Localization principles in set theory

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Introduction.

In this talk I shall survey work published in AML (2010), work which is still under review, as well as some work in progress.

Philosophical Motivation

The axioms of ZFC are supposed to hold in the (absolute) universe of sets. From this point of view ZFC is an **absolutistic** theory.

As a consequence, entities and quantities like $\mathcal{P}(a)$, $|a|$, $|\mathcal{P}(a)|$, for infinite sets a, are required and assumed to be **absolute.**

However, judging by the so far gained experience, this requirement seems to be hopeless and unattainable.

The axioms of ZFC, even augmented with many additional **reasonable** ones, **provably** cannot shed any light on the exact status and size of these absolute entities.

Thus, e.g. $\mathcal{P}(\omega)$ and $|\mathcal{P}(\omega)|$, as absolute entities, seem to be **inherently and definitely elusive.** But being inherently and definitely elusive is practically no different from being **non-existent.**

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So the situation is much like the situation of modern physics where all measurements make sense only **locally**.

Specifically, in the established paradigm of **Relativity Theory,** measurements of all fundamental magnitudes, like time, mass, length, etc, inherently depend on the observer's reference frame. The basic theses are :

- The claim :
  "The length of the rode A is x **in the (absolute) universe**" does not make sense.
- What does make sense is the claim :
  "The length of the rode A is x **in the reference frame** M".
- A reference frame is a **local world** where the basic laws of physics hold with respect to the observer's measurements.
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Coming to the theory of sets, the fundamental magnitudes of this theory are the **infinite cardinalities.** So if we transfer the above relativistic/localistic thesis from the universe of physical objects to the universe of sets, the preceding theses become :

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Localization principles

- Every object (=set) is (theoretically) observed by an observer within some transitive model of T.

We would like to emphasize that we consider a **standard transitive** model as **the correct analogue** of reference frame, because it is a **genuine part** of the world of sets around us.

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At present, our intuitions about sets are best captured by ZFC. So the above speculations give rise to the following **localization principle :**

$Loc(\mathrm{ZFC}) :=$ Every set belongs to some transitive model of ZFC.

Formally :

$$(Loc(\mathrm{ZFC})) \qquad (\forall x)(\exists y)[x \in y \wedge Tr(y) \wedge (y, \in) \models \mathrm{ZFC}].$$

For every first-order axiomatized theory T, the logical complexity of the axiom $Loc(T)$ is Π_2.

Below we shall be mainly concerned with $Loc(\mathrm{ZFC})$, or $Loc(\mathrm{ZFC} + \phi)$, for some extension $\mathrm{ZFC} + \phi$ of ZFC.

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Consequently the **Powerset** axiom and, further, unrestricted **Replacement** are not supposed to hold in V.

What we keep are some absolute facts that constitute **Basic Set Theory** (BST) and comprise the following :

- Extensionality
- Pair
- Union
- Cartesian Product
- existence of ω
- Δ_0-Separation

BST together with $Loc(\mathrm{ZFC})$ is **local ZFC**, denoted **LZFC**, i.e.,

$$\mathrm{LZFC} := \mathrm{BST} + Loc(\mathrm{ZFC})$$

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(i) LZFC *proves : For all* $x_1, \ldots x_n$ *there is a transitive model* $M \models \mathrm{ZFC}$ *such that* $\{x_1, \ldots, x_n\} \subset M$.

(ii) LZFC *proves AC and Found.*

(iii) $\Pi_2(\mathrm{ZFC}) \subseteq \mathrm{LZFC}$.

Yet $\in$-induction or On-induction is not available in LZFC. For instance we cannot define (absolute) cardinalities of sets by

$$|x| = \min\{\alpha \in On : x \sim \alpha\}.$$

For that purpose we should work in $\mathrm{LZFC} + \mathrm{Found}_{On}$, where

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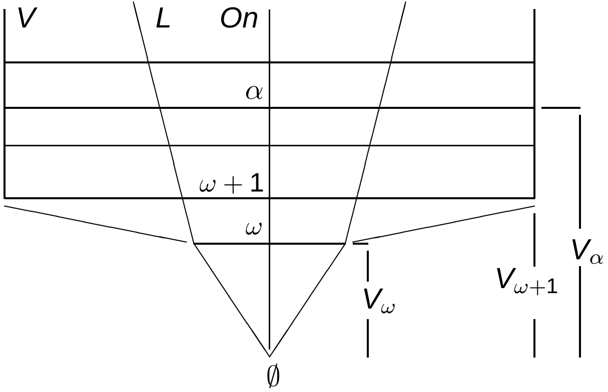
The picture of the universe of LZFC is roughly as follows :



Local ZFC (LZFC)

In LZFC the interest is shifted from (absolute) large cardinals to transitive models of ZFC with special properties.

