Borel Determinacy and the Word Problem for Finitely Generated Groups

Simon Thomas

Rutgers University

July 6th 2011

The word problem for finitely generated groups

For each $n \geq 1$, fix an computable enumeration $\{ w_k(x_1, \cdots, x_n) \mid k \in \mathbb{N} \}$ of the words in $x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}$.

Definition

If $G = \langle a_1, \cdots, a_n \rangle$ is a finitely generated group, then

$$\mathrm{Word}(G) = \{ k \in \mathbb{N} \mid w_k(a_1, \cdots, a_n) = 1 \}$$

Proposition

If $G = \langle a_1, \cdots, a_n \rangle = \langle b_1, \cdots, b_m \rangle$ is a finitely generated group, then

$$\{ k \in \mathbb{N} \mid w_k(a_1, \cdots, a_n) = 1 \} \equiv_T \{ \ell \in \mathbb{N} \mid w_\ell(b_1, \cdots, b_m) = 1 \}.$$

Theorem (Folklore)

For each subset $A \subseteq \mathbb{N}$, there exists a finitely generated group G_A such that $\mathrm{Word}(G_A) \equiv_T A$.

Question

Does there exist a uniform construction $A \mapsto G_A$ with the property that the isomorphism type of G_A only depends upon the Turing degree of A?

Polish Spaces & Borel maps

Definition

If (X, d) *is a complete separable metric space, then the corresponding topological space* $(X, \mathcal{T})$ *is a* ***Polish space****.*

Example

The Cantor space $2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ is a Polish space.

Definition

If X, Y *are Polish spaces, then a function* $f : X \to Y$ *is* ***Borel*** *if* graph(f) *is a Borel subset of* $X \times Y$*.*

Church's Thesis for Real Mathematics

EXPLICIT $=$ BOREL

The Polish space of f.g. groups

- A marked group $(G, \bar{s})$ consists of a f.g. group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators.
- For each $m \geq 1$, let $\mathcal{G}_m$ be the set of isomorphism types of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.
- Then there exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1_G)).$$

- And $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ is the space of f.g. groups.

The Polish space of f.g. groups

- Let $(G, \bar{s}) \in \mathcal{G}_m$ and let d_S be the corresponding word metric. For each $\ell \geq 1$, let
$$B_\ell(G, \bar{s}) = \{g \in G \mid d_S(g, 1_G) \leq \ell\}.$$
- The basic open neighborhoods of $(G, \bar{s})$ in $\mathcal{G}_m$ are given by
$$U_{(G,\bar{s}),\ell} = \{ (H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s}) \}, \qquad \ell \geq 1.$$

Example

For each $n \geq 1$, let $C_n = \langle g_n \rangle$ be cyclic of order n. Then:
$$\lim_{n \to \infty} (C_n, g_n) = (\mathbb{Z}, 1).$$

An inevitable non-uniformity result

Theorem

- *Suppose that $A \mapsto G_A$ is any Borel map from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that* $\mathrm{Word}(G_A) \equiv_T A$ *for all* $A \in 2^{\mathbb{N}}$.
- *Then there exists a Turing degree $\mathbf{d}_0$ such that for all $\mathbf{d} \geq_T \mathbf{d}_0$, there exists an infinite subset $\{ A_n \mid n \in \mathbb{N} \} \subseteq \mathbf{d}$ such that the groups $\{ G_{A_n} \mid n \in \mathbb{N} \}$ are pairwise incomparable with respect to embeddability.*

Today we will prove a slighly weaker version:

Main Theorem

There does not exist a Borel map $A \mapsto G_A$ from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that for all $A, B \in 2^{\mathbb{N}}$,

- $\mathrm{Word}(G_A) \equiv_T A$*; and*
- *if $A \equiv_T B$ then $G_A \cong G_B$.*

Countable Borel equivalence relations

Definition

- *An equivalence relation E on a Polish space X is Borel if E is a Borel subset of $X \times X$.*
- *A Borel equivalence relation E is countable if every E-class is countable.*

Some Examples

- The isomorphism relation $\cong$ is a countable Borel equivalence relation on the space $\mathcal{G}_{fg}$ of f.g. groups.
- The Turing equivalence relation $\equiv_T$ is a countable Borel equivalence relation on $2^{\mathbb{N}}$.

Borel reductions

Definition

Let E, F be Borel equivalence relations on the Polish spaces X, Y respectively.

- $E \leq_B F$ *if there exists a Borel map* $f : X \to Y$ *such that*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

  In this case, f is called a **Borel reduction** *from E to F.*

- $E \sim_B F$ *if both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *if both* $E \leq_B F$ *and* $E \nsim_B F$.

