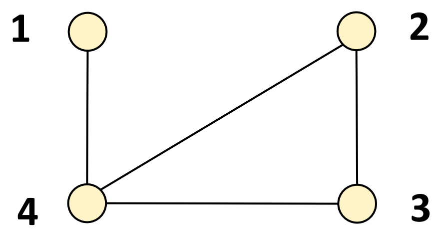


Γραφήματα Τομής (Μεταθετικά)

- A permutation diagram consists of n points on each of two parallel lines and n straight line segments matching the points.



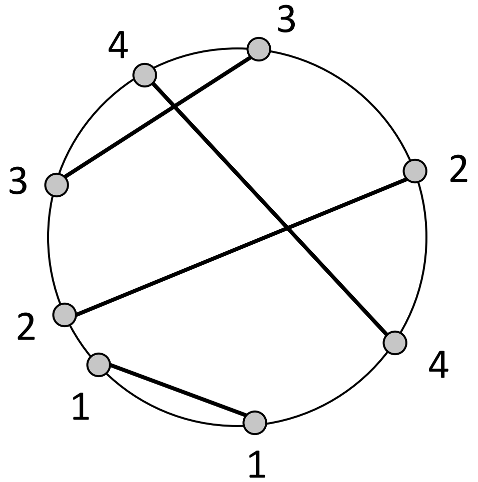
$\pi = [4,1, 3, 2]$

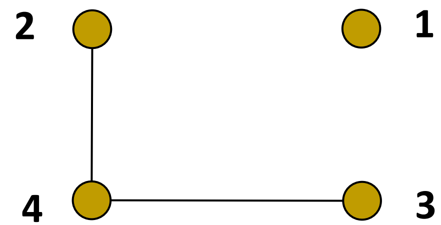


$G[\pi]$

Γραφήματα Τομής (Χορδικών-κύκλων)

- Intersecting chords of a circle





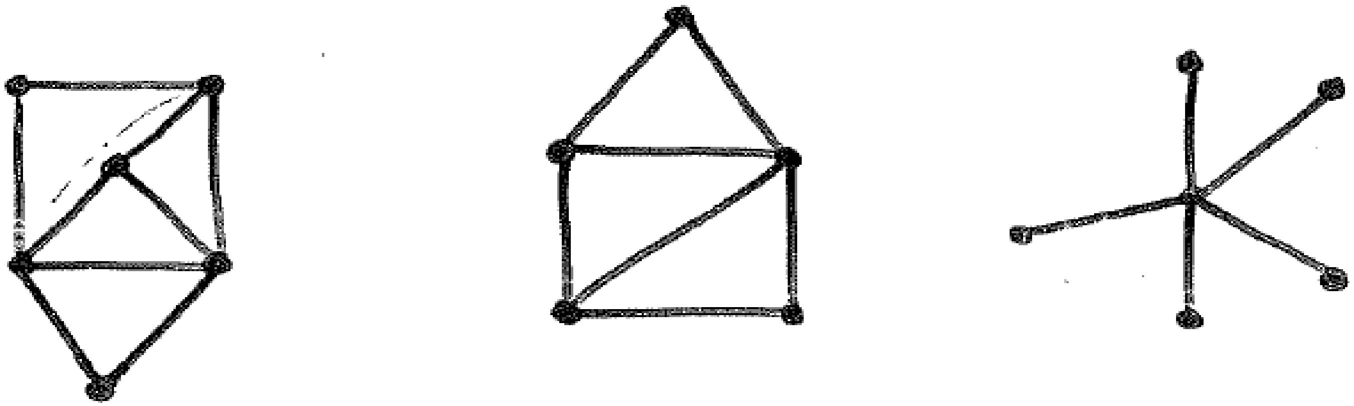
- Proposition. An induced subgraph of an interval graph is an interval graph.

Τριγωνική Ιδιότητα

- Triangulated Graph Property

> every simple cycle of length $l > 3$ possesses a chord

- Triangulated graphs (or chord graphs)

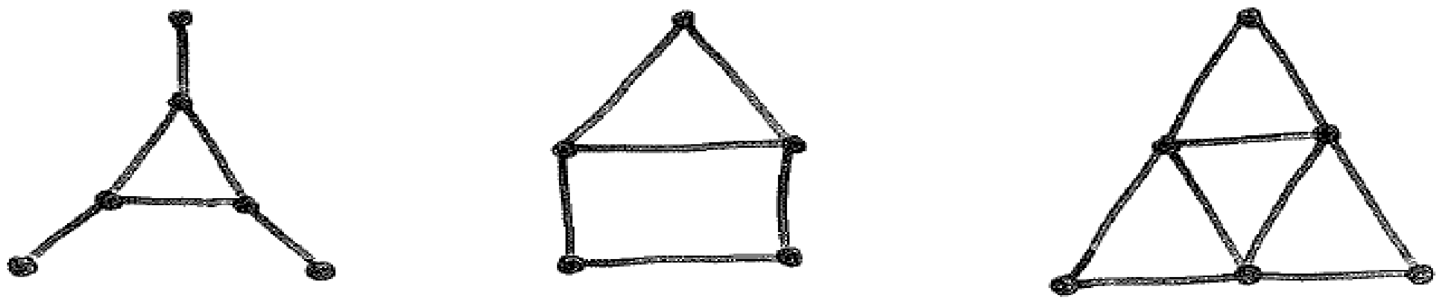


Μεταβατική Ιδιότητα

- Transitive Orientation Property
  Each edge can be assigned a one-way direction in such a way that the resulting oriented graph (V, F):

$$ab \in F \text{ and } bc \in F \Rightarrow ac \in F \ (\forall\ a, b, c \in V)$$

- Graphs which satisfy the transitive orientation property are called comparability graph.



