Polynomial-Time Algorithms for the Subset Feedback Vertex Set Problem on Interval Graphs and Permutation Graphs*

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Abstract

Given a vertex-weighted graph $G = (V, E)$ and a set $S \subseteq V$, a subset feedback vertex set X is a set of the vertices of G such that the graph induced by $V \setminus X$ has no cycle containing a vertex of S. The Subset Feedback Vertex Set problem takes as input G and S and asks for the subset feedback vertex set of minimum total weight. In contrast to the classical Feedback Vertex Set problem which is obtained from the Subset Feedback Vertex Set problem for $S = V$, restricted to graph classes the Subset Feedback Vertex Set problem is known to be NP-complete on split graphs and, consequently, on chordal graphs. However, whereas Feedback Vertex Set is polynomially solvable for AT-free graphs, no such result is known for the Subset Feedback Vertex Set problem on any subclass of AT-free graphs. Here we give the first polynomial-time algorithms for the problem on two subclasses of AT-free graphs: interval graphs and permutation graphs. Moreover, towards the unknown complexity of the problem for AT-free graphs, we give a polynomial-time algorithm for co-bipartite graphs. Thus, we contribute to the first positive results of the Subset Feedback Vertex Set problem when restricted to graph classes for which Feedback Vertex Set is solved in polynomial time.

1 Introduction

For a given set S of vertices of a graph G, a *subset feedback vertex set* X is a set of vertices such that no cycle of $G[V \setminus X]$ contains a vertex from S. The Subset Feedback Vertex Set problem takes as input a graph $G = (V, E)$ and a set $S \subseteq V$ and asks for the subset feedback vertex set of minimum cardinality. In the weighted version every vertex of G has weight and the objective is to compute a subset feedback vertex set of minimum total weight. The Subset Feedback Vertex Set problem is a generalization of the classical Feedback Vertex Set problem in which the goal is to remove a set of vertices X such that $G[V \setminus X]$ has no cycles; by setting $S = V$, the problem coincides with the NP-complete Feedback Vertex Set problem [19]. Both problems find important applications in several aspects that arise in optimization theory, constraint satisfaction, and bayesian inference [1, 2, 14, 15]. Interestingly, the Subset Feedback Vertex Set problem for $|S| = 1$ also coincides with the NP-complete Multiway Cut problem [17] in which the task is to disconnect a predescribed set of vertices [9, 20].

Subset Feedback Vertex Set was first introduced by Even et al. who obtained a constant factor approximation algorithm for its weighted version [14]. The unweighted version in which all vertex weights are equal has been proved to be fixed parameter tractable [13].

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Moreover, the fastest algorithm for the weighted version in general graphs runs in $O^*(1.87^n)$ time[1] by enumerating its minimal solutions [17], whereas for the unweighted version the fastest algorithm runs in $O^*(1.76^n)$ time [16]. As the unweighted version of the problem is shown to be NP-complete even when restricted to split graphs [17], there is a considerable effort to reduce the running time on chordal graphs, a proper superclass of split graphs, and more general on other classes of graphs. Golovach et al. considered the weighted version and gave an algorithm that runs in $O^*(1.68^n)$ time for chordal graphs [21]. Reducing the existing running time even on chordal graphs has proved itself to be quite challenging and only for the unweighted version of the problem a faster algorithm that runs in $O^*(1.62^n)$ time was given [10]. In fact, the $O^*(1.62^n)$-algorithm given in [10] runs for every graph class on which the weighted Feedback Vertex Set problem can be solved in polynomial time that is closed under vertex deletions and edge contractions. Thus, there is an algorithm that runs in $O^*(1.62^n)$ time for the unweighted version of the Subset Feedback Vertex Set problem when restricted to AT-free graphs [10], a graph class that properly contains interval graphs and permutation graphs. Here we design algorithms for the classes of interval graphs and permutation graphs that are much faster even for the weighted version of the problem.

