Strongly chordal and chordal bipartite graphs are sandwich monotone

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The 15th Int'l Computing and Combinatorics Conference COCOON 2009
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Overview

- *Sandwich monotone* graphs
  implies that minimal completion problems are poly-time solvable
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- **Sandwich monotone graphs** implies that minimal completion problems are poly-time solvable
- **Strongly Chordal graphs**
  subclass of chordal graphs
- **Strongly Chordal graphs are sandwich monotone**
  constructive proof
- **Minimal Strongly Chordal completions (characterizations)**
Overview

- *Sandwich monotone* graphs
  implies that minimal completion problems are poly-time solvable

- *Strongly Chordal* graphs
  subclass of chordal graphs

- *Strongly Chordal* graphs are *sandwich monotone*
  constructive proof

- *Minimal* *Strongly Chordal* completions (characterizations)

- *Chordal Bipartite* graphs
  close related to *Strongly Chordal* graphs
    - *Chordal Bipartite* graphs are *sandwich monotone*

- *Minimal Chordal Bipartite* completions (characterizations)
• A graph class $\mathcal{C}$ is sandwich monotone:

Given two graphs $G_0 = (V, E) \in \mathcal{C}$ and $G_{|F|} = (V, E \cup F) \in \mathcal{C}$, there is an edge $f \in F$ such that $G_{|F|} - f \in \mathcal{C}$. 
Sandwich monotone

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\[ G_0 = (V, E) \in \mathcal{C} \]
\[ G_F = (V, E \cup \{f_1, f_2, \ldots, f_{|F|-1}, f_{|F|}\}) \in \mathcal{C} \]

• Sandwich monotonicity $\mathcal{C}$:

Given $G_0 = (V, E)$ and $G_F = (V, E \cup F)$,
A graph class $\mathcal{C}$ is sandwich monotone:
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Sandwich monotonicity $\mathcal{C}$:
Given $G_0 = (V, E)$ and $G_F = (V, E \cup F)$,

$G_F = (V, E \cup \{f_1, f_2, \ldots, f_{|F|-1}, f_{|F|}\}) \in \mathcal{C}$

$G_0 = (V, E) \in \mathcal{C}$

$G_1 = (V, E \cup \{f_1\}) \in \mathcal{C}$
A graph class $\mathcal{C}$ is sandwich monotone:

Given two graphs $G_0 = (V, E) \in \mathcal{C}$ and $G_{|\mathcal{F}|} = (V, E \cup \mathcal{F}) \in \mathcal{C}$, there is an edge $f \in \mathcal{F}$ such that $G_{|\mathcal{F}|} - f \in \mathcal{C}$.

Sandwich monotonicity $\mathcal{C}$:

Given $G_0 = (V, E)$ and $G_F = (V, E \cup \mathcal{F})$, there exist

- $G_1 = (V, E \cup \{f_1\}) \in \mathcal{C}$
- $G_2 = (V, E \cup \{f_1, f_2\}) \in \mathcal{C}$
- $G_F = (V, E \cup \{f_1, f_2, \ldots, f_{|\mathcal{F}|-1}, f_{|\mathcal{F}|}\}) \in \mathcal{C}$
- $G_0 = (V, E) \in \mathcal{C}$
Sandwich monotone

- A graph class $\mathcal{C}$ is sandwich monotone:

  Given two graphs $G_0 = (V, E) \in \mathcal{C}$ and $G_{|F|} = (V, E \cup F) \in \mathcal{C}$, there is an edge $f \in F$ such that $G_{|F|} - f \in \mathcal{C}$.

- Sandwich monotonicity $\mathcal{C}$:

  Given $G_0 = (V, E)$ and $G_F = (V, E \cup F)$,
Why research on sandwich monotonicity?

