Characterising the linear clique-width of a class of graphs by forbidden induced subgraphs*

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Abstract

We study the linear clique-width of graphs that are obtained from paths by disjoint union and adding true twins. We show that these graphs have linear clique-width at most 4, and we give a complete characterisation of their linear clique-width by forbidden induced subgraphs. As a consequence, we obtain a linear-time algorithm for computing the linear clique-width of the considered graphs. Our results extend the previously known set of forbidden induced subgraphs for graphs of linear clique-width at most 3.

1 Introduction and motivation

Clique-width is a well established graph parameter that is algorithmically useful in a similar way as treewidth, since problems that are expressible in a certain kind of monadic second order logic can be solved in linear time on graphs of bounded clique-width [4]. Clique-width can be viewed as more general than treewidth since there are graphs of bounded clique-width but unbounded treewidth, whereas graphs of bounded treewidth have bounded clique-width. The relationship between clique-width and its variant linear clique-width is similar to that between treewidth and pathwidth [6, 10]. Despite their important applications and different attempts to characterise them, the general understanding of clique-width and linear clique-width is still very limited. Even the proof that both parameters are NP-hard to compute is quite recent [6].

So far, we do not know whether the computation of clique-width or linear clique-width is fixed parameter tractable, or whether there is an algorithm with running time $\mathcal{O}(c^n)$ to compute either of these parameters, for c a constant and n the number of vertices of the input graph. We know of only few cases where clique-width or linear clique-width can be computed in polynomial time. Graphs of clique-width at most 2 and at most 3 can be recognised efficiently [2, 5, 8]. Similarly, graphs of linear clique-width at most 2 and at most 3 can be recognised in polynomial time [12]. Furthermore, for graphs whose clique-width or linear clique-width is at most 2, forbidden induced subgraph characterisations are known [5, 8]. No such characterisation is known for graphs whose clique-width or linear clique-width is bounded by 3 or a larger constant.

In this paper we study the linear clique-width of a class of graphs that are obtained from paths by the following two operations: disjoint union and adding true twins. We call these graphs *thickened paths*. Our main aim is not the study of this graph class, but increasing the understanding of and the knowledge on linear clique-width. Hence thickened paths are merely a

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tool to aid in this quest. Since paths have clique-width at most 3, and clique-width is preserved under disjoint union and adding true twins [5], it can be easily seen that the clique-width of thickened paths is at most 3. We prove that thickened paths have linear clique-width at most 4. We completely characterise the thickened paths that have linear clique-width at most 3 and that have linear clique-width 4 by giving forbidden induced subgraphs. Since the proof of this characterisation is constructive and the set of forbidden induced subgraphs is finite, we obtain a simple linear-time algorithm for computing the linear clique-width of thickened paths. Surprisingly, even on this well-structured graph class, the list of forbidden induced subgraphs for linear clique-width at most 3 is quite long and contains non-trivial graphs.

One implication of our results is that we extend the list of known forbidden induced subgraphs of graphs of linear clique-width at most 3. The previously known forbidden induced subgraphs for such graphs are the complements of induced cycles of length at least 5, the square of a path on eight vertices, and all forbidden induced graphs of cocomparability graphs, as graphs of linear clique-width at most 3 are cocomparability graphs [12, 13].

The main implication and importance of our results are the structural proofs of lower bounds on the linear clique-width of certain graphs. The most important obstacle for computing the clique-width or linear clique-width of any interesting graph class exactly is the difficulty in proving lower bounds, and very few lower bound proofs are known [7, 3, 13]. Hence, in addition to contributing towards a possible complete list of forbidden induced subgraphs for graphs of linear clique-width at most 3, our results contribute to developing new lower-bound proof techniques for these graph parameters.

The paper has the following structure. We define thickened paths and give some properties of them in Section 3. In Section 4 we formally define linear clique-width and we prove some upper bounds on the linear clique-width of thickened paths. As a result of independent interest, we consider general graphs with true twins and show that the linear clique-width of a graph does not change when true twins are added to vertices that already have true twins. A similar result is known for false twins, where linear clique-width is invariant with respect to adding false twins [12]. In Sections 5, 6 and 7, we prove structural results about linear expressions for paths and show lower bounds on the linear clique-width of particular thickened paths. Finally, in Section 8, we show that the above considered thickened paths form a set of forbidden induced subgraphs for thickened paths of linear clique-width at most 3, and thereby complete the characterisation of the linear clique-width of thickened paths.

2 Definitions and notation

We consider simple finite undirected graphs. For a graph $G = (V, E)$, $V = V(G)$ denotes the vertex set of G and $E = E(G)$ denotes the edge set of G. An edge between vertices u and v is denoted as uv. If $uv \in E$ then u and v are *adjacent*; otherwise, they are *non-adjacent*. For u and v adjacent vertices, u is a *neighbour* of v, and vice versa. The *(open) neighbourhood* of u, denoted as $N_G(u)$, is the set of neighbours of u in G. By $N_G[u] =_{\text{def}} N_G(u) \cup \{u\}$, we denote the *closed neighbourhood* of u in G. Two vertices u and v are called *false twins* if $N_G(u) = N_G(v)$, and they are called *true twins* if $N_G[u] = N_G[v]$. Note that false twins are non-adjacent and true twins are adjacent. The true twin relation is an equivalence relation and thus defines a partition of V into maximal sets of vertices that are pairwise true twins.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set S of vertices of G, $G[S]$ denotes the subgraph of G *induced* by S: the vertex set of $G[S]$ is S and the edge set contains exactly the edges uv of G with $u, v \in S$. For a vertex x of G, $G-x$ denotes the graph $G[V(G) \setminus \{x\}]$. For two graphs G and H where $V(G) \cap V(H) = \emptyset$, the *disjoint union* of G and

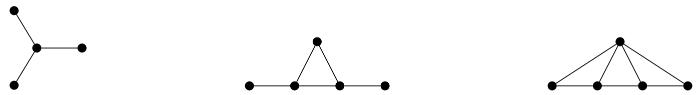

Figure 1: From left to right, the figure shows the three graphs *claw*, *bull* and *gem*, that are forbidden as induced subgraphs of thickened paths.

H is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. When we apply the disjoint union operation, we always implicitly assume that the operand graphs have disjoint vertex sets. It is easy to see how to generalise the disjoint union operation to more than two operand graphs.

A *vertex ordering*, or *(linear) layout*, of a graph $G = (V, E)$ is a bijection between $\{1, \ldots, |V|\}$ and V, and it is denoted as $\beta = \langle x_1, \ldots, x_n \rangle$. For a vertex pair x_i, x_j of G, we say that x_i is *to the left* of x_j with respect to β if $i < j$. We also write $u \prec_\beta v$ for u and v vertices of G if u is to the left of v with respect to β. The *leftmost* vertex of G with respect to β satisfying some condition is the vertex x_i with the smallest index i that satisfies the specified condition. *To the right* and *rightmost* are defined analogously.

For $n \geq 1$ and $a_1, \ldots, a_n$ pairwise different vertices, we denote by $P = (a_1, \ldots, a_n)$ the graph $(\{a_1, a_2, \ldots, a_n\},\ \{a_1a_2, a_2a_3, \ldots, a_{n-1}a_n\})$, that is called *path*, it has *length* $n-1$, and a_1 and a_n are called *endvertices*. A path of length $n-1$ is shortly denoted as P_n, where we do not specify the vertex set. P_1 has exactly one endvertex. We denote by C_n a graph that is isomorphic to $(\{a_1, a_2, \ldots, a_n\},\ \{a_1a_2, \ldots, a_{n-1}a_n, a_na_1\})$ and call such a graph a *cycle* of length n. By $2K_2$, we denote a graph that is isomorphic to $(\{a_1, a_2, a_3, a_4\},\ \{a_1a_2, a_3a_4\})$. The graphs *claw*, *bull*, and *gem* are defined as depicted in Figure 1. Given a set $\mathcal{F}$ of graphs, a graph G is *$\mathcal{F}$-free* if G does not contain any of the graphs in $\mathcal{F}$ as an induced subgraph.

A graph G is *connected* if for every vertex pair u, v of G, G contains a subgraph that is a path and contains u and v; otherwise, G is *disconnected*. The maximal connected subgraphs of a graph are called *connected components*. For graphs G and H, we say that H is "obtained from G by adding a true twin" if there is a vertex x of H such that $V(G) = V(H) \setminus \{x\}$, $H[V(G)] = G$ and there is a vertex y of G such that $N_H[x] = N_H[y]$. Iteration of this operation defines "H is obtained from G by adding true twins". Note the important difference of our definition of adding true twins to the case when true twins are added simultaneously. In the simultaneous case, added true twins are pairwise non-adjacent. The notion "H is obtained from G by adding false twins" is defined analogous to true twins, with the only difference that we require $N_H(x) = N_H(y)$.

3 Thickened paths

A *thickened path* is a graph that is obtained from a disjoint union of paths by adding true twins. Examples of thickened paths are depicted in Figures 2, 3 and 4. It is not difficult to see that thickened paths can be recognised in linear time. Furthermore, every induced subgraph of a thickened path is a thickened path.

Theorem 3.1. *Thickened paths are exactly the* $\{\text{claw}, \text{bull}, \text{gem}, C_k : k \geq 4\}$*-free graphs.*

Proof. It is easy to verify that none of the forbidden graphs is a thickened path. Thus, no thickened path can contain any of these graphs as induced subgraph. For the converse, let G be a graph that is not a thickened path such that every properly induced subgraph of G is a thickened path. Every graph that is not a thickened path contains such a graph as induced subgraph. Note that G must contain at least four vertices. We show that G has one of the listed graphs as induced subgraph. Let x be a vertex of G. Since $G{-}x$ is a thickened path, x is adjacent to at least one vertex of $G{-}x$. Let $H_1, \ldots, H_t$ be the connected components of $G{-}x$ with a vertex from $N_G(x)$. If $t \geq 3$ then G has a claw as induced subgraph formed by x and a vertex adjacent to x from three of the connected components. Next, we assume that $t \leq 2$, which means that x has a neighbour in at most two of the connected components. We distinguish between two cases with respect to t.

