Clique-width of path powers*

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Abstract

We describe the clique-width of path powers by an exact formula, depending only on the number of vertices and the clique number. As a consequence, the clique-width of path powers can be computed in linear time. Path powers are a graph class of unbounded clique-width. Prior to our result, square grids constituted the only known graph class of unbounded clique-width with a similar result. We also show that clique-width and linear clique-width coincide on path powers.

1 Introduction

Clique-width is a highly useful graph parameter [4, 5, 7], since the model-checking problem for monadic second-order logic is tractable on graphs of bounded clique-width [6, 20]. Clique-width is a graph parameter that is notoriously difficult to compute. Very little is known about tractable and intractable cases of the clique-width computation problem, and about the structure of graphs of bounded clique-width. Fellows et al. showed the intractability of the clique-width computation problem for general graphs [8], however, no intractability result for restricted graph classes is known. Conversely, the only known case of a graph class of unbounded clique-width with a tractable clique-width computation is the class of square grids [10].

In this paper, we study clique-width on path powers. Path powers are the powers of induced – or *chordless* – paths. Equivalently, a path power is a graph that admits a vertex ordering so that vertices are adjacent if and only if they are at distance at most k in the vertex ordering for some positive integer k. Path powers are very structured graphs of unbounded clique-width [10]. We show that the clique-width of a path power on n vertices and of clique number ω can be precisely determined as a function in n and ω only. Since pathwidth plus 2 is a general clique-width upper bound [8], the clique-width of a path power is at most $\omega + 1$. For a lower bound, Golumbic and Rotics showed that the clique-width of path powers on at least ω^2 vertices is at least ω [10]. Note that the gap between the known lower and upper bounds of path powers can be arbitrarily large. For example for a k-path power on $2k$ vertices the results of [10] yield a lower bound of 2 and the results of [8] give an upper bound of $k + 2$. We close the remaining gap between the known lower and upper bound and show that the clique-width of a path power

*Preliminary versions of this work were presented at TAMC 2009 [12] and CSR 2011 [14].

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is at least $\omega + 1$ if and only if it has at least $(\omega - 1)\omega + 2$ vertices. The special case of $\omega = 2$ has as consequence the known result that graphs that are not cographs are of clique-width at least 3 [7]: a path power of clique number 2 on $(2-1)2+2$ vertices is an induced path on four vertices, usually denoted as P_4. Since cographs are the graphs that do not have the P_4 as an induced subgraph [2], graphs that are not cographs and therefore have the P_4 as an induced subgraph are of clique-width at least 3.

Path powers are contained in many well-studied graph classes, such as proper interval graphs, interval graphs, chordal graphs and AT-free graphs [1]. Our result therefore has direct consequences for algorithmic problems, such as the design of efficient clique-width computation algorithms for these graph classes. Such an algorithm has to determine the path powers dependency function, that is the main result of this paper, implicitly, by searching a bounded solution space, or explicitly, when built on a correctness proof. In view of our result, it is worth mentioning that hereditary graph classes that do not contain all path powers either have bounded clique-width or are not subclasses of the proper interval graphs [15].

The main challenge of this paper is a tight lower bound. We employ a recent characterisation of clique-width through partition trees [3], rooted trees with a 1-to-1 correspondence between the leaves and the vertices of the input graph and whose inner nodes are labelled with partitions of subsets of the vertex set of the given graph. The construction of partition trees is in two phases: first determining the rooted tree and the leaf-vertex correspondence, and second assigning the partitions to the inner nodes. For the clique-width variant of linear clique-width, the first phase reduces to choosing a vertex ordering for the input graph, and the second phase is the evaluation of an easy-computable function [3, 11, 13, 16]. For clique-width, every rooted tree is eligible, and the second phase is generally intractable [19]. The major part of this paper, Section 4, contributes insights into the first phase and proves a strong result about the structure of good partition trees. In Section 5, we apply this structure result to prove a lower clique-width bound for path powers, and we prove the main result. The matching upper bound is given by an algorithm for computing a linear clique-width expression in Section 3.

2 Preliminaries

Our considered graphs are simple, finite, undirected. For a graph $G = (V, E)$, $V = V(G)$ is the *vertex set* of G and $E = E(G)$ is the *edge set* of G. Edges are denoted as uv, where the vertices u and v are *adjacent* in G, or u is a *neighbour* of v in G. If uv is not an edge of G then u and v are *non-adjacent* in G. For a vertex u of G, the *neighbourhood* of u, $N_G(u)$, is the set of neighbours of u in G. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set $X \subseteq V(G)$, the subgraph of G *induced* by X, $G[X]$, is the subgraph H of G with vertex set X where for every vertex pair u, v of H, $uv \notin E(H)$ implies $uv \notin E(G)$. For $X, Y \subseteq V(G)$, where $X \cap Y = \emptyset$, $X \times Y$ denotes the set $\{xy : x \in X \text{ and } y \in Y\}$.

A *clique* of G is a set of pairwise adjacent vertices of G, and a *maximal clique* of G is a clique of G that is not properly contained in any clique of G. An *independent set* of G is a set of pairwise non-adjacent vertices of G.

Clique-width is an algebraically defined graph parameter. We quickly present the necessary terminology. For a deeper and more comprehensive treatment, we refer to the known sources, such as [3, 4, 5, 7].

Let $k \geq 1$. A *k-labelled graph* is an ordered pair (G, ℓ) where G is a graph and ℓ assigns a label from $\{1, \ldots, k\}$ to each vertex of G. A k-labelled graph has a *k-expression* if it can be built inductively using the following operations: $o(u)$ for creating a k-labelled single-vertex graph with vertex u whose label is o, $\eta_{s,o}$, where $s \neq o$, for adding the missing edges between the vertices with label s and with label o, $\rho_{s \to o}$, where $s \neq o$, for replacing the label of the vertices with label s by label o, and $\oplus$ for the disjoint union of two k-labelled graphs. A k-expression is *linear* if the disjoint union is always applied to at most one k-labelled graph on at least two vertices. For a k-expression α, $\mathrm{val}^\circ(\alpha)$ is the graph inductively built by α and without the vertex labels.

Let G be a graph. The *clique-width* of G, $\mathrm{cwd}(G)$, is the smallest integer k with $k \geq 1$ such that there is a label assignment ℓ so that (G, ℓ) is a k-labelled graph with a k-expression. And the *linear clique-width* of G, $\mathrm{lcwd}(G)$, is the smallest integer k with $k \geq 1$ such that there is an ℓ such that (G, ℓ) is a k-labelled graph with a linear k-expression. Observe that $\mathrm{cwd}(G) \leq \mathrm{lcwd}(G)$.

We use (linear) k-expressions for proving upper bounds on the (linear) clique-width of graphs. We use a characterisation of clique-width for proving lower bounds on the clique-width of graphs.

A *binary rooted tree* is a tree with a root and each of whose inner nodes have exactly two children. Nodes of trees are denoted as $\underline{a}$. A *partial partition* of a set S is a partition of a subset of S. For partitions $\mathcal{P}$ and $\mathcal{R}$ of a set S, $\mathcal{P}$ *refines* $\mathcal{R}$, $\mathcal{P} \sqsubseteq \mathcal{R}$, if each member of $\mathcal{P}$ is contained as a subset in a member of $\mathcal{R}$.

Definition 2.1. *Let G be a graph. A* partition tree *for G is an ordered pair (T, f) where T is a binary rooted tree and f is an assignment of partial partitions of $V(G)$ to the nodes of T that satisfies the following three conditions, where for a node $\underline{a}$ of T, $\overline{f}(\underline{a})$ denotes the union of the members of $f(\underline{a})$:*

1) *for every $u \in V(G)$, there is a leaf $\underline{a}$ of T such that $f(\underline{a}) = \big\{\{u\}\big\}$*

2) *for every inner node $\underline{a}$ of T with $\underline{b}$ and $\underline{c}$ its children, $\overline{f}(\underline{b})$ and $\overline{f}(\underline{c})$ are disjoint and $f(\underline{b}) \cup f(\underline{c}) \sqsubseteq f(\underline{a})$*

3) *for every inner node $\underline{a}$ of T with $\underline{b}$ and $\underline{c}$ its children: for every member pair X, Y of $f(\underline{a})$, if $X \cap \overline{f}(\underline{b})$ contains a vertex u and $Y \cap \overline{f}(\underline{c})$ contains a vertex v such that $uv \in E(G)$ then $X \times Y \subseteq E(G)$.*

The width *of (T, f) is the maximum cardinality $|f(\underline{a})|$ taken over the nodes $\underline{a}$ of T.*

Recall that we consider only simple graphs. Thus, $X \times Y \subseteq E(G)$ in the third condition of Definition 2.1 implicitly requires $X \cap Y = \emptyset$, and since X and Y are both non-empty, $X \neq Y$ directly follows. The members of $f(\underline{a})$ will be called *partition classes* of $f(\underline{a})$.