- Close related to *minimal completion problems*:

Given any graph \( G = (V, E) \) and a graph class \( C \), we are looking for a supergraph \( H = (V, E \cup F) \) such that \( H \in C \) and for every \( F' \subset F: H' = (V, E \cup F') \notin C \)

\( \triangleright \) computing the minimum \( |F| \) is usually difficult (NP-complete).
Why research on sandwich monotonicity?

- Close related to *minimal completion problems:*

  Given any graph $G = (V, E)$ and a graph class $C$, we are looking for a supergraph $H = (V, E \cup F)$ such that $H \in C$ and for every $F' \subset F$: $H' = (V, E \cup F') \notin C$

  ▶ computing the **minimum** $|F|$ is usually difficult (NP-complete).

- if $C$ is sandwich monotone:

  A $C$ completion is minimal if and only if no single *fill edge* (an edge of $F$) can be removed without destroying $C$. 
Why research on sandwich monotonicity? 1/2

- Close related to **minimal completion problems**: Given any graph $G = (V, E)$ and a graph class $C$, we are looking for a supergraph $H = (V, E \cup F)$ such that $H \in C$ and for every $F' \subset F$: $H' = (V, E \cup F') \notin C$

  ▶ computing the **minimum** $|F|$ is usually difficult (NP-complete).

- if $C$ is sandwich monotone:
  A $C$ completion is minimal if and only if no single **fill edge** (an edge of $F$) can be removed without destroying $C$.

  if $C$ is sandwich monotone and $C$ is recognized in poly-time:
  There is a polynomial time algorithm for computing a minimal $C$ completion.
Listing all graphs in $\mathcal{C}$ with edge constraints

COCOON 2008

Input: $G = (V, E)$ and $H = (V, E \cup F)$ such that $H \in \mathcal{C}$

Output: list all graphs in $\mathcal{C}$ between $G$ and $H$
Why research on sandwich monotonicity?

- Listing all graphs in $\mathcal{C}$ with edge constraints

**COCOON 2008**

Input: $G = (V, E)$ and $H = (V, E \cup F)$ such that $H \in \mathcal{C}$

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- Kijima, Kiyomi, Okamoto, and Uno (COCOON 2008)

Efficient solution for chordal graphs
Why research on sandwich monotonicity?

- Listing all graphs in $C$ with edge constraints

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Input: $G = (V, E)$ and $H = (V, E \cup F)$ such that $H \in C$

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Efficient solution for chordal graphs

▷ Observing the previous algorithm...

the proposed solution can be extended to every sandwich monotone class
Which graph classes are sandwich monotone?

- Chordal graphs
- Strongly chordal graphs
- Interval graphs
- Split graphs
- Threshold graphs
- Bipartite graphs
- Chain graphs
- Chordal bipartite graphs

✓: sandwich monotone
×: not sandwich monotone
?: unknown
Which graph classes are sandwich monotone?

- Chordal graphs
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✓: sandwich monotone
×: not sandwich monotone
?: unknown

Today:

✓ (1977)
✓ (2006)
✓ (2007)
✓ (2007)
✓ (2006)
✓ (2007)
Strongly Chordal graphs

- Chordal graphs: do not contain $C_k$, $k \geq 4$
• Chordal graphs: do not contain $C_k$, $k \geq 4$

• Strongly Chordal graphs: chordal graphs having no $k$-sun

$C_4$  $C_5$  $C_6$  

3-sun  4-sun  5-sun  $k$-sun
Strongly Chordal graphs

- **Strongly Chordal graphs:** characterized through simple vertices

![Diagram of a strongly chordal graph]

- Simple elimination ordering: given an ordering $v_1, \ldots, v_n$
  
  For every $i$, $v_i$ is simple in $G_i \equiv G\{v_i, \ldots, v_n\}$, $1 \leq i \leq n$.

Farber, 1983

$G$ is strongly chordal $\iff$ $G$ admits a simple elimination ordering.
Strongly Chordal graphs

- Strongly Chordal graphs: characterized through simple vertices

  - Simple vertex $x$
  - $N(x)$ (clique)
  - $S(x)$ (strongly chordal graph)

- Simple elimination ordering: given an ordering $v_1, \ldots, v_n$
  For every $i$, $v_i$ is simple in $G_i \equiv G[v_i, \ldots, v_n]$, $1 \leq i \leq n$. 