$t = 1$

Since H_1 is a thickened path, we can assume that H_1 is obtained from $P = (a_1, \ldots, a_p)$ by adding true twins. If $p \leq 2$ then H_1 is a complete graph. In this case, $G[V(H_1) \cup \{x\}]$ is obtained from adding true twins to a path of length at most 2, where x and a possible non-neighbour of x are endvertices. So, G is a thickened path, contradicting the choice of G. Thus, $p \geq 3$. Assume that H_1 has no true twins. If there are $1 \leq i < j < k \leq p$ such that x is adjacent to a_i and a_k and non-adjacent to a_j then G contains a cycle of length at least 4 as induced subgraph. Otherwise, the neighbours of x appear consecutively on P. If x has at least four neighbours in H_1 then G contains a gem as induced subgraph. Otherwise, x contains at most three neighbours in H_1. If x has exactly three neighbours in H_1 then x is a true twin of one of the three neighbours, which yields a contradiction to the choice of G. Thus, x has at most two neighbours in H_1. If x has exactly two neighbours in H_1 and one of the two neighbours is an endvertex of P then x is a true twin of the endvertex, contradicting the choice of G. Thus, if x has exactly two neighbours in H_1 then there is $1 < i \leq p-2$ such that $N_G(x) = \{a_i, a_{i+1}\}$, and G contains a bull as induced subgraph. If x has exactly one neighbour in H_1 then this neighbour cannot be an endvertex of P, and G contains a claw as induced subgraph.

Next, assume that H_1 has true twins. If there is a_i of P and a true twin a_i' of a_i in H_1 and x is adjacent to both vertices or non-adjacent to both vertices then a_i and a_i' are true twins in G, and G is a thickened path if and only if $G{-}a_i'$ is a thickened path. This contradicts the choice of G. With a symmetry argument, we can assume without loss of generality that $N_G(x) \subseteq \{a_1, \ldots, a_p\}$. If $G[\{a_1, \ldots, a_p, x\}]$ is not a thickened path then we obtain a contradiction to the choice of G. Hence, $N_G(x) = \{a_1, a_2\}$ or $N_G(x) = \{a_{p-1}, a_p\}$ or $N_G(x) = \{a_i, a_{i+1}, a_{i+2}\}$ for some $1 \leq i \leq p-2$. If $N_G(x) = \{a_1, a_2\}$ and a_1 has a true twin a_1' then $\{x, a_1', a_2, a_3\}$ induces a claw, if a_2 has a true twin a_2' then $\{x, a_1, a_2', a_3, a_2\}$ induces a gem. Analogously for the case when $N_G(x) = \{a_{p-1}, a_p\}$. Let $N_G(x) = \{a_i, a_{i+1}, a_{i+2}\}$. If a_i has a true twin a_i' then $\{a_i', a_i, x, a_{i+2}, a_{i+1}\}$ induces a gem; analogously for the case when a_{i+2} has a true twin. If a_{i+1} has a true twin a_{i+1}' then $\{a_i, x, a_{i+2}, a_{i+1}'\}$ induces a C_4. This completes the case when $t = 1$.

$t = 2$

Let H_1 and H_2 be obtained from $P = (a_1, \ldots, a_p)$ and $Q = (b_1, \ldots, b_q)$ by adding true twins, respectively. Due to the choice of G, $G[V(H_1) \cup \{x\}]$ and $G[V(H_2) \cup \{x\}]$ are thickened paths, so that the choice of G (and arguments similar to the previous case) implies that $H_1 = P$ and $H_2 = Q$. If x is adjacent to a pair of non-adjacent vertices in H_1 or H_2 then G contains a claw as induced subgraph. Otherwise, the neighbours of x in H_1 and in H_2 are pairwise adjacent. It follows that x can have at most two neighbours on P and on Q and one of the two is an endvertex. Without loss of generality, we can assume that $\{a_1, b_1\} \subseteq N_G(x) \subseteq \{a_1, a_2, b_1, b_2\}$. If $p \leq 2$ or $q \leq 2$ then G is a thickened path, so that $p, q \geq 3$. If $N_G(x) = \{a_1, b_1\}$ then G is a thickened path, so that $a_2 \in N_G(x)$ or $b_2 \in N_G(x)$. Then, $\{a_3, a_2, a_1, x, b_1\}$ or $\{a_1, x, b_1, b_2, b_3\}$ induces a bull. This completes the proof. ∎

Thickened paths form a subclass of proper interval graphs, since paths are proper interval graphs and proper interval graphs are closed under adding true twins and taking the disjoint union. Due to the characterisation of proper interval graphs by Wegner [16], Theorem 3.1 shows that thickened paths are exactly the {bull, gem}-free proper interval graphs.

4 Linear clique-width and first results

Linear clique-width is defined using operations on labelled graphs. For $k \geq 0$, a *k-labelled* graph is a graph where each of its vertices is assigned a number from the set $\{1, \dots, k\}$. Let G be a k-labelled graph. We use three operations on labelled graphs:

- for $1 \leq i \leq k$ and v a vertex that does not appear in G, $i(v)$ adds vertex v with label i to G
- for $1 \leq i < j \leq k$, $\eta_{i,j}$ adds all (not yet existing) edges between vertices with label i and vertices with label j to G
- for $i, j \in \{1, \dots, k\}$, $i \neq j$, $\rho_{i \to j}$ changes all occurrences of label i to label j.

A *linear k-expression* is an expression $a_1 \cdots a_r$ where a_i is one of the above defined operations. The (labelled) graph defined by $a_1 \cdots a_r$ is $a_r(\cdots a_1((\emptyset, \emptyset)) \cdots)$, which is the result of iteratively applying the operations $a_1, \dots, a_r$ to the initial empty graph. The *linear clique-width* of a graph G, denoted as $\mathrm{lcwd}(G)$, is the smallest k such that there is a linear k-expression for G, which means that the linear k-expression defines a graph H that is equal to G after removing the assigned labels. The following result is not difficult to verify.

Lemma 4.1. *Let G be a graph and let H be an induced subgraph of G. Then,* $\mathrm{lcwd}(H) \leq \mathrm{lcwd}(G)$.

Let $G = (V, E)$ be a graph. For $A \subseteq V$, a *group* of A is a maximal set of vertices with the same neighbourhood in $V \setminus A$. Note that two groups of A are either equal or disjoint, implying that the group relation defines a partition of A. For a vertex triple u, v, w of G, we say that w *distinguishes* u and v if w is adjacent to the one vertex and non-adjacent to the other. Observe that for $u, v \in A$, u and v do not belong to the same group of A if and only if there is a vertex w with $w \in V \setminus A$ such that w distinguishes u and v. By $\nu_G(A)$, we denote the number of groups of A. Let β be a layout for G. Let x be a vertex of G and let p be the position of x in β, i.e., $p = \beta^{-1}(x)$. The *set of vertices to the left of x with respect to β* is $\{\beta(1), \dots, \beta(p-1)\}$ and denoted as $L_\beta(x)$, and the *set of vertices to the right of x with respect to β* is $\{\beta(p+1), \dots, \beta(|V|)\}$ and denoted as $R_\beta(x)$. We write $L_\beta[x]$ and $R_\beta[x]$ if x is included. The *groupnumber of G with respect to β*, denoted as $\mathrm{gn}(G, \beta)$, is the maximum taken over all values $\nu_G(L_\beta[\beta(i)])$ for $1 \leq i \leq |V|$. The *groupnumber* of G, denoted as $\mathrm{gn}(G)$, is the smallest number k such that there is a layout β for G with $\mathrm{gn}(G, \beta) \leq k$. Function ad_β is a function on V and with values from $\{0, 1\}$. Given a vertex x of G, if one of the following conditions is satisfied:

- all (other) vertices in the group of $L_\beta[x]$ that contains x are neighbours of x
- $\{x\}$ is not a group of $L_\beta[x]$ and there are a non-neighbour y of x in the group of $L_\beta[x]$ containing x and a neighbour z of x in $L_\beta(x)$ such that y and z are non-adjacent

then $\mathrm{ad}_\beta(x) = 1$; if none of the conditions is satisfied then $\mathrm{ad}_\beta(x) = 0$. The *groupwidth of a graph G with respect to a layout β for G*, denoted as $\mathrm{gw}(G, \beta)$, is the smallest number k such that $\nu_G(L_\beta(x)) + \mathrm{ad}_\beta(x) \leq k$ for all $x \in V(G)$. The *groupwidth* of a graph G, denoted as $\mathrm{gw}(G)$, is the smallest number k such that there is a layout β for G satisfying $\mathrm{gw}(G, \beta) \leq k$.

Theorem 4.2 ([9, 14, 11])**.** *For every graph G,* $\mathrm{lcwd}(G) = \mathrm{gw}(G)$*.*

Most of our lower-bound proofs will analyse the groupwidth of layouts. The following technical lemma will be of help.

Lemma 4.3. *Let P be the disjoint union of a set of paths, and let G be a thickened path that is obtained from P by adding true twins. Let β be a layout for G. If $\mathrm{ad}_\beta(x) = 0$ for a vertex x of G then x is an endvertex of a path in P and has no true twin in G.*

Proof. Let x be a vertex of G with $\mathrm{ad}_\beta(x) = 0$. Let B be the group of $L_\beta[x]$ that contains x. Due to the definition of function ad, $|B| \geq 2$ and B contains a vertex y that is non-adjacent to x. Again due to the definition of ad, all neighbours of x in $L_\beta(x)$ are adjacent to y. If x has a true twin x' then x has a neighbour that distinguishes x and y. If $x' \in L_\beta(x)$ then we obtain a contradiction to the above observations. If $x' \in R_\beta(x)$ then x and y are distinguished by x' and therefore cannot be in the same group of $L_\beta[x]$. Hence, x has no true twin. If x is not an endvertex of some path in P then x has two neighbours in P, say u and v. Clearly, one of them, say v, is not adjacent to y in G. Similar to the case of true twin, we observe a contradiction and thus conclude that x is an endvertex of P and has no true twin in G. ∎

The result of Lemma 4.3 provides a simple tool for determining the linear clique-width for a class of thickened paths.

Lemma 4.4. *Let P be the disjoint union of a set of paths. Let G be a thickened path that is obtained from P by adding true twins such that every endvertex of a path in P has a true twin in G. Then,* $\mathrm{lcwd}(G) = \mathrm{gn}(G) + 1$*.*

Proof. Let β be a layout for G. If there is a vertex x of G with $\mathrm{ad}_\beta(x) = 0$ then x is an endvertex of a path in P and has no true twin in G according to Lemma 4.3. However, such vertices do not exist in G due to the assumption about G. Hence, $\mathrm{gw}(G, \beta) = \mathrm{gn}(G, \beta) + 1$. It follows that $\mathrm{gw}(G) = \mathrm{gn}(G) + 1$, and the claim of the lemma follows by applying Theorem 4.2. ∎

The linear clique-width is exactly known only for a few classes of graphs. The following observation is necessary for lower bounds.

Proposition 4.5 ([8])**.** $\mathrm{lcwd}(P_1) = 1$*;* $\mathrm{lcwd}(P_2) = \mathrm{lcwd}(P_3) = 2$*;* $\mathrm{lcwd}(P_4) = 3$*;* $\mathrm{lcwd}(2K_2) = 3$*.*

Graphs of linear clique-width at most 2 are characterised as the graphs that do not contain P_4, $2K_2$ and a graph that is not a thickened path as induced subgraph [8]. This characterisation directly implies the following result.

