The Existence Property among Set Theories

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Plan of the Talk

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1. Intuitionism

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2. The Existence Property and other properties

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2. The Existence Property and other properties
3. The Existence Property and Collection

Existentialism

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The correctness of an existential claim $(\exists x \in A)\varphi(x)$ is to be guaranteed by warrants from which both an object $x_0 \in A$ and a further warrant for $\varphi(x_0)$ are constructible.

Bishop: Man and God

When a man proves a positive integer to **exist**, he should show how to **find it**.

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If God has mathematics of his own that needs to be done, let him do it himself.

Why Intuitionistic Theories?

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- Philosophical Reasons: **Brouwer, Dummett, Martin-Löf,..**
- Computational content: Witness and program extraction from proofs.
- Intuitionistically proved theorems hold in more generality: The internal logic of topoi is intuitionistic logic.
- Axiomatic Freedom Adopt axioms that are classically refutable but intuitionistically preserve algorithmic truth (E.g. All $f : \mathbb{R} \rightarrow \mathbb{R}$ are continuous).

Formalization of intuitionistic logic

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Heyting 1930: intuitionistic predicate logic and arithmetic

Negative translation: **Kolmogorov** 1925, **Gentzen** and **Gödel** 1933.

Kleene's 1945 realizability for **HA**

a realizer of	**has the form**
A atomic	any e providing A is true.
$A \wedge B$	(a, b), where a is a realizer of A and b is a realizer of B.
$A \vee B$	$(0, a)$, where a is a realizer of A, or $(1, b)$, where b is a realizer of B
$\exists x\, B(x)$	(n, b), where b is a realizer of $B(\bar{n})$.

Kleene's 1945 realizability

a realizer of	**has the form**
$A \rightarrow B$	e, where e is the Gödel number of a Turing machine M_e such that M_e halts with a realizer for B whenever a realizer of A is run on M_e.
$\neg A$	any e providing there is **no** realizer for A.
$\forall x\, B(x)$	e, where e is a Gödel number of a Turing machine M_e such that M_e outputs a realizer for $A(\bar{n})$ when run on n.

Basic Assumptions

Let T be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the n-th element of ω.

The Disjunction Property

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1. T has the **disjunction property**, **DP**, if whenever

$$T \vdash \psi \vee \theta$$

holds for sentences ψ and θ of T, then

$$T \vdash \psi \quad \text{or} \quad T \vdash \theta.$$

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1. T has the **numerical existence property**, **NEP**, if whenever

$$T \vdash (\exists x \in \omega)\phi(x)$$

holds for a formula $\phi(x)$ with at most the free variable x, then

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2. T has the **existence property**, **EP**, if whenever
$$T \vdash \exists x \phi(x)$$
holds for a formula $\phi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that
$$T \vdash \exists! x\, \vartheta(x) \quad \text{and} \quad T \vdash \exists x\, [\vartheta(x) \wedge \phi(x)].$$

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- **Gödel** (1932) observed that intuitionistic propositional logic has the **DP**.
- **Gentzen** (1934): Intuitionistic predicate logic has the **DP** and **EP**.
- **Kleene** (1945): **HA** has the **DP** and **NEP**.
- **Joan Moschovakis** (1965): **DP**, **NEP** and **EP** for (many) systems of intuitionistic analysis.

Remarks about Classical Set Theories

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- Ignoring the trivial counterexamples, classical theories never have the **DP** or the **NEP**.
- **Z** (Zermelo set theory), **ZF**, and **ZF** are known **not** to have the **EP**.
- **ZFC** proves that $\mathbb{R}$ is well-orderable, but it cannot prove that there is a **definable** well-ordering of $\mathbb{R}$.

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2. (Feferman, Lévy) **EP** fails for Π^1_2 in **ZF** and **ZFC**.
3. (Y. Moschovakis) **ZF** $+$ Projective Determinacy has the **projective existence property** ($\varphi(x),\ \vartheta(x)$ projective).

