CONTOURLET BASED NATURAL SCENE STATISTICS USING STUDENT'S T DISTRIBUTION

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ABSTRACT

In this work, we consider contourlet based image decomposition as a convenient framework for localized representation of images simultaneously in space, frequency (scale) and orientation. Since the subband marginal distributions of natural images in the contourlet domain are highly non-Gaussian with leptokurtotic behavior, we propose the Student's t probability density function (pdf) as a prior for modeling the contourlet coefficients of natural images. As an extension, we also consider the bivariate form of Student's t pdf in order to capture the across scales dependencies of the contourlet transform. We validate our proposal by adopting subjective and objective measures.

***Index Terms*—** Natural scene statistics, Student's t distribution, contourlet transform, univariate and bivariate model

1. INTRODUCTION

Since modeling the statistics of natural images is a challenging task (partly because of the high dimensionality of the signal), the power of statistical models can be substantially improved by transforming the signal from the pixel domain to a new representation [1]-[5]. For example, in the watermarking problem, based on a specific modeling we can construct a class of statistical detectors with even higher detection sensitivity [5], [6], [18]. In this framework, the accurate characterization of transform coefficient distribution has been a fundamental approach [1].

Multiscale decomposition has been a valid framework for algorithm development and coding standards evolution [1]. Transform domains like DWT (Discrete Wavelet Transform) have very attractive properties, where images exhibit highly kurtotic behavior with non-Gaussian subband marginal distributions [1]-[3].

In this framework, the contourlet domain has also attracted much attention due to its additional advantages [3] compared with e.g. the wavelet domain. In general, the marginal distribution of subband image contourlet coefficients is non-Gaussian, symmetric and sharply peaked around zero with heavy-tails [3]. In addition, the wavelet's restriction of fixed number of directions can be potentially overcome in contourlets with more directions, trying to see the smoothness along contours [3].

The sparseness of contourlet coefficients makes it reasonable to assume that essentially only a few large detail coefficients contain information about the underlying image. It is then legitimate to impose a prior that is designed to model the sparsity of these coefficients. Thus, a variety of parametric models have been proposed all these years, including the SaS (Symmetric alpha Stable) family of distributions, the Laplacian, or the GGD distribution [1], [5], [9], [12].

Many of the aforementioned non-Gaussian statistical models can be unified under the flexible density family known as the GSM (Gaussian Scale Mixture) model [9]. Thus, trying to investigate the validity of the proposed modeling, we consider a suitable representative of this class, meaning the Student's t-distribution in the framework of modeling the subband marginal distributions of natural images in the contourlet domain [4], [13]. The heavy tail characteristic justifies the use of Student's t for signals that are impulsive in nature.

The remainder of this work is organized as follows. In Section 2, an overview of contourlet domain image modeling is provided. In Section 3, we describe our motivation, whereas in Section 4 we give details about our proposal, along with experimental results. Finally, conclusions and future work are provided in Section 5.

2. CONTOURLET DOMAIN IMAGE MODELING

2.1. Statistical properties of contourlet coefficients

It is well known that contourlet coefficients exhibit non Gaussian properties [3]-[5]. Undoubtedly, it is true that the kurtosis in the various contourlet sub-bands shows a clear departure from Gaussianity, which is confirmed by the values clearly greater than 3 in all subbands. In Figure 1, based on 200 images of the dataset in [15], we can verify the above departure from normality, since the median value of all kurtosis values either in the first scale or in higher scales, are greater than 3. Non-stationary processes like natural images in the transform domain cannot be regarded as i.i.d. signals, even though they can be regarded as decorrelated signals. Nevertheless, the tractability of i.i.d. models and the low complexity of the obtained solution is a good motivation to consider this model.