The latter bear rough analogies with **strongly inaccessible cardinals**.

Specifically a transitive $M \models \mathrm{ZFC}$ is a **first-order analogue** of a strongly inaccessible cardinal.

Both are transitive sets "closed" with respect to the two most powerful axioms of ZFC, Replacement and Powerset.

Namely, if κ is a strongly inaccessible cardinal and M is a transitive model of ZFC, then they are similar in the following sense :

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- (1) κ is closed under every function $f : \kappa \to \kappa$, in the sense that for every $\alpha \in \kappa$, $f''\alpha$ is bounded in κ.

  (2) M is closed under every *first-order definable* function $f : M \to M$, in the sense that, by Replacement, for every $x \in M$, $f''x \in M$.

- (1) κ is closed with respect to exponentiation : For every cardinal $\lambda < \kappa$, $2^\lambda < \kappa$.

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Models of LZFC

Concerning transitive models of LZFC we have the following :

Proposition

(i) Let a be a transitive set which is the union of the transitive models of ZFC contained in it, that is, $a = \bigcup\{x \in a : x \models \mathrm{ZFC}\}$. If a satisfies Pair, then $(a, \in) \models \mathrm{LZFC}$.

(ii) In particular, if $(a, \in)$ is a directed set of models of LZFC, such that $\cup a = a$, then $(a, \in) \models \mathrm{LZFC}$.

In the preceding result we can even replace models of ZFC with models of LZFC. Namely the following holds.

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Consistency of LZFC

Despite its relativistic motivation, the principle $Loc(\mathrm{ZFC})$ is compatible with ZFC itself.

The consistency strength of $\mathrm{ZFC} + Loc(\mathrm{ZFC})$ is relatively low.

If

IC^{∞} := there is a proper class of strongly inaccessible cardinals,

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(i) $\mathrm{LZFC} \subset \mathrm{ZFC} + IC^{\infty}$.

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It's worth mentioning that IC^{∞} is equivalent to what in category theory is called "the axiom of universes", the origin of which goes back to Grothendieck.

Roughly a "Grothendieck universe" is a transitive set closed under pairing, powerset and replacement. The axiom of universes says that every set belongs to a Grothendieck universe.

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Mahlo models

$Loc(\mathrm{ZFC})$ can be naturally iterated along $n \in \omega$ as follows :

$Loc_0(\mathrm{ZFC}) = Loc(\mathrm{ZFC})$,
$Loc_{n+1}(\mathrm{ZFC}) = Loc(\mathrm{ZFC} + Loc_n(\mathrm{ZFC}))$.

Inductively, for every $n \in \omega$,

$$Loc_{n+1}(\mathrm{ZFC}) \ \Rightarrow \ Loc_n(\mathrm{ZFC}).$$

If $M \models \mathrm{ZFC} + Loc_0(\mathrm{ZFC})$, M might be called **quasi 1-Mahlo**, since in this case the class of models of ZFC belonging to M is **unbounded**.

M would be **1-Mahlo** if the subclass of models of M was **stationary** instead of just unbounded.

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Ordinary stationarity is a relative notion. Absoluteness is obtained if one is confined to the collection of **definable** clubs and **definable** stationary subsets of a model M.

Definition

Let $M \models \mathrm{ZFC}$. A set $X \in Def(M)$ is said to be **unbounded** in M, if $(\forall x \in M)(\exists y \in X)(x \subseteq y)$.

A set $X \in Def(M)$ is said to be **closed**, if

$$(\forall y \in M)(y \subseteq X \wedge (y, \subseteq) \text{ is a chain } \Rightarrow \cup y \in X).$$

A set $X \in Def(M)$ is said to be a **club** of M if it is unbounded and closed.

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A typical club of M is e.g. the set

$$\{M_\alpha : \alpha \in On \cap M\},$$

where $M_\alpha = V_\alpha^M$. For every $M \models \mathrm{ZFC}$, let

$$Club(M) = \{x \in Def(M) : x \text{ is a club in } M\},$$

and

$$Stat(M) = \{x \in Def(M) : x \text{ is stationary in } M\}.$$

Since $Def(M)$ is absolute, it follows that $Club(M)$ and $Stat(M)$ are absolute too.

The usual closure conditions for clubs (properly adjusted) hold also for the present version.

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We come to the definition of α-Mahlo models of ZFC.