Proposition 4.6 ([8])**.** *Let G be a thickened path. If G contains P_4 or $2K_2$ as induced subgraph then* $\mathrm{lcwd}(G) \geq 3$*, otherwise,* $\mathrm{lcwd}(G) \leq 2$*.*

It can be checked in linear time whether a graph contains P_4 or $2K_2$ as induced subgraph [1]. Proposition 4.6, together with Theorem 3.1, shows that thickened paths of linear clique-width at most 2 are exactly the $\{P_4, 2K_2, C_4, \text{claw}\}$-free graphs. Threshold graphs are the $\{P_4, 2K_2, C_4\}$-free graphs [15]. It follows that thickened paths of linear clique-width at most 2 are exactly the claw-free threshold graphs.

It has been shown that the linear clique-width is invariant under adding false twins [12]. A simple example shows that the same is not true for true twins: a graph on a single vertex has linear clique-width 1 and adding a true twin yields a graph of linear clique-width 2. Nevertheless, we are able to show a true twin counterpart of the result for false twins.

Lemma 4.7. *Let G be a graph. Let u, v, w be pairwise different vertices of G with $N_G[u] = N_G[v] = N_G[w]$. Then,* $\mathrm{lcwd}(G) \leq \mathrm{lcwd}(G{-}w)$.

Proof. Let $G{-}w$ have n vertices. Let β be a layout for $G{-}w$. Since u and v are true twins in $G{-}w$, we can assume without loss of generality that $u \prec_\beta v$. Let $i =_{\mathrm{def}} \beta^{-1}(v)$. Let $\alpha =_{\mathrm{def}} \langle \beta(1), \ldots, \beta(i), w, \beta(i+1), \ldots, \beta(n) \rangle$. We show that $\mathrm{gw}(G, \alpha) \leq \mathrm{gw}(G{-}w, \beta)$. Let x be a vertex of G. We first look at the groups defined by α, and then we determine the groupwidth of α. Let $B_1, \ldots, B_r$ be the groups of $L_\alpha(x)$ in G. Let a and b be vertices in $L_\alpha(x)$ and let $a \in B_j$ and $b \in B_{j'}$. By definition of groups, it holds that $j \neq j'$ if and only if there is a vertex in $R_\alpha[x]$ that distinguishes a and b. We consider three cases.

- $x \preccurlyeq_\alpha v$
  Note that $L_\alpha(x) = L_\beta(x)$ and $R_\alpha[x] = R_\beta[x] \cup \{w\}$. Since v and w are true twins and $v \in R_\beta[x]$, it holds that there is a vertex in $R_\alpha[x]$ that distinguishes a and b if and only if there is a vertex in $R_\alpha[x] \setminus \{w\} = R_\beta[x]$ that distinguishes a and b. Hence, $B_1, \ldots, B_r$ are the groups of $L_\beta(x)$ in $G{-}w$.

- $w \prec_\alpha x$
  Note that $L_\alpha(x) = L_\beta(x) \cup \{w\}$ and $R_\alpha[x] = R_\beta[x]$. Without loss of generality, we can assume that $v \in B_1$. Since v and w are true twins in G, it holds that no vertex in $R_\alpha[x]$ distinguishes v and w, so that $w \in B_1$. It follows that $(B_1 \setminus \{w\}), B_2, \ldots, B_r$ are the groups of $L_\beta(x)$ in $G{-}w$.

- $x = w$
  Note that $L_\alpha(x) = L_\alpha[v] = L_\beta[v]$ and $R_\alpha[x] = R_\alpha(v) = R_\beta(v) \cup \{w\}$. Without loss of generality, we can assume that $v \in B_1$. Since u and v are true twins in G, it holds with the same arguments as in the previous case that $u \in B_1$. Furthermore, since v distinguishes a vertex pair a, b of G, where $\{a, b\} \cap \{v, w\} = \emptyset$, if and only if w distinguishes a and b, it follows that $B_1 \setminus \{v\}, B_2, \ldots, B_r$ are the groups of $L_\beta(v)$ in $G{-}w$. And due to the above proven facts, $B_1 \setminus \{v\}, B_2, \ldots, B_r$ are the groups of $L_\alpha(v)$ in G. Note that $u \in B_1$ implies $B_1 \setminus \{v\} \neq \emptyset$.

We determine $\mathrm{ad}_\alpha(x)$. Since $\mathrm{ad}_\alpha(x)$ depends only on $L_\alpha[x]$ and its groups and since u, v and w are true twins, it holds for all $x \notin \{v, w\}$ with the above results about the groups that $\mathrm{ad}_\alpha(x) = \mathrm{ad}_\beta(x)$. Now, let $x \in \{v, w\}$. Since u, v and w are true twins, it holds that x is in the group of $L_\alpha[x]$ that contains u. In particular, $\{x\}$ is not a group of $L_\alpha[x]$ and x has a neighbour, namely u, in its group. If the group of x contains no non-neighbour of x then $\mathrm{ad}_\alpha(x) = \mathrm{ad}_\beta(v) = 1$. If the group of x contains a non-neighbour y of x then u and y are non-adjacent, and since u and x are adjacent, it again follows that $\mathrm{ad}_\alpha(x) = \mathrm{ad}_\beta(v) = 1$. We summarise that $\nu_G(L_\alpha(x)) + \mathrm{ad}_\alpha(x) = \nu_{G-w}(L_\beta(x)) + \mathrm{ad}_\beta(x)$ for all vertices x of G with $x \neq w$ and $\nu_G(L_\alpha(w)) + \mathrm{ad}_\alpha(w) = \nu_{G-w}(L_\beta(v)) + \mathrm{ad}_\beta(v)$. Thus, the claim follows, and therefore, $\mathrm{gw}(G) \leq \mathrm{gw}(G{-}w)$. The lemma follows by applying Theorem 4.2. ∎

Let G be a graph. Let $\langle M_1, \ldots, M_r \rangle$ be the partition of $V(G)$ into maximal sets of pairwise true twins. A *true twin reduction* of G is a graph $H = G[M]$ where $M =_{\mathrm{def}} \bigcup_{1 \leq i \leq r} M'_i$ and for every $1 \leq i \leq r$, $M'_i = M_i$ if $|M_i| \leq 2$, and $M'_i \subseteq M_i$ with $|M'_i| = 2$ if $|M_i| \geq 3$. Informally, a true twin reduction of G is obtained from G by iteratively deleting a true twin for which the graph contains at least two further copies.

Theorem 4.8. *Let G be a graph and let H be a true twin reduction of G. Then,* $\mathrm{lcwd}(G) = \mathrm{lcwd}(H)$.

Proof. Since H is an induced subgraph of G, $\mathrm{lcwd}(H) \leq \mathrm{lcwd}(G)$ by Lemma 4.1. For the converse, note that there is a sequence $(x_1, \ldots, x_s)$ of vertices of G such that $G_0 =_{\mathrm{def}} G$, $G_i =_{\mathrm{def}} G_{i-1} - x_i$ for every $1 \leq i \leq s$ and $G_s = H$. Furthermore, and independent of the chosen vertex sequence, it holds that for every $1 \leq i \leq s$, there are two vertices y_i' and y_i'' in G_{i-1} where $|\{x_i, y_i', y_i''\}| = 3$, i.e., the three vertices are pairwise different, such that x_i, y_i' and y_i'' are pairwise true twins in G_{i-1}. We apply Lemma 4.7 and obtain that $\mathrm{lcwd}(G_{i-1}) \leq \mathrm{lcwd}(G_i)$. It follows that $\mathrm{lcwd}(G) = \mathrm{lcwd}(G_0) \leq \mathrm{lcwd}(G_s) = \mathrm{lcwd}(H)$. ∎

The result of Theorem 4.8 shows that for determining the linear clique-width of a graph with true twins, we can always restrict to induced subgraphs that contain only sets of pairwise true twins of size at most 2. Thus, in the course of the paper, we will only encounter graphs for which a vertex has either no true twin or exactly one true twin.

As the last result in this section, we give an upper bound on the linear clique-width of thickened paths. We show that the linear clique-width of thickened paths is at most 4. In particular, thickened paths form a class of graphs of bounded linear clique-width.

Lemma 4.9. *Let G be a thickened path. Then,* $\mathrm{lcwd}(G) \leq 4$.

Proof. We apply a true twin reduction to G and obtain G'. Every vertex of G' has at most one true twin. Let H be a graph that is obtained from G' by adding true twins such that every vertex of H has exactly one true twin. It holds that G' is an induced subgraph of H. Thus, due to Lemma 4.1 and Theorem 4.8, $\mathrm{lcwd}(G) \leq \mathrm{lcwd}(G') \leq \mathrm{lcwd}(H)$. Let $C_1, \ldots, C_r$ be the connected components of H. Every connected component of H is obtained from a path by adding true twins. Let C_i be obtained from a path $(x_0^{(i)}, \ldots, x_{k_i}^{(i)})$ by adding true twins. For every $1 \leq i \leq r$ and $0 \leq j \leq k_i$, let $y_j^{(i)}$ be the uniquely defined true twin of $x_j^{(i)}$. Let $\beta =_{\mathrm{def}} \langle x_0^{(1)}, y_0^{(1)}, x_1^{(1)}, \ldots, y_{k_1}^{(1)}, x_0^{(2)}, \ldots, y_{k_r}^{(r)} \rangle$. We approximate the groupwidth of β. Let u be a vertex of H. We determine $\nu_H(L_\beta(u))$. We distinguish between two cases. As the first case, let $u = x_j^{(i)}$ for some $1 \leq i \leq r$ and $0 \leq j \leq k_i$. If $j = 0$ then no vertex in $L_\beta(u)$ has a neighbour in $R_\beta[u]$, and thus, $\nu_H(L_\beta(u)) \leq 1$. So, let $j \geq 1$. The vertices in $L_\beta(u)$ with neighbours in $R_\beta[u]$ are exactly the neighbours of u in $L_\beta(u)$, which are $x_{j-1}^{(i)}$ and $y_{j-1}^{(i)}$. Since these two vertices are true twins, it holds that $L_\beta(u)$ contains at most two groups, namely the group that contains only vertices without a neighbour in $R_\beta[u]$ and the group $\{x_{j-1}^{(i)}, y_{j-1}^{(i)}\}$. Thus, $\nu_H(L_\beta[u]) \leq 2$.

