Polynomial-Time Algorithms for the Subset Feedback Vertex Set Problem on Interval Graphs and Permutation Graphs

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Abstract. Given a vertex-weighted graph $G = (V, E)$ and a set $S \subseteq V$, a subset feedback vertex set X is a set of the vertices of G such that the graph induced by $V \setminus X$ has no cycle containing a vertex of S. The Subset Feedback Vertex Set problem takes as input G and S and asks for the subset feedback vertex set of minimum total weight. In contrast to the classical Feedback Vertex Set problem which is obtained from the Subset Feedback Vertex Set problem for $S = V$, restricted to graph classes the Subset Feedback Vertex Set problem is known to be NP-complete on split graphs and, consequently, on chordal graphs. Here we give the first polynomial-time algorithms for the problem on two subclasses of AT-free graphs: interval graphs and permutation graphs. Moreover towards the unknown complexity of the problem for AT-free graphs, we give a polynomial-time algorithm for co-bipartite graphs. Thus we contribute to the first positive results of the Subset Feedback Vertex Set problem when restricted to graph classes for which Feedback Vertex Set is solved in polynomial time.

1 Introduction

For a given set S of vertices of a graph G, a *subset feedback vertex set* X is a set of vertices such that every cycle of $G[V \setminus X]$ does not contain a vertex from S. The Subset Feedback Vertex Set problem takes as input a graph $G = (V, E)$ and a set $S \subseteq V$ and asks for the subset feedback vertex set of minimum cardinality. In the weighted version every vertex of G has a weight and the objective is to compute a subset feedback vertex set with the minimum total weight. The Subset Feedback Vertex Set problem is a generalization of the classical Feedback Vertex Set problem in which the goal is to remove a set of vertices X such that $G[V \setminus X]$ has no cycles. Thus by setting $S = V$ the problem coincides with the NP-complete Feedback Vertex Set problem [18]. Both problems find important applications in several aspects that arise in optimization theory, constraint satisfaction, and bayesian inference [1, 2, 14, 15]. Interestingly the Subset Feedback Vertex Set problem for $|S| = 1$ also coincides with the NP-complete Multiway Cut problem [17] in which the task is to disconnect a predescribed set of vertices [9, 19].

Subset Feedback Vertex Set was first introduced by Even et al. who obtained a constant factor approximation algorithm for its weighted version [14].

The unweighted version in which all vertex weights are equal has been proved to be fixed parameter tractable [13]. Moreover the fastest algorithm for the weighted version in general graphs runs in $O^*(1.87^n)$ time[1] by enumerating its minimal solutions [17], whereas for the unweighted version the fastest algorithm runs in $O^*(1.76^n)$ time [16]. As the unweighted version of the problem is shown to be NP-complete even when restricted to split graphs [17], there is a considerable effort to reduce the running time on chordal graphs, a proper superclass of split graphs, and more general on other classes of graphs. Golovach et al. considered the weighted version and gave an algorithm that runs in $O^*(1.68^n)$ time for chordal graphs [20]. Reducing the existing running time even on chordal graphs has been proved to be quite challenging and only for the unweighted version of the problem a faster algorithm was given that runs in $O^*(1.62^n)$ time [10]. In fact the $O^*(1.62^n)$-algorithm given in [10] runs for every graph class which is closed under vertex deletions and edge contractions, and on which the weighted Feedback Vertex Set problem can be solved in polynomial time. Thus there is an algorithm that runs in $O^*(1.62^n)$ time for the unweighted version of the Subset Feedback Vertex Set problem when restricted to AT-free graphs [10], a graph class that properly contains permutation graphs and interval graphs. Here we show that for the classes of permutation graphs and interval graphs we design a much faster algorithm even for the weighted version of the problem.

As Subset Feedback Vertex Set is a generalization of the classical Feedback Vertex Set problem, let us briefly give an overview of the complexity of Feedback Vertex Set on related graph classes. Concerning the complexity of Feedback Vertex Set it is known to be NP-complete on bipartite graphs [32] and planar graphs [18], whereas it becomes polynomial-time solvable on the classes of bounded clique-width graphs [8], chordal graphs [11, 31], interval graphs [27], permutation graphs [4–6, 25], cocomparability graphs [26], and, more generally, AT-free graphs [24]. Despite the many positive and negative results of the Feedback Vertex Set problem, very few similar results are known concerning the complexity of Subset Feedback Vertex Set. Clearly for graph classes for which the Feedback Vertex Set problem is NP-complete, so does the Subset Feedback Vertex Set problem. However as the Subset Feedback Vertex Set problem is more general that Feedback Vertex Set problem, it is natural to study its complexity for graph classes for which Feedback Vertex Set is polynomial-time solvable. In fact restricted to graph classes there is only a negative result for the Subset Feedback Vertex Set problem regarding its NP-completeness on split graphs [17]. Such a result, however, implies that there is an interesting algorithmic difference between the two problems, as the Feedback Vertex Set problem is known to be polynomial-time computable for split graphs [11, 31].

Both interval graphs and permutation graphs have unbounded clique-width [22] and, therefore, excluding any application of algorithmic metatheorems related to MSOL formulation [12]. Let us also briefly explain that extending the approach of [24] for the Feedback Vertex Set problem when restricted to

[1] The O^* notation is used to suppress polynomial factors.

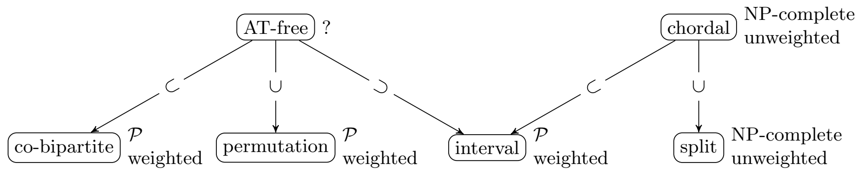


Fig. 1. The computational complexity of the Subset Feedback Vertex Set problem restricted to the considered graph classes. All polynomial-time results ($\mathcal{P}$) are obtained in this work, whereas the NP-completeness result of split graphs is due to [17].

AT-free graphs is not straightforward. A graph is *AT-free* if for every triple of pairwise non-adjacent vertices, the neighborhood of one of them separates the two others. The class of AT-free graphs is well-studied and it properly contains interval, permutation, and cocomparability graphs [7, 21]. One of the basic tools in [24] relies on growing a small representation of an independent set into a suitable forest. Although such a representation is rather small on AT-free graphs (and, thus, on interval graphs or permutation graphs), when considering Subset Feedback Vertex Set it is not necessary that the fixed set induces an independent set which makes it difficult to control how the partial solution may be extended. Therefore the methodology described in [24] cannot be trivially extended towards the Subset Feedback Vertex Set problem.

Our Results. Here we initiate the study of Subset Feedback Vertex Set restricted on graph classes from the positive perspective. We consider its weighted version and give the first positive results on permutation graphs and interval graphs, both being proper subclasses of AT-free graphs. As already explained, we are interested towards subclasses of AT-free graphs since for chordal graphs the problem is already NP-complete [17]. Permutation graphs and interval graphs are unrelated to split graphs and are both characterized by a linear structure with respect to a given vertex ordering [7, 21, 31]. For both classes of graphs we design polynomial-time algorithms based on dynamic programming of subproblems defined by passing the vertices of the graph according to their natural linear ordering. One of our key ingredients is that during the pass of the dynamic programming we augment the considered vertex set and we allow the solutions to be chosen only from a specific subset of the vertices rather than the whole vertex set. Although for interval graphs such a strategy leads to a simple algorithm, the case of permutation graphs requires further descriptions of the considered subsolutions by augmenting the considered part of the graph with a small number of additional vertices. Moreover towards the unknown complexity of the problem for the class of AT-free graphs, we consider the class of co-bipartite graphs (complements of bipartite graphs) and settle its complexity status. More precisely we show that the number of minimal solutions of a co-bipartite graph is polynomial which implies a polynomial-time algorithm of the Subset Feedback Vertex Set problem for the class of co-bipartite graphs. Our overall results are summarized in Figure 1. Therefore, we contribute to provide the first positive results of the Subset Feedback Vertex Set problem on subclasses of AT-free graphs.

2 Preliminaries

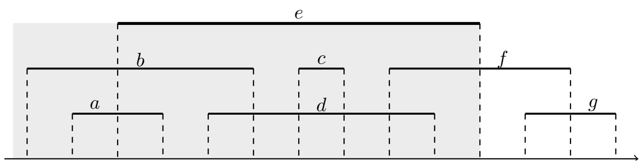
We refer to [7, 21] for our standard graph terminology. A *path* is a sequence of distinct vertices $\langle v_1 v_2 \cdots v_k \rangle$ where each pair of consecutive vertices $v_i v_{i+1}$ forms an edge of G. If in addition $v_1 v_k$ is an edge then we obtain a *cycle*. In this paper, we distinguish between paths (or cycles) and *induced paths* (or *induced cycles*). By an induced path (or cycle) of G we mean a chordless path (or cycle). A chordless cycle on four vertices is referred to as *square*. A *weighted graph* $G = (V, E)$ is a graph, where each vertex $v \in V$ is assigned a *weight* that is a positive integer number. We denote by $w(v)$ the weight of each vertex $v \in V$. For a vertex set $A \subset V$, the weight of A is $\sum_{v \in A} w(v)$.

The *Subset Feedback Vertex Set* (SFVS) problem is defined as follows: given a weighted graph G and a vertex set $S \subseteq V$, find a vertex set $X \subset V$, such that all cycles containing vertices of S, also contains a vertex of X and $\sum_{v \in X} w(v)$ is minimized. In the unweighted version of the problem all weights are equal and positive. A vertex set X is defined as *minimal* subset feedback vertex set if no proper subset of X is a subset feedback vertex set for G and S. The classical Feedback Vertex Set (FVS) problem is a special case of the subset feedback vertex set problem with $S = V$. Note that a *minimum weight* subset feedback vertex set is dependent on the weights of the vertices, whereas a *minimal* subset feedback vertex set is only dependent on the vertices and not their weights. Clearly, both in the weighted and the unweighted versions, a minimum subset feedback vertex set must be minimal.

An induced cycle of G is called S-cycle if a vertex of S is contained in the cycle. We define an *S-forest* of G to be a vertex set $Y \subseteq V$ such that no cycle in $G[Y]$ is an S-cycle. An S-forest Y is *maximal* if no proper superset of Y is an S-forest. Observe that X is a minimal subset feedback vertex set if and only if $Y = V \setminus X$ is a maximal S-forest. Thus, the problem of computing a minimum weighted subset feedback vertex set is equivalent to the problem of computing a maximum weighted S-forest. Let us denote by $\mathcal{F}_S$ the class of S-forests. In such terms, given the graph G and the subset S of V, we are interested in finding a $\max_w \{Y \subseteq V \mid G[Y] \in \mathcal{F}_S\}$, where $\max_w$ selects a vertex set having the maximum sum of its weights. It is not difficult to see that for any clique C of G, a maximal S-forest of G contains at most two vertices of $S \cap C$.

3 Computing SFVS on interval graphs

Here we present a polynomial-time algorithm for the SFVS problem on interval graphs. A graph is an *interval graph* if there is a bijection between its vertices and a family of closed intervals of the real line such that two vertices are adjacent if and only if the two corresponding intervals intersect. Such a bijection is called an *interval representation* of the graph, denoted by $\mathcal{I}$. Notice that every induced subgraph of an interval graph is an interval graph. Moreover it can be decided in linear time whether a given graph is an interval graph, and if so, an interval representation can be generated in linear time [**?**]. Hereafter we assume that the input graph is connected; otherwise, we apply the described algorithm in



$V_e = \{a, b, c, d, e\}$
$V_{<e} = \{a, b, c, d\}$
$V_{\ll e} = \emptyset$
$V_f = \{a, b, c, d, e, f\}$
$V_{<f} = \{a, b, c, d, e\}$
$V_{\ll f} = \{a, b, c\}$

Fig. 2. An interval graph given by its interval representation and the corresponding sets of V_e and V_f. Observe that $<f = e$ whereas $\ll f = c$. Also notice that the intervals that are properly contained within the gray area form the set V_e.

each connected component and take the overall solution as the union of the sub-solutions.

As already mentioned, instead of finding a subset feedback vertex set X of minimum weight of (G, S) we concentrate on the equivalent problem of finding a maximum weighted S-forest Y of (G, S). We first define the necessary vertex sets. Let G be a weighted interval graph and let $\mathcal{I}$ be its interval representation. The left and right endpoints of an interval i, $1 \leq i \leq n$, are denoted by $\ell(i)$ and $r(i)$, respectively. The intervals are numbered from 1 to n according to their ascending $r(i)$. For technical reasons, we add an interval with label 0 that does not belong to S, has weight zero, and augment $\mathcal{I}$ to $\mathcal{I}^+$ by setting $\ell(0) = -1$ and $r(0) = 0$. Notice that interval 0 is non-adjacent to any vertex of G. Clearly if Y is a maximum weighted S-forest for $G[\mathcal{I}^+]$ then $Y \setminus \{0\}$ is a maximum weighted S-forest for $G[\mathcal{I}]$. Moreover it is known that any induced cycle of an interval graph is an induced triangle [27, 31].