Μεταβατική Ιδιότητα

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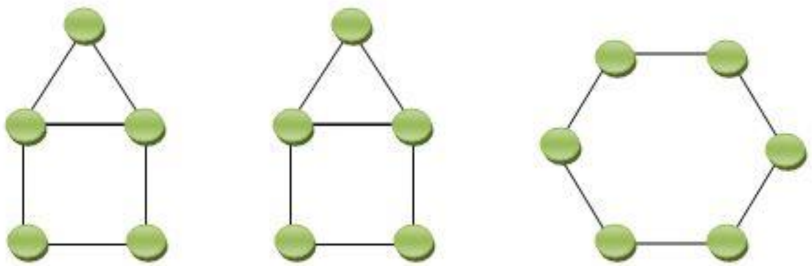
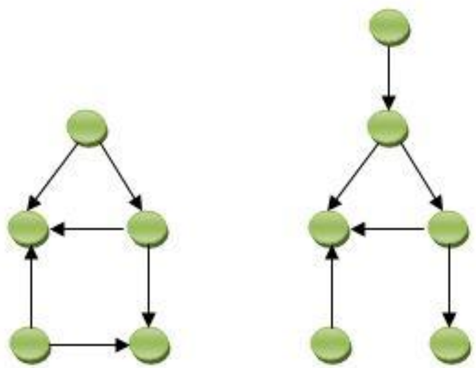
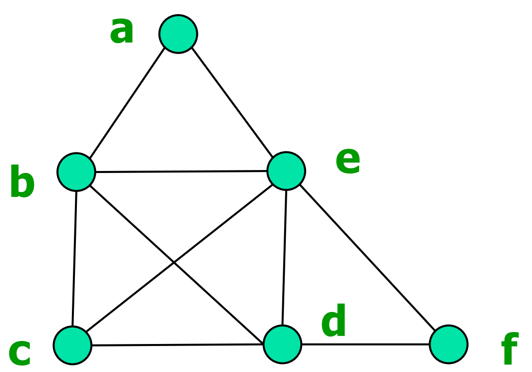




Βασικοί Αριθμοί Γραφήματος

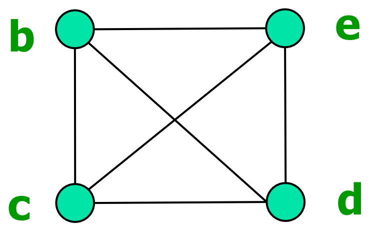
- Clique number ω(G)
  the number of vertices in a maximum clique of G

- Stability number α(G)
  the number of vertices in a stable set of max cardinality



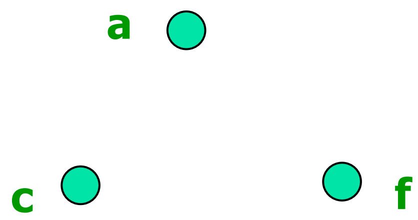
⇒

Max κλίκα του G



ω(G) = 4

Max stable set of G



α(G) = 3

Βασικοί Αριθμοί Γραφήματος

- A clique cover of size k is a partition

  $$V = C_1 + C_2 + \ldots + C_k$$

  such that C_i is a clique.

- A proper coloring of size c (*proper c-coloring*) is a partition

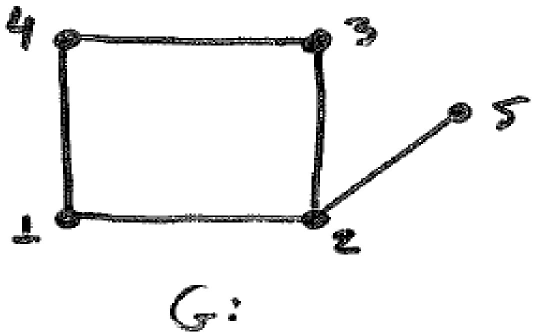
  $$V = X_1 + X_2 + \ldots + X_c$$

  such that X_i is a stable set.

Βασικοί Αριθμοί Γραφήματος

- Clique cover number κ(G)
  the size of the smallest possible clique cover of G

- Chromatic number χ(G)
  the smallest possible c for which there exists a proper c-coloring of G.

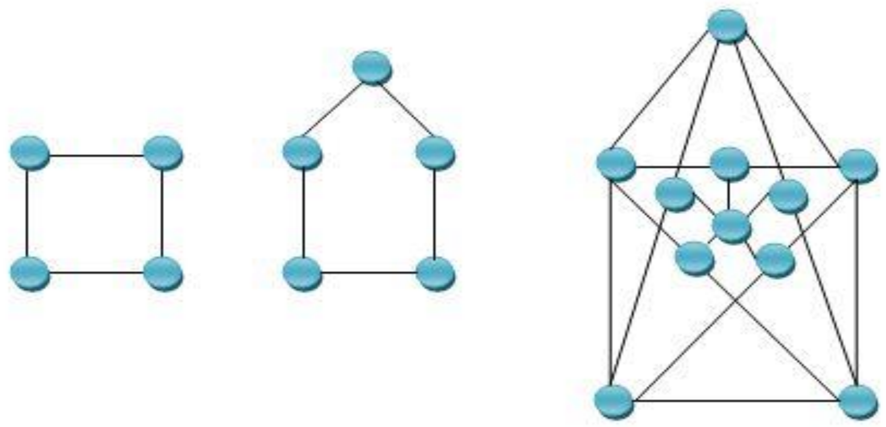


Clique cover V = {2,5} + {3,4} + {1}
c-Coloring V = {1,3,5} + {2,4}

κ(G)=3 χ(G)=2

Βασικοί Αριθμοί Γραφήματος

- For any graph G: $\omega(G) \leq \chi(G)$
  $\alpha(G) \leq \kappa(G)$



- Obriasly : $\alpha(G) = \omega(\breve{G})$ and $\kappa(G) = \chi(\breve{G})$

Τέλεια Γραφήματα

- Let $G = (V, E)$ be an undirected graph:

(P$_1$) $\omega(G_A) = \chi(G_A) \quad \forall A \subseteq V$

(P$_2$) $\alpha(G_A) = \kappa(G_A) \quad \forall A \subseteq V$

G is called Perfect

Τέλεια Γραφήματα

- χ-Perfect property

  For each induced subgraph G_A of G

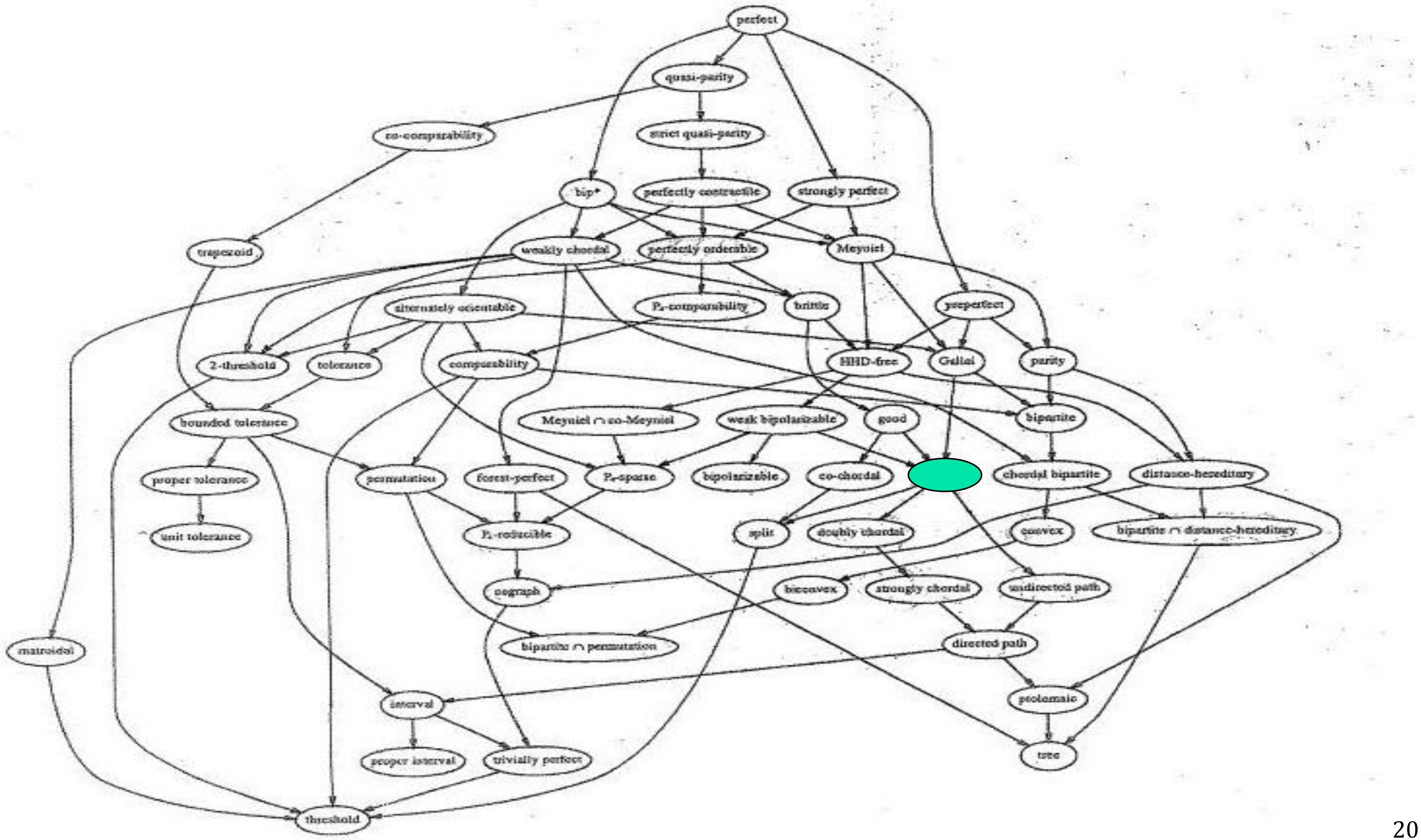
  $$\chi(G_A) = \omega(G_A)$$

- α-Perfect property

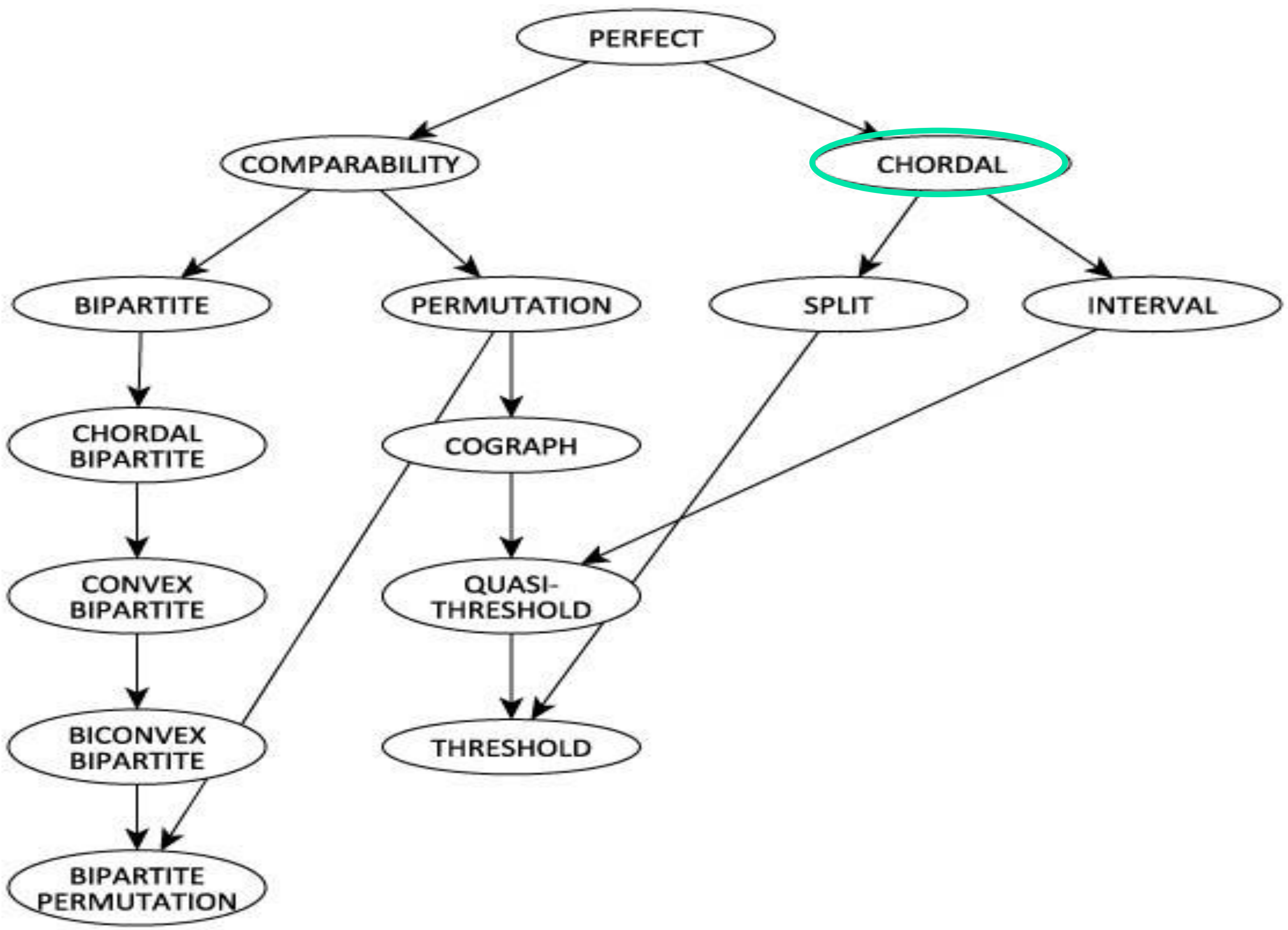
  For each induced subgraph G_A of G

  $$\alpha(G_A) = \kappa(G_A)$$

Κλάσεις Τέλειων Γραφημάτων



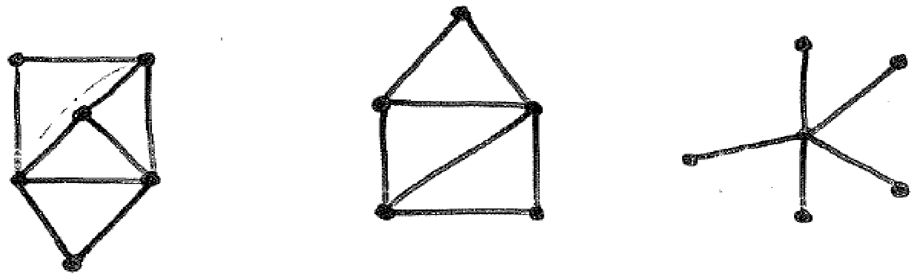
Κλάσεις Τέλειων Γραφημάτων



Τριγωνικά Γραφήματα

- G triangulated ⇔ G has the triangulated graph property

  Every simple cycle of length $l > 3$ possesses a chord.



- Triangulated graphs, or

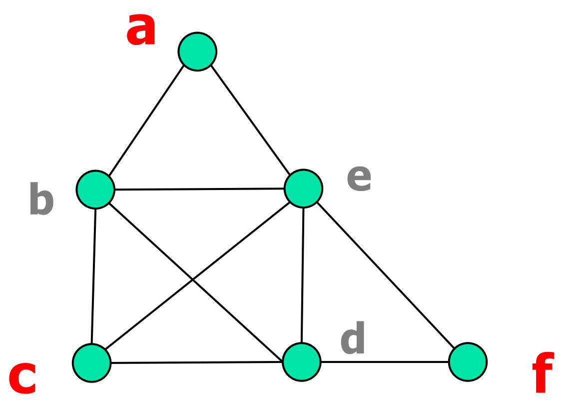
  Chordal graphs, or

  Perfect Elimination graphs

Τριγωνικά Γραφήματα

- Dirac showed that:

every chordal graph has a simplicial node, a node all of whose neighbors form a clique.

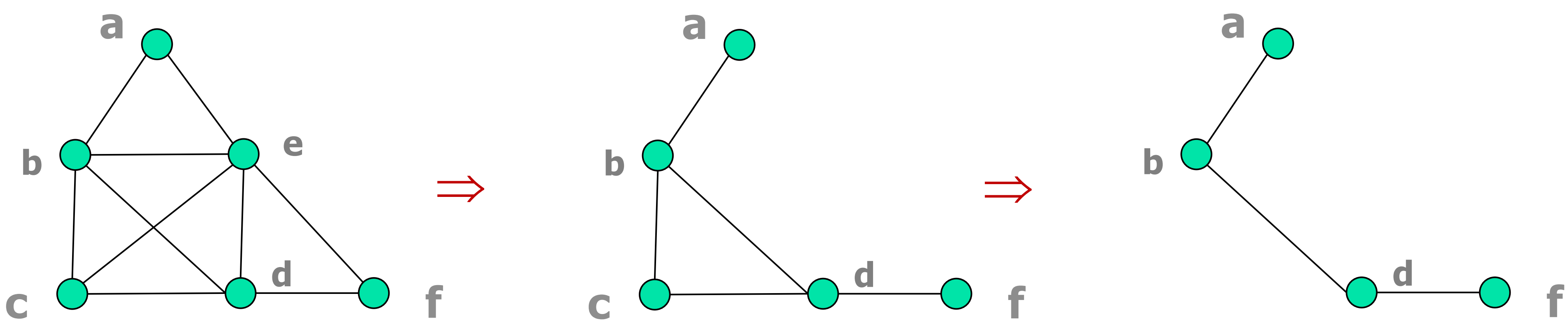


a, c, f simplicial nodes

b, d, e non siplicial

Τριγωνικά Γραφήματα

- It follows easily from the triangulated property that deleting nodes of a chordal graph yields another chordal graph.



Τριγωνικά Γραφήματα

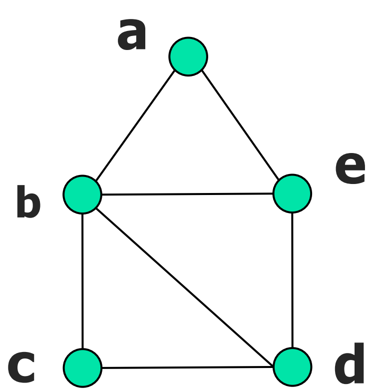
- Recognition Algorithm

  This observation leads to the following easy and simple recognition algorithm:

  - Find a simplicial node of G
  - Delete it from G, resulting G’
  - Recourse on the resulting graph G’, until no node remain

Τριγωνικά Γραφήματα

- node-ordering : perfect elimination ordering, or
  perfect elimination scheme



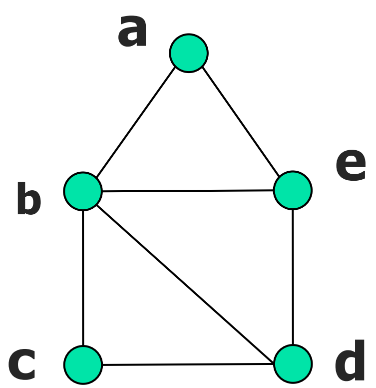
(a, c, b, e, d)   (c, d, e, a, b)   (c, a, b, d, e) ...