As the second case, let $u = y_j^{(i)}$ for some $1 \leq i \leq r$ and $0 \leq j \leq k_i$. If $j = 0$ then $L_\beta(u)$ contains at most two groups, namely $L_\beta(x_0^{(i)})$ (in case $i \geq 2$) and $\{x_0^{(i)}\}$. Thus, $\nu_H(L_\beta(u)) \leq 2$. Then, let $j \geq 1$. The vertices with a neighbour in $R_\beta[u]$ are exactly $x_{j-1}^{(i)}$, $y_{j-1}^{(i)}$ and $x_j^{(i)}$. So, $L_\beta(u)$ has the group that contains all vertices without a neighbour in $R_\beta[u]$ and the group that contains $x_{j-1}^{(i)}$ and $y_{j-1}^{(i)}$. Vertex $x_j^{(i)}$ may be in a singleton group or it may be in the same group as $x_{j-1}^{(i)}$; the latter holds if $j = k_i$. Then, $\nu_H(L_\beta(u)) \leq 3$. We summarise that $\nu_H(L_\beta(u)) + \mathrm{ad}_\beta(u) \leq 3 + 1 = 4$. Thus, $\mathrm{gw}(H) \leq \mathrm{gw}(H, \beta) \leq 4$, which concludes the proof due to Theorem 4.2. ∎

In the following sections, we will classify the thickened paths of linear clique-width exactly 4 and thereby obtain a complete characterisation of the linear clique-width of thickened paths.

5 Connected thickened paths of length 4 and 5

We consider thickened paths that are obtained from adding true twins to a path of length 4 or 5. We consider the two cases separately. Each case begins with a structural result about layouts of the paths and is concluded with a result about added true twins that increase the linear clique-width. We start with paths of length 4.

Lemma 5.1. *Let $P = (a_1, \ldots, a_5)$. Let $\beta = \langle x_1, \ldots, x_5 \rangle$ be a layout for P such that* $\mathrm{gw}(P, \beta) \le 3$ *and $a_1 \prec_\beta a_5$. Then,*

$$\{x_1, x_2, x_3\} \in \Big\{ \{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\} \Big\} .$$

Proof. Let $X_3 =_{\mathrm{def}} \{x_1, x_2, x_3\}$. Suppose that $\{a_1, a_5\} \subseteq X_3$. Independent of whether $a_2 \in X_3$ or $a_3 \in X_3$ or $a_4 \in X_3$, set X_3 has three groups. The assumption about the groupwidth of β implies that $\mathrm{ad}_\beta(x_4) = 0$. However, this is a contradiction to the statement of Lemma 4.3, since x_4 is not endvertex of P. If $a_1, a_5 \notin X_3$ then $X_3 = \{a_2, a_3, a_4\}$, and the statement holds. Let X_3 contain either a_1 or a_5. Due to assumption $a_1 \prec_\beta a_5$, it holds that $a_1 \in X_3$ and $a_5 \notin X_3$. Suppose that the statement does not hold, which means that $X_3 = \{a_1, a_2, a_4\}$. Then, $\nu_P(X_3) = 3$, and thus $\mathrm{ad}_\beta(x_4) = 0$. According to Lemma 4.3, x_4 is an endvertex of P, which means that $x_4 = a_5$. Since $x_5 = a_3$, $\{a_1, a_5\}$ is a group of $L_\beta[x_4]$. And since a_1 and a_5 are non-adjacent and a_4 is adjacent to a_5 and non-adjacent to a_1, the definition of function ad shows that $\mathrm{ad}_\beta(a_5) = 1$, which gives a contradiction. Hence, the statement of the lemma holds. ∎

Proposition 5.2. *Let $P = (a_1, \ldots, a_5)$. Let G be obtained from P by adding a true twin to each vertex. Then,* $\mathrm{lcwd}(G) \ge 4$.

Proof. Let $a'_1, \ldots, a'_5$ be the added true twins of respectively $a_1, \ldots, a_5$. For a contradiction, suppose that there is a layout β for G with $\mathrm{gw}(G, \beta) \le 3$. We apply Lemma 4.4 and see that $\mathrm{gn}(G, \beta) \le 2$. Without loss of generality, we can assume due to symmetry arguments that $a_1 \prec_\beta a_5$ and that $a_i \prec_\beta a'_i$ for all $1 \le i \le 5$. Let β' be the restriction of β to the vertices $a_1, \ldots, a_5$. Then, $\beta' = \langle x_1, \ldots, x_5 \rangle$ is a layout of groupwidth at most 3 for P and we can apply Lemma 5.1. Suppose that $\{x_1, x_2, x_3\} = \{a_2, a_3, a_4\}$. Since $a_1, a_5 \notin L_\beta[x_3]$, it holds that $\nu_G(L_\beta[x_3]) \ge 3$, which implies $\mathrm{gn}(G, \beta) \ge 3$. Thus, $\{x_1, x_2, x_3\} = \{a_1, a_2, a_3\}$ or $\{x_1, x_2, x_3\} = \{a_1, a_3, a_4\}$. Note that $x_4 \in \{a_2, a_4, a_5\} \setminus \{x_1, x_2, x_3\}$. Suppose that $\{x_1, x_2, x_3\} = \{a_1, a_2, a_3\}$. Then, $x_4 \in \{a_4, a_5\}$ and $\{a'_4, a'_5\} \subseteq R_\beta(x_4)$. We consider the groups of $L_\beta[x_4]$ in G: a_3 is distinguished from a_1 and a_2 by a'_4, and a_4 or a_5 is distinguished from a_1, a_2, a_3 by a'_5. Hence, $\mathrm{gn}(G, \beta) \ge \nu_G(L_\beta[x_4]) \ge 3$. Finally, suppose that $\{x_1, x_2, x_3\} = \{a_1, a_3, a_4\}$. By our assumptions about β, it holds that the true twin of x_3 is not contained in $L_\beta[x_3]$. Then, a_1 and a_3 are distinguished by a'_1 or a'_3 or a'_4, and a_4 is distinguished from a_1 and a_3 by a_5. Thus, $\mathrm{gn}(G, \beta) \ge \nu_G(L_\beta[x_3]) \ge 3$. Hence, we obtain only contradictions, so that $\mathrm{gw}(G) \ge 4$. ∎

Next, we consider thickened paths that are obtained from P_6 by adding true twins. Analogous to the previous case, we first show a result about the structure of layouts, and then we consider graphs with true twins.

Lemma 5.3. *Let $P = (a_1, \ldots, a_6)$. Let β be a layout for P such that* $\mathrm{gw}(P, \beta) \le 3$ *and $a_2 \prec_\beta a_5$. Then,*

$$\{a_2, a_3, a_4\} \prec_\beta \{a_5, a_6\} \quad \text{or} \quad \{a_3, a_4, a_6\} \prec_\beta a_2 \prec_\beta a_5 .$$

Proof. Denote by $\beta_{1,5}$ and $\beta_{2,6}$ the restriction of β respectively to the vertices $a_1, \ldots, a_5$ and to the vertices $a_2, \ldots, a_6$. Note that $\beta_{1,5}$ and $\beta_{2,6}$ are layouts for paths of length 4, and we can apply Lemma 5.1. We obtain five possible situations in β for the two layouts, which are given in the following table.

	for $\beta_{1,5}$	for $\beta_{2,6}$	
(1)	$\{a_2, a_3, a_4\} \prec_\beta \{a_1, a_5\}$	* $\{a_3, a_4, a_5\} \prec_\beta \{a_2, a_6\}$	
(2)	$\{a_1, a_3, a_4\} \prec_\beta \{a_2, a_5\}$	$\{a_2, a_4, a_5\} \prec_\beta \{a_3, a_6\}$	(5)
(3)	$\{a_1, a_2, a_3\} \prec_\beta \{a_4, a_5\}$	$\{a_2, a_3, a_4\} \prec_\beta \{a_5, a_6\}$	(6)
	$\{a_5, a_4, a_3\} \prec_\beta \{a_1, a_2\}$ *	* $\{a_6, a_5, a_4\} \prec_\beta \{a_2, a_3\}$	
(4)	$\{a_5, a_3, a_2\} \prec_\beta \{a_1, a_4\}$	$\{a_6, a_4, a_3\} \prec_\beta \{a_2, a_5\}$	(7)

The cases that are marked with the sign * contradict the assumption $a_2 \prec_\beta a_5$, so they are not possible. The remaining seven cases are assigned a number. In the next step, we check left and right column pairs against each other and determine whether they contradict each other; the result is given in the next table, where the entries give the problematic vertex pairs.

	(1)	(2)	(3)	(4)
(5)	a_3, a_5	a_3, a_5	a_3, a_5	a_3, a_4
(6)				a_4, a_5
(7)			a_2, a_4	a_2, a_4

Five combinations remain without contradiction and give the following refined information about β.

(1, 6)	$\{a_2, a_3, a_4\} \prec_\beta \{a_1, a_5, a_6\}$
(2, 6)	$\{a_1, a_3, a_4\} \prec_\beta a_2 \prec_\beta \{a_5, a_6\}$
(3, 6)	$\{a_1, a_2, a_3\} \prec_\beta a_4 \prec_\beta \{a_5, a_6\}$
(1, 7)	$\{a_3, a_4, a_6\} \prec_\beta a_2 \prec_\beta \{a_1, a_5\}$
(2, 7)	$\{a_1, a_3, a_4, a_6\} \prec_\beta a_2 \prec_\beta a_5$

Note that for the combination $(1, 7)$ we are also taking into account the initial assumption about β. Then, we delete vertex a_1 and simplify the cases, which shows the result of the lemma. ∎

Proposition 5.4. *Let $P = (a_1, \ldots, a_6)$. Let G be a thickened path that has one of the following three properties:*

1) G is obtained from P by adding a true twin to a_3, a_4, a_5, a_6

2) G is obtained from P by adding a true twin to a_1, a_2, a_5, a_6

3) G is obtained from P by adding a true twin to a_1, a_3, a_4, a_6.

Then, $\mathrm{lcwd}(G) \geq 4$.

Proof. Let a_i' be the added true twin of a_i. For a contradiction, suppose that there is a layout β for G with $\mathrm{gw}(G, \beta) \leq 3$. Without loss of generality, we can assume $a_i \prec_\beta a_i'$ for each a_i with a true twin in G. Furthermore, if G has the second or third property then we can assume without loss of generality and by symmetry arguments that $a_2 \prec_\beta a_5$. Let β' be the restriction of β to $a_1, \ldots, a_6$. We apply Lemma 5.3 to β', and we consider the possible cases.

G has the third property
We apply Lemma 4.4 and see that $\mathrm{gn}(G, \beta) \leq 2$.