We consider the two relations on V that are defined by the endpoints of the intervals as follows: $i \leq_\ell j \Leftrightarrow \ell(i) \leq \ell(j)$ and $i \leq_r j \Leftrightarrow r(i) \leq r(j)$. Since all endpoints of the collection's intervals are distinct, $\leq_\ell$ and $\leq_r$ are total orders on V. For a set of vertices $U \subseteq V$ we write $\ell\text{-}\min U$ to denote the *minimum* vertex of U with respect to $\leq_\ell$ and we write $r\text{-}\max U$ to denote the *maximum* vertex of U with respect to $\leq_r$. For a vertex $i \in V \setminus \{0\}$ we let $V_i =_{\text{def}} \{h \in V : h \leq_r i\}$. We define two types of predecessors of the interval i with respect to $\leq_r$, which correspond to the subproblems that our algorithm wants to solve: $<i =_{\text{def}} r\text{-}\max(V_i \setminus \{i\})$ and $\ll i =_{\text{def}} r\text{-}\max(V_i \setminus (\{i\} \cup \{h \in V : \{h, i\} \in E\}))$. Observe that for two vertices $i, x \in V$ with $r(i) < r(x)$, $x \in V \setminus V_i$. An example of an interval representation that depicts the corresponding notation of V_i is shown in Figure 2. By definition we get the following described partitions of V_i and $V_{<i}$.

Observation 1. Let $i \in V \setminus \{0\}$ and let $j \in V \setminus V_i$ such that $\{i, j\} \in E$. Then, (1) $V_i = V_{<i} \cup \{i\}$ and (2) $V_{<i} = V_{\ll j} \cup \{h \in V_{<i} : \{h, j\} \in E\}$.

Next we define the sets that our dynamic programming algorithm uses in order to compute the S-forest of G that has maximum weight.

A-**sets**: Let $i \in V$. Then, $A_i =_{\text{def}} \max_w\{X \subseteq V_i : G[X] \in \mathcal{F}_S\}$.

B-**sets**: Let $i \in V$, $x \in V \setminus V_i$. Then, $B_i^x =_{\text{def}} \max_w\{X \subseteq V_i : G[X \cup \{x\}] \in \mathcal{F}_S\}$.

C-**sets**: Let $i \in V$ and let $x, y \in V \setminus (V_i \cup S)$ such that $x <_\ell y$ and $\{x, y\} \in E$. Then, $C_i^{x,y} =_{\text{def}} \max_w\{X \subseteq V_i : G[X \cup \{x, y\}] \in \mathcal{F}_S\}$.

Since $V_0 = \{0\}$ and $w(0) \leq 0$, $A_0 = \emptyset$ and, since $V_n = V$, $A_n = \max_w\{X \subseteq V : G[X] \in \mathcal{F}_S\}$. The following lemmas state how to recursively compute all A-sets, B-sets and C-sets besides A_0.

Lemma 1. *Let $i \in V \setminus \{0\}$. Then $A_i = \max_w \left\{A_{<i}, B^i_{<i} \cup \{i\}\right\}$.*

Proof. By Observation 1 (1), $V_i = V_{<i} \cup \{i\}$. Two cases hold: either $i \notin A_i$ or $i \in A_i$. In the former we have $A_i = A_{<i}$, whereas in the latter i cannot induce an S-cycle in $B^i_{<i}$ by definition, which implies that $A_i = B^i_{<i} \cup \{i\}$. □

Lemma 2. *Let $i \in V$ and let $x \in V \setminus V_i$. Moreover, let $x' = \ell\text{-}\min\{i, x\}$ and let y' be the remaining vertex of $\{i, x\}$.*

(1) If $\{i, x\} \notin E$, then $B^x_i = A_i$.

(2) If $\{i, x\} \in E$, then $B^x_i = \begin{cases} \max_w \left\{B^x_{<i}, B^{x'}_{\ll y'} \cup \{i\}\right\}, & \text{if } i \in S \text{ or } x \in S \\ \max_w \left\{B^x_{<i}, C^{x',y'}_{<i} \cup \{i\}\right\}, & \text{if } i, x \notin S. \end{cases}$

Proof. Assume first that $\{i, x\} \notin E$. Then $r(i) < \ell(x)$, so that x has no neighbor in $G[V_i \cup \{x\}]$. Thus no subset of $V_i \cup \{x\}$ containing x induces an S-cycle of G, implying that $B^x_i = A_i$.

Next assume that $\{i, x\} \in E$. If $i \notin B^x_i$ then according to Observation 1 (1) it follows that $B^x_i = B^x_{<i}$. So let us assume in what follows that $i \in B^x_i$. Observe that $B^x_i \setminus \{i\} \subseteq V_{<i}$, by Observation 1 (1). We distinguish two cases according to whether i or x belong to S.

- Let $i \in S$ or $x \in S$. Assume there is a vertex $h \in B^x_i \setminus \{i\}$ such that $\{h, y'\} \in E$. Then we know that $\ell(y') < r(h)$ and by definition we have $\ell(x') < \ell(y')$ and $r(h) < r(x')$. This particularly means that h is adjacent to x'. This however leads to a contradiction since $\langle h, x', y'\rangle$ is an induced S-triangle of G. Thus for any vertex $h \in B^x_i \setminus \{i\}$ we know that $\{h, y'\} \notin E$. By Observation 1 (2) notice that $B^x_i \setminus \{i\} \subseteq V_{\ll y'}$. Also observe that the neighborhood of y' in $G[V_{\ll y'} \cup \{x', y'\}]$ is $\{x'\}$. Thus no subset of $V_{\ll y'} \cup \{x', y'\}$ that contains y' induces an S-cycle of G. Therefore $B^x_i = B^{x'}_{\ll y'} \cup \{i\}$.
- Let $i, x \notin S$. Since $V_i = V_{<i} \cup \{i\}$ and $x' <_\ell y'$, we get $B^x_i = C^{x',y'}_{<i} \cup \{i\}$.

Therefore in all cases we reach the desired equations. □

Lemma 3. *Let $i \in V$ and let $x, y \in V \setminus (V_i \cup S)$ such that $x <_\ell y$ and $\{x, y\} \in E$. Moreover, let $x' = \ell\text{-}\min\{i, x, y\}$ and let $y' = \ell\text{-}\min(\{i, x, y\} \setminus \{x'\})$.*

1. *If $\{i, y\} \notin E$, then $C^{x,y}_i = B^x_i$.*
2. *If $\{i, y\} \in E$, then* $C^{x,y}_i = \begin{cases} C^{x,y}_{<i}, & \text{if } i \in S \\ \max_w \left\{C^{x,y}_{<i}, C^{x',y'}_{<i} \cup \{i\}\right\}, & \text{if } i \notin S. \end{cases}$

Proof. Assume first that $\{i, y\} \notin E$. Then $r(i), \ell(x) < \ell(y) < r(x)$, so that the neighborhood of y in $G[V_i \cup \{x, y\}]$ is $\{x\}$. Thus no subset of $V_i \cup \{x, y\}$ that contains y induces an S-cycle of G. By definitions it follows that $C^{x,y}_i = B^x_i$.

Assume next that $\{i, y\} \in E$. Then $\ell(x) < \ell(y) < r(i) < r(x), r(y)$, so that $\langle i, x, y\rangle$ is an induced triangle of G. If $i \notin C^{x,y}_i$ then by Observation 1 (1) we have $C_i^{x,y} = C_{<i}^{x,y}$. Suppose that $i \in C_i^{x,y}$. If $i \in S$ then $\langle i, x, y \rangle$ is an induced S-triangle of G, contradicting the fact that $i \in C_i^{x,y}$. By definition, $x \notin S$ and $y \notin S$. Hence $i \in C_i^{x,y}$ implies that $S \cap \{i, x, y\} = \emptyset$. We show that under the assumptions $\{i, y\} \in E$ and $i \in C_i^{x,y}$, we have $C_i^{x,y} = C_{<i}^{x',y'} \cup \{i\}$. Notice that $C_i^{x,y} \setminus \{i\} \subseteq V_{<i}$, so that every solution of $C_i^{x,y}$ is a solution of $C_{<i}^{x',y'} \cup \{i\}$.

To complete the proof we show that every solution of $C_{<i}^{x',y'} \cup \{i\}$ is indeed a solution of $C_i^{x,y}$. Let z' be the vertex of $\{i, x, y\} \setminus \{x', y'\}$. Observe that by the leftmost ordering we have $\ell(x') < \ell(y') < \ell(z')$. By definition the vertices of $V_{<i}$ that induce an S-triangle in $G[V_{<i} \cup \{x', y'\}]$ do not belong in $C_i^{x,y}$. Assume for contradiction that for an S-triangle that contains z' and a subset of $V_{<i} \cup \{x', y', z'\}$ its non-empty intersection with $V_{<i}$ is a subset of $C_i^{x,y}$. Let $\langle v_1, v_2, z' \rangle$ be an induced S-triangle where $v_1, v_2 \in V_{<i} \cup \{x', y'\}$. Since $x', y', z' \notin S$, without loss of generality, assume that $v_1 \in S$. This particularly means that $v_1 \in V_{<i}$. Regarding the vertex ordering notice that the S-triangle implies that $\ell(z') < r(v_1)$. By the fact that $v_1 \in V_{<i}$ we have $r(v_1) < r(x'), r(y'), r(z')$. Since $\ell(x') < \ell(y') < \ell(z')$, the previous inequalities imply that $\{v_1, x'\}, \{v_1, y'\} \in E$. Thus $\langle v_1, x', y' \rangle$ is an induced S-triangle in $C_i^{x',y'}$, leading to a contradiction. Therefore $C_i^{x,y} = C_{<i}^{x',y'} \cup \{i\}$ as desired. $\square$

Now we are equipped with our necessary tools to obtain the main result of this section, namely a polynomial-time algorithm for SFVS on interval graphs.

Theorem 1. Subset Feedback Vertex Set *can be solved in $O(n^3)$ time on interval graphs.*

4 Computing SFVS on permutation graphs

Let $\pi = \pi(1), \ldots, \pi(n)$ be a permutation over $\{1, \ldots, n\}$. The position of an integer i in π is denoted by $\pi^{-1}(i)$. Given a permutation π, the *inversion graph* of π, denoted by $G(\pi)$, has vertex set $\{1, \ldots, n\}$ and two vertices i, j are adjacent if $(i - j)(\pi(i) - \pi(j)) < 0$. A graph is a *permutation graph* if it is isomorphic to the inversion graph of a permutation [7, 21]. For our purposes, we assume that a permutation graph is given as a permutation π and equal to the defined inversion graph. Permutation graphs are the intersection graphs of segments between two horizontal parallel lines, that is, there is a one-to-one mapping from the segments onto the vertices of a graph such that there is an edge between two vertices of the graph if and only if their corresponding segments intersect. We refer to the two horizontal lines as *top* and *bottom* lines. This representation is called a *permutation diagram* and a graph is a permutation graph if and only if it has a permutation diagram. Given a permutation graph, its permutation diagram can be constructed in linear time [29].

We assume that we are given a connected permutation graph $G = (V, E)$ such that $G = G(\pi)$ along with $S \subseteq V$ and a weight function $w : V \to \mathbb{R}^+$ as input. We add an isolated vertex in G and augment π to π' so that $\pi'(0) = 0$. Further we assign zero value for 0's weight and assume that $0 \notin S$. It is important to note that any induced cycle of a permutation graph is either an induced triangle or an induced square [4–6, 25, 31].

We consider the two relations on V defined as follows: $i \leq_t j$ if and only if $i \leq j$ and $i \leq_b j$ if and only if $\pi^{-1}(i) \leq \pi^{-1}(j)$ for all $i, j \in V \cup \{0\}$. It is not difficult to see that both $\leq_t$ and $\leq_b$ are total orders on V; they are exactly the orders in which the integers appear on the top and bottom line, respectively, in the permutation diagram. Moreover we write $i <_t j$ or $i <_b j$ if and only if $i \neq j$ and $i \leq_t j$ or $i \leq_b j$, respectively. We extend $\leq_t$ and $\leq_b$ to support sets of vertices as follows. For two sets of vertices L and R we write $L \leq_t R$ (resp., $L \leq_b R$) if for any two vertices $u \in L$ and $v \in R$, $u \leq_t v$ (resp., $u \leq_b v$).

Two vertices $i, j \in \{0, 1, \ldots, n\}$ with $i \leq_t j$ are called *crossing pair*, denoted by ij, if $j \leq_b i$. We denote by $\mathcal{X}$ the set of crossing pairs in G. Let $\mathcal{I} = \{ii \mid i \in \{1, \ldots, n\}\}$, so that $\mathcal{X} \setminus \mathcal{I}$ contains exactly the edges of G. Given two crossing pairs $gh, ij \in \mathcal{X}$ we define two partial orders $\leq_\ell$ and $\leq_r$:

$$gh \leq_\ell ij \;\Leftrightarrow\; g \leq_t i \text{ and } h \leq_b j \qquad \text{and} \qquad gh \leq_r ij \;\Leftrightarrow\; g \leq_b i \text{ and } h \leq_t j.$$

Given a vertex set $X \subseteq V$ we denote by $\mathcal{X}[X]$ the set of all crossing pairs of G formed exclusively from vertices of X. It is not difficult to see that the *minimum* crossing pair of $\mathcal{X}[X]$ with respect to $\leq_\ell$ and the *maximum* crossing pair contained in $\mathcal{X}[X]$ with respect to $\leq_r$ are both well defined; we write $\ell\text{-}\min$ and $r\text{-}\max$ to denote them respectively.