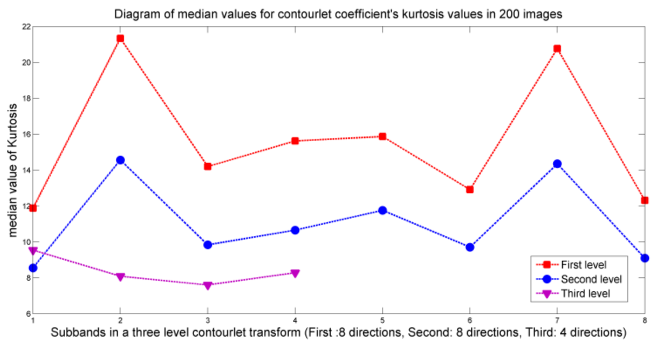

Figure 1. Median values of kurtosis values for 200 images. Contourlet transform of three levels of decomposition (8, 8 and 4 directions in each scale). Median values are getting smaller as we add more levels of decomposition, but they remain greater than the value of 3.

Most commonly, the distributions of contourlet coefficients are modeled as Generalized Gaussian density (GGD) and recently as SaS distributions [5], [8], [18]. Both models are families of distributions that are in general non-Gaussian and heavy tailed. GGD advantages are related with the availability of analytical expressions for their pdfs as well as of efficient parameter estimators [8]. Researchers in [5] claim that the SaS family of distributions are more flexible and rich. But, no closed-form expressions for the general SaS pdf are known except for the Gaussian and Cauchy members. Thus, several times due to the lack of a closed form, the Cauchy is used (e.g. in the case of transform based watermarking) [12] as a valid alternative. For example, in the watermarking problem the lack of a closed-form expression for the alpha stable distribution can result in computationally expensive solutions [18].
In addition, when one statistical model can be transformed into another one by imposing constraints on the parameters of the first model, then we define these models as nested. For example, the set of all SaS distributions has, nested within, the set of Cauchy distributions [12]. Notice that the same happens for the Student-t distribution since it has nested within it the set of Cauchy and Normal distributions [16].

3. MOTIVATION

The motivation for this work, is the GSM model [2], [3], which has been very useful for accounting for both the marginal and joint statistics of subband decomposition coefficients of natural images [2], [3] where the vector is formed by clustering a set of neighboring e.g. wavelet coefficients within a subband, or across neighbouring subbands in scale and orientation.

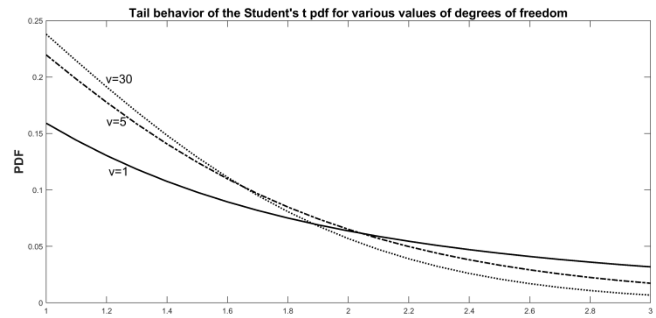

Figure 2. Tail behavior of various forms of Student's t, including the Cauchy (ν=1) and the Gaussian distribution (large values of degrees of freedom).

A GSM vector is defined as the product of a zero mean Gaussian vector and an independent positive scalar random variable [10], [11]. If we define $\boldsymbol{\tau}$ as a d-dimensional zero mean Gaussian vector and z as a positive scalar variable, then we can construct $\mathbf{x} = \sqrt{z}\boldsymbol{\tau}$ independent of z. The density of $\mathbf{x}$ is determined by the covariance matrix, $\boldsymbol{\Sigma}$, of the Gaussian vector and the density of z [3]:

$$p(x) = \int_z N_{\mathbf{x}}\left(0, z\Sigma\right) p_z\left(z\right) dz$$
$$= \int_z \frac{1}{\left(2\pi z\right)^{d/2} \left|\Sigma\right|^{1/2}} \exp\left\{\frac{\mathbf{x}^T \Sigma^{-1} \mathbf{x}}{2z}\right\} p_z\left(z\right) dz \tag{1}$$

where $p_z\left(z\right)$ is the probability density of the mixing variable z. As a family of probability densities, GSM includes many common kurtotic distributions. For example, if z follows an inverse gamma distribution, then the resulting GSM density reduces to a multivariate Student's-t distribution [2].