– $\{a_2, a_3, a_4\} \prec_\beta \{a_5, a_6\}$
Let a be rightmost from $\{a_2, a_3, a_4\}$ with respect to β. We consider the groups of $L_\beta[a]$. Observe that a_5 distinguishes a_4 from a_2 and a_3. Then, a_2 and a_3 are in the same group of $L_\beta[a]$, which implies that $\{a_1, a'_1, a'_4\} \subseteq L_\beta[a]$, and $a \in \{a_2, a_3\}$. If $a = a_3$ then a_1, a_2, a_4 are in pairwise different groups of $L_\beta(a)$, so that $a = a_2$. If $a'_3 \in R_\beta[a]$ then a_1, a_3, a_4 are in pairwise different groups of $L_\beta(a)$. Let $a'_3 \in L_\beta(a)$. Let a' be rightmost from $\{a'_1, a'_3, a'_4\}$ with respect to β. Then, a_1, a_3, a_4 are in pairwise different groups of $L_\beta(a')$.

– $\{a_3, a_4, a_6\} \prec_\beta a_2 \prec_\beta a_5$
Let a be rightmost from $\{a_3, a_4, a_6\}$ with respect to β. Then, $a'_i \notin L_\beta[a]$ for some $i \in \{3, 4, 6\}$, so that $\{a_2, a_5, a'_i\} \subseteq R_\beta(a)$. This shows that a_3, a_4, a_6 are in pairwise different groups of $L_\beta[a]$.

In both cases, we have seen that $\mathrm{gn}(G, \beta) \geq 3$, which shows a contradiction to the choice of β.

G has the second property
Similar to the previous case, we see that $\mathrm{gn}(G, \beta) \leq 2$ due to Lemma 4.4.

– $\{a_2, a_3, a_4\} \prec_\beta \{a_5, a_6\}$
Let a be leftmost from $\{a_5, a_6\}$ with respect to β. Then, $\{a'_5, a'_6\} \subseteq R_\beta(a)$, and thus, a_3, a_4, a are in pairwise different groups of $L_\beta[a]$.

– $\{a_3, a_4, a_6\} \prec_\beta a_2 \prec_\beta a_5$
We distinguish between two cases. Let $a_1 \prec_\beta a_2$. Let a be rightmost from $\{a_1, a_3, a_4, a_6\}$ with respect to β and let b be the vertex following a in β. Observe that $\{a_1, a_3, a_4, a_6\} \subseteq L_\beta(b)$ and $a_2 \in R_\beta[b]$. If $a \in \{a_3, a_4\}$ then the three vertices in $\{a_1, a_3, a_4, a_6\} \setminus \{a\}$ are in pairwise different groups of $L_\beta(a)$. If $a \in \{a_1, a_6\}$ then $a' \in R_\beta[b]$ for a' the true twin of a in G. Then, a_1, a_4, a_6 or a_1, a_3, a_6 are in pairwise different groups of $L_\beta(b)$.

Let $a_2 \prec_\beta a_1$. Then, $\{a_2, a_3, a_4\} \subseteq L_\beta[a_2]$ and $\{a_1, a_5\} \subseteq R_\beta(a_2)$, and a_2, a_3, a_4 are in pairwise different groups of $L_\beta[a_2]$.

We have seen for both cases that $\mathrm{gn}(G, \beta) \geq 3$, which implies a contradiction to the choice of β.

G has the first property
This case requires a more detailed analysis, since G does not have the nice symmetry properties as in the previous two cases. First, suppose that $a_2 \prec_\beta a_5$.

– $\{a_2, a_3, a_4\} \prec_\beta \{a_5, a_6\}$
Let $i \in \{5, 6\}$ be such that $\{a_3, a_4, a_i\} \subseteq L_\beta(a'_i)$ and $\{a'_5, a'_6\} \subseteq R_\beta[a'_i]$. Observe that a_3, a_4, a_i are in pairwise different groups of $L_\beta(a'_i)$. Furthermore, $a'_i \neq a_1$, and due to Lemma 4.3, $\mathrm{ad}_\beta(a'_i) = 1$. Thus, $\mathrm{lcwd}(G, \beta) \geq \nu_G(L_\beta(a'_i)) + \mathrm{ad}_\beta(a'_i) \geq 3 + 1$.

– $\{a_3, a_4, a_6\} \prec_\beta a_2 \prec_\beta a_5$
Let a be a vertex of G such that $a \neq a_1$ and $\{a_3, a_4, a_6\} \subseteq L_\beta(a)$ and $\{a_2, a'_i\} \subseteq R_\beta[a]$ for some $i \in \{3, 4, 6\}$. Then, a_3, a_4, a_6 are in pairwise different groups of $L_\beta(a)$. Due to Lemma 4.3, $\mathrm{ad}_\beta(a) = 1$, and thus, $\mathrm{lcwd}(G, \beta) \geq 4$.

In both cases, we obtain a contradiction to the choice of β. Hence, $a_5 \prec_\beta a_2$.

– $\{a_5, a_4, a_3\} \prec_\beta \{a_2, a_1\}$
Let a be such that $a \neq a_1$ and $\{a_3, a_4, a_5\} \subseteq L_\beta(a)$ and $a_2 \in R_\beta[a]$. Note that a_3 is not in the same group of $L_\beta(a)$ as a_4 or a_5. If a_3, a_4, a_5 are not in pairwise different groups of $L_\beta(a)$

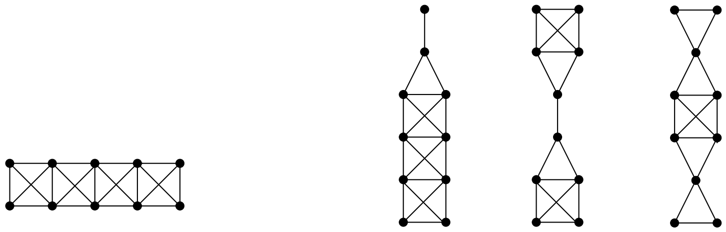

Figure 2: The four different connected thickened paths of linear clique-width at least 4 that are considered in Propositions 5.2 and 5.4.

then a_4 and a_5 are in the same group of $L_\beta(a)$. This means that $\{a_3, a'_3, a_4, a_5, a_6, a'_6\} \subseteq L_\beta(a)$. If $a'_4 \in R_\beta[a]$ then a_3, a_4, a_6 are in pairwise different groups of $L_\beta(a)$. Otherwise, $\{a_3, a'_3, a_4, a'_4, a_6, a'_6\} \subseteq L_\beta(a)$. Let b be rightmost from $\{a'_3, a'_4, a'_6\}$ with respect to β. Then, $\{a_3, a_4, a_6\} \subseteq L_\beta(b)$, and since $\{a_2, b\} \subseteq R_\beta[b]$, it holds that a_3, a_4, a_6 are in pairwise different groups of $L_\beta(b)$.

– $\{a_4, a_3, a_1\} \prec_\beta a_5 \prec_\beta a_2$
If there is $i \in \{3, 4, 6\}$ such that $a'_i \in R_\beta[a_2]$ then a_1, a_3, a_4 or a_3, a_4, a_5 are in pairwise different groups of $L_\beta(a_2)$. Otherwise, $\{a'_3, a'_4, a'_6\} \subseteq L_\beta(a_2)$, and there is $i \in \{3, 4, 6\}$ such that $\{a_3, a_4, a_6\} \subseteq L_\beta(a'_i)$ and $\{a_2, a'_i\} \subseteq R_\beta[a'_i]$. Then, a_3, a_4, a_6 are in pairwise different groups of $L_\beta(a'_i)$.

We have seen that $L_\beta(a)$ or $L_\beta(b)$ or $L_\beta(a_2)$ or $L_\beta(a'_i)$ contains at least three groups, and since $a, b, a_2, a'_i \neq a_1$, it follows that $\mathrm{lcwd}(G, \beta) \geq 4$. ∎

The results of this section are summarised in Figure 2. The depicted thickened paths with true twins are the graphs for which we have shown a lower bound on the linear clique-width in this section.

6 Connected thickened paths of length at least 6

In the previous section, we have considered only a finite set of thickened paths. In this section, we will consider an infinite set of thickened paths. The proof techniques in this section are similar to the ones used in the previous section, and the main structural lemma of this section relies on Proposition 5.3 of the previous section.

Lemma 6.1. *Let $k \geq 7$. Let $P = (a_1, \ldots, a_k)$. Let β be a layout for P such that $\mathrm{gw}(P, \beta) \leq 3$ and $a_2 \prec_\beta a_{k-1}$. Then,*

$$\{a_1, a_2, a_3, a_4\} \prec_\beta a_5 \prec_\beta \cdots \prec_\beta a_{k-2} \prec_\beta \{a_{k-1}, a_k\}\,.$$

Proof. For $1 \leq i < j \leq k$, denote by $\beta_{i,j}$ the restriction of β to the vertices $a_i, \ldots, a_j$. We prove the statement by induction over k. For the induction base, let $k = 7$. Consider $\beta_{1,6}$ and $\beta_{2,7}$. The assumption about β, that is $a_2 \prec_\beta a_6$, and application of Lemma 5.3 gives the following three possible situations for $\beta_{1,6}$:

$$\{a_2, a_3, a_4\} \prec_\beta \{a_5, a_6\} \quad \text{or} \quad \{a_5, a_4, a_3\} \prec_\beta \{a_2, a_1\} \quad \text{or} \quad \{a_4, a_3, a_1\} \prec_\beta a_5 \prec_\beta a_2\,.$$

Note that our assumption $a_2 \prec_\beta a_6$ saves one of the four possible cases. Analogously, we obtain three cases for $\beta_{2,7}$:

$$\{a_3, a_4, a_5\} \prec_\beta \{a_6, a_7\} \quad \text{or} \quad \{a_4, a_5, a_7\} \prec_\beta a_3 \prec_\beta a_6 \quad \text{or} \quad \{a_5, a_4, a_2\} \prec_\beta a_6 \prec_\beta a_3\,.$$

Similar to the proof of Lemma 5.3, we compare the different partial orderings. We assign numbers 1, 2, 3 to the upper row of partial orderings, and we assign numbers $1'$, $2'$, $3'$ to the lower row of partial orderings. We compare the nine possible pairs and determine conflicts. The pairs $(1, 2')$ and $(3, 2')$ do not match because of a_3, a_5, the pair $(1, 3')$ does not match because of a_3, a_6, and the pairs $(2, 3')$ and $(3, 3')$ do not match because of a_2, a_3. The pairs $(2, 1')$, $(3, 1')$ and $(2, 2')$ imply the following subordering: $\{a_3, a_4, a_5\} \prec_\beta \{a_2, a_6\}$. Let a be leftmost from $\{a_2, a_6\}$ with respect to β. Then, a_3, a_4, a_5 are in pairwise different groups of $L_\beta(a)$, and since $\mathrm{ad}_\beta(a) = 1$ due to Lemma 4.3, $\nu_P(L_\beta(a)) + \mathrm{ad}_\beta(a) \geq 4$ yields a contradiction to the choice of β. So, only pair $(1, 1')$ remains, and this implies $\{a_2, a_3, a_4\} \prec_\beta a_5 \prec_\beta \{a_6, a_7\}$. If $a_1 \in R_\beta[a_5]$ then a_2, a_3, a_4 are in pairwise different groups of $L_\beta(a_5)$, and since $\mathrm{ad}_\beta(a_5) = 1$ due to Lemma 4.3, $\nu_P(L_\beta(a_5)) + \mathrm{ad}_\beta(a_5) \geq 4$ yields a contradiction to the assumption about β. Thus, $\{a_1, a_2, a_3, a_4\} \prec_\beta a_5$.