We next define the predecessors of a crossing pair with respect to $\leq_r$, which correspond to the subproblems that our dynamic programming algorithm wants to solve. Let $ij \in \mathcal{X} \setminus \{00\}$ be a crossing pair. We define the set of vertices that induce the part of the subproblem that we consider at each crossing pair as follows: $V_{ij} =_{\text{def}} \{h \in V : hh \leq_r ij\}$. Let x be a vertex such that $i <_b x$ or $j <_t x$. By definition notice that x does not belong in V_{ij}. The predecessors of the crossing pair ij are defined as follows:

$$\leqslant ij =_{\text{def}} r\text{-}\max \mathcal{X}[V_{ij} \setminus \{j\}], \qquad \preccurlyeq ij =_{\text{def}} r\text{-}\max \mathcal{X}[V_{ij} \setminus \{i\}],$$

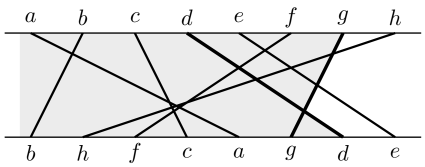
$$< ij =_{\text{def}} r\text{-}\max \mathcal{X}[V_{ij} \setminus \{i, j\}],$$

$$\ll ij =_{\text{def}} r\text{-}\max \mathcal{X}[V_{ij} \setminus (\{i, j\} \cup \{h \in V : \{h, i\} \in E \text{ or } \{h, j\} \in E\})], \text{ and}$$

$$< ij \ll xx =_{\text{def}} r\text{-}\max \mathcal{X}[V_{ij} \setminus \{h \in V : \{h, x\} \in E\}].$$

Although it seems somehow awkward to use one the symbols $\{\leqslant, \preccurlyeq, <, \ll, <\ll\}$ for the defined predecessors, we stress that such predecessors are required only to describe the necessary subset V_{gh} of V_{ij}. Moreover it is not difficult to see that each of the symbol gravitates towards a particular meaning with respect to the top and bottom orderings as well as the non-adjacency relationship. An example of a permutation graph that illustrates the defined predecessors is given in Figure 3. With the above defined predecessors of ij, we show how V_{ij} can be partitioned into smaller sets of vertices with respect to a suitable predecessor.

Observation 2. Let $ij \in \mathcal{X}$ and let $x \in V \setminus V_{ij}$. Then,
(1) $V_{ij} = V_{\leqslant ij} \cup \{j\} = V_{\preccurlyeq ij} \cup \{i\} = V_{<ij} \cup \{i, j\}$,
(2) $V_{<ij} = V_{\ll jj} \cup \{h \in V_{<ij} : \{h, j\} \in E\} = V_{\ll ii} \cup \{h \in V_{<ij} : \{h, i\} \in E\}$,
(3) $V_{\ll ii} = V_{\ll ij} \cup \{h \in V_{\ll ii} : \{h, j\} \in E\}$,
(4) $V_{\ll jj} = V_{\ll ij} \cup \{h \in V_{\ll jj} : \{h, i\} \in E\}$, and
(5) $V_{<ij} = V_{<ij \ll xx} \cup \{h \in V_{<ij} : \{h, x\} \in E\}$.



$\lll dg = df$
$\leqslant dg = gg$
$<dg = af$
$\ll dg = ac$
$<dg \ll hh = bb$

$V_{dg} = \{a, b, c, d, f, g\}$
$V_{\lll dg} = \{a, b, c, d, f\}$
$V_{\leqslant dg} = \{a, b, c, f, g\}$
$V_{<dg} = \{a, b, c, f\}$
$V_{\ll dg} = \{a, b, c\}$

Fig. 3. A permutation graph given by its permutation diagram and the set V_{dg} of the crossing pair dg together with the corresponding predecessors of dg. Observe that the line segments that are properly contained within the gray area form the set V_{dg}.

It is clear that for any edge $\{i,j\} \in E$ either $i <_t j$ and $j <_b i$ hold, or $j <_t i$ and $i <_b j$ hold. If further $ij \in \mathcal{X} \setminus \mathcal{I}$ then we know that $i <_t j$ and $j <_b i$.

Our dynamic programming algorithm relies on similar sets that we used for the case of interval graphs. That is, we need to describe appropriate sets that define the solutions to be chosen only from a specific part of the considered subproblem. Although for interval graphs we showed that adding two vertices into such sets is enough, for permutation graphs we need to consider at most two newly crossing pairs which corresponds to consider four newly vertices.

A**-sets**: Let $ij \in \mathcal{X}$. Then, $A_{ij} = \max_w\{X \subseteq V_{ij} : G[X] \in \mathcal{F}_S\}$.

B**-sets**: Let $ij \in \mathcal{X}$ and let $x \in V \setminus V_{ij}$. Then, $B_{ij}^{xx} =_{\text{def}} \max_w\{X \subseteq V_{ij} : G[X \cup \{x\}] \in \mathcal{F}_S\}$. Moreover, let $xy \in \mathcal{X} \setminus \mathcal{I}$ such that $j <_t y$, $i <_b x$, and $x, y \notin S$. Then, $B_{ij}^{xy} =_{\text{def}} \max_w\{X \subseteq V_{ij} : G[X \cup \{x,y\}] \in \mathcal{F}_S\}$.

C**-sets**: Let $ij \in \mathcal{X}$, $xy \in \mathcal{X} \setminus \mathcal{I}$, and $z \in V \setminus (V_{ij} \setminus \{x,y\})$ such that $xy <_\ell zz$, at least one of x, y is adjacent to z, $j <_t y$, $i <_b x$, and $x, y, z \notin S$. Then, $C_{ij}^{xy,zz} =_{\text{def}} \max_w\{X \subseteq V_{ij} : G[X \cup \{x,y,z\}] \in \mathcal{F}_S\}$. Moreover, let $zw \in \mathcal{X} \setminus \mathcal{I}$ such that $xy <_\ell zw$, $\{x,w\}, \{y,z\} \in E$, $j <_t \{y,w\}$, $i <_b \{x,z\}$, and $x, y, z, w \notin S$. Then, $C_{ij}^{xy,zw} =_{\text{def}} \max_w\{X \subseteq V_{ij} : G[X \cup \{x,y,z,w\}] \in \mathcal{F}_S\}$.

Observe that, since $V_{00} = \{0\}$ and $w(0) \leq 0$, $A_{00} = \emptyset$ and, since $V_{\pi(n)n} = V$, $A_{\pi(n)n} = \max_w\{X \subseteq V : G[X] \in \mathcal{F}_S\}$. The following lemmas state how to recursively compute all A-sets, B-sets, and C-sets other than A_{00}. We first consider the crossing pairs ii for the sets A_{ii}, B_{ii}^{xx}, B_{ii}^{xy}, $C_{ii}^{xy,zz}$, and $C_{ii}^{xy,zw}$.

Lemma 4. *Let $i \in V \setminus \{0\}$. Then $A_{ii} = A_{<ii} \cup \{i\}$.*

Proof. By Observation 2 (1), $A_{<ii} \cup \{i\} \in A_{ii}$. Notice that i is non-adjacent to any vertex of V_{ii}. Thus no subset of V_{ii} that contains i induces an S-cycle. $\square$

Lemma 5. *Let $i \in V$ and let $x \in V \setminus V_{ii}$. 1. If $\{i,x\} \notin E$ then $B_{ii}^{xx} = A_{ii}$. 2. If $\{i,x\} \in E$ then $B_{ii}^{xx} = B_{<ii}^{xx} \cup \{i\}$.*

Proof. Assume first that $\{i,x\} \notin E$. Since $x \in V \setminus V_{ii}$ we know that $i <_t x$ or $i <_b x$. Moreover as $\{i,x\} \notin E$ we have $i <_t x$ and $i <_b x$. Then x has no neighbor in $G[V_{ii} \cup \{x\}]$. Thus no subset of $V_{ii} \cup \{x\}$ that contains x induces an S-cycle in G. Hence $B_{ii}^{xx} = A_{ii}$ follows. Next assume that $\{i,x\} \in E$. Then the neighborhood of i in $G[V_{ii} \cup \{x\}]$ is $\{x\}$. This means that no subset of $V_{ii} \cup \{x\}$ that contains i induces an S-cycle in G, so that $i \in B_{ii}^{xx}$. By Observation 2 (1) it follows that $B_{ii}^{xx} = B_{<ii}^{xx} \cup \{i\}$. $\square$

Lemma 6. *Let $i \in V$ and let $xy \in \mathcal{X} \setminus \mathcal{I}$ such that $i <_t y$, $i <_b x$, and $x, y \notin S$.*

1. *If $\{i,y\} \notin E$ then $B^{xy}_{ii} = B^{xx}_{ii}$.*
2. *If $\{i,x\} \notin E$ then $B^{xy}_{ii} = B^{yy}_{ii}$.*
3. *If $\{i,x\},\{i,y\} \in E$ then $B^{xy}_{ii} = \begin{cases} B^{xy}_{<ii} & , \text{ if } i \in S \\ B^{xy}_{<ii} \cup \{i\}, & \text{if } i \notin S. \end{cases}$*

Lemma 7. *Let $i \in V$, $xy \in \mathcal{X} \setminus \mathcal{I}$, and let $z \in V \setminus (V_{ii} \setminus \{x,y\})$ such that $xy <_\ell zz$, at least one of x, y is adjacent to z, $i <_t y$, $i <_b x$, and $x, y, z \notin S$.*

1. *If $\{i,z\} \notin E$ then $C^{xy,zz}_{ii} = B^{xy}_{ii}$.*
2. *If $\{i,z\} \in E$ then $C^{xy,zz}_{ii} = \begin{cases} C^{xy,zz}_{<ii}, & \text{if } i \in S \\ C^{xy,zz}_{<ii} \cup \{i\}, & \text{if } i \notin S. \end{cases}$*

Lemma 8. *Let $i \in V$ and let $xy, zw \in \mathcal{X} \setminus \mathcal{I}$ such that $xy <_\ell zw$, $\{x,w\},\{y,z\} \in E$, $i <_t \{y,w\}$, $i <_b \{x,z\}$, and $x,y,z,w \notin S$.*

1. *If $\{i,w\} \notin E$ then $C^{xy,zw}_{ii} = C^{xy,zz}_{ii}$.*
2. *If $\{i,z\} \notin E$ then $C^{xy,zw}_{ii} = C^{xy,ww}_{ii}$.*
3. *If $\{i,z\},\{i,w\} \in E$ then $C^{xy,zw}_{ii} = \begin{cases} C^{xy,zw}_{<ii}, & \text{if } i \in S \\ C^{xy,zw}_{<ii} \cup \{i\}, & \text{if } i \notin S. \end{cases}$*

Lemmas 4–8 describe the subsolutions for each crossing pair ii. Next we give the recursive formulations for A_{ij}, B^{xx}_{ij}, B^{xy}_{ij}, $C^{xy,zz}_{ij}$, and $C^{xy,zw}_{ij}$ whenever $ij \in \mathcal{X} \setminus \mathcal{I}$ which particularly means that i and j are distinct vertices in G.

Lemma 9. *Let $ij \in \mathcal{X} \setminus \mathcal{I}$. Then,*

$$A_{ij} = \begin{cases} \max_w \left\{ A_{\lessgtr ij}, A_{\leqslant ij}, B^{ii}_{\ll jj} \cup \{i,j\}, B^{jj}_{\ll ii} \cup \{i,j\} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ A_{\lessgtr ij}, A_{\leqslant ij}, B^{ij}_{<ij} \cup \{i,j\} \right\}, & \text{if } i,j \notin S. \end{cases}$$

With the next two lemmas we describe recursively the sets B^{xx}_{ij} and B^{xy}_{ij}.

Lemma 10. *Let $ij \in \mathcal{X} \setminus \mathcal{I}$ and let $x \in V \setminus V_{ij}$. Moreover let $x'y' = \ell\text{-}\min \mathcal{X}[\{i,j,x\}]$ and let z' be the remaining vertex of $\{i,j,x\}$.*

1. *If $\{i,x\},\{j,x\} \notin E$ then $B^{xx}_{ij} = A_{ij}$.*
2. *If $\{i,x\} \in E$ and $\{j,x\} \notin E$ then*
$$B^{xx}_{ij} = \begin{cases} \max_w \left\{ B^{xx}_{\lessgtr ij}, B^{xx}_{\leqslant ij}, B^{ii}_{\ll jj} \cup \{i,j\}, B^{jj}_{\ll ix} \cup \{i,j\} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ B^{xx}_{\lessgtr ij}, B^{xx}_{\leqslant ij}, B^{ij}_{<ij \ll xx} \cup \{i,j\} \right\}, & \text{if } i,j \notin S, x \in S \\ \max_w \left\{ B^{xx}_{\lessgtr ij}, B^{xx}_{\leqslant ij}, C^{x'y',z'z'}_{<ij} \cup \{i,j\} \right\}, & \text{if } i,j,x \notin S. \end{cases}$$
3. *If $\{i,x\} \notin E$ and $\{j,x\} \in E$ then*
$$B^{xx}_{ij} = \begin{cases} \max_w \left\{ B^{xx}_{\lessgtr ij}, B^{xx}_{\leqslant ij}, B^{ii}_{\ll xj} \cup \{i,j\}, B^{jj}_{\ll ii} \cup \{i,j\} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ B^{xx}_{\lessgtr ij}, B^{xx}_{\leqslant ij}, B^{ij}_{<ij \ll xx} \cup \{i,j\} \right\}, & \text{if } i,j \notin S, x \in S \\ \max_w \left\{ B^{xx}_{\lessgtr ij}, B^{xx}_{\leqslant ij}, C^{x'y',z'z'}_{<ij} \cup \{i,j\} \right\}, & \text{if } i,j,x \notin S. \end{cases}$$

4. *If* $\{i,x\},\{j,x\} \in E$ *then*
$$B_{ij}^{xx} = \begin{cases} \max_w \left\{B_{\lessdot ij}^{xx}, B_{\leqslant ij}^{xx}\right\}, & \text{if } i \in S \text{ or } j \in S \text{ or } x \in S \\ \max_w \left\{B_{\lessdot ij}^{xx}, B_{\leqslant ij}^{xx}, C_{<ij}^{x'y',z'z'} \cup \{i,j\}\right\}, & \text{if } i,j,x \notin S. \end{cases}$$

Let $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ such that $\{i,y\},\{j,x\} \in E$. It is not difficult to see that if we remove the vertices of a crossing pair $uv \in \mathcal{X}[\{i,j,x,y\}]$ from $\{i,j,x,y\}$ then the remaining two vertices are adjacent.