An example and the main notions of partition trees are discussed in the Appendix.

Theorem 2.2 ([3])**.** *For $k \geq 1$, a graph G has clique-width at most k if and only if G has a partition tree of width at most k.*

For proving a lower bound on the clique-width of a graph, we prove a lower bound on the width of a partition tree for the graph, and the lower-bound result translates according to Theorem 2.2. The lower-bound proof mainly exploits the power and properties of the third condition of Definition 2.1. This condition is called the *Compatibility condition* [3].

The intuition behind Theorem 2.2 and the Compatibility condition can be seen as follows. Let (T, f) be a partition tree for a graph G, and let $\underline{a}$ be an inner node of T with $\underline{b}$ and $\underline{c}$ its children. Consider the subgraph defined by the subtree of T rooted at $\underline{a}$. The two subtrees rooted at $\underline{b}$ and $\underline{c}$ partition the given vertex set. In connection to clique-width, every vertex in such a partition receives a label from $\{1, \ldots, k\}$. According to Definition 2.1, two partition classes of $f(\underline{b})$ and $f(\underline{c})$ are either merged into one partition class of $f(\underline{a})$, or belong to two different partition classes of $f(\underline{a})$. In the former case we cannot have an edge between the vertices of the different partition classes because that would require a new label to the corresponding vertices. For the later case, if we have an edge between the vertices of the different partition classes then each vertex of the one partition class is adjacent to every vertex of the other partition class. Both situations are described formally in the Compatibility condition as shown in [3]. Let X be a partition class of $f(\underline{a})$. Then the Compatibility condition implies:

- the vertices in $X \cap \overline{f}(\underline{b})$ and $X \setminus \overline{f}(\underline{b})$ are non-adjacent in G
- the vertices in $X \cap \overline{f}(\underline{b})$ have the same neighbours in $V(G) \setminus \overline{f}(\underline{b})$, and the vertices in X have the same neighbours in $V(G) \setminus \overline{f}(\underline{a})$.

We often apply the latter two implications to show a violation of the Compatibility condition, in order to refute some assumptions. A violation is shown by providing a vertex that is not in $\overline{f}(\underline{b})$ or in $\overline{f}(\underline{a})$ and that is adjacent to some vertex and non-adjacent to another vertex in $X \cap \overline{f}(\underline{b})$ or X, respectively. A particular and interesting case is when $\overline{f}(\underline{b})$ itself is a partition class of $f(\underline{a})$. Then, $\overline{f}(\underline{b})$ is a module of G, also called a *homogeneous set*.

3 Path powers and a clique-width upper bound

We define the graph class of path powers, and we show two upper bounds on the linear clique-width of path powers.

Let G be a graph on n vertices, and let k be a positive integer, i.e., $k \geq 1$. We say that G is a *k-path power* if G admits a vertex ordering $\langle u_1, \ldots, u_n \rangle$ such that for every $1 \leq i < j \leq n$, $u_i u_j \in E(G)$ if and only if $j - i \leq k$. A vertex ordering satisfying this adjacency condition is called a *k-path layout*. It follows that a graph is a k-path power if and only if it has a k-path layout. A graph is a *path power* if it is a k-path power for some positive integer k. As an observation: a k-path power on at most $k+1$ vertices is a complete graph.

Information on path powers and their relationship to other graph classes, such as the superclass of proper interval graphs, their properties and important results can be found in the monographs by Brandstädt, Le, Spinrad [1] and Golumbic [9], and in the work of Roberts [21].

Our upper-bound result below is shown by two algorithmic strategies, that exploit the linear and the 2-dimensional structure of path powers.

Lemma 3.1. *Let k, r, n be positive integers where $k+2 \leq n \leq r(k+1)+1$. Let P be a k-path power on n vertices. Then,* $\mathrm{lcwd}(P) \leq \min\{k+2, r+1\}$.

Proof. We prove the two inequalities separately. Let $\langle u_1, \ldots, u_n \rangle$ be a k-path layout for P, that exists. We prove the first inequality, $\mathrm{lcwd}(P) \leq k+2$, by the following linear $(k+2)$-expressions,

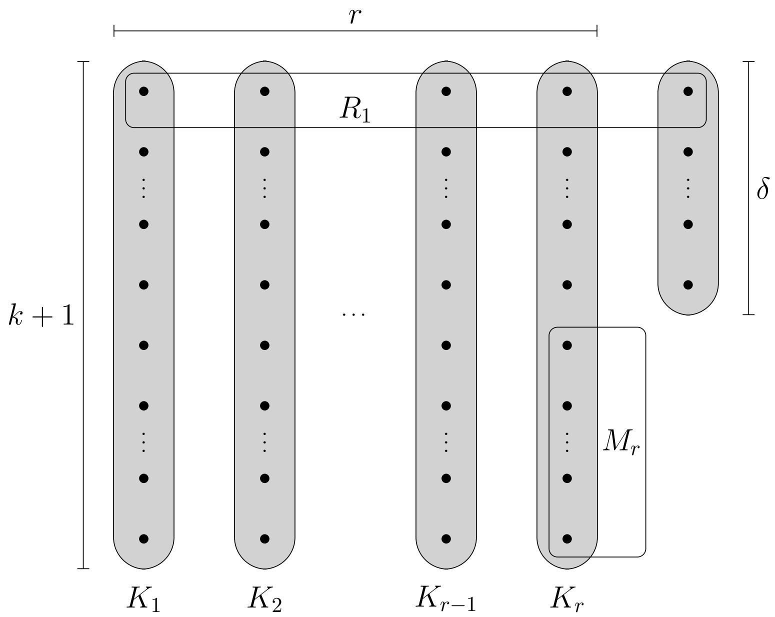


Figure 1: The 2-dimensional structure of a k-path power on $n = r(k+1)+\delta$ vertices. Each column K_j is a maximal clique, and $M_j \subseteq K_j$. The edges between consecutive columns are shown in Figure 3(a).

for $k+2 \leq i \leq n$:

$$\begin{aligned} \alpha_{k+1} &=_{\text{def}} \eta_{\{2,\ldots,k+2\},\{2,\ldots,k+2\}}\big(2(u_1) \oplus 3(u_2) \oplus \cdots \oplus (k+2)(u_{k+1})\big) \\ \alpha_i &=_{\text{def}} \eta_{\{2,\ldots,k+1\},\{k+2\}}\big(\rho_{(k+2)\to(k+1)\to\cdots\to 2\to 1}(\alpha_{i-1}) \oplus (k+2)(u_i)\big)\,. \end{aligned}$$

We use $\eta_{\{2,\ldots,k+2\},\{2,\ldots,k+2\}}$, and similar, as a short notation of a sequence of $\eta_{s,o}$-operations for all $s, o \in \{2,\ldots,k+2\}$ where $s \neq o$, and $\rho_{(k+2)\to(k+1)\to\cdots\to 2\to 1}$ is short for $\rho_{(k+2)\to\cdots\to 3\to 2}(\rho_{2\to 1})$, expanded recursively. It is a straightforward verification to show that $\text{val}^\circ(\alpha_{k+1})$ is the complete graph $P[\{u_1,\ldots,u_{k+1}\}]$ and $\text{val}^\circ(\alpha_i)$ is equal to $P[\{u_1,\ldots,u_i\}]$. Since α_n is a linear $(k+2)$-expression, $\text{lcwd}(P) \leq k+2$ is proved.