For the induction step, let $k \geq 8$ and assume that the statement of the lemma holds for paths of length $k-2$. We consider $P{-}a_k$ and $P{-}a_1$ and apply the induction hypothesis to $\beta_{1,k-1}$ and $\beta_{2,k}$. Because of $a_2 \prec_\beta a_{k-1}$ for β, we obtain for $\beta_{1,k-1}$:

$$\{a_1, a_2, a_3, a_4\} \prec_\beta a_5 \prec_\beta \cdots \prec_\beta a_{k-3} \prec_\beta \{a_{k-2}, a_{k-1}\}\,.$$

With this result, we observe that $a_2 \prec_\beta a_{k-1}$ also implies $a_3 \prec_\beta a_{k-1}$, and so, we obtain for $\beta_{2,k}$:

$$\{a_2, a_3, a_4, a_5\} \prec_\beta a_6 \prec_\beta \cdots \prec_\beta a_{k-2} \prec_\beta \{a_{k-1}, a_k\}\,.$$

The two partial orderings of β show the claimed result. ■

Proposition 6.2. *Let $k \geq 7$. Let $P = (a_1, \ldots, a_k)$. Let G be a thickened path that has one of the following three properties:*

1) G is obtained from P by adding a true twin to a_3 and a_{k-2}

2) G is obtained from P by adding a true twin to a_1, a_2 and a_{k-2}

3) G is obtained from P by adding a true twin to a_1, a_2, a_{k-1} and a_k.

Then, $\mathrm{lcwd}(G) \geq 4$.

Proof. Let a_i' be the added true twin of a_i. For a contradiction, suppose that there is a layout β for G with $\mathrm{gw}(G, \beta) \leq 3$. Without loss of generality, we can assume that $a_i \prec_\beta a_i'$ for each a_i with a true twin in G. Let β' be the restriction of β to $a_1, \ldots, a_k$. We apply Lemma 6.1 to β' and obtain the following two cases:

$$\begin{gathered}\{a_1, a_2, a_3, a_4\} \prec_{\beta'} a_5 \prec_{\beta'} \cdots \prec_{\beta'} a_{k-2} \prec_{\beta'} \{a_{k-1}, a_k\} \quad \text{or} \\ \{a_k, a_{k-1}, a_{k-2}, a_{k-3}\} \prec_{\beta'} a_{k-4} \prec_{\beta'} \cdots \prec_{\beta'} a_3 \prec_{\beta'} \{a_2, a_1\}\,.\end{gathered}$$

We consider the three rightmost vertices with respect to β'. If G has the first or third property, we can restrict to the first case for β' by a symmetry argument. If G has the second property then the situation for the three rightmost vertices is equal to one of the two other cases. We extend the second property by a second possible situation: true twins for a_3, a_{k-1} and a_k are added. This makes also the second property into a symmetric property and clearly does not change the statement of the proposition. So, we can restrict to the first case for β'. We consider the groups of $L_\beta(a_{k-2})$. Observe that a_{k-1} and a_k have no neighbours in $\{a_1, \ldots, a_{k-3}\}$, and a_{k-2} has exactly one neighbour in $\{a_1, \ldots, a_{k-3}\}$. Thus, $\nu_G(L_\beta(a_{k-2})) \geq 2$. Let γ' be the restriction of β to the vertices a_{k-2}, a_{k-1}, a_k and their corresponding true twins. Let G' be the subgraph of G induced by the vertices in γ'. Note that a_{k-2} is the leftmost vertex with respect to γ'. We define a new graph. Let H be the thickened path that is obtained from the path (b_1, b_2, b_3, b_4) by adding the true twins b_1', b_2', b_3' to respectively b_1, b_2, b_3. Let $\gamma = \langle b_1, b_1', b_3, b_3', b_2, b_2' \rangle$ be a layout, such that the concatenation $\gamma \circ \langle b_4 \rangle$ is a layout for H. It is easy to verify that $\mathrm{gw}(H, \gamma \circ \langle b_4 \rangle) \leq 3$. We define another graph, H', as follows: take the disjoint union of $H - b_4$ and G' and add all edges between b_3 and its true twin b_3' and a_{k-2} and its possible true twin. Note that H' can be obtained from a path of length 5 by adding true twins. Now, observe that the concatenation $\gamma \circ \gamma'$ is a layout for H', and by the assumptions about the groupwidth of β, it holds that $\mathrm{gw}(H', \gamma \circ \gamma') \leq 3$. This follows from the above proved $\nu_G(L_\beta(a_{k-2})) \geq 2$ and the construction of H'. This means that $\mathrm{gw}(H') \leq 3$. However, H' contains as induced subgraph a thickened path with one of the properties as in Proposition 5.4, which implies $\mathrm{gw}(H') \geq 4$. We obtain a contradiction, and thus, we conclude the claim of the proposition. ■

Proposition 6.3. *Let $k \in \{7, 8, 9\}$. Let $P = (a_1, \ldots, a_k)$. Let G be a thickened path that has one of the following three properties:*

1) $k = 7$ and G is obtained from P by adding a true twin to a_4, a_5 and a_7

2) $k = 8$ and G is obtained from P by adding a true twin to a_4 and a_5

3) $k = 9$ and G is obtained from P by adding a true twin to a_5.

Then, $\mathrm{lcwd}(G) \geq 4$.

Proof. Suppose that there is a layout β for G such that $\mathrm{gw}(G, \beta) \leq 3$. We can assume that $a_i \prec_\beta a_i'$ for each a_i' that is the true twin of a_i. Let β' be the restriction of β to the vertices of P. First, we assume that G has the second or third property. By a symmetry argument, we can assume without loss of generality that $a_2 \prec_\beta a_{k-1}$. Applying Lemma 6.1 and the above assumption, we obtain $\{a_3, a_4, a_5\} \prec_\beta \{a_5', a_6\}$. Let a be leftmost from $\{a_5', a_6\}$ with respect to β. Observe that a_3, a_4, a_5 are in pairwise different groups of $L_\beta(a)$ because of $\{a_5', a_6\} \subseteq R_\beta[a]$. Due to Lemma 4.3, $\mathrm{ad}_\beta(a) = 1$, so that $\nu_G(L_\beta(a)) + \mathrm{ad}_\beta(a) \geq 4$ yields a contradiction.

Now, assume that G has the first property. Similar to the above case, we apply Lemma 6.1 and obtain $\{a_3, a_4, a_5\} \prec_\beta \{a_5', a_6\}$ or $\{a_7, a_5, a_4\} \prec_\beta a_3$. Assume the first case. Let a be leftmost from $\{a_5', a_6\}$ with respect to β. Then, $\{a_3, a_4, a_5\} \subseteq L_\beta(a)$ and $\{a_5', a_6\} \subseteq R_\beta[a]$, and thus, a_3, a_4, a_5 are in pairwise different groups of $L_\beta(a)$. Assume the second case. Let a be leftmost from $\{a_3, a_4', a_5', a_7'\}$ with respect to β such that $\{a_4, a_5, a_7\} \subseteq L_\beta(a)$. Note that $a_3 \in R_\beta[a]$ and $\{a_7', a_4', a_5'\} \cap R_\beta[a] \neq \emptyset$. Then, a_4, a_5, a_7 are in pairwise different groups of $L_\beta(a)$. Since $\mathrm{ad}_\beta(a) = 1$ due to Lemma 4.3, we conclude that $\mathrm{gw}(G, \beta) \geq 4$, which is a contradiction. ■

We summarise the results of this section, that are also depicted in Figure 3.

Corollary 6.4. *Let $k \geq 7$ and let $P = (a_1, \ldots, a_k)$. Let $T \subseteq \{1, \ldots, k\}$ be non-empty. Let G be obtained from P by adding a true twin to a_i for every $i \in T$. If T satisfies one condition from each column of the table*

(1)	$T \cap \{3, \ldots, k-4\} \neq \emptyset$	$T \cap \{5, \ldots, k-2\} \neq \emptyset$	(3)
(2)	$\{1, 2\} \subseteq T$	$\{k-1, k\} \subseteq T$	(4)

or if $\{k-3, k-2, k\} \subseteq T$ then $\mathrm{lcwd}(G) \geq 4$.

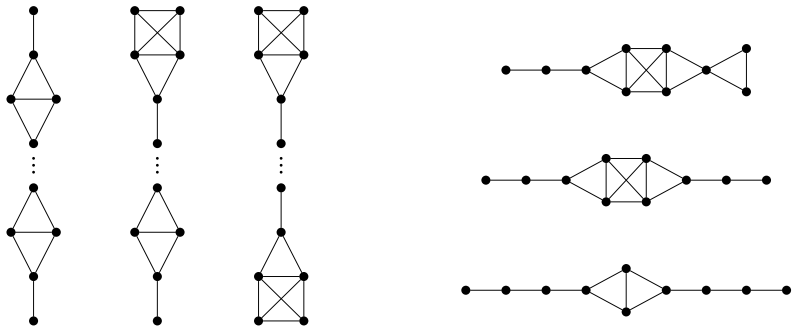

Figure 3: The different connected thickened paths of linear clique-width at least 4 that are considered in Propositions 6.2 and 6.3.

Proof. We first consider the conditions of the table. Let T satisfy condition 1 or 2 and condition 3 or 4. This gives four cases to consider. If T satisfies the pair (2, 4) then G has an induced subgraph with the third property of Proposition 6.2. If T satisfies the pair (2, 3) or (1, 4) then G contains an induced subgraph with the second property of Proposition 6.2. Note in this case that the induced subgraph may be obtained from a path of length smaller than $k-1$. Let T satisfy the pair (1, 3). Let s be the smallest number in $T \cap \{3, \ldots, k-4\}$ and let t be the largest number in $T \cap \{5, \ldots, k-2\}$. Clearly, $s \leq t$. If $t - s \geq 2$ then G contains an induced subgraph with the first property of Proposition 6.2, if $t - s = 1$ then $k \geq 8$, $s > 3$ and $t < k-2$, and G contains an induced subgraph with the second property of Proposition 6.3, if $s = t$ then $\{3, \ldots, k-4\} \cap \{5, \ldots, k-2\} \neq \emptyset$, which means that $k \geq 9$, and $5 \leq s \leq k-4$, and thus, G contains an induced subgraph with the third property of Proposition 6.3.

