J Glob Optim
DOI 10.1007/s10898-007-9152-2

ORIGINAL PAPER

A filled function method for constrained global optimization

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Received: 20 March 2005 / Accepted: 15 February 2007

Abstract In this paper, a filled function method for solving constrained global optimization problems is proposed. A filled function is proposed for escaping the current local minimizer of a constrained global optimization problem by combining the idea of filled function in unconstrained global optimization and the idea of penalty function in constrained optimization. Then a filled function method for obtaining a global minimizer or an approximate global minimizer of the constrained global optimization problem is presented. Some numerical results demonstrate the efficiency of this global optimization method for solving constrained global optimization problems.

Keywords Filled function · Filled function method · Constrained global optimization

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1 Introduction

We consider the following inequality constrained global optimization problem:

$$(P) \qquad \min \quad f(x) \qquad\qquad (1.1)$$
$$\text{s.t.} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m,$$
$$x \in X,$$

where $f : X \to R$, $g_i : X \to R$, $i = 1, 2, \ldots, m$, and X is a box.

The existence of local minimizer other than global one makes global optimization a great challenge. As one of the main methods to solve general unconstrained global optimization problem without special structural property, filled function method has attracted extensive attention, see, e.g., [5,6,9,11,18–20]. The essential idea of filled function method is to construct an auxiliary function called filled function via the current local minimizer of the original optimization problem, with the property that the current local minimizer is a local maximizer of the constructed filled function and a better initial point of the primal optimization problem can be obtained by searching the constructed filled function locally. Both the theoretical and the algorithmic studies show that the filled function method is competitive with the other existing global optimization methods, such as tunneling function methods [1,10] and stochastic methods [14,15].

Constrained global optimization is an even tougher area to tackle. For some global optimization problems with nice special structural property, such as concave minimization problem, D.C. programming problems, reverse convex programming problems and monotone programming problems, there are a number of well developed methods, see, e.g., [2,7,8,13,17]. For general constrained global optimization, i.e., constrained global optimization without special structural property, there is lack of well developed methods. In this paper, we generalize the applicable area of filled function method to general constrained global optimization. The key idea here is to combine the filled function method in unconstrained global optimization proposed in Ref. [18] with penalty function methods in constrained optimization. It is should be noted that all the existing filled function methods proposed in Refs. [5,6,9,11,12,18–22] focus only on solving unconstrained global optimization problems or box constrained global optimization problems, whereas the filled function method proposed in this paper focuses on solving general constrained global optimization problems.

The rest of this paper is organized as follows. In Sect. 2, a filled function for constrained global optimization problem (P) is introduced, and then some basic properties of the proposed filled function are discussed. Especially, it is shown that an improved feasible point can be obtained by solving locally an unconstrained or a box constrained optimization problem constructed via the given filled function. In Sect. 3, we propose a global optimization method named filled function method using the presented filled function to obtain a global minimizer of the constrained global optimization problem (P). Finally, some illustrative numerical examples are given in Sect. 4.

2 Filled function of constrained global optimization and its properties

To begin with, we have the following assumptions.

Assumption 2.1 f and g_i, $i = 1, \ldots, m$, are continuously differentiable.

Let

$$S = \{x \in X \mid g_i(x) \le 0,\ i = 1, \dots, m\}, \tag{2.1}$$

$$S^\circ = \{x \in \mathrm{int}X \mid g_i(x) < 0,\ i = 1, \dots, m\}, \tag{2.2}$$

where int$A$ denotes the interior of set A.

Assumption 2.2 $S^\circ \neq \emptyset$, cl$S^\circ = S$, where cl$A$ denotes the closure of set A.

By Assumption 2.2, we know that for any $x_0 \in S$, there exists a sequence $\{x_n\} \subset S^\circ$, such that $\lim_{n\to\infty} x_n = x_0$.

Let

$$Y = \{x \in S \mid x \text{ is a local minimizer of problem}(P)\}. \tag{2.3}$$

Assumption 2.3 The set $\{f(x) \mid x \in Y\}$ is a finite set.

Definition 2.1 A point $x \in X$ is said to be a vertex of box X if $x = \lambda x_1 + (1 - \lambda)x_2$ with $x_1, x_2 \in X$ and $\lambda \in (0, 1)$ implies that $x = x_1 = x_2$.

Let $x^* \in S$ be the current local minimizer of problem (P).