Definition

(LZFC) α-**Mahlo** models of ZFC are defined inductively as follows :

(i) x is 0-**Mahlo** if x is transitive and $x \models \mathrm{ZFC}$.

(ii) x is $(\alpha+1)$-**Mahlo** if x is transitive, $x \models \mathrm{ZFC}$ and

$$\{y \in x : (y, \in) \text{ is an } \alpha\text{-Mahlo model}\}$$

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(iii) For limit α, x is α-**Mahlo** if it is β-Mahlo for all $\beta < \alpha$.

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(i) x is 0-**Mahlo** if x is transitive and $x \models \mathrm{ZFC}$.

(ii) x is $(\alpha+1)$-**Mahlo** if x is transitive, $x \models \mathrm{ZFC}$ and

$$\{y \in x : (y, \in) \text{ is an } \alpha\text{-Mahlo model}\}$$

is a stationary subset of x.

(iii) For limit α, x is α-**Mahlo** if it is β-Mahlo for all $\beta < \alpha$.

Existence of Mahlo models

Concerning **existence** of α-Mahlo models we have :

Proposition

(ZFC) *If κ is an α-Mahlo cardinal, for some $\alpha < \kappa$, then V_κ is α-Mahlo.*

Concerning the **internal truths** of Mahlo models we have :

Proposition

(ZFC) *If κ is a Mahlo cardinal, then $V_\kappa \models Loc_n(\mathrm{ZFC})$ for every $n \in \omega$.*

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(LZFC) *If M is 1-Mahlo, then $M \models Loc_n(\mathrm{ZFC})$ for every $n \in \omega$.*

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The large cardinals next to Mahlo are the **weakly compact** ones.

However these have several equivalent characterizations over ZFC. Most of them do not make sense for models.

But one of them seems to fit nicely to our context. This is **Π^1_1-indescribability.**

Definition

(LZFC) A transitive model $M \models \mathrm{ZFC}$ is said to be **Π^1_1-indescribable** if for every $U \in Def(M)$ and every Π^1_1 sentence ϕ, if $(M, \in, U, Def(M)) \models \phi$, then there is a transitive model $N \in M$ such that $U \cap N \in Def(N)$ and

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That Π^1_1-indescribable models (can be consistently assumed to) exist is a consequence of the following :

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Natural extensions of ZFC are $\mathrm{ZFC} + \phi$, where ϕ is $V = L$, $V = L(x)$, $|\mathcal{P}(\omega)| = \omega_1$, etc.

The question is if principles like $Loc(\mathrm{ZFC} + \phi)$, though local in essence, have global consequences.

Note that $V = L$ makes perfect sense also in the context of LZFC, and $L = \bigcup_{\alpha \in On} L_\alpha$, where each L_α is a set in LZFC, because of the absoluteness of "$x = L_\alpha$".

Below let the constant c denote some definable set of ZFC, like $\mathcal{P}(\omega)$, ω_1, etc., called "term".

Lemma

(LZFC) *Let c be a term such that c is shown in LZFC to be a set. Then*

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More generally, given a set of sentences Γ, we may extend LZFC to

$$\mathrm{LZFC}_\Gamma = \mathrm{LZFC} + \{Loc(\mathrm{ZFC} + \phi) : \phi \in \Gamma\}$$

and consider its consistency and its consequences on V.

The following is a general fact concerning the consistency of LZFC_Γ.

Proposition

If Γ is a set of sentences such that $\{\phi, \neg\phi\} \subseteq \Gamma$ for some Σ_1^{ZFC} or Π_1^{ZFC} sentence ϕ, then LZFC_Γ is inconsistent.

Given a term c and a transitive model M, let c^M denote the relativization of c with respect to M.

Let us call a term c **stable** if for every transitive M, $c \subseteq M \Rightarrow c^M = c$. For instance $\mathcal{P}(\omega)$ and $H(\omega_1)$ are stable terms, while ω_1 is not.

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By analogy we refer to properties of models like the ones defined above (Mahloness and Π_1-indescribability), as **large model properties**.

We can define even stronger **“large” models** by ways which as above resemble (but not blindly mimic) those producing large cardinals. Such are :

- **Elementarily extendible models**
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A transitive model M is said to be **elementarily extendible** if there is a transitive model N such that $M \in N$ and $M \prec N$.

(Actually $M \in N$ is redundant : If $M \prec N$, then $M \in N$.)