Lemma 11. *Let* $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ *such that* $j <_t y$, $i <_b x$ *and* $x,y \notin S$. *Moreover, if* $\{i,y\},\{j,x\} \in E$ *then let* $x'y' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y\}]$ *and let* $z'w' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y\} \setminus \{x',y'\}]$.

1. *If* $\{i,y\} \notin E$ *then* $B_{ij}^{xy} = B_{ij}^{xx}$.
2. *If* $\{j,x\} \notin E$ *then* $B_{ij}^{xy} = B_{ij}^{yy}$.
3. *If* $\{i,y\},\{j,x\} \in E$ *then*
$$B_{ij}^{xy} = \begin{cases} \max_w \left\{B_{\lessdot ij}^{xy}, B_{\leqslant ij}^{xy}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{B_{\lessdot ij}^{xy}, B_{\leqslant ij}^{xy}, C_{<ij}^{x'y',z'w'} \cup \{i,j\}\right\}, & \text{if } i,j \notin S. \end{cases}$$

Lemma 12. *Let* $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ *and let* $z \in V \setminus V_{ij}$ *such that* $xy <_\ell zz$, *at least one of* x,y *is adjacent to* z, $j <_t y$, $i <_b x$, *and* $x,y,z \notin S$. *Moreover, if* $\{i,z\} \in E$ *or* $\{j,z\} \in E$ *then let* $x'y' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y,z\}]$ *and let* $z'w' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y,z\} \setminus \{x',y'\}]$.

1. *If* $\{i,z\},\{j,z\} \notin E$ *then* $C_{ij}^{xy,zz} = B_{ij}^{xy}$.
2. *If* $\{i,z\} \in E$ *or* $\{j,z\} \in E$ *then*
$$C_{ij}^{xy,zz} = \begin{cases} \max_w \left\{C_{\lessdot ij}^{xy,zz}, C_{\leqslant ij}^{xy,zz}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{C_{\lessdot ij}^{xy,zz}, C_{\leqslant ij}^{xy,zz}, C_{<ij}^{x'y',z'w'} \cup \{i,j\}\right\}, & \text{if } i,j \notin S. \end{cases}$$

The next lemma shows how to recursively compute $C_{ij}^{xy,zw}$. Note that in each case we describe $C_{ij}^{xy,zw}$ as a predefined smaller set of a subsolution that is either in the same form or has already been described in one of the previous lemmas.

Lemma 13. *Let* $ij, xy, zw \in \mathcal{X} \setminus \mathcal{I}$ *such that* $xy <_\ell zw$, $\{x,w\},\{y,z\} \in E$, $j <_t \{y,w\}$, $i <_b \{x,z\}$, *and* $x,y,z,w \notin S$. *Moreover, if* $\{i,w\},\{j,z\} \in E$, *let* $x'y' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y,z,w\}]$ *and let* $z'w' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y,z,w\} \setminus \{x',y'\}]$.

1. *If* $\{i,w\} \notin E$ *then* $C_{ij}^{xy,zw} = C_{ij}^{xy,zz}$.
2. *If* $\{j,z\} \notin E$ *then* $C_{ij}^{xy,zw} = C_{ij}^{xy,ww}$.
3. *If* $\{i,w\},\{j,z\} \in E$ *then*
$$C_{ij}^{xy,zw} = \begin{cases} \max_w \left\{C_{\lessdot ij}^{xy,zw}, C_{\leqslant ij}^{xy,zw}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{C_{\lessdot ij}^{xy,zw}, C_{\leqslant ij}^{xy,zw}, C_{<ij}^{x'y',z'w'} \cup \{i,j\}\right\}, & \text{if } i,j \notin S. \end{cases}$$

It is important to notice that all described formulations are given recursively based on Lemmas 4–13. Now we are in position to state our claimed polynomial-time algorithm for the SFVS problem on permutation graphs.

Theorem 2. Subset Feedback Vertex Set *can be solved in* $O(n+m^3)$ *time on permutation graphs.*

5 Concluding remarks

From the complexity point of view, since FVS is polynomial-time solvable on the class of AT-free graphs [24], a natural problem is to settle the complexity of SFVS on AT-free graphs. Interestingly most problems that are hard on AT-free graphs are already hard on co-bipartite graphs (see for e.g., [28]). Also notice that SFVS remains NP-complete on bipartite graphs, as FVS is NP-complete on bipartite graphs [32]. Co-bipartite graphs are the complements of bipartite graphs and are unrelated to permutation graphs or interval graphs. Here we show that SFVS admits a simple solution on co-bipartite graphs, therefore excluding such an approach through a hardness result on co-bipartite graphs.

Theorem 3. *The number of maximal S-forests of a co-bipartite graph is at most $22n^4$ and these can be enumerated in time $O(n^4)$.*

We believe that such an approach towards AT-free graphs should deal first with the complexity of the unweighted version of SFVS. Moreover it is interesting to settle the complexity of SFVS on other related graph classes such as strongly chordal graphs or subclasses of AT-free graphs like trapezoid graphs or complements of triangle-free graphs. Regarding graphs of bounded structural parameter and due to the nature of the dynamic programming used for SFVS on interval and permutation graphs, it is interesting to consider graphs of bounded maximum induced matching width introduced in [3].

Another interesting open question is concerned with problems related to *terminal-sets* such as the Multiway Cut problem in which we want to disconnect a given set of terminals by removing vertices of minimum total weight. As already mentioned in the Introduction, the Multiway Cut problem reduces to the SFVS problem by adding a vertex s with $S = \{s\}$ that is adjacent to all terminals and whose weight is larger than the sum of the weights of all vertices in the original graph [17]. Notice that through such an approach in order to solve even the unweighted Multiway Cut problem one needs to solve the weighted SFVS problem. This actually implies that Multiway Cut is polynomial-time solvable in permutation graphs and interval graphs by using our algorithms for the SFVS problem. However polynomial-time algorithms for the Multiway Cut problem were already known for permutation and interval graphs by a more general terminal-set problem [23, 30]. Nevertheless it is still interesting to consider the computational complexity of the unweighted Multiway Cut problem on subclasses of AT-free graphs such as cocomparability graphs.

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A Appendix: Omitted proofs of Section 3

Observation 1. Let $i \in V \setminus \{0\}$ and let $j \in V \setminus V_i$ such that $\{i, j\} \in E$. Then, (1) $V_i = V_{<i} \cup \{i\}$ and (2) $V_{<i} = V_{\ll j} \cup \{h \in V_{<i} : \{h, j\} \in E\}$.

Proof. The first statement follows by the definitions of V_i and $<i$. For the second statement observe that $V_{<i}$ can be partitioned into the non-neighbors of j in $V_{<i}$ and the neighbors of j in $V_{<i}$. The first set corresponds to $V_{\ll j}$ whereas the second set is exactly the set $\{h \in V_{<i} : \{h, j\} \in E\}$. ☐

Theorem 1. Subset Feedback Vertex Set *can be solved in $O(n^3)$ time on interval graphs.*

Proof. We briefly describe such an algorithm based on Lemmas 1, 2, and 3. In a preprocessing step we compute $< i$ and $\ll i$ for each interval $i \in V \setminus \{0\}$. We scan all intervals from 0 to n in an ascending order with respect to $<_\ell$. For every interval i that we visit, we compute first A_i according to Lemma 1 and then compute B_i^x and $C_i^{x,y}$ for every x, y such that $\ell(i) < \ell(x) < \ell(y)$ according to Lemmas 2 and 3, respectively. At the end we output A_n as already explained. The correctness of the algorithm follows from Lemmas 1–3.

Regarding the running time, notice that computing $< i$ and $\ll i$ can be done in $O(n)$ time since the intervals are sorted with respect to their end-points. The computation of a single A-set, B-set or C-set takes constant time. Therefore the overall running time of the algorithm is $O(n^3)$. ☐

B Appendix: Omitted proofs of Section 4

Observation 2. Let $ij \in \mathcal{X}$ and let $x \in V \setminus V_{ij}$. Then,
(1) $V_{ij} = V_{\leqslant ij} \cup \{j\} = V_{\lessdot ij} \cup \{i\} = V_{<ij} \cup \{i, j\}$,
(2) $V_{<ij} = V_{\ll jj} \cup \{h \in V_{<ij} : \{h, j\} \in E\} = V_{\ll ii} \cup \{h \in V_{<ij} : \{h, i\} \in E\}$,
(3) $V_{\ll ii} = V_{\ll ij} \cup \{h \in V_{\ll ii} : \{h, j\} \in E\}$,
(4) $V_{\ll jj} = V_{\ll ij} \cup \{h \in V_{\ll jj} : \{h, i\} \in E\}$, and
(5) $V_{<ij} = V_{<ij \ll xx} \cup \{h \in V_{<ij} : \{h, x\} \in E\}$.

Proof. Let ij_1 be the predecessor $\leqslant ij$. By the r-max choice of j_1, there is no vertex j' such that $j_1 <_t j' <_t j$. Thus $V_{ij_1} \cup \{j\}$ is the set V_{ij}. The rest of the equalities in the first statement follow in a similar way.

Let $i_1 j_1$ be the predecessor $\ll jj$. Then both i_1 and j_1 are non-adjacent to j and have the maximum values such that $i_1 <_b j$ and $j_1 <_t j$, respectively. This particularly means that $i_1 <_t j_1 <_t j$ and $j_1 <_b i_1 <_b j$. Thus any vertex $i' \in V_{ij} \setminus \{i, j\}$ with $j_1 <_t i' <_t j$ or $i_1 <_b i' <_b j$ must be adjacent to j which implies that $V_{<ij} \setminus V_{\ll jj}$ contains exactly the neighbors of j in $V_{<ij}$. These arguments imply the second, third, and fourth statements.

For the last statement, notice that $V_{<ij}$ can be partitioned into the neighbors and the non-neighbors of x. By definition $V_{<ij \ll xx}$ contains the non-neighbors of x so that every vertex of $V_{<ij} \setminus V_{<ij \ll xx}$ is adjacent to x. ☐

Lemma 6. *Let $i \in V$ and let $xy \in \mathcal{X} \setminus \mathcal{I}$ such that $i <_t y$, $i <_b x$, and $x, y \notin S$.*

1. *If $\{i, y\} \notin E$ then $B_{ii}^{xy} = B_{ii}^{xx}$.*
2. *If $\{i, x\} \notin E$ then $B_{ii}^{xy} = B_{ii}^{yy}$.*
3. *If $\{i, x\}, \{i, y\} \in E$ then $B_{ii}^{xy} = \begin{cases} B_{<ii}^{xy} & , \text{ if } i \in S \\ B_{<ii}^{xy} \cup \{i\}, & \text{if } i \notin S. \end{cases}$*

Proof. By $i <_t y$, $i <_b x$, and the fact that xy is a crossing pair, we have $\{x, i\} <_t y$ and $\{y, i\} <_b x$. Assume first that i is non-adjacent to at least one of x and y. Let $\{i, y\} \notin E$. Then $\{i, x\} <_t y$ and $i <_b y <_b x$, so that the neighborhood of y in $G[V_{ii} \cup \{x, y\}]$ is $\{x\}$. Thus no subset of $V_{ii} \cup \{x, y\}$ that contains y induces an S-cycle of G which implies that $B_{ii}^{xy} = B_{ii}^{xx}$. Completely symmetric arguments apply if $\{i, x\} \notin E$ showing the second statement.

Next assume that $\{i, x\}, \{i, y\} \in E$. Then $x <_t i <_t y$ and $y <_b i <_b x$, so that the neighborhood of i in $G[X \cup \{x, y\}]$ is $\{x, y\}$. We distinguish two cases according to whether i belongs to S. Suppose that $i \in S$. Then $\langle i, x, y \rangle$ is an induced S-triangle of G, so that $i \notin B_{ii}^{xy}$. Thus by Observation 2 (1), $B_{ii}^{xy} = B_{<ii}^{xy}$ holds if $i \in S$.

Suppose next that $i \notin S$. We will show that no subset of $V_{ii} \cup \{x, y\}$ that contains i induces an S-cycle of G. Recall that i is non-adjacent to any vertex of V_{ii} and the only induced cycles of a permutation graph is either a triangle or a square. Assume that $\langle v_1, v_2, i \rangle$ is an induced S-triangle of G where $v_1, v_2 \in V_{<ii} \cup \{x, y\}$. Then $\{v_1, v_2\} = \{x, y\}$ leading to a contradiction, because $i, x, y \notin S$. So let us assume that $\langle v_1, v_2, v_3, i \rangle$ is an induced S-square of G where $v_1, v_2, v_3 \in V_{<ii} \cup \{x, y\}$. By the fact that i only adjacent to x and y in $G[V_{ii} \cup \{x, y\}]$ we have that v_1, v_3 correspond to the vertices x and y. This however leads to a contradiction since $\{x, y\} \in E$ and $\{v_1, v_3\} \notin E$ by the induced S-square. Therefore no subset of $V_{ii} \cup \{x, y\}$ that contains i induces an S-cycle of G, so that $i \in B_{ii}^{xy}$. By Observation 2 (1) $B_{ii}^{xy} = B_{<ii}^{xy} \cup \{i\}$ holds and this completes the proof. $\square$

Lemma 7. *Let $i \in V$, $xy \in \mathcal{X} \setminus \mathcal{I}$, and let $z \in V \setminus (V_{ii} \setminus \{x, y\})$ such that $xy <_\ell zz$, at least one of x, y is adjacent to z, $i <_t y$, $i <_b x$, and $x, y, z \notin S$.*

1. *If $\{i, z\} \notin E$ then $C_{ii}^{xy,zz} = B_{ii}^{xy}$.*
2. *If $\{i, z\} \in E$ then $C_{ii}^{xy,zz} = \begin{cases} C_{<ii}^{xy,zz}, & \text{if } i \in S \\ C_{<ii}^{xy,zz} \cup \{i\}, & \text{if } i \notin S. \end{cases}$*

Proof. Since $z \in V \setminus (V_{ii} \setminus \{x, y\})$, we have $i <_t z$ or $i <_b z$. Assume first that $\{i, z\} \notin E$. Observe that this means that $i <_t z$ and $i <_b z$. Then z is non-adjacent to any vertex of V_{ii} so that the neighborhood of z in $G[V_{ii} \cup \{x, y, z\}]$ is a subset of $\{x, y\}$. Since $x, y, z \notin S$, no subset of $V_{ii} \cup \{x, y, z\}$ that contains z induces an S-cycle in G. Thus $C_{ii}^{xy,zz} = B_{ii}^{xy}$.