Definition 2.2 A function $p(x)$ is said to be a filled function of problem (P) at a local minimizer x^* if it is continuously differentiable and satisfies the following conditions:

1° x^* is a strict local maximizer of $p(x)$ on X;
2° (i) any point $\bar{x}$ with $\nabla p(\bar{x}) = 0$ and $\bar{x} \neq x^*$ satisfies $f(\bar{x}) < f(x^*)$ and $\bar{x} \in S$;
  (ii) any local minimizer $\bar{x}$ of $p(x)$ on X satisfies
    a. $f(\bar{x}) < f(x^*)$ and $\bar{x} \in S$, or
    b. $\bar{x}$ is a vertex of X.
3° If x^* is not a global minimizer of problem (P), then there exists a point $\bar{x} \in S^\circ$, such that $\bar{x}$ is a local minimizer of $p(x)$ on X with $f(\bar{x}) < f(x^*)$.
4° For any $x_1,\ x_2 \in X$ with
  (i) $f(x_1) \ge f(x^*)$ or $x_1 \in X \setminus S$
    and
  (ii) $f(x_2) \ge f(x^*)$ or $x_2 \in X \setminus S$,
    it holds that $\|x_2 - x^*\| > \|x_1 - x^*\|$ if and only if $p(x_2) < p(x_1)$.

By the above definition, we know that if $p(x)$ is a filled function of problem (P) at a local minimizer x^*, then any infeasible point or any $x \in X$ with $x \neq x^*$ and $f(x) \ge f(x^*)$, except the vertexes of X, is neither a local minimizer of $p(x)$ on X, nor a stationary point of $p(x)$. Also we note that the current local minimizer x^* is a global minimizer of problem (P) if and only if any local minimizer of function $p(x)$ on X is a vertex of X. Condition 4° reveals some kind of algorithmic consideration for an applicable filled function.

In what follows, we will introduce a function $p_{r,c,q,x^*}(x)$ which satisfies Definition 2.2. To begin with, we design two continuously differentiable functions $f_{r,c}(t)$ and $g_r(t)$. Function $f_{r,c}(t)$ should has the following properties: $f_{r,c}(t)$ equals to a constant c when $t > 0$, and $f_{r,c}(t)$ equals to zero when t is less than a negative number $-r$. $f_{r,c}(t)$ plays the role of filling in the construction of filled function. $g_r(t)$ should have the following properties: It is larger than a positive number d when $t > 0$ and equals to zero when t is less than a negative number $-r$, in addition, it is increasing. $g_r(t)$ can be regarded as a modified penalty function and used for dealing with constraints. More specifically, for any given $c > 0$ and $r > 0$, let

$$f_{r,c}(t) = \begin{cases} c, & t \geq 0 \\ -\frac{2c}{r^3}t^3 - \frac{3c}{r^2}t^2 + c, & -r < t \leq 0 \\ 0, & t \leq -r \end{cases}, \tag{2.4}$$

$$g_r(t) = \begin{cases} t+2 & t \geq 0 \\ \frac{r-4}{r^3}t^3 + \frac{2r-6}{r^2}t^2 + t + 2 & -r < t < 0 \\ 0 & t \leq -r \end{cases}. \tag{2.5}$$

Note that the requirement for continuous differentiability of $f_{r,c}(t)$ and $g_r(t)$ justifies the use of cubic polynomials in $f_{r,c}(t)$ and $g_r(t)$.

Consequently we have that

$$f'_{r,c}(t) = \begin{cases} 0, & t \geq 0 \\ -\frac{6c}{r^3}t^2 - \frac{6c}{r^2}t, & -r < t \leq 0 \\ 0, & t \leq -r \end{cases}, \tag{2.6}$$

$$g'_r(t) = \begin{cases} 1 & t \geq 0 \\ \frac{3r-12}{r^3}t^2 + \frac{4r-12}{r^2}t + 1 & -r < t < 0 \\ 0 & t \leq -r \end{cases}. \tag{2.7}$$

Obvilousy $f_{r,c}(t)$ and $g_r(t)$ are continuously differentiable on R.

Define

$$p_{r,c,q,x^*}(x) = \frac{1}{\|x - x^*\|^2 + 1} f_{r,c}\left(g_r\left(f(x) - f(x^*)\right) + \sum_{i=1}^{m} g_{\frac{r}{q}}\left(g_i(x)\right) - 2r \right), \tag{2.8}$$

where $c > 0$, $r > 0$, and $q > 0$ are parameters.

Remark 2.1 Function $p_{r,c,q,x^*}(x)$ is essentially different from the filled function $H_{q,r,x^*}(x)$ proposed in Ref. [18]. Here $p_{r,c,q,x^*}(x)$ includes $\sum_{i=1}^{m} g_{\frac{r}{q}}\left(g_i(x)\right)$ as a penalty term to penalize unfeasible points, while $H_{q,r,x^*}(x)$ just plays the role of filling.

The following theorems shows that $p_{r,c,q,x^*}(x)$ satisfies Definition 2.2.