Classical theories and EP

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Theorem
An extension T of **ZF** has the **EP** if and only if T proves that all sets are ordinal definable, i.e., $T \vdash V = OD$.

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CST based on intuitionistic logic

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Many sorted system: **numbers, sets, functions**

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* **Extensionality**

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* **Extensionality**
- **Pairing**, **Union**, **Infinity** (or $\mathbb{N}$ is a set)
- **Bounded Separation**

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- **Exponentiation**: A, B sets $\Rightarrow A^B$ set.

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- **Exponentiation**: A, B sets $\Rightarrow A^B$ set.
- **Replacement**

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* **Full Separation**
* **Powerset**

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* **Powerset**
* $\#$ **Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x, y) \quad \rightarrow \quad \exists b\ (\forall x \in a)\, (\exists y \in b)\ \varphi(x, y)$$

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$$(IND_{\in}) \qquad \forall a\ (\forall x \in a\ \varphi(x) \ \rightarrow\ \varphi(a)) \ \rightarrow\ \forall a\ \varphi(a),$$

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Myhill's $\mathbf{IZF}_R$:

IZF with **Replacement** instead of **Collection**

Constructive Zermelo-Fraenkel set theory, CZF

* **Extensionality**

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$$(\forall x \in a)\, \exists y\ \varphi(x,y) \ \rightarrow$$
$$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\ \varphi(x,y) \ \wedge\ (\forall y \in b)\, (\exists x \in a)\ \varphi(x,y)\,]$$

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- $*$ **Set Induction scheme**

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- **Non-explicit** set existence axioms: e.g. in classical set theory Axioms of Choice
- **Non-explicit set existence axioms** in intuitionistic set theory: e.g. Axioms of Choice, (Strong) Collection, Subset Collection, Regular Extension Axiom

Some History

Let $\mathbf{IZF}_R$ result from **IZF** by replacing Collection with Replacement, and let **CST** be Myhill's constructive set theory.

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Theorem 3. (Friedman, Scedrov)
IZF does not have the **EP**.

Realizability Theorem

Realizability with truth.

Theorem: (R)

For every theorem θ of **CZF***, there exists an application term s such that*

$$\mathbf{CZF} \vdash \ (s \Vdash_t \theta).$$

Moreover, the proof of this soundness theorem is effective in that the application term s can be effectively constructed from the **CZF** *proof of θ.*

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Theorem: (R)

The **DP** *and the* **NEP** *hold true for* **CZF**, **CZF** $+$ **REA** *and* **CZF** $+$ *Large Set Axioms.*
One can also add Subset Collection and the following choice principles:

AC$_\omega$, **DC**, **RDC**, **PAx**.

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2. R.: **Metamathematical Properties of Intuitionistic Set Theories with Choice Principles.** In: S. B. Cooper, B. Löwe, A. Sorbi (eds.): *New Computational Paradigms: Changing Conceptions of What is Computable* (Springer, New York, 2008) 287–312.

Failure of **EP** *for* **IZF**

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- Let $B(z)$ be a formula expressing that z is an uncountable cardinal. Let $B^*(z)$ result from $B(z)$ by replacing every atomic subformula D of $B(z)$ by

$$D \;\vee\; \forall uv(u \in v \;\vee\; \neg\, u \in v).$$

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EP fails for **IZF** for the following instance:

$$\exists y\, [\forall x \in 1\, \exists z\, B^*(z) \;\rightarrow\; \forall x \in 1\, \exists z \in y\, B^*(z)]\,.$$

Problems

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- (Beeson 1985) Does any reasonable set theory **with Collection** have the existential definability property?