Assume next that $\{i, z\} \in E$. This means that either $i <_t z$ and $z <_b i$ hold, or $z <_t i$ and $i <_b z$ hold. Since $i <_t y$ and $i <_b x$, we get either $i <_t \{y, z\}$ and $z <_b i <_b x$, or $z <_t i <_t y$ and $i <_b \{x, z\}$. Moreover since xy is a crossing pair and $xy <_\ell zz$, exactly one of following holds:

– $\{i,x\} <_t \{y,z\}$ and $y <_b z <_b i <_b x$;
– $x <_t z <_t i <_t y$ and $\{i,y\} <_b \{x,z\}$.

This means that $y,z \in N(i)$ and x is adjacent to z, or $x,z \in N(i)$ and y is adjacent to z. We distinguish two cases depending on whether i belongs to S.

– Let $i \in S$. We will show that $i \notin C_{ii}^{xy,zz}$. If both x and y are adjacent to i then $\langle i,x,y\rangle$ is an induced S-triangle in G. Thus either $y,z \in N(i)$ and x is adjacent to z, or $x,z \in N(i)$ and y is adjacent to z. Assume the former, that is, $y,z \in N(i)$, $x \notin N(i)$, and x is adjacent to z. If $\{y,z\} \in E$ then $\langle i,y,z\rangle$ is an induced S-triangle and if $\{y,z\} \notin E$ then $\langle i,y,x,z\rangle$ is an induced S-square. Similarly if $x,z \in N(i)$, $y \notin N(i)$, and y is adjacent to z we obtain an induced S-cycle in G. Therefore in all cases $i \notin C_{ii}^{xy,zz}$ and by Observation 2 (1) we get $C_{ii}^{xy,zz} = C_{<ii}^{xy,zz}$.
– Let $i \notin S$. We will show that $i \in C_{ii}^{xy,zz}$. Assume for contradiction that there is an induced S-triangle $\langle v_1,v_2,i\rangle$ or S-square $\langle v_1,v_2,v_3,i\rangle$ in G where $v_1,v_2,v_3 \in V_{<ii} \cup \{x,y,z\}$. Notice that $N(i) \cap V_{<ii} = \emptyset$ so that $\{v_1,v_2\} \subset \{x,y,z\}$ or $\{v_1,v_3\} \subset \{x,y,z\}$, respectively. In the former case we reach a contradiction because $i,x,y,z \notin S$. In the latter case for the same reason notice that $v_2 \in S$ which implies that $v_2 \in V_{<ii}$. If $\{v_1,v_3\} = \{x,y\}$ then we reach a contradiction to the S-square $\langle v_1,v_2,v_3,i\rangle$ because $\{x,y\} \in E$. Thus $\{v_1,v_3\} = \{y,z\}$ or $\{v_1,v_3\} = \{x,z\}$. Without loss of generality assume that $\{v_1,v_3\} = \{y,z\}$. Then $\{y,z\} \notin E$, for otherwise we reach again a contradiction to the given S-square. Observe that $\{y,z\} \notin E$ implies that $\{x,z\} \in E$ by the hypothesis for z. This however shows that $\langle y,v_2,x\rangle$ or $\langle y,v_2,z,x\rangle$ induce an S-triangle or an S-square of G without i depending on whether x is adjacent to v_2, so that $v_2 \notin C_{ii}^{xy,zz}$. Therefore in all cases we reach a contradiction which means that $i \in C_{ii}^{xy,zz}$ and by Observation 2 (1), $C_{ii}^{xy,zz} = C_{<ii}^{xy,zz} \cup \{i\}$ holds.

In each case we have showed the described equations and this completes the proof. ◻

Lemma 8. *Let $i \in V$ and let $xy, zw \in \mathcal{X}\backslash\mathcal{I}$ such that $xy <_\ell zw$, $\{x,w\},\{y,z\} \in E$, $i <_t \{y,w\}$, $i <_b \{x,z\}$, and $x,y,z,w \notin S$.*

1. *If $\{i,w\} \notin E$ then $C_{ii}^{xy,zw} = C_{ii}^{xy,zz}$.*
2. *If $\{i,z\} \notin E$ then $C_{ii}^{xy,zw} = C_{ii}^{xy,ww}$.*
3. *If $\{i,z\},\{i,w\} \in E$ then $C_{ii}^{xy,zw} = \begin{cases} C_{<ii}^{xy,zw}, & \text{if } i \in S \\ C_{<ii}^{xy,zw} \cup \{i\}, & \text{if } i \notin S. \end{cases}$*

Proof. Observe that $x,y,z,w \in V \setminus V_{ii}$ because $i <_t \{y,w\}$ and $i <_b \{x,z\}$. Assume first that $\{i,w\} \notin E$. Since $i <_t w$ and $i <_b w$, w has no neighbor in V_{ii}. Thus the neighborhood of w in $G[V_{ii} \cup \{x,y,z,w\}]$ is a subset of $\{x,y,z\}$. We will show that $w \notin C_{ii}^{xy,zw}$. Assume that a subset of $V_{ii} \cup \{x,y,z,w\}$ that contains w induces an S-cycle in G. If $\langle v_1,v_2,w\rangle$ is an induced S-triangle of G

then $\{v_1, v_2\} \subset \{x, y, z\}$ which leads to a contradiction, because $x, y, z, w \notin S$. Suppose that $\langle v_1, v_2, v_3, w\rangle$ is an induced S-square of G. Then $\{v_1, v_3\} \subset \{x, y, z\}$ and, since $x, y, z, w \notin S$ we know that $v_2 \in S$ and $v_2 \in V_{ii}$.

- Assume that $\{v_1, v_3\} = \{x, y\}$ or $\{v_1, v_3\} = \{y, z\}$. Then we reach a contradiction to the induced S-square, because $\{x, y\}, \{y, z\} \in E$.
- Assume that $\{v_1, v_3\} = \{x, z\}$. If $\{x, z\} \in E$ then $\langle v_1, v_2, v_3, w\rangle$ does not induce an S-square. If $\{x, z\} \notin E$ then $\langle x, v_2, y\rangle$ or $\langle x, v_2, z, y\rangle$ induce an S-triangle or an S-square in G which reach to a contradiction to $v_2 \notin C_{ii}^{xy,zz}$.

Therefore, if a subset of $V_{ii} \cup \{x, y, z, w\}$ that contains w induces an S-cycle of G, then its non-empty intersection with V_{ii} is not a subset of $C_{ii}^{xy,zz}$ which implies that $C_{ii}^{xy,zw} = C_{ii}^{xy,zz}$. The case for $\{i, z\} \notin E$ is completely symmetric showing the second statement.

Let $\{i, z\}, \{i, w\} \in E$. Then either $i <_t \{z, w\}$ and $\{z, w\} <_b i$, or $\{z, w\} <_t i$ and $i <_b \{z, w\}$. Since $xy <_\ell zw$, $i <_t \{y, w\}$, and $i <_b \{x, z\}$, the following hold:

- $x <_t z <_t i <_t \{y, w\}$ and
- $y <_b w <_b i <_b \{x, z\}$.

Thus the neighborhood of i in $G[V_{ii} \cup \{x, y, z, w\}]$ is $\{x, y, z, w\}$. Assume that $i \in S$. Then $\langle i, x, y\rangle$ is an S-triangle of G which implies $i \notin C_{ii}^{xy,zw}$. By Observation 2 (1) we get $C_{ii}^{xy,zw} = C_{<ii}^{xy,zw}$. Let us assume that $i \notin S$. We will show that if a subset of $V_{ii} \cup \{x, y, z, w\}$ that contains i induces an S-cycle of G, then its non-empty intersection with $V_{<ii}$ is not a subset of $C_{ii}^{xy,zw}$.

- Let $v_1, v_2 \in V_{<ii} \cup \{x, y, z, w\}$ such that $\langle v_1, v_2, i\rangle$ is an induced S-triangle of G. Then $\{v_1, v_2\} \subset \{x, y, z, w\}$, a contradiction, because $i, x, y, z, w \notin S$.
- Let $v_1, v_2, v_3 \in V_{<ii} \cup \{x, y, z, w\}$ such that $\langle v_1, v_2, v_3, i\rangle$ is an induced S-square of G. Then $\{v_1, v_3\} \subset \{x, y, z, w\}$ and, since $i, x, y, z, w \notin S$, $v_2 \in S$. Thus $v_2 \in V_{<ii}$. Because v_1, v_3 are non-adjacent, we have $\{v_1, v_3\} = \{x, z\}$ or $\{v_1, v_3\} = \{y, w\}$. In both cases we reach a contradiction since $\langle x, v_2, z, y\rangle$ or $\langle y, v_2, w, z\rangle$ induce S-squares in G.

Thus if $i \notin S$ then $i \in C_{ii}^{xy,zw}$. Therefore by Observation 2 (1) we obtain $C_{ii}^{xy,zw} = C_{<ii}^{xy,zw} \cup \{i\}$. □

Lemma 9. *Let $ij \in \mathcal{X} \setminus \mathcal{I}$. Then,*

$$A_{ij} = \begin{cases} \max_w \left\{ A_{\lessgtr ij}, A_{\leqslant ij}, B^{ii}_{\ll jj} \cup \{i,j\}, B^{jj}_{\ll ii} \cup \{i,j\} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ A_{\lessgtr ij}, A_{\leqslant ij}, B^{ij}_{<ij} \cup \{i,j\} \right\}, & \text{if } i,j \notin S. \end{cases}$$

Proof. Let $j \notin A_{ij}$. Then by Observation 2 (1) it follows that $A_{ij} = A_{\lessgtr ij}$. Similarly if $i \notin A_{ij}$ then $A_{ij} = A_{\leqslant ij}$. For the rest of the proof we assume that $i, j \in A_{ij}$. Notice that by Observation 2 (1) we have $A_{ij} \setminus \{i, j\} \subseteq V_{<ij}$. We distinguish two cases according to whether i or j belong to S.

– Assume that $i, j \notin S$. Then $A_{ij} = B^{ij}_{<ij} \cup \{i, j\}$ holds which completes the second description in the formula.
– Assume that $i \in S$ or $j \in S$. Let $h \in A_{ij} \setminus \{i, j\}$ such that $\{h, i\}, \{h, j\} \in E$. Then $\langle h, i, j \rangle$ is an induced S-triangle in G, resulting a contradiction to $i, j \in A_{ij}$. Thus for every $h \in A_{ij} \setminus \{i, j\}$ we know that $\{h, i\} \notin E$ or $\{h, j\} \notin E$. Let $g, h \in A_{ij} \setminus \{i, j\}$ such that $\{g, j\}, \{h, i\} \in E$ and $\{g, i\}, \{h, j\} \notin E$. Observe that $\{g, h\} <_b i$ and $\{g, h\} <_t j$. Since ij is a crossing pair we know that $i <_t j$ and $j <_b i$. If $i <_t g$ or $j <_b h$ then g is adjacent to i or h is adjacent to j, leading to a contradiction. Thu $g <_t i <_t h$ and $h <_b j <_b g$ hold which imply that $\{g, h\} \in E$. Hence $\langle g, h, i, j \rangle$ is an induced S-square in G, a contradiction. This means that all vertices of $A_{ij} \setminus \{i, j\}$ are non-adjacent to i or j or both. Then by Observation 2 (2) it follows that either $A_{ij} \setminus \{i, j\} \subseteq V_{\ll jj}$ or $A_{ij} \setminus \{i, j\} \subseteq V_{\ll ii}$.
Suppose that the former holds, that is $A_{ij} \setminus \{i, j\} \subseteq V_{\ll jj}$. The neighborhood of j in $G[V_{\ll jj} \cup \{i, j\}]$ is $\{i\}$. Thus no subset of $V_{\ll jj} \cup \{i, j\}$ that contains j induces an S-cycle in G. This means that $A_{ij} = B^{ii}_{\ll jj} \cup \{i, j\}$ as described in the first description in the given formula. If $A_{ij} \setminus \{i, j\} \subseteq V_{\ll ii}$ then completely symmetric we have $A_{ij} = B^{jj}_{\ll ii} \cup \{i, j\}$.