As Subset Feedback Vertex Set is a generalization of the classical Feedback Vertex Set problem, let us briefly give an overview of the complexity of Feedback Vertex Set on related graph classes. Feedback Vertex Set is known to be NP-complete on bipartite graphs [34] and planar graphs [19], whereas it becomes polynomial-time solvable on the classes of bounded clique-width graphs [8], chordal graphs [11, 33], interval graphs [28], permutation graphs [4, 5, 6, 26], cocomparability graphs [27], and, more generally, AT-free graphs [25]. In contrast to the many positive and negative results on the complexity of Feedback Vertex Set, concerning the complexity of Subset Feedback Vertex Set very few similar results are known. In fact, there is only a negative result regarding the NP-completeness of Subset Feedback Vertex Set on split graphs [17]. Such a result, however, implies that there is an interesting algorithmic difference between the two problems, as the Feedback Vertex Set problem is known to be polynomial-time computable on split graphs [11, 33]. Clearly, for graph classes on which the Feedback Vertex Set problem is NP-complete, so does the Subset Feedback Vertex Set problem. As the Subset Feedback Vertex Set problem is more general than the Feedback Vertex Set problem, it is natural to study its complexity on graph classes on which Feedback Vertex Set is polynomial-time solvable.

Both interval graphs and permutation graphs have unbounded clique-width [23] and, thus, they lie beyond the scope of known algorithmic metatheorems related to MSOL formulation [12]. Let us also briefly explain that extending the approach of [25] for the Feedback Vertex Set problem when restricted to AT-free graphs is not straightforward. A graph is *AT-free* if for every triple of pairwise non-adjacent vertices, the neighborhood of one of them separates the two others. The class of AT-free graphs is well-studied and it properly contains interval, permutation, and cocomparability graphs [7, 22]. One of the basic tools in [25] relies on growing a small representation of an independent set into a suitable forest. Although such a representation is rather small on AT-free graphs (and, thus, on interval graphs or permutation graphs), when considering Subset Feedback Vertex Set it is not necessary that the fixed set induces an independent set which makes it difficult to control how the partial solution may be extended. Therefore, the methodology described in [25] cannot be trivially extended towards the Subset Feedback Vertex Set problem.

Our Results. Here we initiate the study of Subset Feedback Vertex Set restricted to graph classes from the positive perspective. We consider its weighted version and give the first positive results on interval graphs and permutation graphs, both being proper subclasses of

[1]The O^* notation is used to suppress polynomial factors.

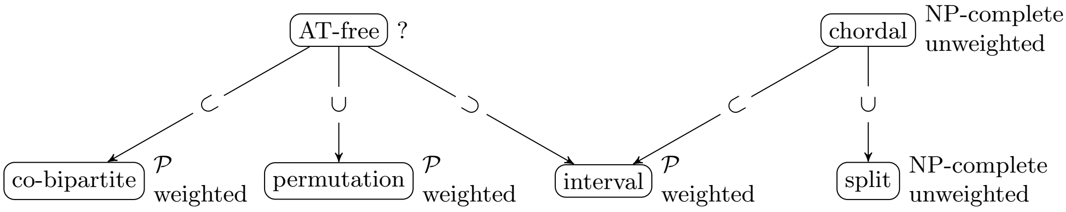


Figure 1: The computational complexity of the Subset Feedback Vertex Set problem restricted to the illustrated graph classes. All polynomial-time results ($\mathcal{P}$) are obtained in this work, whereas the NP-completeness result of split graphs is due to [17].

AT-free graphs. As already explained, we are interested in subclasses of AT-free graphs, since for chordal graphs the problem is already NP-complete [17]. Interval graphs and permutation graphs are unrelated to split graphs and are both characterized by a linear structure implied by certain vertex orderings [7, 22, 33]. For both classes of graphs we design polynomial-time algorithms based on dynamic programming of subproblems implied by their natural linear ordering. One of our key ingredients is that we augment our subproblems with a few additional vertices which are always included in the subsolutions. Although for interval graphs such a strategy leads to a simple algorithm, the case for permutation graphs requires augmenting with more vertices, resulting in more numerous and complex recursive relations.

Moreover, towards the unknown complexity of the problem on the class of AT-free graphs, we consider the class of co-bipartite graphs (complements of bipartite graphs) and settle its complexity status. Interestingly, most problems that are hard on AT-free graphs are already hard on co-bipartite graphs (see for e.g., [29]). Co-bipartite graphs are the complements of bipartite graphs and are unrelated to interval graphs and permutation graphs. We show that Subset Feedback Vertex Set admits a simple solution on co-bipartite graphs, and, thus, we eliminate the possibility of obtaining hardness on AT-free graphs through hardness on co-bipartite graphs. Therefore, we provide the first positive results regarding the complexity of the Subset Feedback Vertex Set problem on subclasses of AT-free graphs. Our overall results are summarized in Figure 1.