4. PROPOSED DISTRIBUTION

4.1. Student's t distribution modeling

In what follows, we examine properties of the class of Student's t family of distributions, and we show that these densities can accurately characterize both the marginal and joint distributions of natural image contourlet coefficients. More specifically, we define the univariate t-pdf in order to characterize the marginal statistics of contourlet subbands and as extension we provide the bivariate form of Student's t distribution trying to model the across scales dependencies between the contourlet coefficient's parents and children [4].
The proposed definition of the univariate distribution of Student's t is:

$$St_1\left(\mathrm{x}_i \mid 0, \lambda, \nu\right) = \frac{\Gamma\left(\left(\nu+1\right)/2\right)}{\Gamma\left(\nu/2\right)} \left(\frac{\lambda}{\pi\nu}\right)^{\frac{1}{2}} \left(1 + \frac{\lambda}{\nu}\mathrm{x}_i\right)^{-\frac{\nu+1}{2}} \tag{2}$$

where $i = 1,...,N$, ν is the number of degrees of freedom, $-\infty < x < +\infty$ and Γ is the Gamma function $\Gamma(\alpha) = \int_0^{\infty} x^{a-1} e^{-x} dx$. The Student's t distribution is symmetric around zero, which is consistent with the fact that its mean is 0 and skewness is also 0. Notice that the Cauchy distribution is a member of Student's t distribution with degrees of freedom equal to 1. Thus, in Figure 2, we can observe the tail behavior of various forms of t prior, due to various values of degrees of freedom. This is an indication that Student's-t is capable of expressing the fat tail and excess kurtosis more accurately than e.g. the normal distribution.

The proposed definition of the bivariate Student's t distribution with zero mean is:

$$St_2\left(\mathrm{x}_{1i,}\mathrm{x}_{2i,} \mid \lambda, \nu\right) = \frac{\Gamma\left((\nu+2)/2\right)}{\Gamma\left(\nu/2\right)} \frac{1}{\pi\nu} \left(1 + \frac{\left[x_{1i}^2 + x_{2i}^2\right]}{\nu}\right)^{-\frac{\nu+2}{2}} \quad (3)$$

TABLE I

Mean values of the degrees of freedom for 200 images in the three finest scales.

Direction	**Scale**	**Mean value of dof (ν)**
1	**I**	0.4272
2		0.5144
3		0.4757
4		0.3938
1	**II**	1.0090
2		0.6572
3		0.7441
4		1.1311
5		0.9841
6		0.7847
7		0.6215
8		0.9104
1	**III**	1.5585
2		1.0433
3		1.2316
4		1.6443
5		1.5241
6		1.2862
7		1.0811
8		1.4589

The bivariate Student's t distribution has analogous parameterization, where $i = 1,...,N$ and ν is the number of degrees of freedom, $-\infty < x < +\infty$ and Γ is the Gamma function. Notice that the multivariate Student's t distribution can be generated in a number of ways [16], where the choices for a corresponding bivariate density are discussed by Kotz and Nadarajah [17].

In order to find the employed parameters of the proposed model, we resort to the iterative EM (Expectation Maximization) algorithm as described in [7]. Notice that the form of hyper-parameterization in the proposed parametric model (random variable by itself) governs the detailed behavior of the model and thus allows a certain degree of adaptability of the model to different types of source material [1], [6].

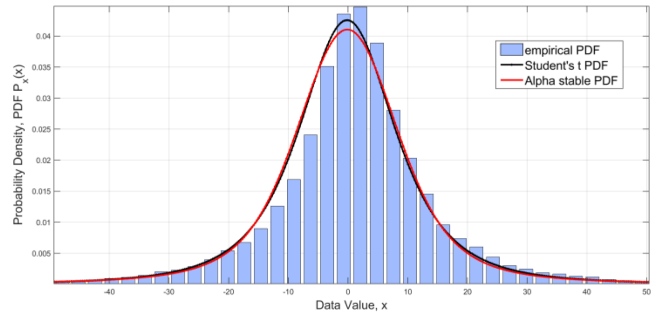

Figure 3. PDFs of empirical image data from Lena image (scale II, direction 5) and the fitted distribution of Student's t and alpha-stable distribution.