Therefore the corresponding formulas given in the statement follow. □

Lemma 10. *Let $ij \in \mathcal{X} \setminus \mathcal{I}$ and let $x \in V \setminus V_{ij}$. Moreover let $x'y' = \ell\text{-}\min \mathcal{X}[\{i, j, x\}]$ and let z' be the remaining vertex of $\{i, j, x\}$.*

1. *If $\{i, x\}, \{j, x\} \notin E$ then $B^{xx}_{ij} = A_{ij}$.*
2. *If $\{i, x\} \in E$ and $\{j, x\} \notin E$ then*
$$B^{xx}_{ij} = \begin{cases} \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}, B^{ii}_{\ll jj} \cup \{i, j\}, B^{jj}_{\ll ix} \cup \{i, j\}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}, B^{ij}_{<ij \ll xx} \cup \{i, j\}\right\}, & \text{if } i, j \notin S, x \in S \\ \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}, C^{x'y', z'z'}_{<ij} \cup \{i, j\}\right\}, & \text{if } i, j, x \notin S. \end{cases}$$
3. *If $\{i, x\} \notin E$ and $\{j, x\} \in E$ then*
$$B^{xx}_{ij} = \begin{cases} \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}, B^{ii}_{\ll xj} \cup \{i, j\}, B^{jj}_{\ll ii} \cup \{i, j\}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}, B^{ij}_{<ij \ll xx} \cup \{i, j\}\right\}, & \text{if } i, j \notin S, x \in S \\ \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}, C^{x'y', z'z'}_{<ij} \cup \{i, j\}\right\}, & \text{if } i, j, x \notin S. \end{cases}$$
4. *If $\{i, x\}, \{j, x\} \in E$ then*
$$B^{xx}_{ij} = \begin{cases} \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}\right\}, & \text{if } i \in S \text{ or } j \in S \text{ or } x \in S \\ \max_w \left\{B^{xx}_{\lessdot ij}, B^{xx}_{\leqslant ij}, C^{x'y', z'z'}_{<ij} \cup \{i, j\}\right\}, & \text{if } i, j, x \notin S. \end{cases}$$

Proof. Let us assume first that $\{i, x\}, \{j, x\} \notin E$. Since $i <_t j$, $j <_b i$, and $x \in V \setminus V_{ij}$, we know that $i <_t j <_t x$ and $j <_b i <_b x$. Thus the neighborhood of x in $G[V_{ij} \cup \{x\}]$ is $\emptyset$. Hence no subset of $V_{ij} \cup \{x\}$ that contains x induces an S-cycle of G and it follows that $B^{xx}_{ij} = A_{ij}$ as described in the first statement.

Assume next that $\{i, x\} \in E$ or $\{j, x\} \in E$. Let $j \notin B_{ij}^{xx}$. By Observation 2 (1) we get $B_{ij}^{xx} = B_{\leqslant ij}^{xx}$. Similarly, if $i \notin B_{ij}^{xx}$ then $B_{ij}^{xx} = B_{\leqslant ij}^{xx}$. So suppose next that $i, j \in B_{ij}^{xx}$. Notice that $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{<ij}$ by Observation 2 (1). We distinguish the following cases.

- Assume that $\{i, x\} \in E$ and $\{j, x\} \notin E$. Since $x \notin V_{ij}$, $j <_t x$ or $i <_b x$. If $i <_b x$ then $x <_t i$ as $\{i, x\} \in E$ but then $x <_t j$ and $j <_b i <_b x$ so that $\{j, x\} \in E$, leading to a contradiction. Thus $j <_t x$ holds. Since $\{i, x\} \in E$ and $\{j, x\} \notin E$, we have $j <_b < x <_b i$ and $i <_t < j <_t x$. We further reduce to the situations depending on whether i, j, x belong to S.
  - Let $i \in S$ or $j \in S$. Let $h \in B_{ij}^{xx} \setminus \{i, j\}$ such that $\{h, i\}, \{h, j\} \in E$. Then $\langle h, i, j \rangle$ is an induced S-triangle in G, a contradiction. So $\{h, i\} \notin E$ or $\{h, j\} \notin E$ for every $h \in B_{ij}^{xx} \setminus \{i, j\}$. Let $g, h \in B_{ij}^{xx} \setminus \{i, j\}$ such that $\{g, j\}, \{h, i\} \in E$. Since $\{g, h\} <_b i$ and $\{g, h\} <_t j$ by the choice of $g, h \in B_{ij}$, it follows that $g <_t i <_t h$ and $h <_b j <_b g$. Thus $\{g, h\} \in E$. This however results in an induced S-square $\langle g, h, i, j \rangle$ in G. This means that for every $h \in B_{ij}^{xx} \setminus \{i, j\}$ either $\{h, i\} \notin E$ or $\{h, j\} \notin E$. By Observation 2 (2) it follows that either $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\ll jj}$ or $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\ll ii}$.
    In the former case notice that both j and x in $G[V_{\ll jj} \cup \{i, j, x\}]$ are adjacent only to i. Thus no subset of $V_{\ll jj} \cup \{i, j, x\}$ that contains j or x induces an S-cycle of G so that $B_{ij}^{xx} = B_{\ll jj}^{ii} \cup \{i, j\}$ as described.
    In the latter case we have $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\ll ii}$. Let $h \in B_{ij}^{xx} \setminus \{i, j\}$. We show that $\{h, x\} \notin E$. Assume for contradiction that $\{h, x\} \in E$. This means that either $h <_t x$ and $x <_b h$, or $x <_t h$ and $h <_b x$. Observe that $h <_t j$ and $h <_b i$. Since $j <_b < x <_b i$ and $i <_t < j <_t x$, we get the following:
    - $h <_t i <_t j <_t x$ and
    - $j <_b x <_b h <_b i$.

    Thus $\{h, j\} \in E$. This however shows that $\langle h, j, i, x \rangle$ is an induced S-square in G, leading to a contradiction. Thus $\{h, x\} \notin E$ for every $h \in B_{ij}^{xx} \setminus \{i, j\}$. Then by Observation 2 (3) it follows that $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\ll ix}$. This means that i and x are only adjacent to j in $G[V_{\ll ix} \cup \{i, j, x\}]$. Hence no subset of $V_{\ll ix} \cup \{i, j, x\}$ that contains i or x induces an S-cycle in G, so that $B_{ij}^{xx} = B_{\ll ix}^{jj} \cup \{i, j\}$ as described.
  - Let $i, j \notin S$ and $x \in S$. Let $h \in B_{ij}^{xx} \setminus \{i, j\}$. We show that $\{h, x\} \notin E$. Assume for contradiction that $\{h, x\} \in E$. Then either $h <_t x$ and $x <_b h$ hold, or $x <_t h$ and $h <_b x$ hold. Since $\{i, x\} \in E$, $\{j, x\} \notin E$, and ij is a crossing pair, we have
    - $\{h, i\} <_t j <_t x$ and
    - $j <_b x <_b h <_b i$

    implying that $\{h, j\} \in E$. If $\{h, i\} \in E$ then $\langle h, i, j \rangle$ is an induced S-triangle whereas if $\{h, i\} \notin E$ then $\langle h, j, i, x \rangle$ is an induced S-square. Thus we reach a contradiction so that $\{h, x\} \notin E$. Then by Observation 2 (5) we get $B_{ij}^{xx} = B_{<ij \ll xx}^{ij} \cup \{i, j\}$, as described.
  - Let $i, j, x \notin S$. By the fact $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{<ij}$, we have $B_{ij}^{xx} = B_{<ij}^{x'y',z'z'} \cup \{i, j\}$.

– Assume that $\{i,x\} \notin E$ and $\{j,x\} \in E$. This case is symmetric to the one above, so that the following hold:
  - If $i \in S$ or $j \in S$ then either $B^{xx}_{ij} = B^{ii}_{\ll xj} \cup \{i,j\}$ or $B^{xx}_{ij} = B^{jj}_{\ll ii} \cup \{i,j\}$.
  - If $x \in S$ then $B^{xx}_{ij} = B^{ij}_{<ij \ll xx} \cup \{i,j\}$.
  - If $i,j,x \notin S$ then $B^{xx}_{ij} = B^{x'y',z'z'}_{<ij}$.

– Assume that both $\{i,x\},\{j,x\} \in E$. Then no vertex of $\{i,x,y\}$ can belong to S as $\langle i,j,x\rangle$ is an induced triangle in G. Since $B^{xx}_{ij} \setminus \{i,j\} \subseteq V_{<ij}$, we get $B^{xx}_{ij} = B^{x'y',z'z'}_{<ij} \cup \{i,j\}$.

Therefore every case results in the described statement of the formulas as required. $\square$

Lemma 11. *Let $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ such that $j <_t y$, $i <_b x$ and $x,y \notin S$. Moreover, if $\{i,y\},\{j,x\} \in E$ then let $x'y' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y\}]$ and let $z'w' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y\} \setminus \{x',y'\}]$.*

1. *If $\{i,y\} \notin E$ then $B^{xy}_{ij} = B^{xx}_{ij}$.*
2. *If $\{j,x\} \notin E$ then $B^{xy}_{ij} = B^{yy}_{ij}$.*
3. *If $\{i,y\},\{j,x\} \in E$ then*

$$B^{xy}_{ij} = \begin{cases} \max_w \left\{B^{xy}_{\lessdot ij}, B^{xy}_{\leqslant ij}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{B^{xy}_{\lessdot ij}, B^{xy}_{\leqslant ij}, C^{x'y',z'w'}_{<ij} \cup \{i,j\}\right\}, & \text{if } i,j \notin S. \end{cases}$$

Proof. Assume that $\{i,y\} \notin E$. Then $i <_b y$ since $i <_t x <_t y$. Thus

– $i <_t j$, $\{j,x\} <_t y$, and
– $j <_b i <_b y <_b x$,

so that the neighborhood of y in $G[V_{ij} \cup \{x,y\}]$ is $\{x\}$. Thus no subset of $V_{ij} \cup \{x,y\}$ that contains y induces an S-cycle in G. Therefore $B^{xy}_{ij} = B^{xx}_{ij}$ as described. If $\{j,x\} \notin E$ then i is non-adjacent to x and similar to the previous case we obtain $B^{xy}_{ij} = B^{yy}_{ij}$.

Assume that $\{i,y\},\{j,x\} \in E$. We distinguish cases depending on whether i or j belong to the solution. Assume first that at least one of i or j does not belong to B^{xy}_{ij}. Let $j \notin B^{xy}_{ij}$. By Observation 2 (1) we have $B^{xy}_{ij} = B^{xy}_{\lessdot ij}$. If $i \notin B^{xy}_{ij}$ then in a similar fashion we get $B^{xy}_{ij} = B^{xy}_{\leqslant ij}$.

Next assume that $i,j \in B^{xy}_{ij}$. Notice that by Observation 2 (1), we have $B^{xy}_{ij} \setminus \{i,j\} \subseteq V_{<ij}$. Let us show that both i and j do not belong to S. If $\{i,x\} \in E$ or $\{j,y\} \in E$ then $\langle i,x,y\rangle$ or $\langle j,x,y\rangle$ induce a triangle in G, since $\{i,y\},\{j,x\} \in E$. Otherwise, $\{i,x\},\{j,y\} \notin E$, so that $\langle i,j,x,y\rangle$ is an induced square in G. Thus in any case there is an S-cycle in G whenever $i \in S$ or $j \in S$ which lead to a contradiction to the fact $i,j \in B^{xy}_{ij}$. Hence $i,j \notin S$. Since $B^{xy}_{ij} \setminus \{i,j\} \subseteq V_{<ij}$, it follows $B^{xy}_{ij} = C^{x'y',z'w'}_{<ij} \cup \{i,j\}$ as required. $\square$

Lemma 12. *Let $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ and let $z \in V \setminus V_{ij}$ such that $xy <_\ell zz$, at least one of x, y is adjacent to z, $j <_t y$, $i <_b x$, and $x, y, z \notin S$. Moreover, if $\{i, z\} \in E$ or $\{j, z\} \in E$ then let $x'y' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y, z\}]$ and let $z'w' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y, z\} \setminus \{x', y'\}]$.*
1. *If $\{i, z\}, \{j, z\} \notin E$ then $C_{ij}^{xy,zz} = B_{ij}^{xy}$.*
2. *If $\{i, z\} \in E$ or $\{j, z\} \in E$ then*

$$C_{ij}^{xy,zz} = \begin{cases} \max_w \left\{ C_{\lessgtr ij}^{xy,zz}, C_{\leqslant ij}^{xy,zz} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ C_{\lessgtr ij}^{xy,zz}, C_{\leqslant ij}^{xy,zz}, C_{<ij}^{x'y',z'w'} \cup \{i, j\} \right\}, & \text{if } i, j \notin S. \end{cases}$$

Proof. Assume first that $\{i, z\}, \{j, z\} \notin E$. Then $i <_t j$, $\{j, x\} <_t \{y, z\}$, $\{i, y\} <_b \{x, z\}$, and $j <_b i$. This means that the neighborhood of z in $G[V_{ij} \cup \{x, y, z\}]$ is a subset of $\{x, y\}$. We will show that no subset of $V_{ij} \cup \{x, y, z\}$ that contains z induces an S-cycle of G.

- Let $\langle v_1, v_2, z \rangle$ be an induced S-triangle such that $v_1, v_2 \in V_{ij} \cup \{x, y\}$. Then $\{v_1, v_2\} = \{x, y\}$ which leads to a contradiction, because $x, y, z \notin S$.
- Let $\langle v_1, v_2, v_3, z \rangle$ be an induced S-square such that $v_1, v_2, v_3 \in V_{ij} \cup \{x, y\}$. Then $\{v_1, v_3\} = \{x, y\}$ which leads to a contradiction, because $\{x, y\} \in E$.

Thus no subset of $V_{ij} \cup \{x, y, z\}$ that contains z induces an S-cycle of G. Therefore $C_{ij}^{xy,zz} = B_{ij}^{xy}$ holds.

Assume that $\{i, z\} \in E$ or $\{j, z\} \in E$. We distinguish cases depending on whether i or j belong to $C_{ij}^{xy,zz}$. If $j \notin C_{ij}^{xy,zz}$ or $i \notin C_{ij}^{xy,zz}$ then by Observation 2 (1) we get $C_{ij}^{xy,zz} = C_{\lessgtr ij}^{xy,zz}$ or $C_{ij}^{xy,zz} = C_{\leqslant ij}^{xy,zz}$, respectively. The remaining case is $i, j \in C_{ij}^{xy,zz}$. Here we will show the described formula given in the second statement. By Observation 2 (3), notice that $C_{ij}^{xy,zz} \setminus \{i, j\} \subseteq V_{<ij}$.