Our paper is organized as follows. In Section 2 we give basic definitions and notations of maximal solutions (S-forests) and present the enumeration algorithm for the complements of bipartite graphs. Sections 3 and 4 contain the main results, namely the polynomial-time algorithms for interval graphs and permutation graphs respectively. We conclude with Section 5 in which we discuss some open problems and directions for future work.

2 Preliminaries

All graphs in this text are undirected and simple. A graph is denoted by $G = (V, E)$ with vertex set V and edge set E. We use the convention that $n = |V|$ and $m = |E|$. For a vertex subset $X \subseteq V$, the *subgraph of G induced by X* is $G[X] = (X, \{uv \in E : u, v \in X\})$. The *neighbourhood* of a vertex x of G is $N(x) = \{v \in V : xv \in E\}$ and the *degree* of x is $|N(x)|$. If $X \subseteq V$, then $N(X) = \bigcup_{x \in X} N(x) \setminus X$. A *weighted graph* $G = (V, E)$ is a graph, where each vertex $v \in V$ is assigned *weight* that is a positive integer number. We denote by $w(v)$ the weight of each vertex $v \in V$. For a vertex subset $X \subseteq V$, the weight of X is $\sum_{v \in X} w(v)$.

Given a relation $\preceq$ on elements, we extend $\preceq$ to support sets of elements as follows. For two sets of elements L and R we write $L \preceq R$ if for any two elements $u \in L$ and $v \in R$, we have $u \preceq v$.

A *clique* is a set of pairwise adjacent vertices, while an *independent set* is a set of pairwise non-adjacent vertices. A *path* is a sequence of vertices $\langle v_1 v_2 \cdots v_k \rangle$ where each pair of consecutive vertices $v_i v_{i+1}$ forms an edge of G. If additionally $v_1 v_k$ is an edge, then we obtain a *cycle*. In this paper, we distinguish between paths (or cycles) and *induced paths* (or *induced cycles*). By an induced path (or cycle) of G we mean a chordless path (or cycle). A cycle on three vertices is referred to as a *triangle* and a chordless cycle on four vertices is referred to as a *square*. Notice that a square refers to an induced cycle. A graph is *connected* if there is a path between any pair of vertices. A *connected component* of G is a maximal connected subgraph of G. A *forest* is a graph that contains no cycles and a *tree* is a forest that is connected.

The *Subset Feedback Vertex Set* (SFVS) problem is defined as follows: given a weighted graph G and a vertex set $S \subseteq V$, find a vertex set $X \subset V$ of minimum weight such that every cycle that contains a vertex of S also contains a vertex of X. An instance of SFVS is denoted by (G, S). In the unweighted version of the problem all weights are equal. A vertex set X is a *minimal* subset feedback vertex set of (G, S) if no proper subset of X is a subset feedback vertex set of (G, S). Thus, a *minimum weight* subset feedback vertex set is dependent on the weights of the vertices, whereas a *minimal* subset feedback vertex set is only dependent on the vertices themselves. Note, however, that a minimum subset feedback vertex set must be minimal in the unweighted as well as the weighted version of the problem, since all weights are positive.

A cycle of G is called an *S-cycle* if a vertex of S is contained in the cycle. We define an *S-forest* of G to be a vertex set $Y \subseteq V$ such that no cycle in $G[Y]$ is an S-cycle. An S-forest Y is *maximal* if no proper superset of Y is an S-forest. Observe that X is a minimal subset feedback vertex set of (G, S) if and only if $Y = V \setminus X$ is a maximal S-forest of G. Thus, the problem of computing a minimum weighted subset feedback vertex set of (G, S) is equivalent to the problem of computing a maximum weighted S-forest of G. Let us denote by $\mathcal{F}_S$ the class of S-forests of G. In such terms, given the graph G and the subset S of V, we are interested in finding a $\max_w \{Y \subseteq V : G[Y] \in \mathcal{F}_S\}$, where $\max_w$ selects a vertex set among the ones of maximum weight. It is not difficult to see that for any clique C of G, an S-forest of G that contains a vertex of $C \cap S$ contains at most two vertices of C.