Universal countable Borel equivalence relations

Definition

A countable Borel equivalence relation E is universal if $F \leq_B E$ for every countable Borel equivalence relation F.

Theorem (Thomas-Velickovic)

The isomorphism relation $\cong$ on $\mathcal{G}_{fg}$ is a universal countable Borel equivalence relation.

Remark

It is currently not known whether the Turing equivalence relation $\equiv_T$ is countable universal.

Universal countable Borel equivalence relations

Corollary

There exists a Borel reduction from $\equiv_T$ *to* $\cong$.

Main Theorem

- *There does not exist a Borel reduction* $A \mapsto G_A$ *from* $\equiv_T$ *to* $\cong$ *such that* $\mathrm{Word}(G_A) \equiv_T A$ *for all* $A \in 2^{\mathbb{N}}$.
- *"Equivalently", there does not exist a continuous reduction from* $\equiv_T$ *to* $\cong$.

Question (Kanovei)

Find natural examples of Borel equivalence relations E, F *such that* $E \leq_B F$ *but there is no continuous reduction from* E *to* F.

Why are such examples hard to find?

Theorem (Folklore)

If X, Y are Polish spaces and $\varphi : X \to Y$ is a Borel map, then there exists a comeager subset $C \subseteq X$ such that $\varphi \restriction C$ is continuous.

Theorem (Lusin)

Let X, Y be Polish spaces and let μ be any Borel probability measure on X. If $\varphi : X \to Y$ is a Borel map, then for every $\varepsilon > 0$, there exists a compact set $K \subseteq X$ with $\mu(K) > 1 - \varepsilon$ such that $\varphi \restriction K$ is continuous.

Another notion of largeness ...

Definition

For each $z \in 2^{\mathbb{N}}$, the corresponding cone is $\mathcal{C}_z = \{\, x \in 2^{\mathbb{N}} \mid z \leq_T x \,\}$.

- Suppose $z_n = \{\, a_{n,\ell} \mid \ell \in \mathbb{N} \,\} \in 2^{\mathbb{N}}$ for each $n \in \mathbb{N}$ and define

$$\oplus z_n = \{\, p_n^{a_{n,\ell}} \mid n, \ell \in \mathbb{N} \,\} \in 2^{\mathbb{N}},$$

  where p_n is the nth prime.

- Then $z_m \leq_T \oplus z_n$ for each $m \in \mathbb{N}$ and so $\mathcal{C}_{\oplus z_n} \subseteq \bigcap_n \mathcal{C}_{z_n}$.

Remark

It is well-known that if $\mathcal{C} \subsetneq 2^{\mathbb{N}}$ is a proper cone, then $\mathcal{C}$ is both null and meager.

Continuous maps on the Cantor space

Theorem (Folklore)

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, then the following are equivalent:

(a) *θ is continuous.*

(b) *There exists $C \in 2^{\mathbb{N}}$ and $e \in \mathbb{N}$ such that $\theta(A) = \varphi_e^{C \oplus A}$.*

Corollary

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is continuous, then there exists a cone $\mathcal{C}$ such that $\theta(A) \leq_T A$ for all $A \in \mathcal{C}$.

Corollary

If $G \mapsto K_G$ is a continuous map from $\mathcal{G}_{fg}$ to $\mathcal{G}_{fg}$, then there exists a cone $\mathcal{C}$ such that if $\mathrm{Word}(G) \in \mathcal{C}$, then $\mathrm{Word}(K_G) \leq_T \mathrm{Word}(G)$.

The “obvious” vs “nonobvious” Turing reductions ...

Definition

If $A, B \in 2^{\mathbb{N}}$, then *A is one-one reducible to B**, written $A \leq_1 B$, if there exists an injective recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,*

$$n \in A \iff f(n) \in B.$$

Example

If $G, H \in \mathcal{G}_{fg}$ and $G \hookrightarrow H$, then $\mathrm{Word}(G) \leq_1 \mathrm{Word}(H)$.

Proof.

Suppose that $G = \langle\, a_1, \cdots, a_n \,\rangle$ and $H = \langle\, b_1, \cdots, b_m \,\rangle$. Let $\varphi : G \to H$ be an embedding and let $\varphi(a_i) = t_i(\bar{b})$. Then

$$w_k(a_1, \cdots, a_n) = 1 \iff w_k(t_1(\bar{b}), \cdots, t_n(\bar{b})) = 1.$$

$\square$

Turing Equivalence vs. Recursive Isomorphism

Definition

The sets $A, B \in 2^{\mathbb{N}}$ are recursively isomorphic, written $A \equiv_1 B$, if both $A \leq_1 B$ and $B \leq_1 A$.