- Rose establishes a connection between chordal graphs and symmetric linear systems.

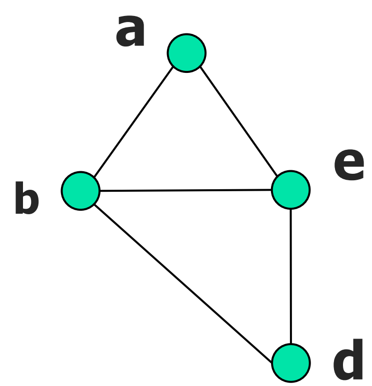
Τριγωνικά Γραφήματα

- Let $\sigma = [v_1, v_2, \ldots, v_n]$ be an ordering of the vertices of a graph $G = (V, E)$.

- σ = peo if each v_i is a simplicial node to graph $G[\{v_i, v_{i+1}, \ldots, v_n\}]$



σ = (c, d, e, a, b)

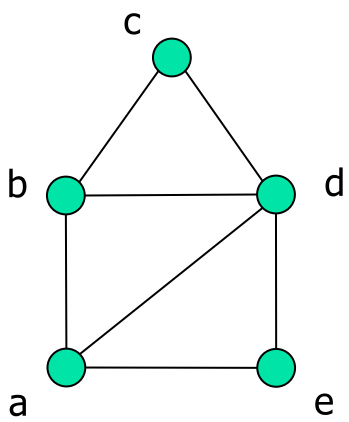


Τριγωνικά Γραφήματα

- Example:

- σ =[1, 7, 2, 6, 3, 5, 4] no simplicial vertex
- G_1 has 96 different peo.

Αλγόριθμος LexBFS



Algorithm LexBFS

Αλγόριθμος LexBFS

Algorithm LexBFS

1. for all v∈V do label(v) : = () ;
2. for i : = |V| down to 1 do

    2.1 choose v∈V with lexmax label (v);

    2.1 σ (i) ← v;

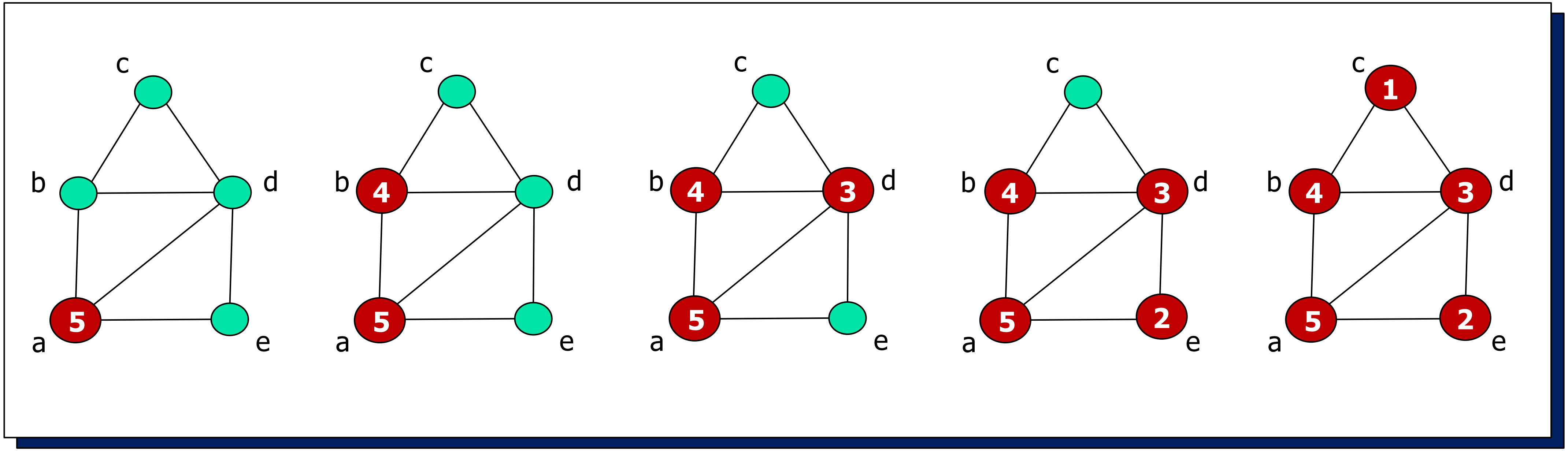
    2.3 for all u ∈V ∩N(v) do

        label (u) ← label (u) || i

    2.4 V ← V \ {v};

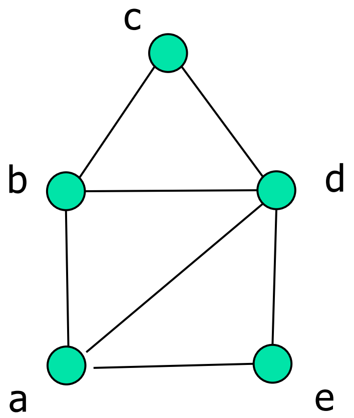
end

Αλγόριθμος LexBFS



σ = [a]	**σ = [b, a]**	**σ = [d, b, a]**	**σ = [e, d, b, a]**	**σ = [c, e, d, b, a]**
L(b) = (4) L(c) = () L(d) = (4) L(e) = (4)	L(c) = (3) L(d) = (43) L(e) = (43)	L(c) = (32) L(e) = (432)	L(c) = (321)	

Αλγόριθμος MCS



Algorithm MCS

Αλγόριθμος MCS

Algorithm MCS

1. for $i := |V|$ down to 1 do

    1.1 choose $v \in V$ with max number of numbered neighbours;