As far as the values of degrees of freedom of the proposed distribution are concerned, in Table I, we observe that these values (for 200 images with size of 512x512 of the dataset in [15]) confirm the status of non-normality of the coefficients. More importantly, various degrees of non-Gaussian characteristics are exhibited in different directional subbands of the contourlet transform.

4.2. AIC (Akaike Information Criterion)

AIC (Akaike Information Criterion) provides a means of model selection based on a goodness of fit (GoF) along with the complexity of the model under consideration [13]. Since different distributions with possibly different scalar parameter numbers can be fairly compared using the AIC approach [13], [14], the most accurate model has the smallest AIC.

Thus, in order to examine which probability model best fits our data, we resort to the (AIC) [13]. Notice that AIC does not assume that one of the candidate models is the "true" or "correct" model. Thus, all the models are treated symmetrically, unlike hypothesis testing. In addition, AIC can be used to compare nested as well as non-nested models and of course, it can also be used to compare models based on different families of probability distributions. The definition of AIC is:

$$AIC = 2q + k \ln\left(\frac{\sum_{i=1}^{k}\left[y_i - p\left(x_i\right)\right]^2}{k}\right) \quad (4)$$

where y_i denotes the histogram of the data, $p\left(x_i\right)$ the pdf of the statistical model, k is the number of bins in the histogram and q is the number of parameters in the statistical model. Towards finding the best statistical model, we select the one with the lowest AIC score.

4.3. Comparison of Marginal Statistic's models

We first investigate the marginal statistics of the contourlet coefficients of natural images. In Fig. 3, we can see a plot of the histogram of one of the eight directions of the finest subband of image Lena. There are many candidate models that could be used in our work. Due to lack of space, we compare our proposal with Symmetric alpha Stable family of distributions as it appears in [5]. In Table II, we can observe the results concerning the AIC for the alpha-stable and the proposed Student's t pdfs of image contourlet coefficients in the three finest scales (for the three known images Baboon, Barbara and Bridge, each of size 512x512).

TABLE II

AIC values for known images (D: Direction, Sc: Scale)

D	Sc	Baboon		Barbara		Bridge	
		St-t	**Stable**	**St-t**	**Stable**	**St-t**	**Stable**
1	**I**	**0.4354**	0.4361	**0.4292**	0.4300	**0.4477**	0.4487
2		**0.4180**	0.4187	**0.4037**	0.4045	**0.4215**	0.4223
3		**0.4320**	0.4333	**0.3911**	0.3916	**0.4414**	0.4420
4		**0.4489**	0.4502	**0.4023**	0.4028	**0.4771**	0.4777
1	**II**	**0.7523**	0.7540	0.7346	**0.7232**	**0.7290**	0.7308
2		**0.7768**	0.7786	0.7407	**0.7296**	**0.7881**	0.7902
3		**0.7489**	0.7509	**0.6890**	0.6898	**0.7526**	0.7547
4		**0.7097**	0.7117	**0.6246**	0.6253	**0.6899**	0.6916
5		**0.7761**	0.7782	0.6479	**0.6367**	**0.7348**	0.7367
6		**0.8061**	0.8083	**0.6219**	0.6232	**0.7782**	0.7796
7		**0.8352**	0.8372	**0.6527**	0.6535	**0.8382**	0.8391
8		**0.8176**	0.8198	**0.6787**	0.6800	**0.7699**	0.7715
1	**III**	**2.4368**	2.4438	2.1780	**2.1321**	**2.3444**	2.3532
2		**2.5676**	2.5736	2.3790	**2.3327**	**2.6026**	2.6101
3		**2.4481**	2.4547	2.3905	**2.3436**	**2.4421**	2.4508
4		**2.2982**	2.3052	2.0288	**1.9830**	**2.1974**	2.2059
5		**2.6045**	2.6115	**2.0302**	2.0323	**2.3826**	2.3909
6		**2.8169**	2.8234	**1.7283**	1.7332	**2.5241**	2.5319
7		**2.9569**	2.9622	**1.8110**	1.8132	**2.7264**	2.7318
8		**2.7620**	2.7704	**2.1578**	2.1597	**2.4986**	2.5070