Theorem (Myhill)

If $A, B \in 2^{\mathbb{N}}$, then $A \equiv_1 B$ if and only if there exists a recursive permutation $\pi \in \mathrm{Sym}(\mathbb{N})$ such that $\pi[A] = B$.

Theorem (Folklore)

The map $A \mapsto A'$ is a Borel reduction from $\equiv_T$ to $\equiv_1$.

Observation

The Borel reduction $A \mapsto A'$ from $\equiv_T$ to $\equiv_1$ is certainly not continuous.

Turing Equivalence vs. Recursive Isomorphism

Definition

Let E, F be Borel equivalence relations on the Polish spaces X, Y. Then the Borel map $\varphi : X \to Y$ is a homomorphism from E to F if

$$x \, E \, y \Longrightarrow \varphi(x) \, F \, \varphi(y).$$

Main Lemma

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to $\equiv_1$, then there exists a cone $\mathcal{C}$ such that θ maps $\mathcal{C}$ into a single $\equiv_1$-class.

Corollary

There does not exist a continuous reduction from $\equiv_T$ to $\equiv_1$.

Turing Equivalence vs. Isomorphism on $\mathcal{G}_{fg}$

Corollary

There does not exist a continuous reduction from $\equiv_T$ to $\cong$.

Proof.

- Suppose $A \mapsto H_A$ is a continuous reduction from $\equiv_T$ to $\cong$.
- Note that $H \mapsto \mathrm{Word}(H)$ is an injective continuous homomorphism from $\cong$ to $\equiv_1$.
- Thus $A \mapsto \mathrm{Word}(H_A)$ is a countable-to-one continuous homomorphism from $\equiv_T$ to $\equiv_1$, which is a contradiction.

$\square$

Determinacy

Definition

For each $X \subseteq 2^{\mathbb{N}}$, let $G(X)$ be the two player game

$$\begin{array}{lcccccccccc} \color{red}{I} & s(0) & & s(2) & & s(4) & & s(6) & & \cdots & \\ \color{red}{II} & & s(1) & & s(3) & & s(5) & & s(7) & & \cdots \end{array}$$

where I wins if and only if $s = (\, s(0)\, s(1)\, s(2)\, s(3)\, \cdots\,) \in X$.

Definition

- *A strategy is a map $2^{<\mathbb{N}} \to 2$ which tells the relevant player which move to make in a given position.*
- *The game $G(X)$ is determined if one of the players has a winning strategy.*

Determinacy

Observation

If X is countable, then player II has a winning strategy in $G(X)$.

Theorem (AC)

There exists a subset $X \subseteq 2^{\mathbb{N}}$ such that $G(X)$ is not determined.

Borel Determinacy (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a Borel subset, then $G(X)$ is determined.

An easy application of Borel Determinacy

Definition

A subset $X \subseteq 2^{\mathbb{N}}$ is $\equiv_T$-invariant if it is a union of $\equiv_T$-classes.

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X or $2^{\mathbb{N}} \smallsetminus X$ contains a cone.

Cf. Kolmogorov's Zero-One Law ...

Proof of Martin's Theorem

- Suppose that $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset.
- Consider the two player game $G(X)$

$$s(0) \quad s(1) \quad s(2) \quad s(3) \quad \cdots$$

  where I wins if and only if $s = (\, s(0)\, s(1)\, s(2)\, \cdots\,) \in X$.

- Then the Borel game $G(X)$ is determined. Suppose, for example, that $\sigma : 2^{<\mathbb{N}} \to 2$ is a winning strategy for I.
- Let $\sigma \leq_T t \in 2^{\mathbb{N}}$ and consider the run of $G(X)$ where
  - II plays $t = (\, s(1)\, s(3)\, s(5)\, \cdots\,)$
  - I uses the strategy σ and plays $(\, s(0)\, s(2)\, s(4)\, \cdots\,)$.
- Then $s \in X$ and $s \equiv_T t$. Hence $t \in X$ and so $\mathcal{C}_\sigma \subseteq X$.

Some easy consequences of Martin's Theorem

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X or $2^{\mathbb{N}} \smallsetminus X$ contains a cone.

Corollary

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant $\leq_T$-cofinal Borel subset, then X contains a cone.

Corollary

If $X \subseteq 2^{\mathbb{N}}$ is an arbitrary $\leq_T$-cofinal Borel subset, then X contains representatives of a cone.

