A Descriptive View of Combinatorial Group Theory

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Introduction

The Basic Theme:

Descriptive set theory provides a framework for explaining the inevitable non-uniformity of many classical constructions in mathematics.

Two Examples from Combinatorial Group Theory:

- *The Higman-Neumann-Neumann Embedding Theorem.*
- *The word problem for finitely generated groups.*

The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann 1949)

Every countable group G can be embedded into a 2-generator group.

Sketch Proof.

- Let $(\, g_n \mid n \in \mathbb{N} \,)$ be a sequence of generators with $g_0 = 1$.
- Let $\mathbb{F}$ be the free group on $\{\, a, b \,\}$ and let $G * \mathbb{F}$ be the free product.
- Then $\{\, b^{-n} a b^n \mid n \in \mathbb{N} \,\}$ and $\{\, g_n a^{-n} b a^n \mid n \in \mathbb{N} \,\}$ freely generate free subgroups of $G * \mathbb{F}$.
- Hence we can construct the *HNN* extension

$$G \hookrightarrow K_G = \langle\, G * \mathbb{F}, t \mid t^{-1} b^{-n} a b^n t = g_n a^{-n} b a^n \,\rangle$$

- Since $g_n \in \langle a, b, t \rangle$ and $t^{-1} a t = b$, it follows that $K_G = \langle a, t \rangle$.

$\square$

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$\square$

A natural question

Observation

It is reasonably clear that the isomorphism type of the 2-generator group K_G usually depends upon both the generating set of G and the particular enumeration that is used.

Question

Does there exist a more uniform construction with the property that the isomorphism type of K_G only depends upon the isomorphism type of G?

The word problem for finitely generated groups

For each $n \geq 1$, fix an computable enumeration $\{ w_k(x_1, \cdots, x_n) \mid k \in \mathbb{N} \}$ of the words in $x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}$.

Definition

If $G = \langle a_1, \cdots, a_n \rangle$ is a finitely generated group, then

$$\mathrm{Word}(G) = \{ k \in \mathbb{N} \mid w_k(a_1, \cdots, a_n) = 1 \}$$

Remark

The word problem for $G = \langle a_1, \cdots, a_n \rangle$ is the problem of deciding whether $k \in \mathrm{Word}(G)$.

Turing Reducibility

Convention

Throughout these talks, the powerset $\mathcal{P}(\mathbb{N})$ will be identified with $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is Turing reducible to B, written $A \leq_T B$, if there exists a B-oracle Turing machine which computes A.

Remark

In other words, there is an algorithm which computes A modulo an oracle which correctly answers questions of the form "Is $n \in B$?"

Turing Reducibility

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is Turing equivalent to B, written $A \equiv_T B$, if both $A \leq_T B$ and $B \leq_T A$.

Definition

If $A \in 2^{\mathbb{N}}$, then the corresponding Turing degree is defined to be

$$\mathbf{a} = \{ B \in 2^{\mathbb{N}} \mid B \equiv_T A \}.$$

Proposition

If $G = \langle a_1, \cdots, a_n \rangle = \langle b_1, \cdots, b_m \rangle$ is a finitely generated group, then

$$\{ k \in \mathbb{N} \mid w_k(a_1, \cdots, a_n) = 1 \} \equiv_T \{ \ell \in \mathbb{N} \mid w_\ell(b_1, \cdots, b_m) = 1 \}.$$

Prescribing the Turing degree of the word problem

Theorem (Folklore)

For each subset $A \subseteq \mathbb{N}$, there exists a finitely generated group G_A such that $\mathrm{Word}(G_A) \equiv_T A$.

- Notation: $[\,x, y\,] = x^{-1}\, y^{-1}\, x\, y$

Sketch Proof.

Let G_A be the group generated by the elements a, b subject to the following defining relations, where $c_n = [\,b, a^{-(n+1)} b\, a^{n+1}\,]$.