These values indicate that our proposal proves to be a valid candidate (due to better fit) compared to e.g. the stable family of distributions. This kind of match appears in many other images. Therefore, for reasons of statistical significance, we apply the same criterion in the case of image dataset [15].

TABLE III

Percentage of fitted histograms based on AIC values for the dataset of 200 images [15]

	Student's t	**Stable**
Percentage	**73.03**	26.97

The results of the application, in a total of 4000 histograms, resulting from the transformation of images of the dataset into 3 levels in 4, 8 and 8 directions (20 histograms in total for each image) towards more detailed subbands, show a significant preference for the case of the proposed distribution. More specifically, almost three-quarters of the fitted histograms showed that these histograms are better fitted using Student's t compared with stable family of distributions, as shown in Table III.

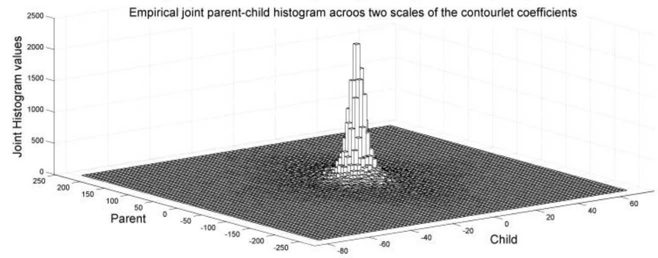

Figure 4: Empirical joint parent-children histogram across two consecutive scales (1, 2) from lower to finer details.

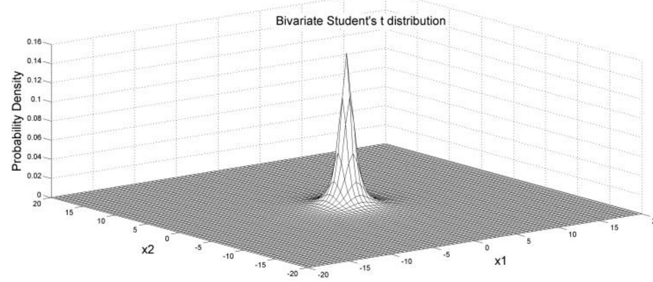

Figure 5: Possible configuration of the bivariate Student's t distribution.

4.4. Modeling as a Bivariate Distribution

Notice that, even though coefficients in different subbands are considered as uncorrelated, this does not mean that are independent. Thus, in this Section, we investigate the joint statistics of contourlet coefficients at different positions, scales and orientations. In order to do that, we introduce the bivariate Student's t pdf, given by Eq. (3), which is able to model the non-gaussian heavy-tailed behavior and the dependencies in a bivariate setting. The proposed bivariate distribution can suitably model the parent-children relationship of the contourlet coefficients across two consecutive scales, as depicted in Figures 4, 5.

5. CONCLUSIONS AND FUTURE WORK

Given the complex structure and enormous variation of the observed image contourlet domain histograms, we proposed and investigated the Student's t distribution for modeling natural images in the contourlet domain. The proposed model accounts for the statistics of a wide variety of visual images. In the future, we intend to use these results to propose new watermarking schemes where the exploitation of hidden variables could be used as a perceptual guidance towards more advanced models during watermark embedding. Since a pdf for modeling a histogram of coefficients is an important, there is also an additional factor that plays critical role in the efficiency of some prior, meaning the form of the prior (e.g. exponential based, stretched exponential, etc), which should be taken into account.

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