Proposition

(i) In LZFC *: Every elementarily extendible model is* Π^1_1*-indescribable.*

(ii) In $\mathrm{LZFC} + \mathrm{Found}_{On}$ *: The converse of (i) is false. I.e., if there are* Π^1_1*-indescribable models, then there is one which is not elementarily extendible.*

In contrast to Mahloness and Π^1_1-indescribability, which are absolute properties, elementary extendibility is Σ_1, hence not absolute.

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For every large model property $\phi(x)$ there is a natural **strengthened localization principle** :

$$Loc^{\phi}(\mathrm{ZFC}) := \forall x \exists y (x \in y \wedge \phi(y) \wedge (y, \in) \models \mathrm{ZFC}).$$

$Loc^{\phi}(\mathrm{ZFC})$ says that every set belongs to a ϕ-large transitive model of ZFC.

Let $mahlo_{\alpha}(x)$, $\pi^1_1 ind(x)$, $ext(x)$ formalize the properties of α-Mahloness, Π^1_1-indescribability and elementary extendibility, respectively.

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Here are some facts about strengthened localization principles :

Proposition

(ZFC) *If κ is strongly inaccessible, then $V_\kappa \models Loc^{ext}(\mathrm{ZFC})$.*

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Elementary embeddings and critical models

Certain large cardinal properties that involve non-trivial elementary embeddings are naturally adjusted to the context of models of ZFC.

The main difference is that, in contrast to the **internal** elementary embeddings $j : V \to V$ of ZFC, the elementary embeddings $j : M \to N$ of LZFC, where M, N are transitive set models, are generally **external** with respect to both M and N.

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Does the existence of critical or strongly critical models in LZFC contradict $V = L$?

We were able only to show that $V = L$ is refuted if in addition to the existence of strongly critical models we assume something more :

An axiom that goes beyond $Loc(\mathrm{ZFC})$ and gives information about **internal truths** of models, e.g., about how they see the cardinalities of certain sets.

This is the **Tall Model Axiom**, or **TMA** for short :

$$(TMA) \quad (\forall\kappa)(\exists\alpha > \kappa)(\forall\delta \geq \alpha)(\exists M)(\delta \in M \wedge M \models |\kappa| < |\alpha|).$$

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Namely the following holds :

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If $\mathrm{ZFC} + Loc(\mathrm{ZFC}) +$ "$0^{\#}$ exists" *is consistent, then in the universe of this theory* $H(\omega_1) \models \mathrm{LZFC} + TMA + GC$.

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Among strong large cardinal properties one that is particularly fitting to the context of LZFC is **Vopěnka's Principle** (*VP*).

Recall that *VP* is a scheme rather than a single axiom, defined as follows : Given a formula $\phi(x)$ in one free variable, let X_ϕ denote the extension $\{x : \phi(x)\}$ of ϕ. Then :

(VP_ϕ) If X_ϕ is a proper class of structures (of some fixed first-order language), then there are distinct $x, y \in X_\phi$ and an elementary embedding $j : x \to y$ (where $j : x \to y$ may be trivial, i.e., $x \prec y$).

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What if we add *VP* to LZFC ?

It turns out that in such a case ZFC is restored ! Indeed, using a classical ZFC result of **P. Vopěnka, A. Pultr and Z. Hedrlín, (1965),** that a rigid relation exists on any set, we can show the following :

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(i) $\mathrm{LZFC} + VP$ *proves Powerset and Replacement. Therefore the theories* $\mathrm{LZFC} + VP$ *and* $\mathrm{ZFC} + Loc(\mathrm{ZFC}) + VP$ *are identical.*

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The proof is based on the fact that in proper context, *VP* works as a **set existence principle**.

Specifically, *VP* is an implication of the form :

"if X_ϕ is a proper class, then such and such is the case".

Taking the contrapositive we have equivalently :

"if such and such is not the case, then the class X_ϕ is a set".

Using the Vopěnka – Pultr – Hedrlín rigidity result, we show that, given a set a, the classes $\mathcal{P}(a)$ and $F_\phi''a$, yielded by Powerset and Replacement, respectively, are sets.

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The following question has recently come up :

Is there any connection between **large model** properties of a model M and **large cardinal** properties held **in** M ?

So far we do not have any positive result of this type.
In general, if $\phi(x)$ is a large cardinal property and there is an analogous large model property $\phi^*(x)$, one does not expect that

$$\phi^*(M) \Rightarrow M \models \exists x\ \phi(x).$$

E.g. we can show that :

$$mahlo(M) \not\Rightarrow M \models \exists \kappa\ \kappa \text{ is a Mahlo cardinal.}$$

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We close with certain notions and questions about the global structure of models which are presently under investigation.