• **Strongly Chordal graphs:**
  characterized through **simple** vertices

\[
\text{simple vertex } x
\]

\[
N(x) \text{ (clique)}
\]

\[
S(x) \text{ (strongly chordal graph)}
\]

• **Simple elimination ordering:** given an ordering \(v_1, \ldots, v_n\)
  For every \(i\), \(v_i\) is simple in \(G_i \equiv G[v_i, \ldots, v_n]\), \(1 \leq i \leq n\).

**Farber, 1983**

\(G\) is strongly chordal \(\iff\) \(G\) admits a simple elimination ordering.
- Simple vertices $\Rightarrow$ simple partition

\[
\begin{align*}
N(x) &\subset N(N(x)) & S(x) &\subset N(N(x)) \\
S(x) &\subset N(N(x))
\end{align*}
\]
Strongly Chordal graphs

- Simple vertices $\Rightarrow$ simple partition

- $N(N_1) \subset N(N_2) \subset N(N_3) \Rightarrow S_1 \prec S_2 \prec S_3$
Our goal is to prove the following:

Given: \( G = (V, E) \) and \( G' = (V, E \cup F) \) two strongly chordal graphs,

there is an edge \( f \in F \) such that \( G' - f \) is strongly chordal.
Our goal is to prove the following:

- Given: $G = (V, E)$ and $G' = (V, E \cup F)$ two strongly chordal graphs,
- there is an edge $f \in F$ such that $G' - f$ is strongly chordal.

For the proof we apply induction on $|V|:

- For $|V| \leq 3$: all graphs are strongly chordal (the statement holds)
- Assume that the statement is true for $|V| - 1$.
- Prove that the statement holds for $|V|$. 
Strongly Chordal graphs are Sandwich Monotone

\[ G = (V, E) \]

\[ G' = (V, E \cup F) \]

- \( G' \): Pick a simple vertex
Strongly Chordal graphs are Sandwich Monotone

\[ G = (V, E) \]
\[ G' = (V, E \cup F) \]

- if \( x \) is incident to an added edge \( f \in F \)
Strongly Chordal graphs are Sandwich Monotone

$G = (V, E)$

$G' = (V, E \cup F)$

- then $G' - f$ has no $C_4$ or 3-sun $\Rightarrow G' - f \checkmark$
Strongly Chordal graphs are Sandwich Monotone

\[ G = (V, E) \]
\[ G' = (V, E \cup F) \]

- otherwise, \( F \subset N(x) \cup S(x) \)
Strongly Chordal graphs are Sandwich Monotone

\[ G = (V, E) \Rightarrow H = (V, E \cup E') \quad G' = (V, E \cup F) \]

- Add certain type of edges \( E' \) to \( G = (V, E) \Rightarrow H = (V, E \cup E') \)

\( E' : N_G(x) \) clique and between \( N(x) - S(x) \)
Strongly Chordal graphs are Sandwich Monotone

\[ G = (V, E) \Rightarrow H = (V, E \cup E') \quad G' = (V, E \cup F) \]

- Remove \( x \) from both graphs \( H \) and \( G' \)
**Strongly Chordal graphs are Sandwich Monotone**

- Both graphs are on $|V| - 1$ vertices
- Both graphs are strongly chordal
  
  $G = (V, E) \Rightarrow H = (V, E \cup E')$
  $G' = (V, E \cup F)$

  $\Rightarrow$ apply induction
Strongly Chordal graphs are Sandwich Monotone

\[ G = (V, E) \Rightarrow H = (V, E \cup E') \quad G' = (V, E \cup F) \]