Theorem 2.1 *For any $c > 0$, $q > 0$, and $0 < r \leq 1$, x^* is a strict local maximizer of $p_{r,c,q,x^*}(x)$ on X.*

Proof Since x^* is a local minimizer of the original problem (P), there exists $\delta_0 > 0$, such that $f(x) \geq f(x^*)$ for any $x \in S \cap N(x^*, \delta_0)$, where $N(x^*, \delta_0)$ denotes the neighborhood of x^* with radius δ_0, i.e., $N(x^*, \delta_0) = \{x \in R^n \mid \|x - x^*\| < \delta_0\}$. Then, for any $x \in X \cap N(x^*, \delta_0)$, we have $f(x) \geq f(x^*)$ or $x \in X \setminus S$, which yields

$$g_r(f(x) - f(x^*)) + \sum_{i=1}^{m} g_{\frac{r}{q}}(g_i(x)) \geq 2.$$

Thus, when $q > 0$ and $0 < r \leq 1$, we have that

$$f_{r,c}\Big(g_r(f(x) - f(x^*)) + \sum_{i=1}^{m} g_{\frac{r}{q}}(g_i(x)) - 2r\Big) = c.$$

Therefore, we have that

$$p_{r,c,q,x^*}(x) = \frac{c}{\|x - x^*\|^2 + 1} < c = p_{r,c,q,x^*}(x^*)$$

for any $x \in X \cap (N(x^*, \delta_0) \setminus \{x^*\})$ when $0 < r \leq 1$. Thus, x^* is a strict local maximizer of $p_{r,c,q,x^*}(x)$ on X. □

In what follows we assume that $c \geq 1,\ q \geq 1$, and $r > 0$.

Theorem 2.2 (i) *Any $x \in X$ with $x \neq x^*$ and $\nabla p_{r,c,q,x^*}(x) = 0$ satisfies that $f(x) < f(x^*)$ and $x \in S$;*

(ii) *when $r \leq 1$, any local minimizer $\bar{x} \in X$ of $p_{r,c,q,x^*}(x)$ on X satisfies that $f(\bar{x}) < f(x^*)$ and $\bar{x} \in S$, or $\bar{x}$ is a vertex of X.*

Proof (i) By (2.8), we have that

$$\begin{aligned}\nabla p_{r,c,q,x^*}(x) = &-\frac{2(x - x^*)}{\Big(\|x - x^*\|^2 + 1\Big)^2} f_{r,c}\Big(g_r\Big(f(x) - f(x^*)\Big) + \sum_{i=1}^{m} g_{\frac{r}{q}}\Big(g_i(x)\Big) - 2r\Big) \\ &+ \frac{1}{\|x - x^*\|^2 + 1} \cdot f'_{r,c}\Big(g_r\Big(f(x) - f(x^*)\Big) + \sum_{i=1}^{m} g_{\frac{r}{q}}\Big(g_i(x)\Big) - 2r\Big) \\ &\cdot\Big(g'_r\Big(f(x) - f(x^*)\Big)\nabla f(x) + \sum_{i=1}^{m} g'_{\frac{r}{q}}\Big(g_i(x)\Big)\nabla g_i(x)\Big). \end{aligned} \tag{2.9}$$

For any $x \in X$ satisfying $f(x) \geq f(x^*)$ or $x \in X \setminus S$, we have that

$$g_r\Big(f(x) - f(x^*)\Big) + \sum_{i=1}^{m} g_{\frac{r}{q}}\Big(g_i(x)\Big) \geq 2.$$

When $r \leq 1$, we have that

$$g_r\Big((f(x) - f(x^*)\Big) + \sum_{i=1}^{m} g_{\frac{r}{q}}\Big(g_i(x)\Big) - 2r \geq 0,$$

which yields

$$f_{r,c}\Big(g_r\Big(f(x) - f(x^*)\Big) + \sum_{i=1}^{m} g_{\frac{r}{q}}\Big(g_i(x)\Big) - 2r\Big) = c$$

and

$$f'_{r,c}\Big(g_r\Big(f(x) - f(x^*)\Big) + \sum_{i=1}^{m} g_{\frac{r}{q}}\Big(g_i(x)\Big) - 2r\Big) = 0.$$

Thus, we have that

$$\nabla p_{r,c,q,x^*}(x) = -\frac{2c(x - x^*)}{(\|x - x^*\|^2 + 1)^2}.$$

Therefore, for any $x \in X$ with $x \neq x^*$ and $f(x) \geq f(x^*)$, or $x \in X \setminus S$, we have that $\nabla p_{r,c,q,x^*}(x) \neq 0$, furthermore, $\nabla^T p_{r,c,q,x^*}(x)d(x) < 0$, where $d(x)$ is defined as

$$d(x) = \frac{x - x^*}{\|x - x^*\|}.$$

Thus, any point $\bar{x} \neq x^*$ with $\nabla p_{r,c,q,x^*}(\bar{x}) = 0$ satisfies that

$$f(\bar{x}) < f(x^*) \text{ and } \bar{x} \in S.$$

(ii) If $\bar{x}$ is a local minimizer of $p_{r,c,q,x^*}(x)$ on X, and $\bar{x}$ is neither a point satisfying

$$f(\bar{x}) < f(x^*) \text{ and } \bar{x} \in S,$$

nor a vertex of X, then there exist two different points $z_1, z_2 \in X$ and a positive number α with $0 < \alpha < 1$ such that $\bar{x} = \alpha z_1 + (1 - \alpha)z_2$ and $p_{r,c,q,x^*}(\bar{x}) = \dfrac{c}{\|\bar{x} - x^*\|^2 + 1}$ when $0 < r \leq 1$.