Before reaching the details of our algorithms, let us also explain that we are interested in computing a minimum weighted subset feedback vertex set on connected graphs with $n \leq m$. If G is disconnected then we simply construct the union of the subsolutions taken over all its connected components. Moreover, if a connected component Q of G is a tree then there is no cycle (and, thus, no S-cycle) that passes through a vertex of Q, which means that no vertex of Q belongs to an optimal solution. Thus, we assume throughout the remaining part that the input graph G is connected and $n \leq m$. We note that such restrictions are not required for our described algorithms, rather than for the analysis of their running times.

2.1 Maximal S-forests of co-bipartite graphs

Here we show that the number of minimal solutions of a co-bipartite graph is polynomial, which implies a polynomial-time algorithm for the Subset Feedback Vertex Set problem on the class of co-bipartite graphs.

Theorem 2.1. *The number of maximal S-forests of a co-bipartite graph is at most $22n^4$ and these can be enumerated in time $O(n^4)$.*

Proof. Let $G = (V, E)$ be a co-bipartite graph and let (A, B) be a partition of V such that such that $G[A]$ and $G[B]$ are cliques. We further partition V as (A_S, A_R, B_S, B_R) where $A_S = A \cap S$, $A_R = A \setminus S$, $B_S = B \cap S$, and $B_R = B \setminus S$. Let (X, Y, Z, W) be the partition of the vertex set of a maximal S-forest of G such that $X \subseteq A_S$, $Y \subseteq A_R$, $Z \subseteq B_S$ and $W \subseteq B_R$

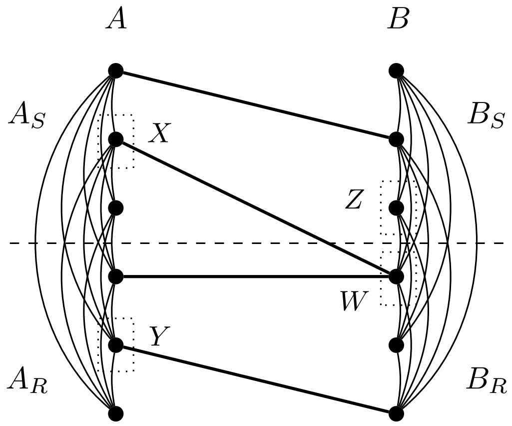


Figure 2: Illustrating the partition (X, Y, Z, W) of a maximal S-forest of a co-bipartite graph.

(see Figure 2). The key observation here is that since for any clique C of G, an S-forest of G that contains a vertex of $C \cap S$ contains at most two vertices of C, we have $|X| \leq 2$ and $|Z| \leq 2$. By examining the nine cases corresponding to the combinations of $|X|$ and $|Z|$, we can show that there are at most $22n^4$ maximal S-forests. In order to keep the main results in focus, we have moved the resulting exhaustive list of subcases in Appendix A. Taking into account that any maximal S-forest has at most n vertices, these arguments can be applied to obtain an enumeration algorithm that runs in time $O(n^4)$. $\square$

3 Computing SFVS on interval graphs

Here we present a polynomial-time algorithm for the SFVS problem on interval graphs. A graph is an *interval graph* if there is a bijection between its vertices and a family of closed intervals of the real line such that two vertices are adjacent if and only if the two corresponding intervals intersect. Such a bijection is called an *interval representation* of the graph, denoted by $\mathcal{I}$. We identify the intervals of the given representation with the vertices of the graph, interchanging these appropriately. Whether a given graph is an interval graph can be decided in linear time and if so, an interval representation can be generated in linear time [18]. Notice that every induced subgraph of an interval graph is an interval graph. Moreover, it is known that any induced cycle of an interval graph is a triangle [28, 33].