- By induction \( \Rightarrow \) (i) \( G' - x - f \) ✓
- By the edges \( E' \) \( \Rightarrow \) (ii) either \( f \in S(x) \) or \( f \in N_i S_i \)
Strongly Chordal graphs are Sandwich Monotone

\[ G' - x - f: \checkmark \]

\[ G = (V, E) \Rightarrow H = (V, E \cup E') \quad G' = (V, E \cup F) \]

- Construct \( G' - x - f \) from \( H - x \) and \( G' - x \)
Strongly Chordal graphs are Sandwich Monotone

\[ G' - x - f: \checkmark \]

\[ G = (V, E) \Rightarrow H = (V, E \cup E') \quad G' = (V, E \cup F) \]

- Add \( x \) to \( G' - x - f \)
Strongly Chordal graphs are Sandwich Monotone

\[ \begin{align*}
G & = (V, E) \\
G' & = (V, E \cup F) \\
H & = (V, E \cup E') \\
G' - x - f & : \checkmark
\end{align*} \]

- Add \( x \) to \( G' - x - f \) \( \Rightarrow \) \( x \) is simple in \( G' - f \)
  
  (either \( f \in S(x) \) or \( f \in N_i S_i \))
Strongly Chordal graphs are Sandwich Monotone

\[ G = (V, E) \Rightarrow H = (V, E \cup E') \quad G' = (V, E \cup F) \]

- Add \( x \) to \( G' - x - f \) \( \Rightarrow \) \( x \) is simple in \( G' - f \) \( \Rightarrow \) \( G' - f \) \( \checkmark \)
  (either \( f \in S(x) \) or \( f \in N_i S_i \))
Sketch of the proof (algorithm):

1. **Pick** a simple vertex \( x \) in \( G' \)

2. **if** \( x \) is incident to an added edge \( f \in F \) **then**
   
   2.1 \( G' - f \) ✓

   **else**

   2.2 **Add** \( E' \) to \( G = (V, E) \Rightarrow H = (V, E \cup E') \)

   2.3 \( G' - x - f \): by induction on \( H - x \) and \( G' - x \)

   2.4 **Add** \( x \) in \( G' - x - f \) as simple \( \Rightarrow G' - f \) ✓
Sketch of the proof (algorithm):

1. Pick a simple vertex \( x \) in \( G' \)
   
   *there is always a simple vertex*

2. if \( x \) is incident to an added edge \( f \in F \) then
   2.1 \( G' - f \) ✓

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1. Pick a simple vertex \( x \) in \( G' \)

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2. if \( x \) is incident to an added edge \( f \in F \) then

   2.1 \( G' - f \checkmark \)

      *by contradiction \( G' - f \) has no \( C_4 \) or 3-sun*

   else

   2.2 Add \( E' \) to \( G = (V, E) \Rightarrow H = (V, E \cup E') \)

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Sketch of the proof (algorithm):

1. Pick a simple vertex \( x \) in \( G' \)
   
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2. if \( x \) is incident to an added edge \( f \in F \) then
   
   2.1 \( G' - f \sqrt{✓} \)
   
   *by contradiction \( G' - f \) has no \( C_4 \) or 3-sun*

   else

   2.2 Add \( E' \) to \( G = (V, E) \Rightarrow H = (V, E \cup E') \)

   *prove that (i) \( F \not\subseteq E' \) and (ii) \( H \) is strongly chordal*

   2.3 \( G' - x - f \): by induction on \( H - x \) and \( G' - x \)

   2.4 Add \( x \) in \( G' - x - f \) as simple \( \Rightarrow G' - f \sqrt{✓} \)
Sketch of the proof (algorithm):

1. **Pick a simple vertex** \(x\) in \(G'\)
   *there is always a simple vertex*

2. **if** \(x\) is incident to an added edge \(f \in F\) **then**
   
   2.1 \(G' - f \checkmark\)
   *by contradiction \(G' - f\) has no \(C_4\) or 3-sun*