Since, X is a box, we have that $[z_1, z_2] := \{\lambda z_1 + (1 - \lambda)z_2 \mid 0 \leq \lambda \leq 1\} \subset X$. Let $d_0 = z_2 - z_1, \alpha_0 = \min\{\alpha, 1 - \alpha\}$. Then, we can easily verify that for any s with $|s| \leq \alpha_0$, it holds

$$\bar{x} + sd_0 \in [z_1, z_2] \subset X.$$

Let $z_{1,\lambda} = \bar{x} + \lambda d_0$ and $z_{2,\lambda} = \bar{x} - \lambda d_0$, where $0 < |\lambda| \leq \alpha_0$. Then $z_{1,\lambda}, z_{2,\lambda} \in X$ and

$$\begin{aligned} p_{r,c,q,x^*}(z_{1,\lambda}) &\leq \frac{c}{\|z_{1,\lambda} - x^*\|^2 + 1} \\ &= \frac{c}{\|(\bar{x} - x^*) + \lambda d_0\|^2 + 1}, \\ p_{r,c,q,x^*}(z_{2,\lambda}) &\leq \frac{c}{\|z_{2,\lambda} - x^*\|^2 + 1} \\ &= \frac{c}{\|(\bar{x} - x^*) - \lambda d_0\|^2 + 1}. \end{aligned}$$

By

$$\begin{aligned} \|(\bar{x} - x^*) + \lambda d_0\|^2 + \|(\bar{x} - x^*) - \lambda d_0\|^2 &= \|\bar{x} - x^*\|^2 + \lambda^2\|d_0\|^2 \\ &\quad + 2\lambda\langle d_0, \bar{x} - x^*\rangle + \|\bar{x} - x^*\|^2 + \lambda^2\|d_0\|^2 \\ &\quad - 2\lambda\langle d_0, \bar{x} - x^*\rangle \\ &= 2\|\bar{x} - x^*\|^2 + 2\lambda^2\|d_0\|^2 \\ &> 2\|\bar{x} - x^*\|^2, \end{aligned}$$

we have that $\|(\bar{x} - x^*) + \lambda d_0\|^2$ or $\|(\bar{x} - x^*) - \lambda d_0\|^2$ is larger than $\|\bar{x} - x^*\|^2$. Thus, $p_{r,c,q,x^*}(z_{1,\lambda})$ or $p_{r,c,q,x^*}(z_{2,\lambda})$ is less than $p_{r,c,q,x^*}(\bar{x})$. Since, λ can approach 0 to any extent, we obtain that $\bar{x}$ is not a local minimizer of $p_{r,c,q,x^*}(x)$ on X. This is a contradiction. Therefore, if $\bar{x}$ is a local minimizer of $p_{r,c,q,x^*}(x)$ on X, then we have $f(\bar{x}) < f(x^*)$ and $\bar{x} \in S$, or $\bar{x}$ is a vertex of X. □

Theorem 2.3 *If x^* is not a global minimizer of the original problem (P), then there exist $r_0 > 0$, $q_0 > 0$, and $\bar{x} \in S^\circ$, such that $\bar{x}$ is a local minimizer of $p_{r,c,q,x^*}(x)$ on X with $f(\bar{x}) < f(x^*)$ when $r \leq r_0$ and $q \geq q_0$.*

Proof (i) If x^* is not a global minimizer of the original problem (P), then there exists $\bar{x}_0 \in S$ such that $f(\bar{x}_0) < f(x^*)$. Let $r_0 = \frac{f(x^*)-f(\bar{x}_0)}{2}$, then $r_0 > 0$. By Assumption 2.2, there exists $\bar{x} \in S^\circ$ such that

$$f(\bar{x}) - f(\bar{x}_0) < r_0.$$

For this r_0, there exists $q_0 > 0$, such that $g_i(\bar{x}) < -\frac{r_0}{q_0}$ for any $i = 1, \ldots, m$. Thus, when $r \le r_0$ and $q \ge q_0$, we have

$$\begin{aligned} f(\bar{x}) - f(x^*) &= f(\bar{x}) - f(\bar{x}_0) + f(\bar{x}_0) - f(x^*) \\ &< r_0 - 2r_0 \\ &\le -r, \end{aligned}$$

and $g_i(\bar{x}) < -\frac{r}{q}$ for any $i = 1, \ldots, m$, which yield $p_{r,c,q,x^*}(\bar{x}) = 0$. Note that $p_{r,c,q,x^*}(x) \ge 0$ for any $x \in X$ and $p_{r,c,q,x^*}(x^*) = c$, it follows that $\bar{x} \ne x^*$ is a global minimizer of $p_{r,c,q,x^*}(x)$ on X. Consequently, by $\bar{x} \in S^\circ$, we have that $\nabla p_{r,c,q,x^*}(\bar{x}) = 0$, which yields $f(\bar{x}) < f(x^*)$. □

Theorem 2.4 *For any* $x_1,\ x_2 \in X$ *with*

(i) $f(x_1) \ge f(x^*)$ *or* $x_1 \notin S$
and
(ii) $f(x_2) \ge f(x^*)$ *or* $x_2 \notin S$,

we have that when $0 < r \le 1$, $\|x_2 - x^*\| > \|x_1 - x^*\|$ *if and only if* $p_{r,c,q,x^*}(x_2) < p_{r,c,q,x^*}(x_1)$.