We prove the second inequality, $\text{lcwd}(P) \leq r+1$. Since the clique-width of an induced subgraph is not larger than the clique-width of the graph itself, we can assume $n = r(k+1)+1$. We partition the vertex set of P into r "columns" of size $k+1$ each, leaving out the last vertex: $K_0 =_{\text{def}} \emptyset$ for convenience, and for $1 \leq j \leq r$, let $K_j =_{\text{def}} \{u_{(j-1)(k+1)+1},\ldots,u_{j(k+1)}\}$. Figure 1 shows the columns K_j of the considered graph. We also define a process order π on the vertex set of P: for $1 \leq i \leq n$, let

$$\pi(i) \;=_{\text{def}}\; \begin{cases} (i-1)(k+1)+1 & ,\ \text{if } i \leq r+1 \\ 2 & ,\ \text{if } i = r+2 \\ \pi(i-1)+(k+1) & ,\ \text{if } i \geq r+3 \text{ and } \pi(i-1)+k+1 \leq n \\ \pi(i-1)-(r-1)(k+1)+1 & ,\ \text{if } i \geq r+3 \text{ and } \pi(i-1)+k+1 > n\,. \end{cases}$$

The definitions of $K_1,\ldots,K_r$ and π are illustrated in Figures 1 and 2. What is important to note about the process order π is: for each $1 \leq i \leq n$, if vertex $u_{\pi(i)}$ is in column K_j

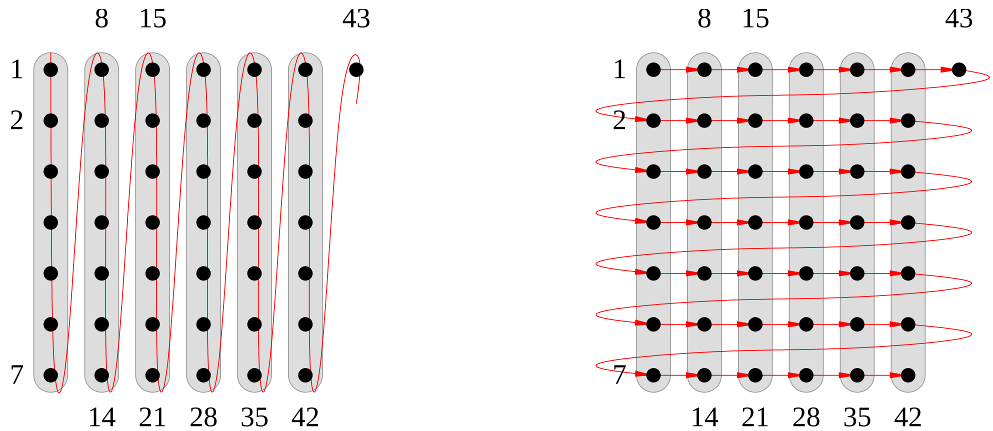


Figure 2: We consider a vertex layout of a 6-path power on 43 vertices, and the figures illustrate the partitions defined in the second part of the proof of Lemma 3.1, where $r = 6$. The attributed numbers are vertex names, or positions in the 6-path layout. The left-side figure illustrates the partition of the vertex set into the six "columns", that are represented by the filled regions. The red curve aligns the vertices according to the given 6-path layout. The right-side figure illustrates the definition of process order π, by following the red curve.

then the vertices in $\{u_{\pi(i+1)}, \ldots, u_{\pi(n)}\}$ that are from columns K_{j-1} and K_j are exactly the neighbours of $u_{\pi(i)}$ among the vertices in $\{u_{\pi(i+1)}, \ldots, u_{\pi(n)}\}$ (see also Figure 3 (a)). In other words: $N_P(u_{\pi(i)}) \cap \{u_{\pi(i+1)}, \ldots, u_{\pi(n)}\} = \{u_{\pi(i+1)}, \ldots, u_{\pi(n)}\} \cap (K_{j-1} \cup K_j)$.

We define the following linear $(r+1)$-expressions, for $1 \leq i < n$ where $i \neq r$ and $i \neq r+1$ and $u_{\pi(i)} \in K_j$ for some $1 \leq j \leq r$, following the order $\alpha_n, \ldots, \alpha_{r+2}, \alpha_{r+1}, \alpha_r, \alpha_{r-1}, \ldots, \alpha_1$:

$$\begin{aligned}
\alpha_n &=_{\text{def}} \; r(u_{n-1}) \\
\alpha_i &=_{\text{def}} \; \rho_{(r+1)\to j}(\eta_{\{j-1,j\},\{r+1\}}(\alpha_{i+1} \oplus (r+1)(u_{\pi(i)}))) \\
\alpha_{r+1} &=_{\text{def}} \; \eta_{\{r-1,r\},\{r+1\}}(\alpha_{r+2} \oplus (r+1)(u_{n-k-1})) \\
\alpha_r &=_{\text{def}} \; \rho_{(r+1)\to r}(\eta_{r,(r+1)}(\alpha_{r+1} \oplus (r+1)(u_n)))\,.
\end{aligned}$$

Recall from the definition of π: $\pi(r) = n-k-1$ and $\pi(r+1) = n$. For ease of presentation, the definition of α_i may involve a "non-label" 0, that occurs in case of $j = 1$: then, $\eta_{\{j-1,j\},\{r+1\}}$ is implicitly replaced by $\eta_{j,(r+1)}$.

The final linear $(r+1)$-expression for P is α_1. The correctness is verified by applying above's observation about the neighbours of $u_{\pi(i)}$ and the fact that the already processed vertices of column j have label j, ignoring u_n and its label r. ∎

Remarkable about Lemma 3.1 is the two fully different constructions in the proof. Given a k-path power P on at most $r(k+1)+1$ vertices, the good upper bound on the linear clique-width of P is determined by the relationship between k and r: if r is larger than k then $k+2$ is a good upper bound, and otherwise, $r+1$ is a good upper bound. Naturally, the proof of Lemma 3.1 cannot certify the necessity or even optimality of the second construction. The results of the next sections, however, can be seen as such a proof.

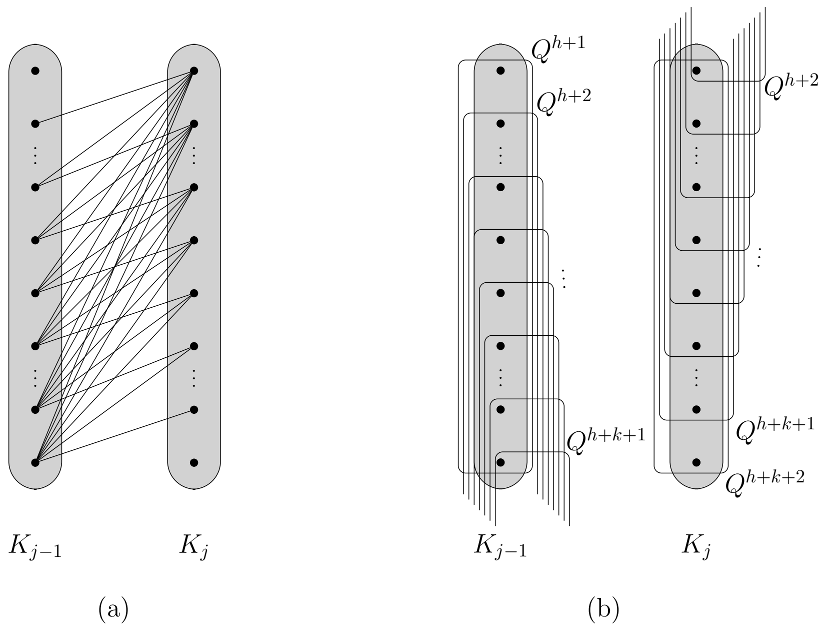


Figure 3: (a) The edges between the vertices of columns K_{j-1} and K_j. (b) The structure of the $k+2$ maximal cliques $Q^{h+1}, \dots, Q^{h+k+2}$ containing vertices from K_{j-1} and K_j, where $h = (j-2)(k+1)$.

4 Structure of good partition trees

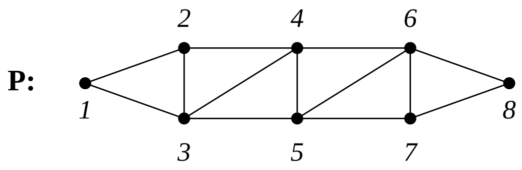
We aim for a lower bound on the clique-width of path powers. We prove this lower bound in two steps and for partition trees. In this section, we consider a first step, that provides a result about the structure of partition trees of small width. We lay out this section as a blueprint for similar such results and consider a more general setting.

We fix the following stipulations for this section. Let G be a graph, and let P be an induced subgraph of G on n vertices. Let $k \geq 2$, and we assume that P is a k-path power. Throughout this section, we assume $n \geq 3k+1$. Let $\langle u_1, \dots, u_n \rangle$ be a k-path layout for P. Since $\langle u_n, \dots, u_1 \rangle$ is also a k-path layout for P, it is easy to verify that $\varphi : V(P) \to V(P), u_i \mapsto u_{n-i+1}$ is an automorphism for P.

The maximal cliques of P play a central role here. For $1 \leq h \leq n-k$, let $Q^h =_{\text{def}} \{u_h, \dots, u_{h+k}\}$. Since the maximal cliques of P are exactly the sets of $k+1$ vertices appearing consecutively in $\langle u_1, \dots, u_n \rangle$, $Q^1, \dots, Q^{n-k}$ are exactly the maximal cliques of P. In Figure 3 (b) we list all maximal cliques between $2(k+1)$ consecutive vertices of a k-path layout.

Let $X \subseteq V(G)$. We say that a maximal clique Q of P is *full* in X if $Q \subseteq X$, and we say that X *has* a full maximal clique of P if there is a maximal clique of P that is full in X. Analogously, we say that a maximal clique Q of P is *empty* in X if $Q \cap X = \emptyset$, and we say that X *has* an empty maximal clique of P if there is a maximal clique of P that is empty in X.