   **else**

   2.2 **Add** \(E'\) to \(G = (V, E) \Rightarrow H = (V, E \cup E')\)
   *prove that (i) \(F \notin E'\) and (ii) \(H\) is strongly chordal*

2.3 \(G' - x - f\): **by induction on** \(H - x\) and \(G' - x\)
   *either \(f \in S(x)\) or \(f \in N_iS_i\)*

2.4 **Add** \(x\) in \(G' - x - f\) **as simple** \(\Rightarrow G' - f \checkmark\)
Sketch of the proof (algorithm):

1. Pick a simple vertex \( x \) in \( G' \)
   
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   prove that (i) \( F \not\subset E' \) and (ii) \( H \) is strongly chordal

2.3 \( G' - x - f \): by induction on \( H - x \) and \( G' - x \)

   either \( f \in S(x) \) or \( f \in N_i S_i \)

2.4 Add \( x \) in \( G' - x - f \) as simple \( \Rightarrow G' - f \checkmark \)

   \( N(x) \) retains the inclusion set property
chords:
- chord of a cycle: an edge between two nonconsecutive vertices
- chord of a $k$-sun: an edge between the indep. set and the clique
Characterizations of minimal strongly chordal completions

- **chords:**
  - chord of a cycle: an edge between two nonconsecutive vertices
  - chord of a $k$-sun: an edge between the indep. set and the clique

Let $G$ be strongly chordal and $e$ be an edge.
$G - e$ is strongly chordal iff $e$ is not the unique chord of a $C_4$ or the unique chord of a 3-sun.

$G'$ is a minimal strongly completion of an arbitrary graph $G$ iff every added edge is the unique chord of a $C_4$ or a 3-sun.
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- *Sandwich monotone* graphs implies that minimal completion problems are poly-time solvable
- *Strongly Chordal* graphs subclass of chordal graphs
- *Strongly Chordal* graphs are sandwich monotone constructive proof
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- *Strongly Chordal* graphs are sandwich monotone constructive proof
- Minimal *Strongly Chordal* completions (characterizations)
- *Chordal Bipartite* graphs close related to Strongly Chordal graphs
  - Chordal Bipartite graphs are sandwich monotone
- Minimal *Chordal Bipartite* completions (characterizations)
Chordal bipartite graphs

- **Chordal bipartite graphs:**
  Bipartite graphs that do not contain $C_k$ for $k \geq 6$

Dahlhaus, 1991

$B = (X, Y, E)$

$G = (X, Y, E \cup C)$

$B$ is chordal bipartite $\iff G$ is strongly chordal
Chordal bipartite graphs

- **Chordal bipartite graphs:**
  Bipartite graphs that do not contain $C_k$ for $k \geq 6$

- **Close related to strongly chordal graphs**
  - Let $B = (X, Y, E)$ a bipartite graph
  - Make $X$ a clique by adding edges in $X$ $\Rightarrow$ $G$

\[ B = (X, Y, E) \quad \Rightarrow \quad G = (X, Y, E \cup C) \]
Chordal bipartite graphs

- **Chordal bipartite graphs:**
  Bipartite graphs that do not contain $C_k$ for $k \geq 6$

- **Close related to strongly chordal graphs**
  - Let $B = (X, Y, E)$ a bipartite graph
  - Make $X$ a clique by adding edges in $X \Rightarrow G$

\[ B = (X, Y, E) \Rightarrow G = (X, Y, E \cup C) \]

**Dahlhaus, 1991**

$B$ is chordal bipartite $\iff$ $G$ is strongly chordal
Chordal bipartite graphs are sandwich monotone

**Sandwich monotone**

\[ B = (X, Y, E) \text{ and } B' = (X, Y, E \cup F) \] two chordal bipartite graphs; there is an \( f \in F \) such that \( B' - f \) is chordal bipartite.
Sandwich monotone

\[ B = (X, Y, E) \text{ and } B' = (X, Y, E \cup F) \text{ two chordal bipartite graphs;} \]
\[ \text{there is an } f \in F \text{ such that } B' - f \text{ is chordal bipartite.} \]