Case 1: Assume that $i \in S$ or $j \in S$. We will show that there is an S-cycle that contains i or j leading to a contradiction to the assumption $i, j \in C_{ij}^{xy,zz}$. Let us assume that $\{i, z\} \in E$; the case for $\{j, z\} \in E$ is completely symmetric. Thus $i <_t z$ and $z <_b i$ hold or $z <_t i$ and $i <_b z$ hold. Moreover we know that $x <_t z$ and $y <_b z$ because $xy <_\ell zz$. Since ij, xy are crossing pairs and $i <_t j <_t y$, $i <_b x$, exactly one of the following holds:

- $x <_t z <_t < i <_t y$ and $\{i, y\} <_b \{x, z\}$;
- $\{i, x\} <_t \{y, z\}$ and $y <_b z <_b i <_b x$.

If the former inequalities hold then it is not difficult to see that $\{i, x\}, \{y, z\} \in E$. And if the latter inequalities hold then $\{i, y\}, \{x, z\} \in E$. Suppose that $\{i, x\}, \{y, z\} \in E$.

- If $\{i, y\} \in E$ then $\langle i, x, y \rangle$ is an induced S-triangle.
- If $\{x, z\} \in E$ then $\langle i, x, z \rangle$ is an induced S-triangle.
- If $\{i, y\}, \{x, z\} \notin E$ then $\langle i, x, y, z \rangle$ is an induced S-square.

Next suppose that $\{i, y\}, \{x, z\} \in E$.

- If $\{i, x\} \in E$ then $\langle i, x, y \rangle$ is an induced S-triangle.

– If $\{y, z\} \in E$ then $\langle i, y, z\rangle$ is an induced S-triangle.
– If $\{i, x\}, \{y, z\} \notin E$ then $\langle i, y, x, z\rangle$ is an induced S-square.

Therefore if $i \in S$ or $j \in S$ then $i, j \notin C_{ij}^{xy,zz}$ so that $C_{ij}^{xy,zz}$ can be expressed as $C_{\leqslant ij}^{xy,zz}$ or $C_{\lessgtr ij}^{xy,zz}$, as already explained previously.

Case 2: Assume that $i, j \notin S$. Let a' be the vertex of $\{i, j, x, y, z\} \setminus \{x', y', z', w'\}$. Observe that $a' \notin S$ since $S \cap \{i, j, x, y, z\} = \emptyset$. We will show that if a subset of $V_{<ij} \cup \{x', y', z', w', a'\}$ that contains a' induces an S-cycle of G, then its non-empty intersection with $V_{<ij}$ is not a subset of $C_{<ij}^{x'y',z'w'}$. Assume for contradiction that a subset of vertices of an induced S-cycle that contains a' belongs to $C_{<ij}^{x'y',z'w'}$. Since the only induced cycles in a permutation graph are triangles or squares we assume that a' is contained in an S-triangle or an S-square.

– Let $\langle v_1, v_2, a'\rangle$ be an induced S-triangle such that $v_1, v_2 \in V_{<ij} \cup \{x', y', z', w'\}$. Since $x', y', z', w' \notin S$, without loss of generality, assume that $v_1 \in S$ which implies that $v_1 \in V_{<ij}$. This means that $v_1 <_t j \leq_t y'$ and $v_1 <_b i \leq_b x'$. By the choices of $x'y'$, $z'w'$, and a' we know that $x' <_t z' <_t a'$ and $y' <_b w' <_b a'$. Since $\{v_1, a'\} \in E$, $a' <_t v_1$ and $v_1 <_b a'$ hold or $v_1 <_t a'$ and $a' <_b v_1$ hold. Thus exactly one of the following holds:
  - $x' <_t z' <_t < a' <_t v_1 <_t y'$ and $\{v_1, y'\} <_b \{x', z', a'\}$;
  - $\{v_1, x'\} <_t \{y', w', a'\}$ and $y' <_b w' <_b < a' <_b v_1 <_b x'$.

  If the former inequalities hold then it is not difficult to see that $\{v_1, x'\}, \{v_1, z'\} \in E$. And if the latter inequalities hold then $\{v_1, y'\}, \{v_1, w'\} \in E$. Suppose that $\{v_1, x'\}, \{v_1, z'\} \in E$.
  - If $\{v_1, y'\} \in E$ then $\langle v_1, x', y'\rangle$ is an induced S-triangle.
  - If $\{x', z'\} \in E$ then $\langle v_1, x', z'\rangle$ is an induced S-triangle.
  - If $\{v_1, y'\}, \{x', z'\} \notin E$ then $\langle v_1, x', y', z'\rangle$ is induced an S-square.

  Next suppose that $\{v_1, y'\}, \{v_1, w'\} \in E$.
  - If $\{v_1, x'\} \in E$ then $\langle v_1, x', y'\rangle$ is an induced S-triangle.
  - If $\{y', w'\} \in E$ then $\langle v_1, y', w'\rangle$ is an induced S-triangle.
  - If $\{v_1, x'\}, \{y', w'\} \notin E$ then $\langle v_1, y', x', w'\rangle$ is induced an S-square.

  Therefore in all cases we obtain that $v_1 \notin C_{<ij}^{x'y',z'w'}$.
– Let $\langle v_1, v_2, v_3, a'\rangle$ be an induced S-square such that $v_1, v_2, v_3 \in V_{<ij} \cup \{x', y', z', w'\}$. If $v_1 \in S$ or $v_3 \in S$ then by the previous argument the S-vertex is not an element of $C_{<ij}^{x'y',z'w'}$. So assume that $v_2 \in S$. Since $x', y', z', w' \notin S$, we have $v_2 \in V_{<ij}$, so that $v_2 <_t j \leq_t y'$ and $v_2 <_b i \leq_b x'$. By the choices of $x'y'$ and a' we know that $x' <_t a'$ and $y' <_b a'$. Moreover the induced S-square imply that either $\{v_2, a'\} <_t \{v_1, v_3\}$ and $\{v_1, v_3\} <_b \{v_2, a'\}$ hold or $\{v_1, v_3\} <_t \{v_2, a'\}$ and $\{v_2, a'\} <_b \{v_1, v_3\}$ hold. Thus exactly one of the following holds:
  - $\{v_2, x'\} <_t a' <_t \{v_1, v_3\}$ and $\{v_1, v_3\} <_b v_2 <_b \{x', a'\}$;
  - $\{v_1, v_3\} <_t v_2 <_t \{y', a'\}$ and $\{v_2, y'\} <_b a' <_b \{v_1, v_3\}$.

  The first inequalities imply that $\{v_1, x'\}, \{v_3, x'\} \in E$ whereas the second inequalities imply that $\{v_1, y'\}, \{v_3, y'\} \in E$. Suppose that $\{v_1, x'\}, \{v_3, x'\} \in E$.

- If $\{v_2, x'\} \in E$ then $\langle v_1, v_2, x'\rangle$ is an induced S-triangle.
- If $\{v_2, x'\} \notin E$ then $\langle v_1, v_2, v_3, x'\rangle$ is an induced S-square.

Similarly if $\{v_1, y'\}, \{v_3, y'\} \in E$ then we obtain an S-triangle $\langle v_1, v_2, x'\rangle$ or an S-square $\langle v_1, v_2, v_3, x'\rangle$. Thus $\{v_1, v_2, v_3\} \cap V_{<ij}$ is not a subset of $C_{<ij}^{x'y',z'w'}$.

Therefore if a subset of $V_{<ij} \cup \{x', y', z', w', a'\}$ that contains a' induces an S-cycle then its non-empty intersection with $V_{<ij}$ is not a subset of $C_{<ij}^{x'y',z'w'}$. This particularly implies that $C_{ij}^{xy,zz} = C_{<ij}^{x'y',z'w'} \cup \{i, j\}$ as described in the second statement. □

Lemma 13. *Let $ij, xy, zw \in \mathcal{X} \setminus \mathcal{I}$ such that $xy <_\ell zw$, $\{x, w\}, \{y, z\} \in E$, $j <_t \{y, w\}$, $i <_b \{x, z\}$, and $x, y, z, w \notin S$. Moreover, if $\{i, w\}, \{j, z\} \in E$, let $x'y' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y, z, w\}]$ and let $z'w' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y, z, w\} \setminus \{x', y'\}]$.*

1. *If $\{i, w\} \notin E$ then $C_{ij}^{xy,zw} = C_{ij}^{xy,zz}$.*
2. *If $\{j, z\} \notin E$ then $C_{ij}^{xy,zw} = C_{ij}^{xy,ww}$.*
3. *If $\{i, w\}, \{j, z\} \in E$ then*

$$C_{ij}^{xy,zw} = \begin{cases} \max_w \left\{ C_{\leqslant ij}^{xy,zw}, C_{\leqslant ij}^{xy,zw} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ C_{\leqslant ij}^{xy,zw}, C_{\leqslant ij}^{xy,zw}, C_{<ij}^{x'y',z'w'} \cup \{i, j\} \right\}, & \text{if } i, j \notin S. \end{cases}$$

Proof. Assume first that $\{i, w\} \notin E$. Then $i <_t w$ and $i <_b w$ hold, since $i <_t j <_t w$. Thus the following hold, as ij, xy, zw are crossing pairs:

- $i <_t j <_t \{y, w\}$, $x <_t \{y, z\}$, $z <_t w$,
- $j <_b i <_b w <_b z$, and $\{i, y\} <_b \{x, w\}$.

Then the neighborhood of w in $G[V_{ij} \cup \{x, y, z, w\}]$ is a subset of $\{x, y, z\}$. We will show that if a subset of $V_{ij} \cup \{x, y, z, w\}$ that contains w induces an S-cycle then its non-empty intersection with V_{ij} is not a subset of $C_{ij}^{xy,zz}$. Since an S-cycle in G is only an S-triangle or an S-square, we distinguish the following two cases:

- Let $\langle v_1, v_2, w\rangle$ be an induced S-triangle such that $v_1, v_2 \in V_{ij} \cup \{x, y, z\}$. Then $\{v_1, v_2\} \subset \{x, y, z\}$ since $N(w) \subseteq \{x, y, z\}$, which leads to a contradiction, because $x, y, z, w \notin S$.
- Let $\langle v_1, v_2, v_3, w\rangle$ be an induced S-square such that $v_1, v_2, v_3 \in V_{ij} \cup \{x, y, z\}$. Then $\{v_1, v_3\} \subset \{x, y, z\}$. Since $x, y, z, w \notin S$, we have $v_2 \in S$ so that $v_2 \in V_{ij}$.
  - Assume that $\{v_1, v_3\} = \{x, y\}$ or $\{y, z\}$. Then we reach a contradiction because $\{x, y\}, \{y, z\} \in E$.
  - Assume that $\{v_1, v_3\} = \{x, z\}$. Then it is clear that $\{x, z\} \notin E$. This however shows that $\langle x, v_2, z, y\rangle$ is an induced S-square, so that $v_2 \notin C_{ij}^{xy,zz}$.

Therefore, if a subset of $V_{ij} \cup \{x,y,z,w\}$ that contains w induces an S-cycle then its non-empty intersection with V_{ij} is not a subset of $C_{ij}^{xy,zz}$. Thus $C_{ij}^{xy,zw} = C_{ij}^{xy,zz}$ holds.

If we assume that $\{j,z\} \notin E$ then similar arguments with the previous case for $\{i,w\} \notin E$ show that $C_{ij}^{xy,zw} = C_{ij}^{xy,ww}$.

Our remaining case is $\{i,w\},\{j,z\} \in E$. If $j \notin C_{ij}^{xy,zw}$ then by Observation 2 (1) we get $C_{ij}^{xy,zw} = C_{\leqslant ij}^{xy,zw}$. Similarly if $i \notin C_{ij}^{xy,zw}$ then $C_{ij}^{xy,zw} = C_{\leqslant ij}^{xy,zw}$. So let us assume that both i,j belong to the solution $C_{ij}^{xy,zw}$, that is $i,j \in C_{ij}^{xy,zw}$. Notice that by Observation 2 (1) we know that $C_{ij}^{xy,zw} \setminus \{i,j\} \subseteq V_{<ij}$. We distinguish two cases depending on whether i,j belong to S. If $i \in S$ or $j \in S$ then the following induced S-cycles show that we reach a contradiction to $i,j \in C_{ij}^{xy,zw}$:

– If $\{i,z\} \in E$ then $\langle i,j,z \rangle$ is an induced S-triangle.
– If $\{j,w\} \in E$ then $\langle i,j,w \rangle$ is an induced S-triangle.
– If $\{i,z\},\{j,w\} \notin E$, then $\langle i,j,z,w \rangle$ is an induced S-square.

Thus if $i \in S$ or $j \in S$ then we know that $j \notin C_{ij}^{xy,zw}$ or $i \notin C_{ij}^{xy,zw}$ which shows the first description of $C_{ij}^{xy,zw}$ in the third statement.

Let $i,j \notin S$ and recall that $i,j \in C_{ij}^{xy,zw}$ and $\{i,w\},\{j,z\} \in E$. Observe that the set $\{i,j,x,y,z,w\} \setminus \{x',y',z',w'\}$ contains two adjacent vertices. Let $a'b'$ be the crossing pair of $\{i,j,x,y,z,w\} \setminus \{x',y',z',w'\}$ so that $a' <_t b'$ and $b' <_b a'$. We will show that if a subset of $V_{<ij} \cup \{x',y',z',w',a',b'\}$ that contains a' or b' induces an S-cycle then its non-empty intersection with $V_{<ij}$ is not a subset of $C_{<ij}^{x'y',z'w'}$. Such an S-cycle is either an S-triangle or an S-square.