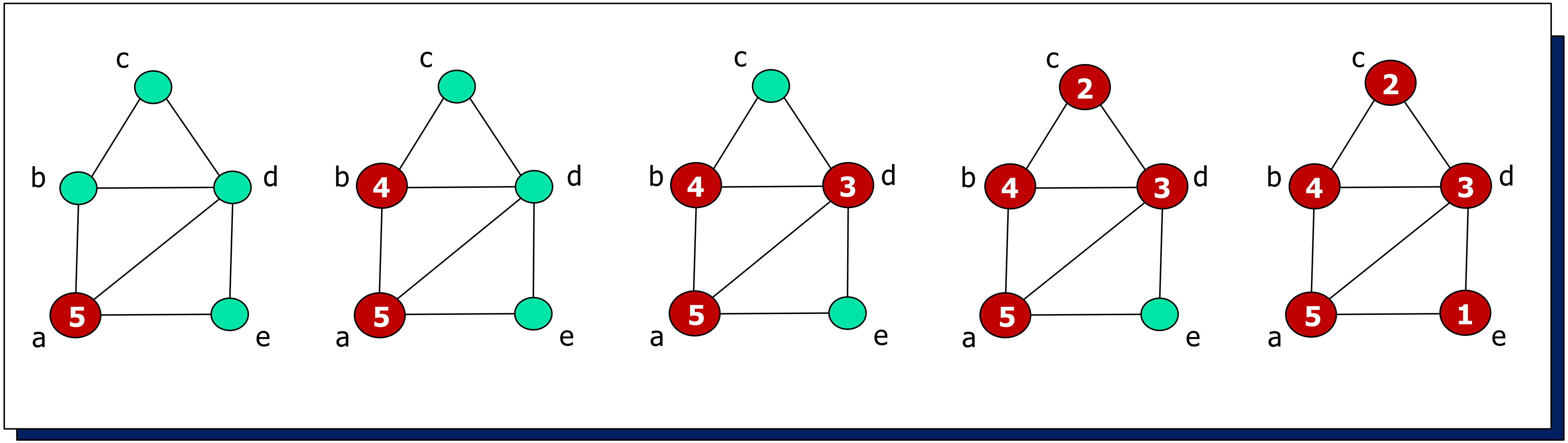
    1.2 number v by i;

    1.3 $\sigma(i) \leftarrow v$;

    1.4 $V \leftarrow V \setminus \{v\}$;

   end

Αλγόριθμος MCS



σ = [a] **σ = [b, a]** **σ = [d, b, a]** **σ = [e, d, b, a]** **σ = [c, e, d, b, a]**

Υπολογισμός PEO - Πολυπλοκότητα

Algorithms LexBFS & MCS

Complexity : O(1 + degree(v))

O(n + m)

Χαρακτηρισμοί - Ιδιότητες

- Definition: A subset **S** of vertices is called a

  Vertex Separator for nonadjacent vertices a, b

  or, equivalently, a-b separator,

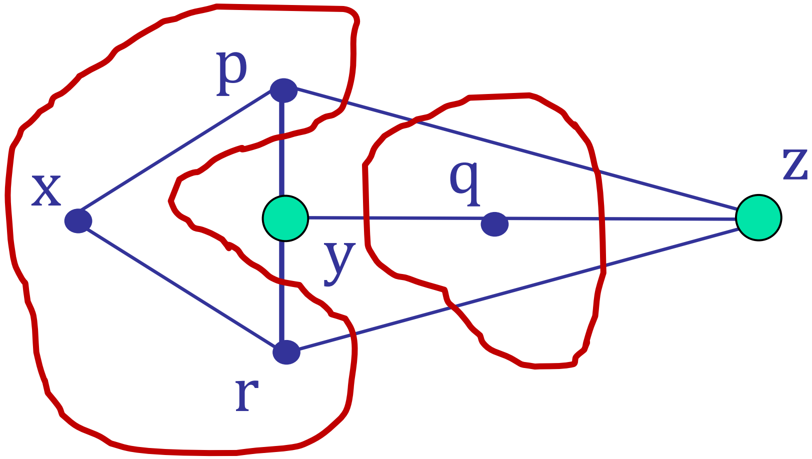
  if in graph G_{V-S} vertices a and b are in different connected components.

  If no proper subset of **S** in an a-b separator, **S** is called

  Minimal Vertex Separator.

Χαρακτηρισμοί - Ιδιότητες

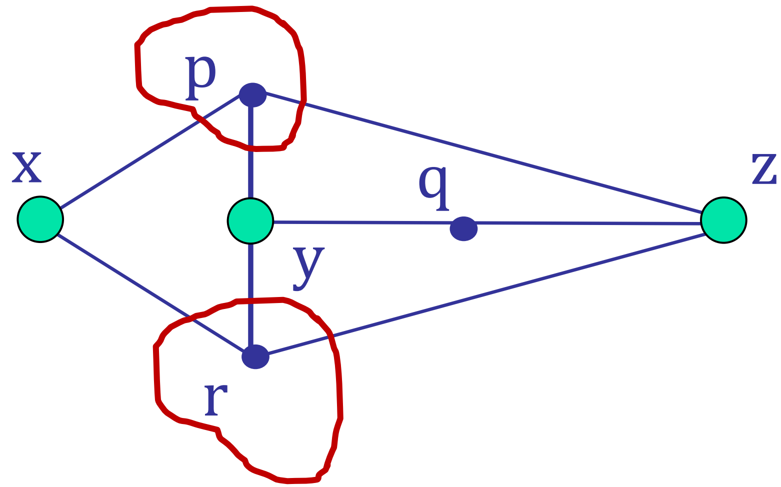
- Example 1:



The set {y, z} is a minimal vertex separator for p and q.

Χαρακτηρισμοί - Ιδιότητες

- Example 2:



The set {x, y, z} is a minimal vertex separator for p and r (p-r separator).