Let (T, f) be a partition tree for G, and let $\underline{a}$ be an inner node of T with $\underline{b}$ and $\underline{c}$ its children. We call $\underline{a}$ a *maximal P-clique split node* of (T, f) if $\overline{f}(\underline{a})$ has a full maximal clique of P and neither $\overline{f}(\underline{b})$ nor $\overline{f}(\underline{c})$ has a full maximal clique of P. We can say that T "splits" each maximal



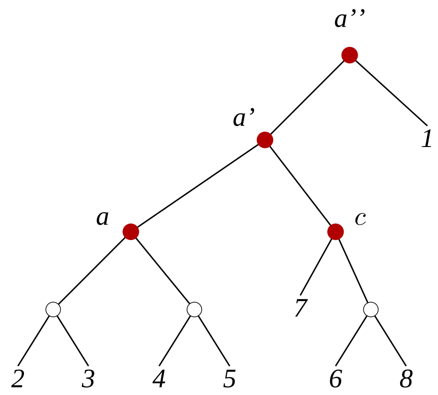


Figure 4: Graph P on the left hand side depicts a 2-path power on eight vertices. The names of the vertices of P correspond to a 2-path layout for P. The right-hand-side figure depicts a partition tree for P with an assignment of the vertices of P to the leaves of the tree. The highlighted nodes, that are a, c, a', a'', represent nodes with full maximal cliques of P. The partition tree has two maximal P-clique split nodes, namely a and c.

clique of P that is full in $\overline{f}(\underline{a})$ at $\underline{a}$. For an example and an illustration of maximal P-clique split nodes, consider the graph and the partition tree depicted in Figure 4. The partition tree has two maximal P-clique split nodes.

We state the main result of this section. The proof of the result and a local-structure consequence are given at the end of this section. The claim establishes a lower bound on the width of a partition tree by considering a combinatorial property.

Proposition 4.1. *Let (T, f) be a partition tree for G. Assume that (T, f) has a maximal P-clique split node $\underline{a}$ with its children $\underline{b}$ and $\underline{c}$ such that both $\overline{f}(\underline{b})$ and $\overline{f}(\underline{c})$ have an empty maximal clique of P. Then, (T, f) is of width at least $k + 2$.*

As an application of Proposition 4.1, consider the example given in Figure 4. Node a of the partition tree is a maximal P-clique split node. The two children of a have both an empty maximal clique: $\{6, 7, 8\}$. Hence the width of the partition tree is at least $k + 2 = 4$ and, thus, the clique-width of the given graph is at least 4.

4.1 Partition-class structure on maximal P-clique split nodes

Let (T, f) be a partition tree for G. Let $\underline{a}$ be a maximal P-clique split node of (T, f). Since each maximal clique of P is full in $\overline{f}(\underline{r})$, for $\underline{r}$ the root of T, $\underline{a}$ does exist. Let $\underline{b}$ and $\underline{c}$ be the children of $\underline{a}$, and let $B =_{\text{def}} \overline{f}(\underline{b})$ and $C =_{\text{def}} \overline{f}(\underline{c})$. Observe $\overline{f}(\underline{a}) = B \cup C$. We fix these stipulations for this subsection.

Lemma 4.2. *Let X be a partition class of $f(\underline{a})$. Then, $X \cap V(P)$ is a clique of $G[B]$ or of $G[C]$.*

Proof. For the proof, it suffices to assume that B and C have no full maximal clique of P, and $B \cup C$ may not have a full maximal clique of P.

We assume that there are $u_t, u_{t'} \in X$ where $1 \le t < t' \le n$. If u_t and $u_{t'}$ are adjacent in P then $\{u_t, u_{t'}\} \subseteq B$ or $\{u_t, u_{t'}\} \subseteq C$ due to the first part of the Compatibility condition, and u_t and $u_{t'}$ are adjacent in $G[B]$ or $G[C]$. Otherwise, u_t and $u_{t'}$ are non-adjacent in P, i.e., $t < t + k < t'$.

Assume $u_t \in B$. If $t \ge k+1$ then $Q^{t-k} \setminus B \ne \emptyset$ and each vertex from $Q^{t-k} \setminus B$ is adjacent to u_t and non-adjacent to $u_{t'}$ in P, and we conclude a contradiction to the second part of the Compatibility condition. Symmetrically, we conclude a contradiction for the case of $t' \le n-k$ by considering the maximal clique $Q^{t'+k}$. Since $Q^1 \setminus B$ is non-empty, each vertex from $Q^1 \setminus B$ implies a contradiction to the second part of the Compatibility condition, if $t < k+1$ and $t' > n-k$. Recall here $(k+1)+k < 2k+2 \le (3k+1)-k+1 \le n-k+1 \le t'$.

If $u_t \in C$, the proof is analogous. ∎

For $1 \le g \le k+1$, let $R_g =_{\text{def}} \{u_p : k{+}1|(p-g)\}$. Following the illustrations of Figures 1 and 2, we can call $R_1, \ldots, R_{k+1}$ the "rows" of P, in analogy to the "columns" of P in the proof of Lemma 3.1. We employ special vertices in rows and columns to determine lower-bound criteria. It is important to observe that $R_1, \ldots, R_{k+1}$ are independent sets of G.

Let $X \subseteq V(G)$, and consider $u_t \in R_g$. If $R_g \cap X \cap \{u_1, \ldots, u_{t-1}\}$ is non-empty, we say that *u_t has a left vertex* in X, and the *close left vertex* of u_t in X is $u_h \in R_g \cap X \cap \{u_1, \ldots, u_{t-1}\}$ of largest possible index h. In other words, the close left vertex refers to the rightmost vertex from X that is on the left of u_t and is in the same row with u_t. It is clear that u_1 cannot have a left vertex, and the same is true for each of $u_2, \ldots, u_{k+1}$. Analogously, the *close right vertex* of u_t in X is $u_{h'} \in R_g \cap X \cap \{u_{t+1}, \ldots, u_n\}$ of smallest possible index h', if it exists. Similarly, the close right vertex refers to the leftmost vertex from X that is on the right of u_t and is in the same row with u_t. For $1 \le h \le n-k$, we denote by $\Phi^h(X)$ and $\Psi^h(X)$ the sets of the respectively close left and close right vertices in X of the vertices from $Q^h \setminus X$.

Lemma 4.3. *Let $1 \le h \le n-k$. Each partition class of $f(\underline{a})$ contains at most one vertex from $\Phi^h(B) \cup \Psi^h(B)$.*

Proof. In the first part of the proof, we consider the vertices in $\Phi^h(B)$ and show that they are contained in different partition classes. It suffices to assume $n \ge k+1$, and no restriction on $\underline{a}$ is required. We assume that there are $u_s, u_t \in R_g$ and $u_{s'}, u_{t'} \in R_{g'}$ for $g \ne g'$ and that $u_s, u_{s'} \in \Phi^h(B)$ and $u_t, u_{t'} \in Q^h \setminus B$ and $h \le t < t' \le h+k$. Recall that u_s and $u_{s'}$ are the close left vertices of respectively u_t and $u_{t'}$, and $s, s' < h$. Observe

$$\left\{ \begin{array}{ll} Q^s \cap R_{g'} & \text{, if } s' < s \\ Q^{s'-k} \cap R_g & \text{, if } s+k < s' \\ Q^{s'} \cap R_g & \text{, if } s < s' \le s+k \end{array} \right\} \cap B \ = \ \emptyset \, .$$

Thus, in each of the three cases, there is a vertex of P that is not in B and that is adjacent to exactly one of u_s and $u_{s'}$. Thus, u_s and $u_{s'}$ do not have the same neighbours in $V(G) \setminus B$, and u_s and $u_{s'}$ cannot appear in the same partition class of $f(\underline{a})$ due to the Compatibility condition.

Applying the same arguments to the automorphically symmetric case of $\Psi^h(B)$, we conclude that the vertices also from $\Psi^h(B)$ appear in pairwise different partition classes of $f(\underline{a})$.

In the second part of the proof, we consider vertices from $\Phi^h(B)$ and $\Psi^h(B)$. We apply Lemma 4.2 and therefore must assume $n \ge 3k+1$ and that B and C have no full maximal clique of P. Since $\Phi^h(B) \subseteq \{u_1, \ldots, u_{h-1}\}$ and $\Psi^h(B) \subseteq \{u_{h+k+1}, \ldots, u_n\}$, no clique of G contains vertices from $\Phi^h(B)$ and from $\Psi^h(B)$, and no partition class of $f(\underline{a})$ contains two vertices from $\Phi^h(B) \cup \Psi^h(B)$. ■

Let $X \subseteq V(G)$, and assume that Q^h is not empty in X. The *top vertex* of Q^h in X is $u_j \in Q^h \cap X$ of smallest possible index j, and the *bottom vertex* of Q^h in X is $u_{j'} \in Q^h \cap X$ of largest possible index j'.