If $\{k-3, k-2, k\} \subseteq T$ then G contains an induced subgraph with the first property of Proposition 6.3. ∎

7 Disconnected thickened paths

In the two previous sections, we have considered connected thickened paths only. In this section, we look at the situation for disconnected thickened paths. It will turn out that already small graphs require larger linear clique-width.

Proposition 7.1. *Let $P_3 = (a_1, a_2, a_3)$, $P_4 = (b_1, b_2, b_3, b_4)$ and $P_5 = (c_1, c_2, c_3, c_4, c_5)$.*

- *Let G_3 be obtained from P_3 by adding a true twin to a_1, a_2, a_3.*
- *Let G_4 be obtained from P_4 by adding a true twin to b_1, b_2.*
- *Let G_5 be obtained from P_5 by adding a true twin to c_3.*

Let H be the disjoint union of two graphs H' and H'' where each of H' and H'' is a copy of G_3 or G_4 or G_5. Then, $\text{lcwd}(H) \geq 4$.

Proof. Suppose that there is a layout β for H such that $\text{gw}(H, \beta) \leq 3$. Denote by d the leftmost vertex of H with respect to β. Without loss of generality, we can assume that d is a vertex of H'. Let $a'_1, a'_2, a'_3, b'_1, b'_2, c'_3$ be the added true twins for G_3, G_4, G_5. The following observation is important for the proof and is an implication of our assumption about the groupwidth of β. Let x be a vertex of H''. It holds that d cannot be in a group of $L_\beta(x)$ that contains a vertex of H'' with a neighbour in $R_\beta[x]$. Thus, if $\mathrm{ad}_\beta(x) = 1$ then $\nu_H(L_\beta(x)) \leq 2$ and at most one group of $L_\beta(x)$ contains vertices of H'' with a neighbour in $R_\beta[x]$. We distinguish the three cases according to whether H'' is a copy of G_3 or G_4 or G_5.

H'' is a copy of G_3
By a symmetry argument, we can assume that $\{a_1, a_1', a_3\} \prec_\beta a_3'$ and $a_2 \prec_\beta a_2'$. There is $i \in \{1,2,3\}$ such that $\{a_1, a_2\} \subseteq L_\beta(a_i')$ or $\{a_1, a_3\} \subseteq L_\beta(a_i')$ and such that $\{a_1', a_3'\} \subseteq R_\beta[a_i']$ or $\{a_2', a_3'\} \subseteq R_\beta[a_i']$. Then, $L_\beta(a_i')$ contains two groups with vertices from H'' that have a neighbour in $R_\beta[a_i']$. Since $\mathrm{ad}_\beta(a_i') = 1$ due to Lemma 4.3, we obtain a contradiction.

H'' is a copy of G_4
Let $\langle x_1, x_2, x_3, x_4 \rangle$ be the restriction of β to the vertices of P_4. Let x be the leftmost vertex of H'' with respect to β such that $\{x_1, x_2\} \subseteq L_\beta(x)$. With this choice of x, $L_\beta(x)$ contains at most three vertices of H''. If $\{x_1, x_2\} \subseteq \{b_1, b_2, b_3\}$ then x_1 and x_2 are in pairwise different groups of $L_\beta(x)$, and x_1 as well as x_2 has a neighbour in $R_\beta[x]$. If $x \neq b_4$ then $\mathrm{ad}_\beta(x) = 1$ due to Lemma 4.3, and we obtain a contradiction. As the other case, let $b_4 \in L_\beta[x]$. Let $b_4 = x$. We obtain the three cases:

$$\{b_1, b_2\} \prec_\beta b_4 \prec_\beta b_3 \quad \text{or} \quad \{b_1, b_3\} \prec_\beta b_4 \prec_\beta b_2 \quad \text{or} \quad \{b_2, b_3\} \prec_\beta b_4 \prec_\beta b_1 \,.$$

One verifies that $b_1' \in R_\beta[x]$ or $b_2' \in R_\beta[x]$ in all three cases and that x_1 and x_2 are not in the same group of $L_\beta(x)$. Furthermore, $\mathrm{ad}_\beta(x) = 1$ in all three cases: in the first case, $\{b_4\}$ is a group of $L_\beta[x]$, in the second case, b_4 is in the group in $L_\beta[x]$ with all vertices in $L_\beta[x]$ that have no neighbour in $R_\beta(x)$, and in the third case, since in case $b_2' \in L_\beta(x)$, b_3 and b_4 are in the same group of $L_\beta[x]$ but they are adjacent. Since we observe a contradiction in every case, we must conclude that $x \neq b_4$ and thus $b_4 \in L_\beta(x)$. Due to Lemma 4.3, $\mathrm{ad}_\beta(x) = 1$. If $b_3 \in R_\beta[x]$ then b_4 has a neighbour in $R_\beta[x]$ and we observe a contradiction as in the above cases. So, let $b_3 \in L_\beta(x)$, which means $\{b_3, b_4\} \subseteq L_\beta(x)$ and $\{b_1, b_1', b_2, b_2'\} \subseteq R_\beta[x]$. Independent of whether $x = b_1$ or $x = b_2$, we obtain a contradiction to the groupwidth of β with the vertex of H'' that follows x in β. This completes this case.

H'' is a copy of G_5
By a symmetry argument, we can assume that $c_1 \prec_\beta c_5$ and $c_3 \prec_\beta c_3'$. Let β' be the restriction of β to the vertices of P_5. Let x be the vertex of P_5 with $|L_{\beta'}[x]| = 3$. We apply Lemma 5.1 to β' and obtain the following three cases:

$R_{\beta'}(x)$	groups of $L_{\beta'}[x]$
$\{c_4, c_5\}$	$\{c_1, c_2\}, \{c_3\}^*$
$\{c_2, c_5\}$	$\{c_1, c_3\}^*, \{c_4\}^*$
$\{c_1, c_5\}$	$\{c_2\}^*, \{c_3\}, \{c_4\}^*$

The symbol * marks the groups that contain vertices with a neighbour in $R_\beta(x)$. Consider the last case of the table. Due to Lemma 4.3, it holds that $\mathrm{ad}_\beta(x) = 1$. However, independent of the actual choice of x, $L_\beta(x)$ contains at least two groups with vertices of H'' that have a neighbour in $R_\beta[x]$. This yields contradiction. For the second case of the table, let y be leftmost from $\{c_2, c_5\}$ with respect to β. If $y = c_2$ then $\mathrm{ad}_\beta(y) = 1$ due to Lemma 4.3, and we observe a contradiction. If $y = c_5$ then Lemma 4.3 does not provide information. However, the definition of function ad shows that $\mathrm{ad}_\beta(c_5) = 1$, so that also this case leads to a contraction. It remains to consider the first case of the table. If $x = c_2$ then $\mathrm{ad}_\beta(x) = 1$ and a contradiction follows analogous to the previous cases. If $x = c_1$ then, analogous to the arguments seen before, the definition of function ad shows that $\mathrm{ad}_\beta(x) = 1$, and we conclude a contradiction. It remains to

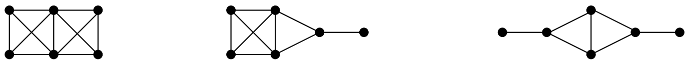

Figure 4: The three thickened paths that are discussed in Proposition 7.1. Any disjoint union of at least two copies of the depicted graphs has linear clique-width at least 4.

consider the case when $x = c_3$. Due to our assumptions about β, c_1 and c_2 are distinguished by c'_3, so that $L_\beta[x]$ contains (at least) three groups. By our assumption about the groupwidth of β, it holds that $\mathrm{ad}_\beta(y) = 0$ for y the leftmost vertex of H'' with respect to β such that $x \in L_\beta(y)$. Clearly, $y = c_5$, and therefore, $\{c'_3, c_4, c_5\} \subseteq R_\beta[y]$. However, the definition of function ad shows that $\mathrm{ad}_\beta(c_5) = 1$, in particular, since c_5 is not in group with any of the vertices c_1, c_2, c_3. We conclude a contradiction.

We have seen that every case yields a contradiction to the assumption that $\mathrm{gw}(H, \beta) \leq 3$, so that $\mathrm{gw}(H) \geq 4$ follows. ∎

It is interesting to observe that Proposition 7.1 implies the result of Proposition 6.2 for connected thickened paths that are obtained from paths of sufficiently large length. The three graphs considered in Proposition 7.1 are depicted in Figure 4.

8 The complete characterisation

In the previous sections, we have shown lower bounds on the linear clique-width. The results show a sufficient condition for a thickened path to have linear clique-width at least 4. In this section, we complete the results by showing that the given conditions are also necessary.

Lemma 8.1. *Let $k \geq 6$ and let $P = (a_1, \ldots, a_k)$. Let $T \subseteq \{1, \ldots, k\}$ be non-empty. Let G be obtained from P by adding a true twin to a_i for every $i \in T$. If T satisfies the following three conditions:*

1) $T \subseteq \{1, 2\} \cup \{k-3, k-2, k-1, k\}$

2) $k-3 \notin T$ or $k-2 \notin T$ or $k \notin T$

3) $1 \notin T$ or $2 \notin T$

then $\mathrm{lcwd}(G) \leq 3$.