Χαρακτηρισμοί - Ιδιότητες

Theorem (Dirac 1961, Fulkerson and Gross 1965)

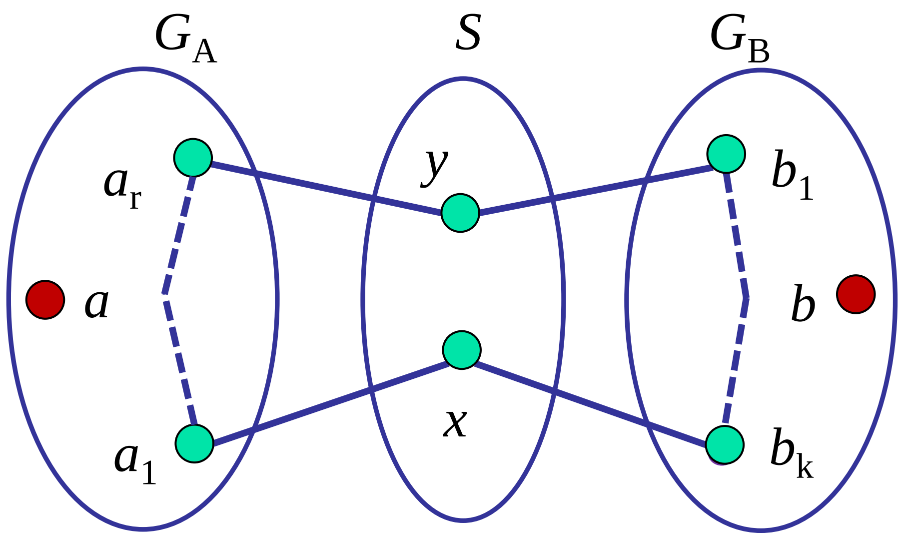
(1) G is triangulated.

(2) G has a peo; moreover, any simplicial vertex can start a perfect order.

(3) Every minimal vertex separator induces a complete subgraph of G.

Proof:

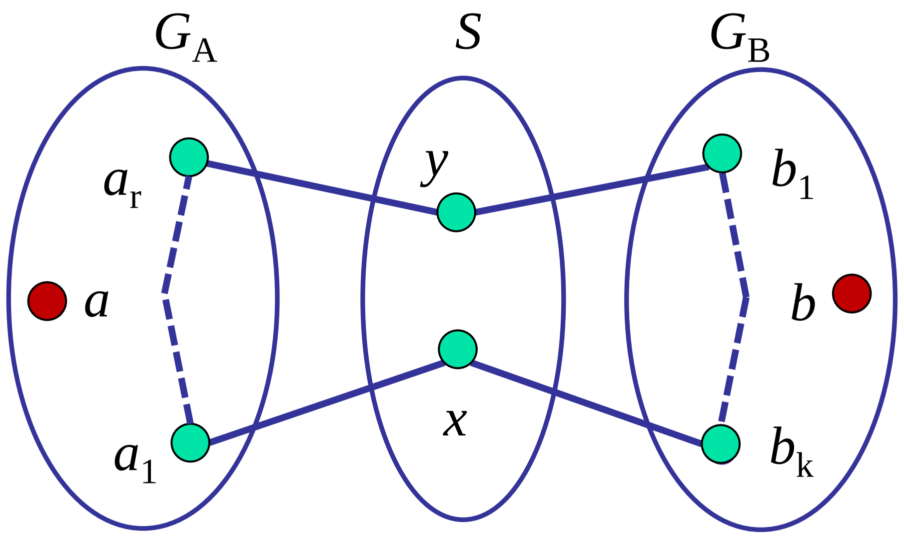
(1) ⇒ (3)



Χαρακτηρισμοί - Ιδιότητες

Let S be an a-b separator.

We will denote G_A, G_B
the connected components of G_{V-S}
containing *a*, *b*.

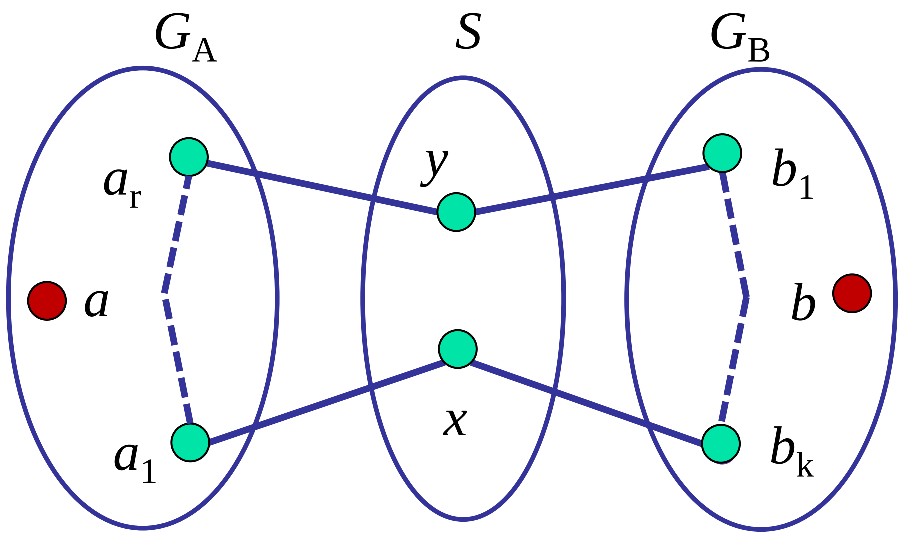


Since S is minimal, every vertex $x \in S$ is a neighbor of a vertex in G_A and a vertex in G_B.

For any $x, y \in S$, $\exists$ minimal paths

$(x,a_1,\ldots, a_i,\ldots,a_r,y)$ $a_i \in G_A$ and $(x,b_k,\ldots, b_i,\ldots,b_1,y)$ $b_i \in G_B$

Χαρακτηρισμοί - Ιδιότητες



Since

$[x, a_1, \ldots a_r, y, b_1, \ldots., b_k, x]$

is a simple cycle of length

$l \geq 4$, $\Rightarrow$ it contains a chord.

For every i, j $a_i b_j \notin E$, (S is a-b separate)

and also $a_i a_j \notin E$, $b_i b_j \notin E$ (by the minimality of the paths)

Thus, $x\, y \in E$.