Lemma 4.4. *Choose h and h' with $1 \leq h < h + k < h' \leq n - k$. The top vertices of Q^h and $Q^{h'}$ in B appear in different partition classes of $f(\underline{a})$, and the bottom vertices of Q^h and $Q^{h'}$ in B appear in different partition classes of $f(\underline{a})$.*

Proof. It suffices to assume $n \geq 2k + 2$ and that B has no full maximal clique of P.

Assume that Q^h and $Q^{h'}$ are non-empty in B; let u and v be the top vertices of respectively Q^h and $Q^{h'}$ in B. If $u \neq u_h$ then $u_h \notin B$ and u_h is adjacent to u and non-adjacent to v. If $u = u_h$ then each vertex in $Q^{h'} \setminus B$ is adjacent to v and non-adjacent to u; recall that $Q^{h'} \setminus B$ is non-empty, since B has no full maximal clique of P. So, in each of the two cases, there is a vertex that is not in B and that is adjacent to exactly one of u and v, and thus, u and v appear in different partition classes due to the Compatibility condition.

The automorphically equivalent case of bottom vertices follows analogously. ■

It is clear that the claims of Lemmas 4.3 and 4.4 analogously hold for $\underline{c}$ and C in place of $\underline{b}$ and B, respectively.

We prove a first case for Proposition 4.1.

Lemma 4.5. *Assume that B and C have an empty maximal clique of P. Assume $|\overline{f}(\underline{a}) \cap V(P)| \geq k + 2$, and assume $\overline{f}(\underline{a}) \cap Q^1 \neq \emptyset$ and $\overline{f}(\underline{a}) \cap Q^{n-k} \neq \emptyset$. Then, $|f(\underline{a})| \geq k + 2$.*

Proof. By a symmetry argument, we may assume $B \cap Q^1 \neq \emptyset$. Choose h, t, t' such that Q^t is empty in B and $Q^{t'}$ is empty in C and Q^h is full in $B \cup C$, that exist. Clearly: $h \neq t$ and $h \neq t'$, and $|Q^h| = |B \cap Q^h| + |C \cap Q^h| = k + 1$. Due to Lemma 4.3, each partition class of $f(\underline{a})$ contains at most one vertex from $\Phi^t(B) \cup \Psi^t(B)$ and at most one vertex from $\Phi^{t'}(C) \cup \Psi^{t'}(C)$, and due to Lemma 4.2, for each partition class X of $f(\underline{a})$, the set $X \cap V(P)$ is a subset of B or of C. The combination of these observations yields $|f(\underline{a})| \geq |\Phi^t(B) \cup \Psi^t(B)| + |\Phi^{t'}(C) \cup \Psi^{t'}(C)|$. By the choice of h, t, t', $|\Phi^t(B) \cup \Psi^t(B)| \geq |B \cap Q^h|$ and $|\Phi^{t'}(C) \cup \Psi^{t'}(C)| \geq |C \cap Q^h|$, and thus, $|f(\underline{a})| \geq |B \cap Q^h| + |C \cap Q^h| = k + 1$.

For a contradiction, we suppose $|f(\underline{a})| \leq k + 1$. Note that this means $|f(\underline{a})| = |\Phi^t(B) \cup \Psi^t(B)| + |\Phi^{t'}(C) \cup \Psi^{t'}(C)| = |B \cap Q^h| + |C \cap Q^h|$. And this particularly means $|\Phi^t(B)| + |\Psi^t(B)| = |B \cap Q^h|$ and $|\Phi^{t'}(C)| + |\Psi^{t'}(C)| = |C \cap Q^h|$.

Recall our assumption of $B \cap Q^1 \neq \emptyset$. This implies that $\Phi^t(B) \neq \emptyset$, which means $|\Phi^t(B)| \geq 1$. For a contradiction, suppose that $t < h$. Then, $B \cap Q^h \subseteq \{u_{t+k+1}, \ldots, u_n\}$, and this implies $|\Psi^t(B)| \geq |B \cap Q^h|$. So, $|\Phi^t(B)| + |\Psi^t(B)| > |B \cap Q^h|$ follows, which contradicts the derived equality of the preceding paragraph. Thus, $h < t$ is the case. This also implies $\Psi^t(B) = \emptyset$, which yields $B \cap Q^{n-k} = \emptyset$ and $|\Phi^t(B)| = |B \cap Q^h|$ in particular. By the assumptions of the lemma, $C \cap Q^{n-k} \neq \emptyset$ therefore follows. Analogous to $h < t$, we derive $t' < h$ and $|\Psi^{t'}(C)| = |C \cap Q^h|$ and $\Phi^{t'}(C) = \emptyset$. As a particular conclusion, we conclude $\Phi^t(B) \subseteq \{u_h, \ldots, u_{t-1}\}$ and $\Psi^{t'}(C) \subseteq \{u_{t'+k+1}, \ldots, u_{h+k}\}$.

We consider h, we claim $h \leq k+1$, and we suppose for a contradiction the contrary, that $h \geq k+2$. Since $h < t$, which we concluded in the preceding paragraph, $\Phi^t(B) \subseteq \{u_{k+2}, \ldots, u_{t-1}\}$ follows. Let X be the partition class of $f(\underline{a})$ containing the top vertex of Q^1 in B. Recall that each partition class of $f(\underline{a})$ contains a vertex from $\Phi^t(B) \cup \Psi^{t'}(C)$. So, $\Phi^t(B) \cap X \neq \emptyset$. Since X contains the top vertex of Q^1 in B and since $X \cap V(P)$ is a clique of $G[B]$ due to Lemma 4.2, $u_1 \notin B$ follows. Hence, X contains a vertex that is adjacent to u_1 and a vertex that is non-adjacent to u_1. This is a contradiction to the Compatibility condition. Analogously, $h+k \geq n-k$ follows.

We conclude $h \leq k+1$ and $h+k \geq n-k$, which implies $h = k+1$ and $n = 3k+1$.

Recall $B \cap Q^{2k+1} = \emptyset$ and $\Phi^t(B) \subseteq \{u_h, \ldots, u_{t-1}\}$, which implies $\Phi^t(B) \subseteq \{u_{k+1}, \ldots, u_{2k}\}$. Also recall for each partition class X of $f(\underline{a})$ containing a vertex from B that $X \cap V(P) \subseteq B$ due to Lemma 4.2 and $X \cap \Phi^t(B) \neq \emptyset$. Since $u_{2k+1} \notin B$, which is an obvious consequence of $B \cap Q^{2k+1} = \emptyset$, each partition class of $f(\underline{a})$ containing a vertex from B contains only neighbours of u_{2k+1} due to the Compatibility condition. Since $N_P(u_{2k+1}) = \{u_{k+1}, \ldots, u_{2k}, u_{2k+2} \cdot \ldots, u_{3k+1}\}$, we can conclude $B \cap V(P) \subseteq \{u_{k+1}, \ldots, u_{2k}\}$. Analogously, we can conclude $C \cap V(P) \subseteq \{u_{k+2}, \ldots, u_{2k+1}\}$, by applying the derived facts $C \cap Q^1 = \emptyset$ and $\Psi^{t'}(C) \subseteq \{u_{k+2}, \ldots, u_{2k+1}\}$. So, $|\overline{f}(\underline{a}) \cap V(P)| = |(B \cup C) \cap V(P)| \leq k+1$ follows, which is the desired contradiction. ■

4.2 Maximal P-clique split node ancestors and the main results

We want to prove Proposition 4.1, that determines a lower bound on the width of partition trees for G purely by considering the assignment of the vertices of G to the leaves of T through f. Lemma 4.5 solves one case. In this subsection, we solve two remaining cases, that are not covered by Lemma 4.5, and we prove Proposition 4.1. At the end, we finish with concluding the local-structure consequence.

For the first two lemmas of this subsection, let (T, f) be a partition tree for G and let $\underline{a}$ be a maximal P-clique split node of (T, f). We begin with an easy remaining case that is not covered by Lemma 4.5.