As already mentioned, instead of finding a subset feedback vertex set X of (G, S) of minimum weight we concentrate on the equivalent problem of finding a maximum weighted S-forest Y of G. We first define the necessary vertex sets. Let G be a weighted interval graph and let $\mathcal{I}$ be an interval representation of G. The vertices of G are considered to be equivalent to their corresponding intervals of $\mathcal{I}$, which are numbered from 1 to n in ascending order of their right endpoints. The left and right endpoints of an interval $i \in \mathcal{I}$ are denoted by $\ell(i)$ and $r(i)$ respectively and every endpoint can be assumed to be distinct from all others and positive. For technical reasons, we add an interval numbered 0 with $\ell(0) = -1$ and $r(0) = 0$ that does not belong to S and has zero weight, thus augmenting $\mathcal{I}$ to $\mathcal{I}^+$. Notice that interval 0 is non-adjacent to all other intervals of $\mathcal{I}^+$. Clearly, if Y is a maximum weighted S-forest of $G[\mathcal{I}^+]$, then $Y \setminus \{0\}$ is a maximum weighted S-forest of $G[\mathcal{I}]$. To simplify notations, hereafter we assume that G is an interval graph that contains the interval 0 and corresponds to $\mathcal{I}^+$.

We consider the two relations on V that are defined by the endpoints of the intervals as follows: $i \leq_\ell j \Leftrightarrow \ell(i) \leq \ell(j)$ and $i \leq_r j \Leftrightarrow r(i) \leq r(j)$. Since $\leq$ is a total order on the real numbers, we get that $\leq_\ell$ and $\leq_r$ are total orders on V. For a set of vertices $U \subseteq V$, we write $\ell\text{-}\min U$ to denote the *minimum* vertex of U with respect to $\leq_\ell$ and we write

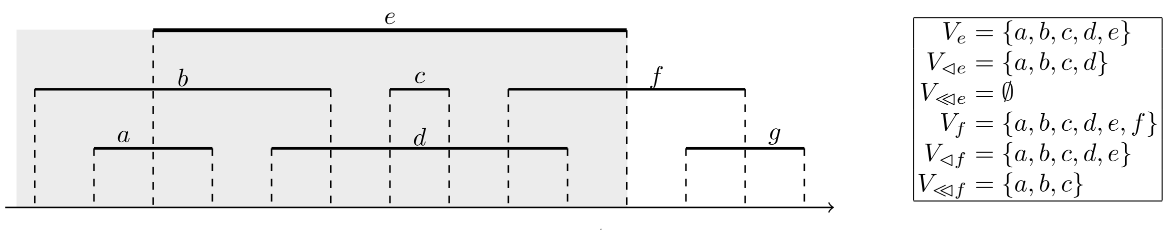


Figure 3: An interval graph given by its interval representation along with its corresponding sets V_e and V_f and their related subsets. Observe that $\lhd f = e$ whereas $\lll f = c$. Also notice that the intervals that are properly contained within the gray area form the set V_e.

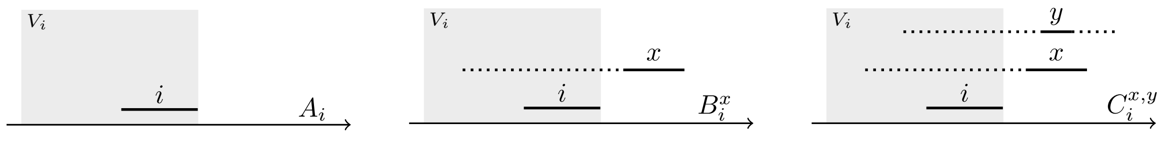


Figure 4: Illustrating the three sets A_i, B_i^x, and $C_i^{x,y}$ defined for interval graphs. Intervals x and y can be extended on their dotted parts, as long as $\ell(x) < \ell(y)$.

r-$\max U$ to denote the *maximum* vertex of U with respect to $\leq_r$. For a vertex $i \in V$, we let $V_i := \{h \in V : h \leq_r i\}$. Then observe that for two vertices $i, x \in V$, we have $i <_r x$ if and only if $x \in V \setminus V_i$.

We define two different types of predecessors of the interval i with respect to $\leq_r$, which correspond to the subproblems that our dynamic programming algorithm wants to solve. These are

$$\lhd i := r\text{-}\max(V_i \setminus \{i\}) \text{ and}$$

$$\lll i := r\text{-}\max(V_i \setminus (\{i\} \cup N(i))).$$

Intuitively, if we consider time increasing from left to right, then $\lhd i$ is the last interval that ends before i ends and $\lll i$ is the last interval that ends before i begins. An example of an interval graph given by an interval representation that depicts the defined vertex subset and predecessor notation is shown in Figure 3. By definition, we get the following partitions of V_i and $V_{<i}$.