- $a\, c_n = c_n\, a$ for all $n \in \mathbb{N}$.
- $b\, c_n = c_n\, b$ for all $n \in \mathbb{N}$.
- $c_n^2 = 1$ for all $n \in \mathbb{N}$.
- $c_n = 1$ for all $n \in A$.

$\square$

Another natural question

Observation

The above construction of G_A is highly dependent on the specific subset $A \subseteq \mathbb{N}$, in the sense that if $A \neq B$ are subsets such that $A \equiv_T B$, then we "usually" have that $G_A \ncong G_B$.

Question

Does there exist a more uniform construction $A \mapsto G_A$ with the property that the isomorphism type of G_A only depends upon the Turing degree of A?

The answers ...

Notation

$\mathcal{G}$ and $\mathcal{G}_{fg}$ denotes the spaces of countable groups and f.g. groups.

"Theorem"

There does not exist an explicit map $G \mapsto K_G$ from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that for all $G, H \in \mathcal{G}$,

- $G \hookrightarrow K_G$; *and*
- *if* $G \cong H$, *then* $K_G \cong K_H$.

"Theorem"

There does not exist an explicit map $A \mapsto G_A$ from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that for all $A, B \in 2^{\mathbb{N}}$,

- $\mathrm{Word}(G_A) \equiv_T A$; *and*
- *if* $A \equiv_T B$ *then* $G_A \cong G_B$.

What is an explicit map?

Question

Which functions $f : \mathbb{R} \to \mathbb{R}$ are explicit?

Church's Thesis for the Reals

EXPLICIT $=$ BOREL

Definition

- *A function $f : \mathbb{R} \to \mathbb{R}$ is Borel if* $\mathrm{graph}(f)$ *is a Borel subset of $\mathbb{R} \times \mathbb{R}$.*
- *Equivalently, $f^{-1}(A)$ is Borel for each Borel subset $A \subseteq \mathbb{R}$.*

The Cantor Space

- The Cantor space $2^{\mathbb{N}}$ is a complete separable metric space with respect to the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x(n) - y(n)|}{2^{n+1}}.$$

- The corresponding topological space is a Polish space with basic open neighborhoods

$$U_s = \{ x \in 2^{\mathbb{N}} \mid x \restriction n = s \}, \qquad \text{where } s \in 2^{<\mathbb{N}}.$$

The Polish space of countably infinite groups

- Let $\mathcal{G}$ be the set of groups with underlying set $\mathbb{N}$.
- We can identify each group

$$G \in \mathcal{G} \longleftrightarrow m_G \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$$

  with the graph of its multiplication operation.
- Then $\mathcal{G}$ is a G_δ subset of the Cantor space $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$; i.e. $\mathcal{G}$ is a countable intersection of open subsets.
- It follows that $\mathcal{G}$ is a Polish subspace of the Cantor space $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$.

The Polish space of f.g. groups

- A marked group $(G, \bar{s})$ consists of a f.g. group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators.
- For each $m \geq 1$, let $\mathcal{G}_m$ be the set of isomorphism types of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.
- Then there exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1_G)).$$

- And $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ is the space of f.g. groups.

The Polish space of f.g. groups

- Let $(G, \bar{s}) \in \mathcal{G}_m$ and let d_S be the corresponding word metric. For each $\ell \geq 1$, let

$$B_\ell(G, \bar{s}) = \{g \in G \mid d_S(g, 1_G) \leq \ell\}.$$

- The basic open neighborhoods of $(G, \bar{s})$ in $\mathcal{G}_m$ are given by

$$U_{(G,\bar{s}),\ell} = \{\, (H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s}) \,\}, \qquad \ell \geq 1.$$

Example

For each $n \geq 1$, let $C_n = \langle g_n \rangle$ be cyclic of order n. Then:

$$\lim_{n \to \infty} (C_n, g_n) = (\mathbb{Z}, 1).$$

A slight digression ...