The Weak Existence Property

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holds for a formula $\phi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that

$$\begin{array}{rcl} T & \vdash & \exists! x\, \vartheta(x), \\ T & \vdash & \forall x\, [\vartheta(x) \rightarrow \exists u\, u \in x], \\ T & \vdash & \forall x\, [\vartheta(x) \rightarrow \forall u \in x\, \phi(u)]. \end{array}$$

Extended E-recursive functions

Extended E*-recursive functions*

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- However, we shall introduce an extended notion of E-computability, christened **$E_\wp$-computability**, rendering the function

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computable as well.
- Classically, $E_\wp$-computability is related to **power recursion**, where the power set operation is regarded to be an initial function. Notion studied by Yiannis Moschovakis and Larry Moss.

Realizability with sets of witnesses

We use the expression $a \neq \emptyset$ to convey the positive fact that the set a is inhabited, that is $\exists x\, x \in a$.
We define a relation

$$a \Vdash_{\mathfrak{wt}} B$$

between sets and set-theoretic formulae.

$$a \bullet f \Vdash_{\mathfrak{wt}} B$$

will be an abbreviation for

$$\exists x [a \bullet f \simeq x \ \wedge \ x \Vdash_{\mathfrak{wt}} B]$$

$$
\begin{array}{rcl}
a \Vdash_{\mathfrak{wt}} A & \text{iff} & A \text{ holds true, whenever } A \text{ is an atomic formula} \\
a \Vdash_{\mathfrak{wt}} A \wedge B & \text{iff} & \jmath_0 a \Vdash_{\mathfrak{wt}} A \;\wedge\; \jmath_1 a \Vdash_{\mathfrak{wt}} B \\
a \Vdash_{\mathfrak{wt}} A \vee B & \text{iff} & a \neq \emptyset \;\wedge\; (\forall d \in a)(\left[\jmath_0 d = 0 \;\wedge\; \jmath_1 d \Vdash_{\mathfrak{wt}} A\right] \;\vee \\
 & & \qquad\qquad\qquad \left[\jmath_0 d = 1 \;\wedge\; \jmath_1 d \Vdash_{\mathfrak{wt}} B\right]) \\
a \Vdash_{\mathfrak{wt}} \neg A & \text{iff} & \neg A \;\wedge\; \forall c\, \neg c \Vdash_{\mathfrak{wt}} A \\
a \Vdash_{\mathfrak{wt}} A \rightarrow B & \text{iff} & (A \rightarrow B) \;\wedge\; \forall c \left[c \Vdash_{\mathfrak{wt}} A \;\rightarrow\; a \bullet c \Vdash_{\mathfrak{wt}} B\right] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\; a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\, A & \text{iff} & a \neq \emptyset \;\wedge\; (\forall d \in a)[\jmath_0 d \in b \;\wedge\; \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d] \\
a \Vdash_{\mathfrak{wt}} \forall x A & \text{iff} & \forall c \;\; a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} \exists x A & \text{iff} & a \neq \emptyset \;\wedge\; (\forall d \in a)\; \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]
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 & & \qquad [\jmath_0 d = 1 \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} B]\big) \\
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a \Vdash_{\mathfrak{wt}} A \rightarrow B & \text{iff} & (A \rightarrow B) \wedge \forall c\,[c \Vdash_{\mathfrak{wt}} A \rightarrow a \bullet c \Vdash_{\mathfrak{wt}} B] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\; a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\, A & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)(\jmath_0 d \in b \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]) \\
a \Vdash_{\mathfrak{wt}} \forall x A & \text{iff} & \forall c\;\; a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} \exists x A & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)\; \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]
\end{array}
$$

$$\Vdash_{\mathfrak{wt}} B \text{ iff } \exists a\, a \Vdash_{\mathfrak{wt}} B.$$

If we use indices of $E_{\wp}$-recursive functions rather than E_{exp}-recursive functions, we notate the corresponding notion of realizability by $a \Vdash^{\wp}_{\mathfrak{wt}} B$.

$$\Vdash_{\mathfrak{wt}} B \text{ iff } \exists a\, a \Vdash_{\mathfrak{wt}} B.$$

If we use indices of $E_{\wp}$-recursive functions rather than $E_{\exp}$-recursive functions, we notate the corresponding notion of realizability by $a \Vdash^{\wp}_{\mathfrak{wt}} B$.