Proof:

\[ B = (X, Y, E) \quad B' = (X, Y, E \cup F) \]

\[ X \quad \circ \quad \circ \quad \ldots \quad \circ \quad \circ \]
\[ Y \quad \circ \quad \circ \quad \ldots \quad \circ \quad \circ \]

\[ B = (X, Y, E) \]
\[ B' = (X, Y, E \cup F) \]
Chordal bipartite graphs are sandwich monotone

**Sandwich monotone**

\[ B = (X, Y, E) \text{ and } B' = (X, Y, E \cup F) \] two chordal bipartite graphs; there is an \( f \in F \) such that \( B' - f \) is chordal bipartite.

**Proof:**

![Graph diagrams](Diagram)

- \( G = (X, Y, E \cup C) \)
- \( G' = (X, Y, E \cup F \cup C) \)

\( G \) and \( G' \) are both strongly chordal.
Chordal bipartite graphs are sandwich monotone

**Sandwich monotone**

\[ B = (X, Y, E) \text{ and } B' = (X, Y, E \cup F) \] two chordal bipartite graphs; there is an \( f \in F \) such that \( B' - f \) is chordal bipartite.

**Proof:**

\[ G = (X, Y, E \cup C) \]

\[ G' = (X, Y, E \cup F \cup C) \]

- \( G \) and \( G' \) are both strongly chordal
- By sandwich monotonicity of strongly chordal \( \Rightarrow \)
  there is an \( f \in F \) such that \( G' - f \) strongly chordal
Chordal bipartite graphs are sandwich monotone

Sandwich monotone

\[ B = (X, Y, E) \text{ and } B' = (X, Y, E \cup F) \text{ two chordal bipartite graphs; there is an } f \in F \text{ such that } B' - f \text{ is chordal bipartite.} \]

Proof:

\[ B = (X, Y, E) \]
\[ B' = (X, Y, E \cup F) \]

\[ B' - f \]

- \( G \) and \( G' \) are both strongly chordal
- By sandwich monotonicity of strongly chordal \( \Rightarrow \) there is an \( f \in F \) such that \( G' - f \) strongly chordal
- Remove all edges in \( X \) \( \Rightarrow \) \( B' - f \) is chordal bipartite
Characterizations of minimal chordal bipartite completions

- Characterizing:
  - edge removal from a chordal bipartite graph
  - minimal chordal bipartite completions
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- Characterizing:
  - edge removal from a chordal bipartite graph
  - minimal chordal bipartite completions

Let $B$ be chordal bipartite and $e$ be an edge

$B - e$ is chordal bipartite iff $e$ is not the unique chord of a $C_6$. 

![Graph Diagram]
Characterizations of minimal chordal bipartite completions

- Characterizing:
  - edge removal from a chordal bipartite graph
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Characterizations of minimal chordal bipartite completions

- Characterizing:
  - edge removal from a chordal bipartite graph
  - minimal chordal bipartite completions

Let $B$ be chordal bipartite and $e$ be an edge

$B - e$ is chordal bipartite iff $e$ is not the unique chord of a $C_6$.

$B'$ is a minimal chordal bipartite completion of a bipartite $B$ iff

every added edge is the unique chord of a $C_6$. 
Conclusions and Open problems

• Strongly chordal graphs and chordal bipartite graphs are sandwich monotone.
  ▶ recognition problem ⇒ \( O(\min\{m \log n, n^2\}) \)
    Paige and Tarjan 1987, Spinrad 1993
  ▶ computing a minimal completion ⇒ \( O(n^4(\min\{m \log n, n^2\})) \)
    by applying a straightforward algorithm; any improvement?

• Other graph classes
  ▶ weakly chordal: neither \( G \) nor \( \bar{G} \) contain \( C_5, C_6, \ldots \)
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Thank you!!