Χαρακτηρισμοί - Ιδιότητες

(3) ⇒ (1) Suppose every minimal separator S is a clique

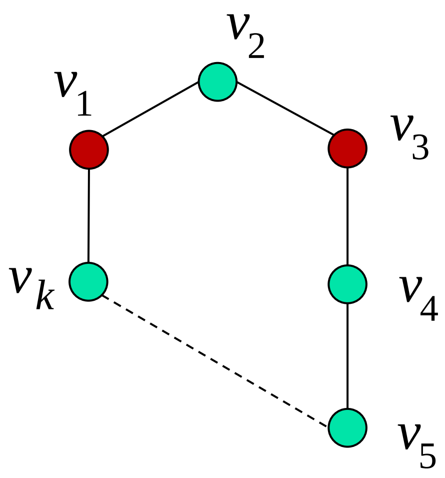
Let $[v_1, v_2, \ldots, v_k, v_1]$ be a chordless cycle.

v_1 and v_3 are nonadjacent.

Any minimal v_1-v_3 separator $S_{1,3}$ contains v_2 and at least one of $v_4, v_5, \ldots, v_k$.

But vertices v_2, v_i ($i = 4, 5, \ldots, k$) are nonadjacent

⇒ $S_{1,3}$ does not induce a clique.



Χαρακτηρισμοί - Ιδιότητες

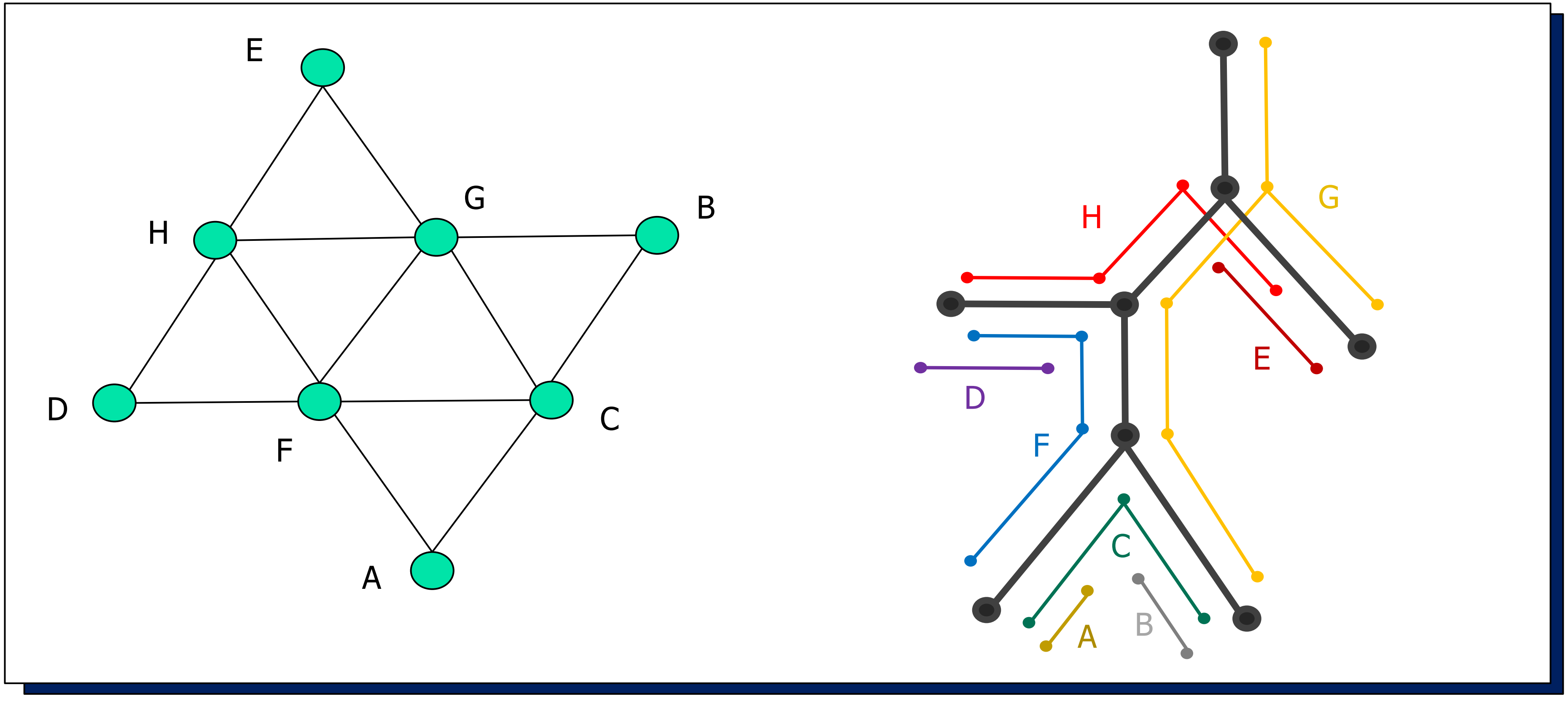
- The chordal graphs are exactly the intersection graphs of subtrees of trees.

That is, for a tree T and subtrees $T_1, T_2, \ldots, T_n$ of T there is a graph G:

- its nodes correspond to subtrees $T_1, T_2, \ldots, T_n$, and
- two nodes are adjacent if the corresponding subtrees share a node of T.

Χαρακτηρισμοί - Ιδιότητες

- Example:



Γραφήματα Διαστημάτων

Theorem: Let G be a graph. The following statements are equivalent.

(i) G is an interval graph.

(ii) G contains no C_4 and Ğ is a comparability graph.

(iii) The maximal cliques of G can be linearly ordered such that, for every vertex x of G the maximal cliques containing vertex x occur consecutively.

Τέλεια Γραφήματα - Προβλήματα

Βασικοί Αλγόριθμοι Γραφημάτων

Πολυπλοκότητα χώρου και χρ

Τέλεια Γραφήματα

- *Κλάσεις*
- *Ιδιότητες*
- *Προβλήματα*

Τεχνικές Διάσπασης (modular

Αλγόριθμοι Προβλημάτων Αναγν ... ιηση

- Coloring
- Max Clique
- Max Stable Set
- Clique Cover
- Matching
- Hamiltonian Path
- Hamiltonian Cycle
- …

- Triangulated
- Comparability
- Interval
- Permutation
- Split
- Cographs
- Threshold graphs
- QT graphs
- …

Τέλεια Γραφήματα - Μεταπτυχιακό