Proof For such x_i, $p_{r,c,q,x^*}(x_i) = \frac{c}{\|x_i - x^*\|^2+1}, i = 1, 2$, when $0 < r \le 1$. Thus, we have that $\|x_2 - x^*\| > \|x_1 - x^*\|$ if and only if $p_{r,c,q,x^*}(x_2) < p_{r,c,q,x^*}(x_1)$ when $0 < r \le 1$. □

3 Algorithm

We introduce here the following box constrained optimization problem named filled function problem:

$$(\bar{P}) \qquad \min_{x \in X} \ p_{r,c,q,x^*}(x).$$

Based on the proposed filled function $p_{r,c,q,x^*}(x)$, we can obtain a filled function method for the constrained global optimization problem (P). The general idea of this filled function method is as follows: if the current local minimizer x^* is not a global minimizer of (P), then we can manage to obtain a point $\bar{x} \in S$ with $f(\bar{x}) < f(x^*)$ by applying some local search schemes to problem $(\bar{P})$. Consequently we can obtain a better local minimizer of (P) by applying local search schemes to problem (P) starting from $\bar{x}$. Finally, a global minimizer or an approximate global minimizer of (P) can be obtained. The corresponding algorithm is denoted by Algorithm FFMC and detailed as follows:

3.1 Algorithm FFMC

Step 0 (a) Choose a small positive numbers μ, and a large positive number M (in the examples of Sect. 4, we take $\mu = 10^{-5}$ and $M = 10^5$). Choose a positive integer number K and directions $e_1, \ldots, e_K$ (in the numerical examples in Sect. 4, we let $K = 2n$

and let $e_i,\ i=1,\ldots,K$, be the coordinate directions, where n is the number of dimensions of the variable). Choose the initial values q_1, c_1, and r_1 for the parameters q, c, and r, respectively (in the numerical examples of Sect. 4, we take $q_1 = 100,\ c_1 = 1$, and $r_1 = 1$).

(b) Choose an initial point $x_1^0 \in X$(here x_1^0 may not be a feasible point), then use penalty function methods to find the first local minimizer x_1^* of the original problem (P). Let $k := 1, j := 1$, and $\lambda := 1$, and go to Step 1.

Step 1 Let

$$p_{r_k,c_k,q_k,x_k^*}(x) = \frac{1}{\|x-x_k^*\|^2+1} f_{r_k,c_k}\left(g_{r_k}\left(f(x)-f(x_k^*)\right)+\sum_{i=1}^{m} g_{\frac{r_k}{q_k}}\left(g_i(x)\right)-2r_k\right), \tag{3.1}$$

where $f_{r,c}(t)$ and $g_r(t)$ are defined in (2.4) and (2.5), respectively. Go to Step 2.

Step 2 If $j \le K$, choose a nonnegative λ with $\lambda \le 1$ such that $y_k^j := x_k^* + \lambda e_j \in X$, and go to Step 3; otherwise, go to Step 5.

Step 3. Search for a local minimizer of the following filled function problem starting from y_k^j (in the numerical examples of Sect. 4, we use the conjugate gradient method to search for a local minimizer of the following problem):

$$\min_{x\in X}\ p_{r_k,c_k,q_k,x_k^*}(x). \tag{3.2}$$

Once a point $y_k^* \in X$ with $f(y_k^*) < f(x_k^*)$ and $g_i(y_k^*) \le 0,\ i=1,\ldots,m$, is obtained in the process of search, set $x_{k+1}^0 := y_k^*,\ k := k+1$, and go to Step 4; otherwise continue the process. Let $\bar{x}_k^*$ be an obtained local minimizer of problem (3.2). If $\bar{x}_k^*$ satisfies $f(\bar{x}_k^*) < f(x_k^*)$ and $\bar{x}_k^* \in S$, then let $x_{k+1}^0 := \bar{x}_k^*, k := k+1$, and go to Step 4; otherwise, let $j := j+1$ and go to Step 2.

Step 4 Find a local minimizer x_k^* of the original constrained problem (P) by local search methods starting from x_k^0 (in the numerical examples of Sect. 4, we use the SQP method to obtain a local minimizer of the original problem (P)). Go to Step 1.

Step 5 If $q_k \le M$, let $q_k := 10q_k$, and $j := 1$, go to Step 1; otherwise, go to Step 6.