We saw that $Loc(\mathrm{ZFC})$ can be combined either with BST, or with ZFC itself. In any case it generates an abundance of models, the structure of which raises a lot of questions.

For every set x, let

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To cope with such questions we need extra assumptions. In particular we need

$$(\mathrm{Found}_{On}) \qquad \exists\alpha \in On\, \phi(\alpha) \to \exists\alpha \in On[\phi(\alpha) \wedge \forall\beta < \alpha \neg\phi(\beta)].$$

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The minimality of models of ZFC is closely related to the **degrees of constructibility of well-orderings**.

Namely, given a model M, let $\leq_c^M$ be the relation :

$$x \leq_c^M y \Leftrightarrow M \models x \leq_c y \Leftrightarrow M \models x \in L(y),$$

and let $[x]_c^M$ be the corresponding degrees.

Then M is a **minimal element** of $\mathcal{M}(x)$ iff for any well-orderings $\preceq_1, \preceq_2$ of $TC(\{x\})$ in M, $[\preceq_1]_c^M = [\preceq_2]_c^M$.

The corresponding question for L is open as far as we know :

Question : Given a set $x \notin L$, does there exist a $\leq_c$-minimal well-ordering $\preceq$ of $TC(\{x\})$, i.e., $\preceq$ such that $L(\preceq)$ is a **minimal inner model of** ZFC **containing** x ?

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The reducibility relation $\leq_{mdl}$

$Loc(\mathrm{ZFC})$ gives rise to the **model reducibility** relation :

$$x \leq_{mdl} y := (\forall M \in \mathcal{M})(y \in M \Rightarrow x \in M)$$

i.e.,

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(i) For every ordinal α, $(\alpha]_{mdl} = L_{ceil(\alpha)}$, hence $L_{ceil(\alpha)}$ is the least element of $\mathcal{M}(\alpha)$.

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In particular we have :

Lemma

(i) For every ordinal α, $(\alpha]_{mdl} = L_{ceil(\alpha)}$, hence $L_{ceil(\alpha)}$ is the least element of $\mathcal{M}(\alpha)$.

(ii) $L = \bigcup\{(\alpha]_{mdl} : \alpha \in On\}$.

(iii) The relation $\leq_{mdl}$ is linear on the ordinals and $[ceil(\alpha)]_{mdl}$ is the immediate successor of $[\alpha]_{mdl}$.

More generally the operator $\mathcal{D}$ defined for every class X by :

$$\mathcal{D}(X) = \bigcup\{(x]_{mdl} : x \in X\}$$

is a **closure operator**, with the following properties :

Lemma

(LZFC) *(i)* $\mathcal{D}(\{x\}) = (x]_{mdl} = \bigcap \mathcal{M}(x)$.

(ii) In particular $\mathcal{D}(\{\emptyset\}) = L_\theta =$ *the least model of* ZFC.

(iii) $\mathcal{D}^2(X) = \mathcal{D}(X)$.

(iv) For every $M \in \mathcal{M}$, $\mathcal{D}(M) = M$.

(v) If $(X, \in)$ *is a directed class, then* $\mathcal{D}(X)$ *is a transitive class-model of* BST*, i.e., a model of Extensionality, Pair, Union, Cartesian Product and* Δ_0*-Separation.*

(vi) In particular, for every x, $(x]_{mdl} \models$ BST.

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It is well-known that every set within a transitive model M of ZFC can be coded by a set of ordinals of M. In LZFC this yields the following :

Lemma

(LZFC) *For every x there is a set $A \subset On$ such that $x \leq_{mdl} A$.*

In view of this fact, one could transfer **tameness** properties from **sets of ordinals**, to arbitrary sets.

Specifically, if X is a reasonable class of tame sets of ordinals (e.g. **non-random**), then $\mathcal{D}(X)$ would be the corresponding class of **tame sets** in general, which satisfies at least the axioms of BST.

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The reducibility relation $\leq_{mdl}$

Given a set x, let $WO(x)$ be the class (in LZFC) of all well-orderings of x. We have the following characterization of $\leq_{mdl}$:

Lemma

$(\mathrm{LZFC} + Found_{On})$ *For all* x, y, $x \leq_{mdl} y$ *iff*

$$x \in \bigcap \{L_{ceil(\preceq)}(\preceq) : \preceq \in WO(TC(\{y\}))\}.$$

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$(\mathrm{LZFC} + Found_{On})$ *For every* x,

$$L_{ceil(x)}(x) \subseteq \bigcap \{L_{ceil(\preceq)}(\preceq) : \preceq \in WO(TC(\{x\}))\}.$$

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