Proof. We assume T to be maximal possible. We distinguish between three cases according to the second condition and two cases according to the third condition. We give a linear 3-expression for G, that is constructed in three steps, the first step creating vertices $a_{k-3}, a_{k-2}, a_{k-1}, a_k$, the second step creating vertices $a_{k-4}, \ldots, a_3$ and the third case creating vertices a_2 and a_1. Note that the second step is empty for $k = 6$. For the first and third step, the expression naturally depends on the actual situation in T. Denote by $a'_1, a'_2, a'_{k-3}, a'_{k-2}, a'_{k-1}, a'_k$ the true twins of respectively $a_1, a_2, a_{k-3}, a_{k-2}, a_{k-1}, a_k$ of G (as they exist). We begin with the three cases for the first step. Instead of giving full linear 3-expressions, we give layouts for the seven vertices. The layouts translate into linear expressions in a canonical way. The three layouts are:

$$\begin{array}{ll} \text{if } k \notin T & : \langle a_{k-3}, a'_{k-3}, a_{k-2}, a'_{k-2}, a_k, a_{k-1}, a'_{k-1} \rangle \\ \text{if } k-2 \notin T: & \langle a_{k-3}, a'_{k-3}, a_k, a'_k, a_{k-2}, a_{k-1}, a'_{k-1} \rangle \\ \text{if } k-3 \notin T: & \langle a_k, a'_k, a_{k-2}, a'_{k-2}, a_{k-1}, a'_{k-1}, a_{k-3} \rangle \,. \end{array}$$

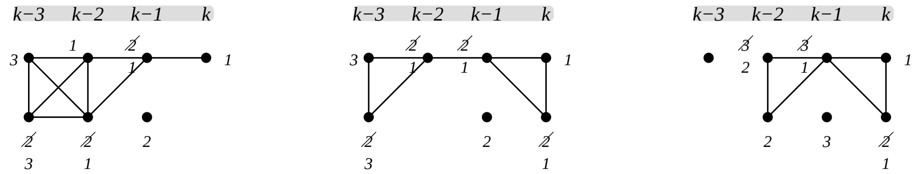

Figure 5: The three figures show the beginnings of linear 3-expressions for thickened paths, as they are constructed in the proof of Lemma 8.1. The left figure considers the case of a thickened path without a true twin of a_k, the middle thickened path contains no true twin of a_{k-2}, and the right thickened path is without a true twin of a_{k-3}. The three figures show a possible label assignment at the moment of adding a'_{k-1}. The crossed labels are the result of a relabel operation.

Translating the layouts into expressions, we can assume without loss of generality that a_{k-3} and its possible true twin a'_{k-3} receive label 3 and the other vertices, all without neighbour in the rest of the graph, receive label 1 at the end of the expression. For illustration, Figure 5 shows the results of linear 3-expressions for the three cases at the moment of adding vertex a'_{k-1}. With these assumptions, the expression continues with a sequence of subexpressions, for $i = k-4, \ldots, 3$:

$$\Big(2(a_i)\,\eta_{2,3}\,\rho_{3\to1}\,\rho_{2\to3}\Big)\,.$$

This sequence iteratively adds the vertices $a_{k-4}, \ldots, a_3$ to the already created graph, following their ordering in P. It is important to remember that none of these vertices has a true twin in G. For the last three vertices, we have to distinguish between the two cases according to whether $1 \in T$ or $2 \in T$. The two subexpressions are given here:

$$\begin{aligned} &\text{if } 1 \in T: && \Big(2(a_2)\,\eta_{2,3}\,\rho_{3\to1}\,\rho_{2\to3}\Big)\Big(2(a_1)\,\eta_{2,3}\,\rho_{2\to3}\Big)\Big(2(a'_1)\,\eta_{2,3}\,\varrho_{2\to3}\Big) \\ &\text{if } 2 \in T: && \Big(3(a_1)\Big)\Big(2(a_2)\,\eta_{2,3}\,\rho_{2\to3}\Big)\Big(2(a'_2)\,\eta_{2,3}\,\rho_{2\to3}\Big)\,. \end{aligned}$$

This completes the construction of the expression for G. If T is not maximal with respect to the three conditions then G is induced subgraph of some thickened path with maximal T, and the result follows with the monotonicity of linear clique-width for induced subgraphs (Lemma 4.1). ■

Note that the result of Lemma 8.1 directly extends to graphs that are isomorphic to a thickened path that satisfies the given conditions. In such a case, condition 1 of the lemma could also be formulated as $T \subseteq \{1, 2, 3, 4, k-1, k\}$, and the two other conditions would be re-formulated correspondingly.

We combine all obtained results to a complete characterisation of the linear clique-width of thickened paths.

Theorem 8.2. *Let G be a thickened path. Let one of the following three conditions be true for G:*

1) G contains a copy of the graphs in Figure 2 as induced subgraph

2) G contains a copy of the graphs in Figure 3 as induced subgraph

3) for G_3, G_4, G_5 the three graphs in Figure 4, G has induced subgraphs H' and H'' such that $V(H') \cap V(H'') = \emptyset$ and H' and H'' are copies of G_3, G_4, G_5.

Then, $\mathrm{lcwd}(G) = 4$*, otherwise,* $\mathrm{lcwd}(G) \le 3$.

Proof. Due to the results of Propositions 5.2 and 5.4, Corollary 6.4, Proposition 7.1 and Lemmata 4.1 and 4.9, if G satisfies one of the three conditions of the theorem then $\mathrm{lcwd}(G) = 4$. For the converse, assume that G does not satisfy any of the three conditions. This particularly means that G contains at most one connected component that contains a copy of G_3, G_4 or G_5 as induced subgraph. Let C be a connected component of G that does not contain a copy of G_3, G_4, G_5 as induced subgraph. We show that there is a linear 3-expression for G that does not use label 1 for creating edges. Let T be such that C is obtained from a path $(a_1, \ldots, a_k)$ by adding a true twin a_i' to each vertex a_i with $i \in T$. Because of G_5, we notice that $T \cap \{3, \ldots, k-2\} = \emptyset$. And if $k \ge 4$ then G_4 implies that $1 \notin T$ or $2 \notin T$ and that $k \notin T$ or $k-1 \notin T$. For the following construction, we can assume without loss of generality that T is maximal with these properties. We partition the construction of an appropriate linear expression for G into two steps. Depending on the case, the expression begins as:

$$\begin{array}{ll} \text{if } k \in T \quad : & \Big(2(a_k)\, 3(a_k')\, \eta_{2,3}\, \rho_{3\to 2}\, 3(a_{k-1})\, \eta_{2,3}\, \rho_{2\to 1}\Big) \\ \text{if } k-1 \in T: & \Big(2(a_{k-1})\, 3(a_{k-1}')\, \eta_{2,3}\, \rho_{2\to 3}\, 2(a_k)\, \eta_{2,3}\, \rho_{2\to 1}\Big)\,. \end{array}$$

We continue as in the second and third step of the proof of Lemma 8.1, which defines a desired linear 3-expression for C. For the case when $k \le 3$, it suffices to observe that C is induced subgraph of some graph that is obtained from a path of length 2 with $T = \{1,3\}$ or $T = \{2,3\}$. This particularly holds since C cannot contain G_3 as induced subgraph.

Now, let C be a connected component of G that contains a copy of G_3, G_4 or G_5 as induced subgraph. Similar to the above case, let T be a set such that C is obtained from a path $(a_1, \ldots, a_k)$ by adding a true twin a_i' to each a_i with $i \in T$. By assumption about G, C does not satisfy condition 1 or 2. If $k \ge 7$ then T does not satisfy the conditions of Corollary 6.4. This means that $\{k-3, k-2, k\} \not\subseteq T$ and $\{1,3,4\} \not\subseteq T$ and that

- $T \cap \{3, \ldots, k-4\} = \emptyset$ and $\{1,2\} \not\subseteq T$, or
- $T \cap \{5, \ldots, k-2\} = \emptyset$ and $\{k-1, k\} \not\subseteq T$.

This exactly means that C satisfies the conditions of Lemma 8.1, and thus, $\mathrm{lcwd}(C) \le 3$. If $k \le 6$ then C contains no copy of a graph depicted in Figure 2 as induced subgraph. Let $k = 6$. We consider the possible cases. Let $T \subseteq \{2,3,4,6\}$. It suffices to show that $\mathrm{lcwd}(C) \le 3$ for the case when $T = \{2,3,4,6\}$. The vertex ordering $\langle a_4, a_4', a_3, a_3', a_1, a_2, a_2', a_5, a_6, a_6' \rangle$ defines a linear 3-expression for G. If $T \subseteq \{1,3,4,5\}$ then an analogous linear 3-expression for G exists. If $1 \notin T$ and $6 \notin T$ then $\mathrm{lcwd}(C) \le 3$ due to Lemma 8.1. If $1 \notin T$ and $6 \in T$ then $2 \notin T$ or $3 \notin T$ or $4 \notin T$ or $5 \notin T$: since $\{3,4,5,6\} \not\subseteq T$ due to condition 1, $\mathrm{lcwd}(C) \le 3$ due to Lemma 8.1 (if $3 \notin T$ or $4 \notin T$) or due to an above shown case (if $5 \notin T$); symmetrically, $\mathrm{lcwd}(C) \le 3$ for $1 \in T$ and $6 \notin T$. Let $\{1,6\} \subseteq T$. Then, $2 \notin T$ or $5 \notin T$ and $3 \notin T$ or $4 \notin T$ (due to condition 1). Thus, $\mathrm{lcwd}(C) \le 3$ due to Lemma 8.1. This completes the case of $k = 6$. If $k \le 5$, it is not difficult to see, since $\{1,2,3,4,5\} \not\subseteq T$, that C is induced subgraph of some thickened path obtained from a path on six vertices by adding true twins and that satisfies the conditions of Lemma 8.1. Thus, $\mathrm{lcwd}(C) \le 3$.

The final linear 3-expression for G is obtained by first constructing the connected component that contains a copy of G_3, G_4 or G_5 as induced subgraph (if such exists), then changing all labels to 1 and then iteratively constructing all other connected components, using a linear 3-expression where label 1 is not used for creating edges. Such expressions exist due to the above paragraphs, and this completes the proof. ∎

The proof of Theorem 8.2 directly translates into a linear-time algorithm for deciding for a given thickened path whether its linear clique-width is at most 3. In combination with Proposition 4.6, we obtain a linear-time algorithm for computing the linear clique-width of thickened paths.

9 Conclusions

We have given a complete characterisation of thickened paths of linear clique-width at most 3 by forbidden induced subgraphs. It is important to note that our obtained set is finite. We see that all connected graphs in Figure 2 and on the right hand side of Figure 3 have ten vertices, and all disconnected graphs from Figure 4 have twelve vertices. Furthermore, for the graphs in Figure 4, G_5 is a subgraph of G_4 and G_4 is a subgraph of G_3. Similar subgraph relationships are observed for the other graphs. These relationships trigger the question whether there is a general result about the linear clique-width of graphs that are sandwiched between two graphs of large linear clique-width. As an example, G_4 of Figure 4 is sandwiched between G_5 and G_3.

Thickened paths are proper interval graphs, so our results provide a class of forbidden induced subgraphs of proper interval graphs of linear clique-width at most 3. A next step towards a complete characterisation of graphs of linear clique-width at most 3 can be to consider subclasses of proper interval graphs that disallow either only bull or only gem as induced subgraph.

How can our results help for better understanding of not only linear clique-width but also clique-width? Expressions for clique-width have a tree structure, and expressions for linear clique-width have a path structure. Fixing an expression for clique-width and restricting to a leaf-root path in the tree defines a linear clique-width expression for a subgraph. The linear clique-width of this subgraph provides a lower bound on the clique-width of the whole graph, and the structure of its expressions provides information about the whole expression. Due to this connection, the study of linear clique-width of certain graph classes potentially gives valuable insights into the nature of clique-width of larger graph classes.

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