Step 6 If $c_k \le M$, let $c_k := 10c_k,\ q_k := q_1, j := 1$, go to Step 1; otherwise, go to Step 7.

Step 7 If $r_k \ge \mu$, let $r_k := \frac{r_k}{10}, c_k := c_1, q_k := q_1$, go to Step 1; otherwise, stop and x_k^* is a global minimizer or an approximate global minimizer of problem (P).

By Assumption 2.3, we know that the set $V := \{f(x) \mid x \in Y\}$ is a finite set, where Y is defined in (2.3). Let

$$V := \{v_1, \ldots, v_l\}, \tag{3.3}$$

$$\eta := \min\{\frac{|v_i - v_j|}{2} \mid v_i, v_j \in V, i \ne j\}. \tag{3.4}$$

For any $v_i \in V$, there exists $x_i \in Y$ such that $f(x_i) = y_i$. If x_i is not a global minimizer of problem (P), then by Assumption 2.2, there exists $\bar{x}_i \in S^\circ$ such that $f(\bar{x}_i) - f(x_i) < -\eta$.

For this η, there exists $\bar{q}_i > 0$ such that $g_j(\bar{x}_i) < -\frac{\eta}{\bar{q}_i}$ for any $j = 1, \ldots, m$. Otherwise, if x_i is a global minimizer of problem (P), then let $\bar{q}_i = 0$. Let

$$\bar{q} := \max\{\bar{q}_i, i = 1, \ldots, l\}. \tag{3.5}$$

The following assumption together with Assumptions 2.1–2.3 can ensure the convergence of Algorithm FFMC.

Assumption 3.1 $\mu \leq \eta$, $M \geq \bar{q}$, and the set $\{e_1, \ldots, e_K\}$ is large enough to ensure that if problem (P) has a better local minimizer than the current local minimizer x_k^*, then x_{k+1}^0 in Step 3 can be obtained.

Remark 3.1 Theoretically we can obtain a global optimizer in finite steps under Assumptions 2.1–2.3 and 3.1. However, in practice, some of these assumptions are frequently violated, especially Assumption 3.1. Hence we often obtain approximate global minimizers instead of global minimizers by implementing Algorithm FFMC to solve constrained global optimization problems.

4 Numerical examples

In this section we give four numerical examples to illustrate the efficiency of Algorithm FFMC. For each example, we take several initial points to start the search process. In the list of tables, we use the same notations as in Algorithm FFMC, and moreover, we use NFFE to denote the number of filled function evaluations. FORTRAN 95 is used to code the algorithm, and the NCONF subroutine, which is based on the SQP method, is used to find local minimizers of the original constrained problem when $k \geq 2$. The conjugate gradient method is used to search for local minimizers of the filled function problems.

Example 4.1

$$\begin{aligned}
\min \quad & f(x) = x_1^2 + x_2^2 - \cos(17x_1) - \cos(17x_2) + 3 \\
\text{s.t.} \quad & g_1(x) = (x_1 - 2)^2 + x_2^2 - 1.6^2 \leq 0, \\
& g_2(x) = x_1^2 + (x_2 - 3)^2 - 2.7^2 \leq 0, \\
& x \in X = \{(x_1, x_2) \mid 0 \leq x_i \leq 2,\ i = 1, 2\}.
\end{aligned}$$

This example is taken from Ref. [16]. It is a nonconvex problem with 22 local optimal solutions in the interior of the feasible region. By Sun and Li [16], we know that $x^* = (0.7255,\ 0.3993)$ is a global minimum of Example 4.1 with global optimal value $f^* = 1.8376$. Table 1 records the numerical results of Example 4.1.

Table 1 Numerical results for Example 4.1

k	x_k^0	$f(x_k^0)$	x_k^*	$f(x_k^*)$	NFFE
1	$\begin{pmatrix}1.000000\\1.000000\end{pmatrix}$	5.550327	$\begin{pmatrix}1.101169\\1.101011\end{pmatrix}$	3.441944	512
2	$\begin{pmatrix}0.8053394\\0.8051808\end{pmatrix}$	3.431020	$\begin{pmatrix}0.7339468\\0.7339482\end{pmatrix}$	2.085320	366
3	$\begin{pmatrix}0.7000732\\0.3899146\end{pmatrix}$	1.914345	$\begin{pmatrix}0.7251187\\0.3991908\end{pmatrix}$	1.837547	Sum: 878
1	$\begin{pmatrix}0.5000000\\0.5000000\end{pmatrix}$	4.704024	$\begin{pmatrix}(0.4395753\\0.3536631\end{pmatrix}$	1.982740	52
2	$\begin{pmatrix}0.7103605\\0.3991448\end{pmatrix}$	1.905220	$\begin{pmatrix}0.7250241\\0.3991590\end{pmatrix}$	1.837506	Sum: 52
1	$\begin{pmatrix}1.500000\\1.500000\end{pmatrix}$	5.633370	$\begin{pmatrix}0.4396234\\0.3538790\end{pmatrix}$	1.982720	52
2	$\begin{pmatrix}0.7103826\\0.399364\end{pmatrix}$	1.907054	$\begin{pmatrix}0.7250276\\0.3991598\end{pmatrix}$	1.837504	Sum: 52
1	$\begin{pmatrix}2.000000\\2.000000\end{pmatrix}$	12.6971	$\begin{pmatrix}0.4396208\\0.3538660\end{pmatrix}$	1.982726	52
2	$\begin{pmatrix}0.7103814\\0.3993511\end{pmatrix}$	1.906942	$\begin{pmatrix}0.7250268\\0.3991596\end{pmatrix}$	1.837504	Sum: 52
1	$\begin{pmatrix}2.000000\\1.000000\end{pmatrix}$	9.123734	$\begin{pmatrix}1.835084\\1.100764\end{pmatrix}$	5.612519	80
2	$\begin{pmatrix}1.516515\\1.182418\end{pmatrix}$	5.586804	$\begin{pmatrix}0.4396137\\0.3538310\end{pmatrix}$	1.982740	52
3	$\begin{pmatrix}0.7103785\\0.3993152\end{pmatrix}$	1.906637	$\begin{pmatrix}0.7250262\\0.3991594\end{pmatrix}$	1.837505	Sum: 132