Pointed Trees

Definition

- *A subset $S \subseteq 2^{<\mathbb{N}}$ is a tree if it is closed under taking initial segments.*
- *If S is a tree, then $[S] \subseteq 2^{\mathbb{N}}$ denotes the set of infinite branches through T.*
- *The tree S is perfect if for each $s \in S$, there exist incomparable $a, b \in S$ with $s \lessdot a, b$.*
- *The perfect tree S is pointed if $S \leq_T y$ for all $y \in [S]$.*

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\leq_T$-cofinal Borel subset, then there exists a pointed tree $S \subseteq 2^{<\mathbb{N}}$ such that $[S] \subseteq X$.

Proof of the Main Lemma

Main Lemma

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to $\equiv_1$, then there exists a cone $\mathcal{C}$ such that θ maps $\mathcal{C}$ into a single $\equiv_1$-class.

- Let $\mathcal{A}$ be a cone such that $\theta(A) \leq_T A$ for all $A \in \mathcal{A}$.
- Then there exists a cone $\mathcal{C} \subseteq \mathcal{A}$ such that either
  - (a) $\theta(A) <_T A$ for all $A \in \mathcal{C}$; or
  - (b) $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.

Case (a): suppose that $\theta(A) <_T A$ for all $A \in \mathcal{C}$.

Theorem (Slaman-Steel)

If $\mathcal{C}$ is a cone and $\theta : \mathcal{C} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T \restriction \mathcal{C}$ to $\equiv_T$ such that $\theta(A) <_T A$ for all $A \in \mathcal{C}$, then there exists a cone $\mathcal{D} \subseteq \mathcal{C}$ such that θ maps $\mathcal{D}$ into a single $\equiv_T$-class.

- Thus θ maps a cone $\mathcal{D}$ into a single $\equiv_T$-class $\mathbf{a}$.
- Let $\mathbf{a} = \bigsqcup_{n \in \mathbb{N}} \mathbf{b}_n$ be the decomposition of $\mathbf{a}$ into $\equiv_1$-classes.
- For each $n \in \mathbb{N}$, let $\mathcal{B}_n = \theta^{-1}(\mathbf{b}_n)$.
- Then there exists $n \in \mathbb{N}$ such that $\mathcal{B}_n$ contains a cone, as required.

Case (b): suppose that $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.

The Non-Selector Theorem

- *If $\mathcal{C}$ is a cone, then there does not exist a Borel homomorphism $\theta : \mathcal{C} \to \mathcal{C}$ from $\equiv_T \restriction \mathcal{C}$ to $\equiv_1 \restriction \mathcal{C}$ such that $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.*
- *In other words, if $\mathcal{C}$ is a cone, then there does not exist a Borel map which selects an $\equiv_1$-class within each $\equiv_T$-class.*

Proof of the Non-Selector Theorem

- Suppose $\theta : \mathcal{C} \to \mathcal{C}$ selects a $\equiv_1$-class within each $\equiv_T$-class.
- Then $\theta[\mathcal{C}]$ is a $\leq_T$-cofinal Borel subset of $2^{\mathbb{N}}$.
- By Martin's Theorem, there exists a pointed tree $S \subseteq 2^{<\mathbb{N}}$ such that $[\,S\,] \subseteq \theta[\mathcal{C}]$.
- Note that if $x, y \in [\,S\,]$, then $x \equiv_T y$ iff $x \equiv_1 y$.
- We can suppose that $(\,\pi_n \mid n \in \mathbb{N}\,) \leq_T S$, where $\{\,\pi_n \mid n \in \mathbb{N}\,\}$ is the group of recursive permutations.
- Let $x \in [\,S\,]$ be the left-most branch, so that $x \equiv_T S$.
- Then we can construct a branch $y \leq_T S$ such that $\pi_n(y) \neq x$ for all $n \in \mathbb{N}$.
- But then $y \equiv_T x$ and $y \not\equiv_1 x$, which is a contradiction!

Proof of the Main Theorem

Main Theorem

There does not exist a Borel reduction $A \mapsto G_A$ from $\equiv_T$ to $\cong$ such that $\mathrm{Word}(G_A) \equiv_T A$ for all $A \in 2^{\mathbb{N}}$.

- Suppose that $A \mapsto G_A$ is a Borel reduction from $\equiv_T$ to $\cong$ such that $\mathrm{Word}(G_A) \equiv_T A$ for all $A \in 2^{\mathbb{N}}$.
- Consider the Borel map $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by $A \mapsto \mathrm{Word}(G_A)$.
- If $A \equiv_T B$, then $G_A \cong G_B$ and so $\mathrm{Word}(G_A) \equiv_1 \mathrm{Word}(G_B)$.
- Thus $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel map which selects an $\equiv_1$-class within each $\equiv_T$-class, which is a contradiction!

The End