Corollary

$$\Vdash_{\mathfrak{wt}} B \text{ iff } \exists a\, a \Vdash_{\mathfrak{wt}} B.$$

If we use indices of $E_{\wp}$-recursive functions rather than E_{exp}-recursive functions, we notate the corresponding notion of realizability by $a \Vdash^{\wp}_{\mathfrak{wt}} B$.

Corollary

(i) $\mathbf{CZF} \vdash (\Vdash_{\mathfrak{wt}} B) \rightarrow B$.

$$\Vdash_{\mathfrak{wt}} B \text{ iff } \exists a\, a \Vdash_{\mathfrak{wt}} B.$$

If we use indices of $E_{\wp}$-recursive functions rather than E_{exp}-recursive functions, we notate the corresponding notion of realizability by $a \Vdash^{\wp}_{\mathfrak{wt}} B$.

Corollary

(i) $\mathbf{CZF} \vdash (\Vdash_{\mathfrak{wt}} B) \rightarrow B$.

(ii) $\mathbf{CZF} + \mathbf{Pow} \vdash (\Vdash^{\wp}_{\mathfrak{wt}} B) \rightarrow B$.

A variant of **wEP**, dubbed **wEP'**, is the following: if

$$T \vdash \forall u\, \exists x A(u, x)$$

holds for a formula $A(u, x)$ having at most the free variables u, x, then there is a formula $B(u, x)$ with exactly u, x free, so that

$$\begin{aligned} T &\vdash \forall u\, \exists! x\, B(u, x), \\ T &\vdash \forall u \forall x\, [B(u, x) \rightarrow \exists z\, z \in x], \\ T &\vdash \forall u \forall x\, [B(u, x) \rightarrow \forall z \in x\, A(u, z)]. \end{aligned}$$

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Theorem CZF and **CZF** $+$ **Pow** both have the weak existence property. Indeed, they both satisfy the stronger property **wEP**$'$.

Even better

Even better

- **THEOREM** If

$$\mathbf{CZF} \vdash \exists x\, A(x)$$

then one can effectively construct a Σ^E formula $B(y)$ such that

$$\mathbf{CZF} \vdash \exists! y\, B(y)$$

$$\mathbf{CZF} \vdash \forall y [\, B(y) \rightarrow \exists x\; x \in y]$$

$$\mathbf{CZF} \vdash \forall y\, [B(y) \rightarrow \forall x \in y\, A(x)]$$

Even better

Even better

- **THEOREM** If

$$\mathbf{CZF} + \mathbf{Pow} \vdash \exists x\, A(x)$$

then one can effectively construct a Σ^P formula $B(y)$ such that

$$\mathbf{CZF} + \mathbf{Pow} \vdash \exists! y\, B(y)$$

$$\mathbf{CZF} + \mathbf{Pow} \vdash \forall y [\, B(y) \rightarrow \exists x\ x \in y]$$

$$\mathbf{CZF} + \mathbf{Pow} \vdash \forall y\, [B(y) \rightarrow \forall x \in y\, A(x)]$$

Conservativity

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THEOREM
CZF is conservative over **IKP**$(\mathcal{E})$ for Σ^E sentences.

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THEOREM
CZF $+$ **Pow** is conservative over **IKP**$(\mathcal{P})$ for Σ^P sentences.

Theorems and a Conjecture

Theorem 1: **CZF** has the **existence property.**

Theorems and a Conjecture

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Theorem 2: **CZF** + **Pow** has the **existence property.**

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Theorem 1: **CZF** has the **existence property.**

Theorem 2: **CZF** $+$ **Pow** has the **existence property**.

Conjecture 3: **CZF** $+$ $\mathrm{Subset\ \ Collection}$ does **not** have the **weak existence property.**