Example 4.2

$$
\begin{aligned}
\min \quad & f(x) = -25(x_1-2)^2 - (x_2-2)^2 - (x_3-1)^2 \\
& \qquad\qquad - (x_4-4)^2 - (x_5-1)^2 - (x_6-4)^2 \\
\text{s.t.} \quad & (x_3-3)^2 + x_4 \geq 4 \\
& (x_5-3)^2 + x_6 \geq 4 \\
& x_1 - 3x_2 \leq 2 \\
& -x_1 + x_2 \leq 2 \\
& x_1 + x_2 \leq 6 \\
& x_1 + x_2 \geq 2 \\
& x \in X = \left\{ x \;\middle|\; \begin{array}{c} 0 \leq x_1 \leq 6 \\ 0 \leq x_2 \leq 8 \\ 1 \leq x_3 \leq 5 \\ 0 \leq x_4 \leq 6 \\ 1 \leq x_5 \leq 5 \\ 0 \leq x_6 \leq 10 \end{array} \right\}.
\end{aligned}
$$

This example is taken from Ref. [4]. We know from Ref. [4] that there exist 18 local minima and that $x^* = (5, 1, 5, 0, 5, 10)$ is a global minimum of Example 4.2 with global optimal value $f^* = -310$. Table 2 records the numerical results of Example 4.2.

Table 2 Numerical results for Example 4.2

k	x_k^0	$f(x_k^0)$	x_k^*	$f(x_k^*)$	NFFE
1	(3, 3, 3, 3, 3, 3)	−36	(5, 1, 5, 3.997948, 5, 3.997948)	−258.0000	1162
2	(5.1, 1, 5, 3.997948, 5, 3.997948)	−273.25	(5.1,1,5,3.997948,5,3.997948)	−273.2500	7146
3	(5.2,1.1,5,3.997948,5,3.997948)	−288.8100	(5,1,5,0,5,0)	−290.0000	7893
4	(5,1.000002,5,0,5,8.132071)	−291.0740	(5,1,5,0,5,10)	−310.0000	Sum: 16201
1	(4,4,4,4,4,4)	−122	(5, 1, 5, 4, 5, 4)	−258.0000	1162
2	(5, 1, 5, 5.002001, 5, 4)	−259.0040	(5, 1, 5, 6, 5, 4)	−262.0000	3371
3	(5, 1, 5, 1.757979, 5, 4)	−263.0267	(5, 1, 5, 0, 5, 4)	−274.0000	9468
4	(5, 1, 5, 0, 5, 5.002001)	−275.0040	(5, 1, 5, 0, 5, 10)	−310.0000	Sum: 14001
1	(3, 3, 4, 4, 3, 5)	−40	(5, 1, 5, 4, 5, 10)	−294.0000	1162
2	(5, 1, 5, 1.757979, 5, 10)	−299.0267	(5, 1, 5, 0, 5, 10)	−310.0000	Sum: 1162
1	(2, 2, 3, 2, 3, 2)	−16	(5, 1, 5, 6, 5, 10)	−298.0000	1162
2	(5, 1, 5, 1.757979, 5, 10)	−299.0267	(5, 1, 5, 0, 5, 10)	−310.0000	Sum: 1162
1	(4, 7, 4, 5, 4, 7)	−153	(5, 1, 5, 6, 5, 10)	−298.0000	1162
2	(5, 1, 5, 1.757979, 5, 10)	−299.0266	(5, 1, 5, 0, 5, 10)	−310.0000	Sum: 1162