Some Isolated Points

- Finite groups
- Finitely presented simple groups

The Next Stage

- $SL_3(\mathbb{Z})$

Question (Grigorchuk)

What is the Cantor-Bendixson rank of $\mathcal{G}$?

The answers revisited ...

Theorem

There does not exist a Borel map $G \mapsto K_G$ from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that for all $G, H \in \mathcal{G}$,

- $G \hookrightarrow K_G$; *and*
- *if* $G \cong H$, *then* $K_G \cong K_H$.

Theorem

There does not exist a Borel map $A \mapsto G_A$ from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that for all $A, B \in 2^{\mathbb{N}}$,

- $\mathrm{Word}(G_A) \equiv_T A$; *and*
- *if* $A \equiv_T B$ *then* $G_A \cong G_B$.

But Greg Cherlin wasn't satisfied ...

Theorem

- *Suppose that $A \mapsto G_A$ is any Borel map from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that* $\text{Word}(G_A) \equiv_T A$ *for all $A \in 2^{\mathbb{N}}$.*
- *Then there exists a Turing degree $\mathbf{d}_0$ such that for all $\mathbf{d} \geq_T \mathbf{d}_0$, there exists an infinite subset $\{ A_n \mid n \in \mathbb{N} \} \subseteq \mathbf{d}$ such that the groups $\{ G_{A_n} \mid n \in \mathbb{N} \}$ are pairwise incomparable with respect to embeddability.*

But Greg Cherlin wasn't satisfied ...

Theorem (*LC*)

- *Suppose that $G \mapsto K_G$ is* any *Borel map from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$.*
- *Then there exists an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise* incomparable with respect to relative constructibility*; i.e., if $G \neq H \in \mathcal{F}$, then $K_G \notin L[K_H]$ and $K_H \notin L[K_G]$.*

Remarks

- (*LC*): There exists a Ramsey cardinal κ.
- In *ZFC*, we can find an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to embeddability.

Why are the Theorems "obviously true"?

Definition

Let E, F be equivalence relations on the Polish spaces X, Y. Then the Borel map $\varphi : X \to Y$ is a homomorphism if

$$x \, E \, y \implies \varphi(x) \, F \, \varphi(y).$$

Theorem

If $\varphi : \langle \mathcal{G}, \cong_{\mathcal{G}} \rangle \to \langle \mathcal{G}_{fg}, \cong_{\mathcal{G}_{fg}} \rangle$ is any Borel homomorphism, then there exists a group $G \in \mathcal{G}$ such that $G \not\to \varphi(G)$.

Heuristic Reason

Since $\cong_{\mathcal{G}}$ is much more complex than $\cong_{\mathcal{G}_{fg}}$, the Borel homomorphism must have a "large kernel" and hence "too many" groups $G \in \mathcal{G}$ will be mapped to a fixed $K \in \mathcal{G}_{fg}$.

Borel reductions

Definition

Let E, F be equivalence relations on the Polish spaces X, Y.

- *$E \leq_B F$ if there exists a Borel map $\varphi : X \to Y$ such that*

$$x \, E \, y \iff \varphi(x) \, F \, \varphi(y).$$

  *In this case, φ is called a **Borel reduction** from E to F.*
- *$E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$.*
- *$E <_B F$ if both $E \leq_B F$ and $E \nsim_B F$.*

The isomorphism relations on $\mathcal{G}$ and $\mathcal{G}_{fg}$

Definition

Let E be an equivalence relation on the Polish space X.

- *E is Borel if E is a Borel subset of $X \times X$.*
- *E is analytic if E is an analytic subset of $X \times X$.*

Example

If $G, H \in \mathcal{G}$, then

$$G \cong H \quad \text{iff} \quad \exists \pi \in \mathrm{Sym}(\mathbb{N}) \;\; \pi[m_G] = m_H.$$

Hence $\cong_{\mathcal{G}}$ is an analytic equivalence relation.

Theorem (Folklore)

The isomorphism relation on $\mathcal{G}$ is analytic but not Borel.