Lemma 4.6. *Assume $n = 3k+1$. Also assume $\overline{f}(\underline{a}) \cap V(P) = Q^{k+1}$. The width of (T, f) is at least $k+2$.*

Proof. Let $\underline{a}'$ be the parent of $\underline{a}$ in T, and let $\underline{c}$ be the child of $\underline{a}'$ that is different from $\underline{a}$. We claim that no partition class of $f(\underline{a}')$ contains vertices from $\overline{f}(\underline{a}) \cap V(P)$ and $\overline{f}(\underline{c}) \cap V(P)$. Let u_t be a vertex from $\overline{f}(\underline{c}) \cap V(P)$, and let X be the partition class of $f(\underline{a}')$ containing u_t. We show $X \cap Q^{k+1} = \emptyset$ by applying the Compatibility condition. Assume $1 \leq t \leq k$. Then, $X \cap \{u_{k+1}, \ldots, u_{t+k}\} = \emptyset$ because of $\{u_{k+1}, \ldots, u_{t+k}\} \subseteq N_P(u_t) \cap \overline{f}(\underline{a})$, and $X \cap \{u_{t+k+1}, \ldots, u_{2k+1}\} = \emptyset$ because of $u_{2k+2} \notin \overline{f}(\underline{a})$ and $\{u_{t+k+1}, \ldots, u_{2k+1}\} \subseteq N_P(u_{2k+2}) \setminus N_P(u_t)$. Similarly the same arguments hold for $2k+2 \leq t \leq 3k+1$.

So, no partition class of $f(\underline{a}')$ contains vertices from $\overline{f}(\underline{a}) \cap V(P)$ and $\overline{f}(\underline{c}) \cap V(P)$. Above arguments also show that the vertices from Q^{k+1} appear in pairwise different partition classes of $f(\underline{a}')$. Thus, $|f(\underline{a}')| \geq k+2$ directly follows. ■

An *ancestor* of $\underline{a}$ is a node on the path from $\underline{a}$ to the root of T, and the *least* ancestor is closest to $\underline{a}$. The next lemma is mainly the complementary case of Lemma 4.5.

Lemma 4.7. *Assume $\overline{f}(\underline{a}) \cap Q^1 = \emptyset$ or $\overline{f}(\underline{a}) \cap Q^{n-k} = \emptyset$. The width of (T, f) is at least $k + 2$.*

Proof. Since the situations of $\overline{f}(\underline{a}) \cap Q^1 = \emptyset$ and $\overline{f}(\underline{a}) \cap Q^{n-k} = \emptyset$ are automorphically equivalent, we may assume $\overline{f}(\underline{a}) \cap Q^{n-k} = \emptyset$. Let $\underline{a}'$ be the least ancestor of $\underline{a}$ in T such that $\overline{f}(\underline{a}') \cap Q^{n-k} \not\subseteq \{u_n\}$. Let $\underline{b}'$ and $\underline{c}'$ be the children of $\underline{a}'$, let $B' =_{\text{def}} \overline{f}(\underline{b}')$ and $C' =_{\text{def}} \overline{f}(\underline{c}')$, and we assume $\overline{f}(\underline{a}) \subseteq B'$. We distinguish between two cases.

As the first case, assume $u_n \in B'$. This means $B' \cap Q^{n-k} = \{u_n\}$ by the choice of $\underline{a}'$. By the first part of the proof of Lemma 4.3, we conclude that the vertices in $\Phi^{n-k}(B')$ appear in pairwise different partition classes of $f(\underline{a}')$. And since $u_{n-1} \notin B'$ and $\Phi^{n-k}(B') \cap N_G(u_{n-1}) = \emptyset$, no partition class of $f(\underline{a}')$ contains u_n and a vertex from $\Phi^{n-k}(B')$.

Choose $z \in C' \cap Q^{n-k}$, and let Z be the partition class of $f(\underline{a}')$ containing z. Since u_n and z are adjacent in P, $u_n \notin Z$ and $Z \cap V(P) \subseteq N_P(u_n) \subseteq Q^{n-k}$. Thus, $Z \cap (\Phi^{n-k}(B') \cup \{u_n\}) = \emptyset$, and $|f(\underline{a}')| \geq k + 2$ follows.

As the second case, assume $u_n \notin B'$. This means $B' \cap Q^{n-k} = \emptyset$ by the choice of $\underline{a}'$. By the first part of the proof of Lemma 4.3, the vertices in $\Phi^{n-k}(B')$ appear in pairwise different partition classes of $f(\underline{a}')$.

For a contradiction, suppose $|f(\underline{a}')| = k + 1$. This means that each partition class of $f(\underline{a}')$ contains a vertex from $\Phi^{n-k}(B')$. Let Q^h be full in $\overline{f}(\underline{a})$, and thus in B'. Let t be smallest with $u_t \in C' \cap \{u_{h+k+1}, \ldots, u_{n-1}\}$, that exists, let X be the partition class of $f(\underline{a}')$ containing u_t, and choose $x \in X \cap \Phi^{n-k}(B')$. Observe that Q^{t-k-1} is empty in C' and x and u_t are non-adjacent in P. Thus, $x \in \{u_1, \ldots, u_{t-k-1}\} \cup \{u_{t+k+1}, \ldots, u_n\}$ and $X \cap V(P) \subseteq N_P(u_{t-k}) \cap \cdots \cap N_P(u_{t-1})$, which means $x = u_{t-k-1}$.

Observe that $\{u_{t-k-1}, u_t\} \subseteq X$ implies $u_{t+1} \in C'$. So, there is a partition class Y of $f(\underline{a}')$ containing u_{t+1}. Since $u_{t+1} \notin N_P(u_{t-k})$, $X \neq Y$ follows. And since $u_{t-k+1}, u_{t-1} \notin C'$, $u_{t-k} \in Y \cap \Phi^{n-k}(B')$, so that $\{u_{t-k}, u_{t+1}\} \subseteq Y$.

We conclude: $u_{t-k} \in B'$ and $u_t \in C'$ and $u_{t-k}u_t \in E(P)$. Condition 3 of Definition 2.1 implies $X \times Y \subseteq E(G)$, which yields a contradiction to $u_{t-k-1}u_{t+1} \notin E(G)$. ∎

We are ready to prove the main result of this section.

Proof of Proposition 4.1. If $\overline{f}(\underline{a}) \cap Q^1 = \emptyset$ or $\overline{f}(\underline{a}) \cap Q^{n-k} = \emptyset$, we apply Lemma 4.7. If $\overline{f}(\underline{a}) \cap Q^1 \neq \emptyset$ and $\overline{f}(\underline{a}) \cap Q^{n-k} \neq \emptyset$ and $|\overline{f}(\underline{a}) \cap V(P)| \geq k + 2$, we apply Lemma 4.5. And if $\overline{f}(\underline{a}) \cap Q^1 \neq \emptyset$ and $\overline{f}(\underline{a}) \cap Q^{n-k} \neq \emptyset$ and $|\overline{f}(\underline{a}) \cap V(P)| = k + 1$ then $n = 3k + 1$ and $\overline{f}(\underline{a}) \cap V(P) = Q^{k+1}$, and we apply Lemma 4.6. ∎

A corollary of Proposition 4.1 is a local-structure consequence about partition trees of width at most $k + 1$ for k-path powers.

Proposition 4.8. *Let (T, f) be a partition tree for P that is of width at most $k + 1$. Let $\underline{a}$ be a maximal P-clique split node of (T, f) with its children $\underline{b}$ and $\underline{c}$. Then, one of $\overline{f}(\underline{b})$ and $\overline{f}(\underline{c})$ has no empty maximal clique of P and $n < (k + 1)(k + 1)$, and if $n \geq k(k + 1)$ then one of $\underline{b}$ and $\underline{c}$ is a leaf of T.*

Proof. As a consequence of Proposition 4.1 and by a symmetry argument, we may assume that $\overline{f}(\underline{b})$ has no empty maximal clique of P.

Let $r =_{\text{def}} \lfloor \frac{n}{k+1} \rfloor$. Let Θ be the set of the top vertices of $Q^{i(k+1)+1}$ in $\overline{f}(\underline{b})$ for each $i \in \{0, \ldots, r - 1\}$. Since $\overline{f}(\underline{b})$ has no empty maximal clique of P, $|\Theta| = r$. Due to Lemma 4.4, the vertices in Θ appear in pairwise different partition classes of $f(\underline{a})$. And due to Lemma 4.2, each partition class of $f(\underline{a})$ is a subset of $\overline{f}(\underline{b})$ or of $\overline{f}(\underline{c})$. So, $|f(\underline{a})| \geq |\Theta|+1 = r+1$. If $r \geq k+1$ then (T,f) is of width at least $k+2$, a contradiction, so that $r \leq k$ is the case, i.e., $n < (k+1)(k+1)$.

Assume $r = k$. Then, $f(\underline{a})$ contains at least k partition classes with vertices from $\overline{f}(\underline{b})$, which means that $\overline{f}(\underline{c})$ itself is a partition class of $f(\underline{a})$. Thus, $\overline{f}(\underline{c})$ is a homogeneous set, so that $|\overline{f}(\underline{c})| = 1$, and $\underline{c}$ is a leaf of T. ■

As a final remark, we want to mention that the result of Proposition 4.8 can be extended to our considered situation about arbitrary graphs G with P as an induced subgraph, when assuming structural restrictions on G, that let us conclude $|\overline{f}(\underline{c})| = 1$ from $\overline{f}(\underline{c})$ being a homogeneous set.