Table 3 Numerical results for Example 4.3

k	x_k^0	$f(x_k^0)$	x_k^*	$f(x_k^*)$	NFFE
1	(0, 0)	0	(0.6115744, 3.442064)	−4.053638	12228
2	(1.544067, 2.509580)	−4.053647	(1.599621,2.820364)	−4.419985	31210
3	(2.165317, 2.254670)	−4.419987	(2.329527, 3.178482)	−5.508009	Sum: 43438
1	(2.500000, 2.500000)	−5.000000	(2.329529, 3.178479)	−5.508008	0
1	(0.6000000, 0.8000000)	−1.400000	(0.6116033, 3.442105)	−4.053708	12228
2	(1.544096, 2.509621)	−4.053716	(1.599621, 2.820364)	−4.419985	31210
3	(2.165317, 2.254670)	−4.419987	(2.329527, 3.178482)	−5.508009	Sum: 43438
1	(1.000000, 1.500000)	−2.500000	(2.329529, 3.178479)	−5.508008	0

Example 4.3

$$
\begin{aligned}
\min\ & -x-y \\
\text{s.t.}\ & y \le 2x^4 - 8x^3 + 8x^2 + 2 \\
& y \le 4x^4 - 32x^3 + 88x^2 - 96x + 36 \\
& 0 \le x \le 3 \\
& 0 \le y \le 4.
\end{aligned}
$$

This problem is taken from Ref. [3]. We know from Ref. [3] that the best known global solution is $x^* = (2.3295, 3.1783)$ with function value $f(x^*) = -5.5079$.

Table 3 records the numerical results of Example 4.3.

Example 4.4

$$
\begin{aligned}
\min\ & 37.293239x_1 + 0.8356891x_1x_5 + 5.3578547x_3^2 - 40792.141\\
\text{s.t.}\ & -0.0022053x_3x_5 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 6.665593 \le 0\\
& 0.0022053x_3x_5 - 0.0056858x_2x_5 - 0.0006262x_1x_4 - 85.334407 \le 0\\
& 0.0071317x_2x_5 + 0.0021813x_3^2 + 0.0029955x_1x_2 - 29.48751 \le 0\\
& -0.0071317x_2x_5 - 0.0021813x_3^2 - 0.0029955x_1x_2 + 9.48751 \le 0\\
& 0.0047026x_3x_5 + 0.0019085x_3x_4 + 0.0012547x_1x_3 - 15.699039 \le 0\\
& -0.0047026x_3x_5 - 0.0019085x_3x_4 - 0.0012547x_1x_3 + 10.699039 \le 0\\
& x \in X = \left\{ x \,\middle|\, \begin{array}{c} 78 \le x_1 \le 102\\ 33 \le x_2 \le 45\\ 27 \le x_3 \le 45\\ 27 \le x_4 \le 45\\ 27 \le x_5 \le 45 \end{array} \right\}.
\end{aligned}
$$

This problem is also taken from Ref. [3]. We know from Ref. [3] that the best known global solution is $x^* = (78, 33, 29.9953, 45, 36.7758)$ with function value $f(x^*) = -30665.5387$.

Table 4 Numerical results for Example 4.4

k	x_k^0	$f(x_k^0)$	x_k^*	$f(x_k^*)$	NFFE
1	$\begin{pmatrix}90\\33.00000\\35.00000\\35.00000\\40.00000\end{pmatrix}$	−27863.90	$\begin{pmatrix}88.73555\\32.98778\\30.22352\\34.99693\\38.97875\end{pmatrix}$	−29698.23	274
2	$\begin{pmatrix}84.23562\\32.98778\\30.22352\\34.99693\\38.97875\end{pmatrix}$	−30012.63	$\begin{pmatrix}84.23562\\32.98778\\30.22352\\34.99693\\38.97875\end{pmatrix}$	−30012.63	45
3	$\begin{pmatrix}80.56554\\32.98511\\30.75649\\35.99625\\38.90526\end{pmatrix}$	−30099.86	$\begin{pmatrix}82.93620\\32.98778\\30.22874\\34.99778\\38.98085\end{pmatrix}$	−30101.58	24
4	$\begin{pmatrix}82.63790\\32.98778\\30.24487\\35.00041\\38.98732\end{pmatrix}$	−30116.75	$\begin{pmatrix}78.00000\\33.00000\\29.99525\\45.00000\\36.77581\end{pmatrix}$	−30665.54	Sum: 353
1	$\begin{Bmatrix}90.00000\\39.00000\\36.00000\\36.00000\\36.00000\end{Bmatrix}$	−27784.34	$\begin{Bmatrix}78.00000\\33.00000\\29.99526\\45.00000\\36.77582\end{Bmatrix}$	−30665.54	0
1	$\begin{Bmatrix}80.00000\\45.00000\\40.00000\\45.00000\\27.00000\end{Bmatrix}$	−27431.03	$\begin{Bmatrix}78.00000\\33.00000\\29.99525\\45.00000\\36.77581\end{Bmatrix}$	−30665.54	0

Table 4 records the numerical results of Example 4.4.

Acknowledgements This work is partly supported by Australia Research Council Project Grant and by Program for New Century Excellent Talents in University of China.

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