– Let $\langle v_1,v_2,b' \rangle$ be an induced S-triangle such that $v_1,v_2 \in V_{<ij} \cup \{x',y',z',w',a'\}$. Since $x',y',z',w',a',b' \notin S$, without loss of generality, assume that $v_1 \in S$ so that $v_1 \in V_{<ij}$. Then $v_1 <_t j <_t \{y,w\}$ and $v_1 <_b i <_b \{x,z\}$. Our goal is to show that $v_1 \notin C_{<ij}^{x'y',z'w'}$. Since $\{v_1,b'\} \in E$ and $x' <_t a' <_t b'$, we get $v_1 <_t b'$ and $b' <_b v_1$. Also notice that $x'y' <_\ell z'w' <_\ell a'b'$ so that $y' <_b w' <_b b'$. Thus the following hold:
  - $\{v_1,x'\} <_t \{y',w',b'\}$ and
  - $y' <_b w' <_b b' <_b \{v_1,x'\}$.

  This shows that $\{v_1,y'\},\{v_1,w'\} \in E$. With these facts we obtain the following S-cycles so that $v_1 \notin C_{<ij}^{x'y',z'w'}$:
  - If $\{v_1,x'\} \in E$ then $\langle v_1,x',y' \rangle$ is an induced S-triangle.
  - If $\{y',w'\} \in E$ then $\langle v_1,y',w' \rangle$ is an induced S-triangle.
  - If $\{v_1,x'\},\{y',w'\} \notin E$ then $\langle v_1,y',x',w' \rangle$ is an induced S-square.
– Let $\langle v_1,v_2,v_3,b' \rangle$ be an S-square such that $v_1,v_2,v_3 \in V_{<ij} \cup \{x',y',z',w',a'\}$. If $v_1 \in S$ or $v_3 \in S$ then similar to the previous argument we can show that the S-vertex does not belong to $C_{<ij}^{x'y',z'w'}$. Assume that $v_2 \in S$. Since $x',y',z',w',a',b' \notin S$, $v_2 \in V_{<ij}$. Thus $v_2 <_t j <_t \{y,w\}$ and $v_2 <_b i <_b \{x,z\}$ which mean that $v_2 <_t \{y',w',b'\}$ and $v_2 <_b \{x',z',a'\}$. Moreover $x'y' <_\ell z'w' <_\ell a'b'$ imply $x' <_t z' <_t a'$ and $y' <_b w' <_b b'$. By the given S-square we have $\{v_1,v_3\} <_t \{v_2,b'\}$ and $\{v_2,b'\} <_b \{v_1,v_3\}$. Since $\{v_2,b'\} \notin E$, we also have $v_2 <_t b'$ and $v_2 <_b b'$. Then the following hold:

- $\{v_1, v_3\} <_t v_2 <_t \{y', b'\}$ and
- $\{v_2, y'\} <_b b' <_b \{v_1, v_3\}$.

Thus $\{v_1, y'\}, \{v_3, y'\} \in E$. Now the following S-cycles show that $v_2 \notin C_{<ij}^{x'y',z'w'}$.

- If $\{v_2, y'\} \in E$ then $\langle v_1, v_2, y' \rangle$ is an induced S-triangle.
- If $\{v_2, y'\} \notin E$ then $\langle v_1, v_2, v_3, y' \rangle$ is an induced S-square.

– Following the same lines as above we can show that if a subset of $V_{<ij} \cup \{x', y', z', w', a', b'\}$ that contains a' induces an S-cycle of G then its non-empty intersection with $V_{<ij}$ is not a subset of $C_{<ij}^{x'y',z'w'}$.

Therefore, if a subset of $V_{<ij} \cup \{x', y', z', w', a', b'\}$ that contains a' or b' induces an S-cycle then its non-empty intersection with $V_{<ij}$ is not a subset of $C_{<ij}^{x'y',z'w'}$. By this fact it follows that $C_{ij}^{xy,zw} = C_{<ij}^{x'y',z'w'} \cup \{i, j\}$ as described in the third statement. □

Theorem 2. Subset Feedback Vertex Set *can be solved in* $O(n+m^3)$ *time on permutation graphs.*

Proof. Let us describe such an algorithm. Given the permutation diagram, that is the ordering $<_t$ and $<_b$ on V, we first compute all crossing pairs ij of $\mathcal{X}$. Observe that the number of such pairs is $n+m$. For each crossing pair ij we compute its predecessors $\{\lessdot, \leqslant, <, \ll, <\!\!\ll\}$ according to the corresponding definition. Note that such a simple application requires $O(n^2)$ time for every crossing pair ij, giving a total running time of $O(n^2m)$. Next we scan all crossing pairs of $\mathcal{X}$ according to their ascending order with respect to $<_r$. For every crossing pair ij we compute A_{ij} according to Lemmas 4 and 9. That is, for every crossing pair xy of $V \setminus V_{ij}$ in descending order with respect to $<_\ell$ we compute B_{ij}^{xy} according to Lemmas 5, 6, 10, and 11. By the recursive formulations of B_{ij}^{xy}, for every crossing pair zw of $V \setminus V_{xy}$ in descending order with respect to $<_\ell$ we compute $C_{ij}^{xy,zw}$ according to Lemmas 7, 8, 12, and 13. In total such computations require $O(n+m^3)$ time. At the end the set $A_{\pi(n)n}$ is the maximum weighted S-forest for G so that $V \setminus A_{\pi(n)n}$ is exactly the minimum subset feedback vertex set of G. □

C Appendix: Omitted proof of Section 5

Theorem 3. *The number of maximal S-forests of a co-bipartite graph is at most $22n^4$ and these can be enumerated in time $O(n^4)$.*

Proof. Let $G = (V, E)$ be a co-bipartite graph and let (A, B) be a partition of V such that such that $G[A]$ and $G[B]$ are cliques. We further partition V as (A_S, A_R, B_S, B_R) where $A_S = A \cap S$, $A_R = A \setminus S$, $B_S = B \cap S$ and $B_R = B \setminus S$. For a vertex v of G and a set $U \in \{A_S, B_S, A_R, B_R\}$ we denote by $N_U(v)$ the neighbors of v in the set U, that is, $N_U(v) =_{\text{def}} N(v) \cap U$. Moreover the *symmetric difference* of two sets L and R is the set $(L \setminus R) \cup (R \setminus L)$ and is denoted by $L \vartriangle R$. Let (X, Y, Z, W) be the partition of the vertex set of a maximal S-forest of G such that $X \subseteq A_S$, $Y \subseteq A_R$, $Z \subseteq B_S$ and $W \subseteq B_R$. It is clear that $|X| \leq 2$ and $|Z| \leq 2$. Thus it is sufficient to consider the following cases with respect to X and Z:

- Let $X = \emptyset$ and $Z = \emptyset$. Then the maximal S-forest contains no vertex of S, so we can safely include all vertices of $V \setminus S$. Thus the following set is a maximal S-forest of G:
  1. $(\emptyset, A_R, \emptyset, B_R)$.
- Let $X = \{a_S\}$ and $Z = \emptyset$. Observe that $|Y| \leq 1$, since $G[X \cup Y]$ is a clique. If $Y = \emptyset$ then including at least two neighbors of a_S that are contained in B_R leads to an S-cycle. Thus we can safely include all non-neighbors of a_S and exactly one neighbor of a_S contained in B_R in the maximal S-forest. If $Y = \{a_R\}$ then including a neighbor of a_S and a neighbor of a_R (may well be the same) that are contained in B_R leads to an S-cycle. If we do not include a neighbor of a_S then we can safely include all other vertices of B_R. However if we include a neighbor of a_S that is non-adjacent to a_R then we can safely include all other vertices that are non-adjacent to both. Thus the following sets induce the corresponding maximal S-forests of G:
  2. $(\{a_S\}, \emptyset, \emptyset, B_R)$, where $N_{B_R}(a_S) = \emptyset$;
  3. $(\{a_S\}, \emptyset, \emptyset, \{b_R\} \cup (B_R \setminus N(a_S)))$, where $b_R \in N_{B_R}(a_S)$;
  4. $(\{a_S\}, \{a_R\}, \emptyset, B_R \setminus N(a_S))$;
  5. $(\{a_S\}, \{a_R\}, \emptyset, \{b_R\} \cup (B_R \setminus N(\{a_S, a_R\})))$, where $b_R \in N_{B_R}(a_S) \setminus N_{B_R}(a_R)$.
- Let $X = \emptyset$ and $Z = \{b_S\}$. Completely symmetric arguments with the previous case imply that the following sets induce the corresponding maximal S-forests of G:
  6. $(\emptyset, A_R, \{b_S\}, \emptyset)$, where $N_{A_R}(b_S) = \emptyset$;
  7. $(\emptyset, \{a_R\} \cup (A_R \setminus N(b_S)), \{b_S\}, \emptyset)$, where $a_R \in N_{A_R}(b_S)$;
  8. $(\emptyset, A_R \setminus N(b_S), \{b_S\}, \{b_R\})$;
  9. $(\emptyset, \{a_R\} \cup (A_R \setminus N(\{b_S, b_R\})), \{b_S\}, \{b_R\})$, where $a_R \in N_{A_R}(b_S) \setminus N_{A_R}(b_R)$.
- Let $X = \{a_S\}$ and $Z = \{b_S\}$. Then both $|Y| \leq 1$ and $|W| \leq 1$. Thus the following sets induce the maximal S-forest of G:
  10. $(\{a_S\}, \emptyset, \{b_S\}, \emptyset)$, where $\{a_S, b_S\} \in E$ and $V \setminus S \subseteq N(a_S) \cap N(b_S)$;
  11. $(\{a_S\}, \{a_R\}, \{b_S\}, \emptyset)$, where $G[\{a_S, a_R, b_S\}]$ is acyclic and $B_R \subseteq N(a_S) \cup N(a_R)$;
  12. $(\{a_S\}, \emptyset, \{b_S\}, \{b_R\})$, where $G[\{a_S, b_S, b_R\}]$ is acyclic and $A_R \subseteq N(b_S) \cup N(b_R)$;
  13. $(\{a_S\}, \{a_R\}, \{b_S\}, \{b_R\})$, where $G[\{a_S, a_R, b_S, b_R\}]$ is acyclic.
- Let $X = \{a_S, a'_S\}$ and $Z = \emptyset$. Then $|Y| = 0$, since $G[X \cup Y]$ is a clique. Adding a vertex of B_R that is adjacent to both a_S and a'_S leads to an S-cycle. If we add a vertex of B_R that is adjacent to either a_S or a'_S then adding another such vertex leads to an S-cycle. Thus we can safely include all other vertices that are non-adjacent to either a_S or a'_S.
  14. $(\{a_S, a'_S\}, \emptyset, \emptyset, B_R \setminus (N(a_S) \cap N(a'_S)))$, where $N_{B_R}(a_S) \vartriangle N_{B_R}(a'_S) = \emptyset$;
  15. $(\{a_S, a'_S\}, \emptyset, \emptyset, \{b_R\} \cup (B_R \setminus N(\{a_S, a'_S\})))$, where $b_R \in N_{B_R}(a_S) \vartriangle N_{B_R}(a'_S)$.

- Let $X = \emptyset$ and $Z = \{b_S, b'_S\}$. Completely symmetric arguments with the previous case imply that the following sets induce the corresponding maximal S-forest of G:
  16. $(\emptyset, A_R \setminus (N(b_S) \cap N(b'_S)), \{b_S, b'_S\}, \emptyset)$, where $N_{A_R}(b_S) \vartriangle N_{A_R}(b'_S) = \emptyset$;
  17. $(\emptyset, \{a_R\} \cup (A_R \setminus (N(b_S) \cup N(b'_S))), \{b_S, b'_S\}, \emptyset)$, where $a_R \in N_{A_R}(b_S) \vartriangle N_{A_R}(b'_S)$.
- Let $X = \{a_S, a'_S\}$ and $Z = \{b_S\}$. Then $|Y| = 0$ and $|W| \leq 1$. Thus the following sets induce the maximal S-forest of G:
  18. $(\{a_S, a'_S\}, \emptyset, \{b_S\}, \emptyset)$, where $G[\{a_S, a'_S, b_S\}]$ is acyclic;
  19. $(\{a_S, a'_S\}, \emptyset, \{b_S\}, \{b_R\})$, where $G[\{a_S, a'_S, b_S, b_R\}]$ is acyclic.
- Let $X = \{a_S\}$ and $Z = \{b_S, b'_S\}$. Then similarly to the previous case we obtain the following:
  20. $(\{a_S\}, \emptyset, \{b_S, b'_S\}, \emptyset)$, where $G[\{a_S, b_S, b'_S\}]$ is acyclic.
  21. $(\{a_S\}, \{a_R\}, \{b_S, b'_S\}, \emptyset)$, where $G[\{a_S, b_S, b'_S, a\}]$ is acyclic.
- Let $X = \{a_S, a'_S\}$ and $Z = \{b_S, b'_S\}$. Then $|Y| = 0$ and $|W| = 0$ so that the following set induces such a maximal S-forest:
  22. $(\{a_S, a'_S\}, \emptyset, \{b_S, b'_S\}, \emptyset)$, where $G[\{a_S, a'_S, b_S, b'_S\}]$ is acyclic.

Because $|X|, |Y|, |Z|, |W| \leq n$, each described maximal S-forest gives at most n^4 maximal S-forests. Therefore in total there are at most $22n^4$ maximal S-forests that correspond to each particular case. Taking into account that any maximal S-forest has at most n vertices, these arguments can be applied to obtain an enumeration algorithm that runs in time $O(n^4)$. $\square$