Observation 3.1. *Let $i \in V \setminus \{0\}$ and let $j \in V \setminus V_{<i}$ such that $ij \in E$. Then,*

(1) $V_i = V_{\lhd i} \cup \{i\}$ and

(2) $V_{\lhd i} = V_{\lll j} \cup (N(j) \cap V_{<i})$.

Proof. The first statement follows by the definitions of V_i and $\lhd i$. For the second statement, observe that $V_{\lhd i}$ can be partitioned into the non-neighbors of j in $V_{\lhd i}$ and the neighbors of j in $V_{\lhd i}$. The first set corresponds to $V_{\lll j}$, whereas the second set is exactly the set $N(j) \cap V_{\lhd i}$. $\square$

Next, we define the sets that our dynamic programming algorithm uses in order to compute the S-forest of G that has maximum weight.

***A*-sets** Let $i \in V$. Then,

$$A_i := \max_w \{Y \subseteq V_i : G[Y] \in \mathcal{F}_S\}.$$

B-sets Let $i \in V$ and let $x \in V \setminus V_i$. Then,

$$B_i^x := \max_w \{Y \subseteq V_i : G[Y \cup \{x\}] \in \mathcal{F}_S\}.$$

C-sets Let $i \in V$ and let $x, y \in V \setminus (V_i \cup S)$ such that $x <_\ell y$ and $xy \in E$. Then,

$$C_i^{x,y} := \max_w \{Y \subseteq V_i : G[Y \cup \{x, y\}] \in \mathcal{F}_S\}.$$

Observe that A_i corresponds to an optimal S-forest of the graph $G[V_i]$. Moreover, B_i^x corresponds to an optimal S-forest of the graph $G[V_i \cup \{x\}]$ such that x belongs to the optimal S-forest and $C_i^{x,y}$ corresponds to an optimal S-forest of the graph $G[V_i \cup \{x, y\}]$ such that both x and y belong to the optimal S-forest (see Figure 4). Since $V_0 = \{0\}$ and $w(0) = 0$, we have $A_0 = \emptyset$ and, since $V_n = V$, we have $A_n = \max_w\{Y \subseteq V : G[Y] \in \mathcal{F}_S\}$. The following lemmas state how to recursively compute all A-sets, B-sets, and C-sets besides A_0.

Lemma 3.2. *Let $i \in V \setminus \{0\}$. Then $A_i = \max_w \left\{A_{\lhd i}, B_{\lhd i}^i \cup \{i\}\right\}$.*

Proof. By Observation 3.1 (1), $V_i = V_{\lhd i} \cup \{i\}$. If $i \notin A_i$, then we get $A_i = A_{\lhd i}$; otherwise, $A_i = B_{\lhd i}^i \cup \{i\}$ holds, since $B_{\lhd i}^i$ is the $\max_w$ subset of $V_{\lhd i}$ such that the graph induced by its union with $\{i\}$ contains no S-cycle by definition. $\square$

To simplify the proofs in the forthcoming lemmas, we use the following observation.

Observation 3.3. *Let $i, x, y \in V$ such that (i) $i <_r \{x, y\}$, (ii) $x <_\ell y$ and (iii) $iy, xy \in E$. Then $\langle i, x, y\rangle$ is a triangle of G.*

Proof. Assuming $r(i) < \ell(y)$ results in non-adjacent vertices i and y, a contradiction to $iy \in E$, so we have $\ell(y) < r(i)$. This inequality along with inequalities (i) and (ii) imply $\ell(x) < r(i) < r(x)$, which means that $ix \in E$. Therefore, $\langle i, x, y\rangle$ is a triangle of G. $\square$

We next describe the set B_i^x. Here we need to distinguish between i and x, the interval with the smallest left endpoint, as the definition of B_i^x allows for both cases. For that reason, we introduce x' and y'.

Lemma 3.4. *Let $i \in V$ and let $x \in V \setminus V_i$. Moreover, let $x' = \ell\text{-}\min\{i, x\}$ and let y' be the remaining vertex of $\{i, x\}$.*

(1) If $ix \notin E$, then $B_i^x = A_i$.