The isomorphism relations on $\mathcal{G}$ and $\mathcal{G}_{fg}$

Theorem

The isomorphism relation on $\mathcal{G}_{fg}$ is a countable Borel equivalence relation.

Definition

The Borel equivalence relation E is countable if every E-class is countable.

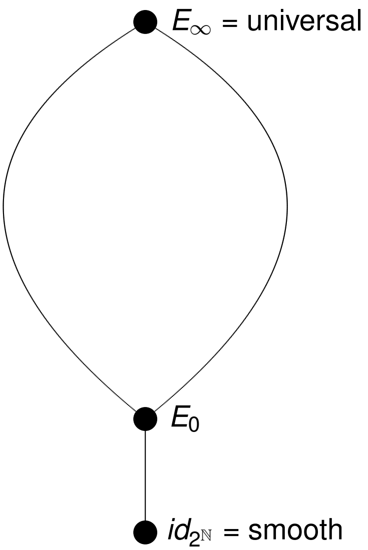
Theorem

$\cong_{\mathcal{G}_{fg}} <_B \cong_{\mathcal{G}}$.

Proof.

Suppose that $f : \mathcal{G} \to \mathcal{G}_{fg}$ is a Borel reduction. Then $\cong_{\mathcal{G}} = f^{-1}(\cong_{\mathcal{G}_{fg}})$ is Borel, which is a contradiction. □

Countable Borel equivalence relations



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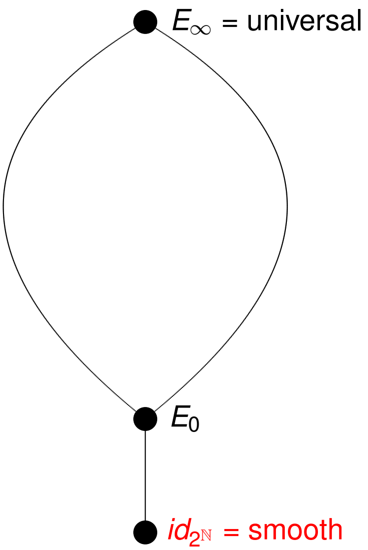

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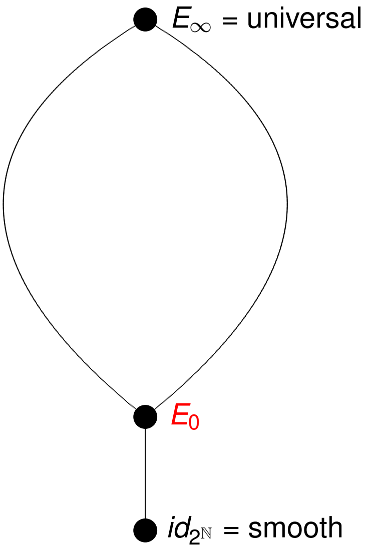
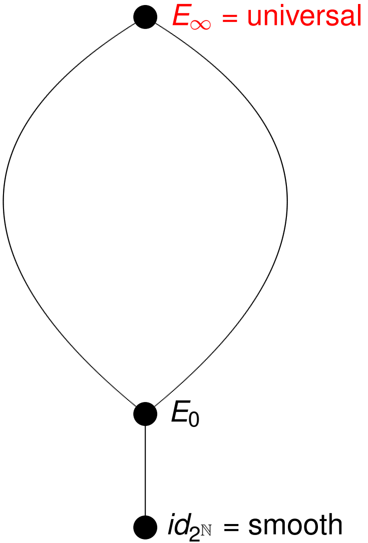


Definition (HKL)

E_0 is the equivalence relation of eventual equality on the space $2^{\mathbb{N}}$ of infinite binary sequences.