5 Clique-width of path powers

We consider k-path powers and determine their minimal graphs of largest clique-width, where minimality is considered with respect to the induced subgraph relation. We apply this result to derive a formula for the clique-width of path powers that depends on the number of vertices and the clique number only.

Similar to the preceding section, we fix some terminology, for the first three results of this section, that aim at determining the minimal k-path powers of largest clique-width. Let $k \geq 2$, let P be a k-path power on n vertices, and assume $n \geq 3k+1$. Choose r and δ with $0 \leq \delta \leq k$ such that $n = r(k+1)+\delta$. Let $\langle u_1, \ldots, u_n \rangle$ be a k-path layout for P, and let $Q^1, \ldots, Q^{n-k}$ be the maximal cliques of P, where $Q^h = \{u_h, \ldots, u_{h+k}\}$.

Let (T,f) be a partition tree for P, and let $\underline{a}$ be a maximal P-clique split node of T with $\underline{b}$ a child of $\underline{a}$. Let $B =_{\text{def}} \overline{f}(\underline{b})$. We assume that B has no empty maximal clique of P. In the next two lemmas, we determine the structure of B.

Lemma 5.1. *Choose $h_1, \ldots, h_r$ satisfying $1 \leq h_1 < \cdots < h_r \leq n-k$ and $h_{i+1} > h_i + k$ for every $1 \leq i < r$. Assume $B \subseteq Q^{h_1} \cup \cdots \cup Q^{h_r}$. Let Θ be the set of the top vertices of $Q^{h_1}, \ldots, Q^{h_r}$ in B. Assume that every partition class of $f(\underline{a})$ with a vertex from B contains a vertex from Θ. Then, $B \cap Q^{h_1}, \ldots, B \cap Q^{h_r}$ are the partition classes of $f(\underline{a})$ containing a vertex from B.*

Proof. Let $X_1, \ldots, X_s$ be the partition classes of $f(\underline{a})$ with vertices from B. Observe $B \subseteq X_1 \cup \cdots \cup X_s$, and $s \leq |\Theta| \leq r$ according to the assumptions of the lemma. Due to Lemma 4.2, $X_1, \ldots, X_s$ are cliques of $P[B]$. Furthermore, $|\Theta| \geq r$, since B has no empty maximal clique of P. So, there is a 1-to-1 correspondence between $\{X_1, \ldots, X_s\}$ and Θ due to Lemma 4.4, and $r = s = |\Theta|$ in particular.

If $\{(B \cap Q^{h_1}), \ldots, (B \cap Q^{h_r})\}$ refines $\{X_1, \ldots, X_s\}$ then the two partitions of B are equal, and the claim of the lemma directly follows. Otherwise, there are $1 \leq p \leq r$ and $X, X' \in \{X_1, \ldots, X_s\}$, where $X \neq X'$, such that $X \cap Q^{h_p} \neq \emptyset$ and $X' \cap Q^{h_p} \neq \emptyset$. We choose p largest possible and X to contain the top vertex of Q^{h_p} in B. Let X' contain the top vertex of Q^{h_q} in B. Observe: $p \neq q$ and X' contains vertices from Q^{h_q} and Q^{h_p}.

For a contradiction, suppose $q < p$. Then, $X' \subseteq \{u_{h_q+1}, \ldots, u_{h_p+k-1}\}$ follows, and $u_{h_q} \notin X'$ in particular. Since X' contains the top vertex of Q^{h_q} in B, u_{h_q} cannot be the top vertex of Q^{h_q} in B, and thus, $u_{h_q} \notin B$. Since the vertices in Q^{h_p} are non-adjacent to u_{h_q}, X' therefore contradicts the Compatibility condition. So, $p < q \leq r$, and $B \cap Q^{h_q} \subseteq X'$ by the choice of p, and $X' \subseteq \{u_{h_p+1}, \ldots, u_{h_q+k-1}\}$, which means $u_{h_q+k} \notin B$. Since X' contains vertices that are adjacent to u_{h_q+k} and that are non-adjacent to u_{h_q+k}, we conclude a contradiction to the Compatibility condition. ∎

For every $1 \leq i \leq r$, let $M_i =_{\text{def}} \{u_{(i-1)(k+1)+\delta+1}, \ldots, u_{i(k+1)}\}$. Note $M_i \subseteq K_i$, where $K_1, \ldots, K_r$ are as defined in the proof of Lemma 3.1. We refer to Figure 1 for an illustrative example of the sets M_i.

Lemma 5.2. *Assume that $f(\underline{a})$ contains at most r partition classes containing a vertex from B. Also assume $r \leq k$ and $\delta \geq 1$. Then, $B \subseteq M_1 \cup \cdots \cup M_r$.*

Proof. For $1 \leq i \leq r$, let $h_i =_{\text{def}} (i-1)(k+1)+1$ and $h'_i =_{\text{def}} h_i + \delta$. Observe $M_i = Q^{h_i} \cap Q^{h'_i}$ for every $1 \leq i \leq r$. Let Θ be the set of the top vertices of $Q^{h_1}, \ldots, Q^{h_r}$ in B, and Θ' be the set of the bottom vertices of $Q^{h'_1}, \ldots, Q^{h'_r}$ in B. Since B has no empty maximal clique of P, $|\Theta| = |\Theta'| = r$ follows, and each partition class of $f(\underline{a})$ containing a vertex from B contains exactly one vertex from Θ and exactly one vertex from Θ' due to Lemma 4.4.

For a contradiction, suppose $\{u_{n-\delta+1}, \ldots, u_n\} \cap B \neq \emptyset$, and let X be a partition class of $f(\underline{a})$ containing a vertex from $\{u_{n-\delta+1}, \ldots, u_n\} \cap B$. Since $X \subseteq \{u_{h_r+1}, \ldots, u_n\}$ due to Lemma 4.2, X contains the top vertex of Q^{h_r} in B, and $u_{h_r} \notin B$, and $X \subseteq \{u_{h_r+1}, \ldots, u_{h_r+k}\} \subseteq Q^{h_r}$, a contradiction. So, $B \subseteq Q^{h_1} \cup \cdots \cup Q^{h_r}$, and $(B \cap Q^{h_1}), \ldots, (B \cap Q^{h_r})$ are the partition classes of $f(\underline{a})$ containing a vertex from B due to Lemma 5.1.

Similarly, if $\{u_1, \ldots, u_\delta\} \cap B \neq \emptyset$, the partition class Y of $f(\underline{a})$ containing the bottom vertex of $Q^{h'_1}$ in B contains a vertex from $\{u_1, \ldots, u_\delta\}$, which yields a contradiction. Thus, $B \subseteq Q^{h'_1} \cup \cdots \cup Q^{h'_r}$, and $(B \cap Q^{h'_1}), \ldots, (B \cap Q^{h'_r})$ are the partition classes of $f(\underline{a})$ containing a vertex from B due to Lemma 5.1.

Since $(B \cap Q^{h_1}) \subseteq (B \cap Q^{h'_1})$, an easy induction shows $(B \cap Q^{h_i}) = (B \cap Q^{h'_i})$ for every $1 \leq i \leq r$. Thus, $(B \cap Q^{h_1} \cap Q^{h'_1}), \ldots, (B \cap Q^{h_r} \cap Q^{h'_r})$ are the partition classes of $f(\underline{a})$ containing a vertex from B. ∎

As the first main result, we show an upper bound on the number of vertices of minimal k-path powers of clique-width above a certain bound. The upper clique-width bound of Lemma 3.1 completes this upper-number-of-vertices bound into the desired threshold result.

Proposition 5.3. *Let (T, f) be a partition tree for P, and we assume $n \geq k(k+1)+2$. The width of (T, f) is at least $k+2$.*

Proof. Let $\underline{a}$ be a maximal P-clique split node of (T, f), and let $\underline{b}$ and $\underline{c}$ be the children of $\underline{a}$. For a contradiction, suppose that (T, f) is of width at most $k+1$. We apply Proposition 4.8, that is applicable because of $n \geq 3k+1$: $n < (k+1)(k+1)$, and we can assume that $\underline{c}$ is a leaf of T and $\overline{f}(\underline{b})$ has no empty maximal clique of P. Due to Lemma 4.2, $f(\underline{a})$ has at most k partition classes containing a vertex from $\overline{f}(\underline{b})$. Due to Lemma 5.2, choosing $r = k$ and $\delta = n - k(k+1)$, $\overline{f}(\underline{b}) \subseteq M_1 \cup \cdots \cup M_k$. Thus, each maximal clique of P that is full in $\overline{f}(\underline{a})$ has at most $k-1$ vertices from $\overline{f}(\underline{b})$ and at least two vertices from $\overline{f}(\underline{c})$, contradicting $|\overline{f}(\underline{c})| = 1$. ∎

We are ready to give the complete characterisation of the clique-width of path powers.

Theorem 5.4. *Let $k \geq 1$, and let P be a k-path power on n vertices. Then,*

$$\mathrm{cwd}(P) \;=\; \mathrm{lcwd}(P) \;=\; \begin{cases} 1 & , \text{ if } n = 1 \\ 2 & , \text{ if } 2 \leq n \leq k+1 \\ \left\lceil \frac{n-1}{k+1} \right\rceil + 1 & , \text{ if } k+2 \leq n \leq k(k+1)+1 \\ k+2 & , \text{ if } n \geq k(k+1)+2\,. \end{cases}$$

Proof. If $k = 1$ or if $n \leq k+2$ then the claim follows from the results in [7]. If $k \geq 2$ and $n \geq k(k+1)+2$ then $\mathrm{cwd}(P) \leq \mathrm{lcwd}(P) \leq k+2$ due to Lemma 3.1 and $\mathrm{lcwd}(P) \geq \mathrm{cwd}(P) \geq k+2$ due to Proposition 5.3 and Theorem 2.2.

Assume $k \geq 2$ and $k+3 \leq n \leq k(k+1)+1$. Choose r such that $r(k+1)+2 \leq n \leq (r+1)(k+1)+1$. Observe $r+1 = \lceil \frac{n-1}{k+1} \rceil$. For the claimed upper bound, $\mathrm{cwd}(P) \leq \mathrm{lcwd}(P) \leq r+2$ due to Lemma 3.1. For the claimed lower bound, observe that P has an r-path power on $r(r+1)+2$ vertices as an induced subgraph. If $r = 1$ then the above shows $\mathrm{lcwd}(P) \geq \mathrm{cwd}(P) \geq r+2 = 3$, and if $r \geq 2$ then $\mathrm{lcwd}(P) \geq \mathrm{cwd}(P) \geq r+2$ due to Proposition 5.3 and Theorem 2.2. ∎

Recall that $k+1$ is equal to the clique-number of a k-path power on at least $k+1$ vertices. So, the formula of Theorem 5.4 admits an expression dependent on the number of vertices and the clique number of a path power only. As special cases about the clique-width of a k-path power P due to Theorem 5.4, we point out: if $n \leq 1(k+1)+1$ then $\mathrm{cwd}(P) \leq 2$, if $n = 1(k+1)+2$ then $\mathrm{cwd}(P) = 3$, and if $n = k(k+1)+1$ then $\mathrm{cwd}(P) = k+1$.

The result of Theorem 5.4 implies three noteworthy consequences. Firstly, clique-width and linear clique-width coincide on path powers. A partial and preliminary explanation can be seen in the structural result of Proposition 4.8. Secondly, the clique-width, and thus the linear clique-width, of path powers can be computed in linear time. Path powers can be recognised in linear time in a straightforward fashion, and the clique number k is also linear-time computable. Thirdly, the path powers of bounded clique-width admit a forbidden induced subgraph characterisation, as it is shown in the second part of the proof of Theorem 5.4.

Corollary 5.5. *Let P be a path power, and let $t \geq 1$. Then, $\mathrm{cwd}(P) \leq t+1$ if and only if P does not have a t-path power on $t(t+1)+2$ vertices as an induced subgraph.*

6 Final remarks

Path powers have no chordless cycles of length at least 4, and square grids have no cycles of length 3. These are the only known graph classes of unbounded clique-width with a precise knowledge of their clique-width. Graphs without chordless cycles of small length are interesting to study, since they may simplify the clique-width computation problem [17].

Lemma 3.1 proved two linear clique-width upper bounds. The one upper bound treated path power as a linear chain of vertices, and the other upper bound treated path powers as 2-dimensional or grid-like graphs. Our main result of Theorem 5.4 distinguishes between these two sides of the structural double nature of path powers: the underlying path structure is of strong influence in k-path powers on at least $(k+1)^2+2$ vertices, and the 2-dimensional structure is of strong influence in k-path powers on at most $k(k+1)+1$ vertices; the range between the two thresholds combines both natures. An algorithm for computing the clique-width of path powers that does not know these thresholds explicitly must determine these thresholds implicitly. This is the case particularly for algorithms for graph classes such as proper interval graphs, interval graphs, chordal graphs, cocomparability graphs, AT-free graphs, all of which contain the path powers as a subclass.

Path powers are a small graph class of highly regular structure. How good is our lower-bound result? Is it possible to improve the result by considering more general graphs? It turns out that this is not the case. In fact, the k-path powers on $k(k+1)+2$ vertices are not only minimal k-path powers of clique-width at least $k+2$, but they are minimal graphs of clique-width at least $k+2$ [18].

Acknowledgement

We are grateful to Bruno Courcelle for discussions on the subject of this paper and his valuable comments. We also thank anonymous reviewers for their helpful comments about improvements of the presentation of the results.

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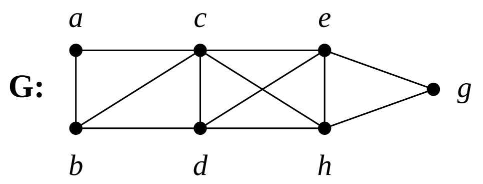
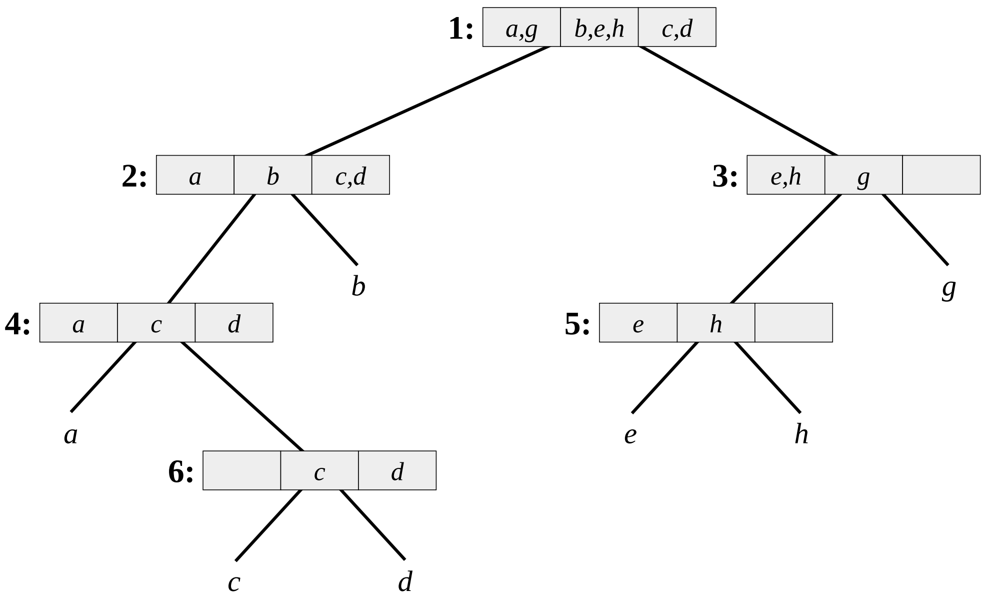
Appendix

We give an example of a partition tree, and explain the main terms of Definition 2.1.

Consider the below partition tree (T, f) of width at most 3 for the depicted graph G:





The labels at the leaves of T establish a 1-to-1 correspondence with the vertex set of G. The inner nodes of T, that are $\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}$, are labelled with partitions of subsets of $\{a, b, c, d, e, h, g\}$, such as $f(\underline{2}) = \big\{\{a\}, \{b\}, \{c, d\}\big\}$. Observe that each node of T is labelled with a set containing at most three partition classes.

By $\overline{f}(\underline{u})$, we denote the union of the partition classes of $f(\underline{u})$, that is, for node $\underline{3}$ of T, $\overline{f}(\underline{3}) = \{e, h, g\}$.

For an example of the consequences of the Compatibility condition, consider node $\underline{1}$ and $f(\underline{1}) = \Big\{\{a, g\}, \{b, e, h\}, \{c, d\}\Big\}$. Since $c \in \overline{f}(\underline{2})$ and $e \in \overline{f}(\underline{3})$ and $ce \in E(G)$, the Compatibility condition of Definition 2.1 requires $\{b, e, h\} \times \{c, d\} \subseteq E(G)$, which is the case indeed. Also observe $ac \in E(G)$ and $\{a, g\} \times \{c, d\} \not\subseteq E(G)$, however, $a, c \in \overline{f}(\underline{2})$. Finally, consider the partition class $\{e, h\}$ of $f(\underline{3})$, that is a homogeneous set of G.