Countable Borel equivalence relations



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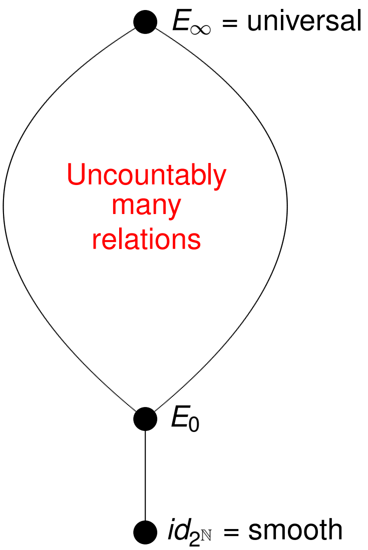
Definition (DJK)

A countable Borel equivalence relation E is universal if $F \leq_B E$ for every countable Borel equivalence relation F.

Countable Borel equivalence relations



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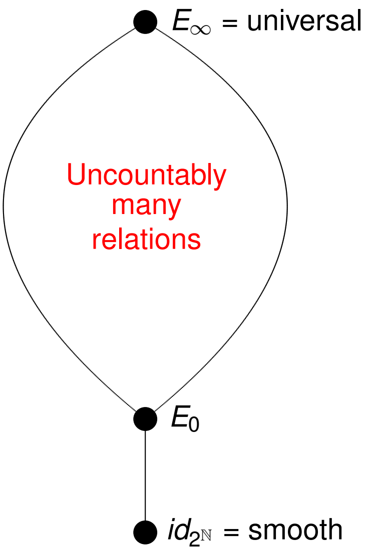
Question

Where do $\cong_{\mathcal{G}_{fg}}$ and $\equiv_T$ fit in?

Countable Borel equivalence relations



Confirming a conjecture of Hjorth-Kechris ...

Theorem (S.T.-Velickovic)

$\cong_{\mathcal{G}_{fg}}$ is a universal countable Borel equivalence relation.

Corollary

$\equiv_T \leq_B \cong_{\mathcal{G}_{fg}}$.

Remark

Unfortunately the Word Problem Theorem isn't so "obviously true" ...

How to prove such theorems?

The Word Problem Theorem

- *Reduce to a problem in Recursion Theory and then apply Martin's Theorem on the determinacy of Borel games.*
- *To be explained in the second talk ...*

The HNN Embedding Theorem

- *Collapse the continuum* $\mathbb{R}$ *to a countable set and then apply a suitable Absoluteness Theorem.*
- *To be explained in the third talk ...*

The obvious follow-up question to the *HNN* Theorem

Question (Cherlin, Hrushovski, ...)

Does there exist a Borel homomorphism $\varphi : \mathcal{G}_3 \to \mathcal{G}_2$ such that $G \hookrightarrow \varphi(G)$ for all $G \in \mathcal{G}_3$?

The Friedman Embedding Theorem

There exists a Borel homomorphism $\psi : \mathcal{G}_{fg} \to \mathcal{G}_2$ such that $G \hookrightarrow \psi(G)$ for all $G \in \mathcal{G}_{fg}$.

Question

What does Friedman know that the group theorists don't know ...
and that might conceivably be useful?

Answer

Absolutely nothing!

The word problem as a group-theoretic invariant

Theorem (Friedman)

There exists a Borel map $A \mapsto (\, g_A, h_A \,)$ *from* $2^{\mathbb{N}}$ *to* $\mathrm{Sym}(\mathbb{N}) \times \mathrm{Sym}(\mathbb{N})$ *such that:*

- *If* $\Gamma \in \mathcal{G}_{fg}$ *and* $\mathrm{Word}(\Gamma) \leq_T A$*, then* $\Gamma \hookrightarrow \langle\, g_A, h_A \,\rangle \in \mathcal{G}_2$.
- *If* $A \equiv_T B$*, then* $\{\, g_A, h_A \,\}$ *and* $\{\, g_B, h_B \,\}$ *generate the same subgroup of* $\mathrm{Sym}(\mathbb{N})$ *and so* $\langle\, g_A, h_A \,\rangle \cong \langle\, g_B, h_B \,\rangle$.

Corollary (Friedman)

Let $\psi : \mathcal{G}_{fg} \to \mathcal{G}_2$ *be the Borel homomorphism defined by*

$$\Gamma \mapsto \mathrm{Word}(\Gamma) \mapsto \langle\, g_{\mathrm{Word}(\Gamma)}, h_{\mathrm{Word}(\Gamma)} \,\rangle.$$

Then $\Gamma \hookrightarrow \psi(\Gamma)$ *for all* $\Gamma \in \mathcal{G}_{fg}$.

Friedman's Idea

Notation

If $A \in 2^{\mathbb{N}}$, then φ_i^A is the i-th partial A-recursive function and

$$\psi_i^A = \begin{cases} \varphi_i^A & \text{if } \varphi_i^A \in \mathrm{Sym}(\mathbb{N}); \\ \mathrm{id}_{\mathbb{N}} & \text{otherwise.} \end{cases}$$

Lemma (Friedman)

If $A \equiv_T B$, then there exists a recursive permutation $\theta \in \mathrm{Sym}(\mathbb{N})$ such that $\psi_i^B = \psi_{\theta(i)}^A$ for all $i \in \mathbb{N}$.

Friedman's Idea

Definition

Define $\pi_A \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ *by* $\pi_A(i, j) = (i, \psi_i^A(j))$.

Lemma (Friedman)

If $A \equiv_T B$, *then there exists a recursive permutation* $\theta \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ *such that* $\theta^{-1} \pi_A \theta = \pi_B$.

Definition

Let $H_A \leqslant \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ *be the subgroup generated by*

$$\{ \pi_A \} \cup \{ \theta \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N}) \mid \theta \text{ is recursive} \}.$$

Friedman's Idea

Notation

For each $g \in \mathrm{Sym}(\mathbb{N})$, define $\tilde{g} \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ by

$$\tilde{g}(i,j) = \begin{cases} (\,0, g(j)\,) & \text{if } i = 0. \\ (\,i, j\,) & \text{otherwise.} \end{cases}$$

Proposition (Friedman)

$\{\, \tilde{g} \mid g \in \mathrm{Sym}(\mathbb{N})$ *and* $g \leq_T A \,\} \leqslant H_A$.

Corollary (Friedman)

If $\Gamma \in \mathcal{G}_{fg}$ *and* $\mathrm{Word}(\,\Gamma) \leq_T A$*, then* $\Gamma \hookrightarrow H_A$.

Galvin's Embedding Theorem

Notation

For each $\pi \in \mathrm{Sym}(\Omega)$, define $\hat{\pi} \in \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \Omega)$ by

$$\hat{\pi}(\, m, n, \omega\,) = \begin{cases} (\, 0, 0, \pi(\omega)\,) & \text{if } m = n = 0; \\ (\, m, n, \omega\,) & \text{otherwise.} \end{cases}$$

Theorem (Galvin)

If $K \leqslant \mathrm{Sym}(\Omega)$ is a countable subgroup, then there exists a 2-generator subgroup $T_K \leqslant \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \Omega)$ such that $\{\, \hat{k} \mid k \in K \,\} \leqslant T_K$.

Definition

Let $\Omega = \mathbb{N} \times \mathbb{N}$ and let K be the group of recursive permutations of $\mathbb{N} \times \mathbb{N}$. Then G_A is the 3-generator group generated by $T_K \cup \{\, \hat{\pi}_A \,\}$.

And to get a 2-generator group? Work a little harder!

An Open Problem

Observation

The standard group-theoretic constructions (e.g. wreath products, free products with amalgamation, HNN extensions, ...) induce continuous homomorphisms $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$.

Conjecture

There does not exist a continuous homomorphism $\varphi : \mathcal{G}_3 \to \mathcal{G}_2$ *such that* $G \hookrightarrow \varphi(G)$ *for all* $G \in \mathcal{G}_3$.