(2) If $ix \in E$, then $B_i^x = \begin{cases} \max_w \left\{B_{\lhd i}^x, B_{\lll y'}^{x'} \cup \{i\}\right\}, & \text{if } i \in S \text{ or } x \in S \\ \max_w \left\{B_{\lhd i}^x, C_{<i}^{x',y'} \cup \{i\}\right\}, & \text{if } i, x \notin S. \end{cases}$

Proof. Assume first that $ix \notin E$. Then $r(i) < \ell(x)$, because we already have $i <_r x$, so x has no neighbor in $G[V_i \cup \{x\}]$. Thus no subset of $V_i \cup \{x\}$ containing x induces an S-cycle of G, implying that $B_i^x = A_i$.

Next assume that $ix \in E$. If $i \notin B_i^x$, then, according to Observation 3.1 (1), it follows that $B_i^x = B_{\lhd i}^x$. So let us assume in what follows that $i \in B_i^x$. Observe that $B_i^x \setminus \{i\} \subseteq V_{\lhd i}$ by Observation 3.1 (1). We distinguish two cases according to whether i and x belong to S.

- Let $i \in S$ or $x \in S$. If there is a vertex $h \in B_i^x \setminus \{i\}$ such that $hy' \in E$ then, by Observation 3.3, $\langle h, x', y'\rangle$ is an S-triangle of G. Thus, for any vertex $h \in B_i^x \setminus \{i\}$, we have that $hy' \notin E$. By Observation 3.1 (2), notice that $B_i^x \setminus \{i\} \subseteq V_{\lll y'}$. Also observe that the neighborhood of y' in $G[V_{\lll y'} \cup \{x', y'\}]$ is $\{x'\}$. Thus, no subset of $V_{\lll y'} \cup \{x', y'\}$ containing y' induces an S-cycle of G. Therefore, $B_i^x = B_{\lll y'}^{x'} \cup \{i\}$.

- Let $i, x \notin S$. Since $V_i = V_{\lhd i} \cup \{i\}$ and $x' <_\ell y'$, we get $B_i^x = C_{\lhd i}^{x',y'} \cup \{i\}$.

Therefore, in every case, we obtain the desired equation. $\square$

We next inductively describe the set $C_i^{x,y}$. As before, we use x' and y' to denote the two leftmost intervals among i, x, and y.

Lemma 3.5. *Let $i \in V$ and let $x, y \in V \setminus (V_i \cup S)$ such that $x <_\ell y$ and $xy \in E$. Moreover, let $x' = \ell\text{-}\min\{i, x\}$ and let $y' = \ell\text{-}\min(\{i, x, y\} \setminus \{x'\})$.*

1. *If $iy \notin E$, then $C_i^{x,y} = B_i^x$.*
2. *If $iy \in E$, then* $C_i^{x,y} = \begin{cases} C_{\lhd i}^{x,y} & \text{, if } i \in S \\ \max_w \left\{ C_{\lhd i}^{x,y}, C_{\lhd i}^{x',y'} \cup \{i\} \right\}, & \text{if } i \notin S. \end{cases}$

Proof. Assume first that $iy \notin E$. Then $r(i) < \ell(y)$, because we already have $i <_r y$, so the neighborhood of y in $G[V_i \cup \{x, y\}]$ is $\{x\}$. Thus, no subset of $V_i \cup \{x, y\}$ containing y induces an S-cycle of G. By the relevant definitions, it follows that $C_i^{x,y} = B_i^x$.

Assume next that $iy \in E$. If $i \notin C_i^{x,y}$, then, by Observation 3.1 (1), we have $C_i^{x,y} = C_{\lhd i}^{x,y}$. Suppose that $i \in C_i^{x,y}$. If $i \in S$ then, by Observation 3.3, $\langle i, x, y \rangle$ is an S-triangle of G, a contradiction to $i \in C_i^{x,y}$. In what follows, we will assume that $i \notin S$ and we will show that $C_i^{x,y} = C_{\lhd i}^{x',y'} \cup \{i\}$.

By definition, $C_i^{x,y} \setminus \{i\}$ and $C_{\lhd i}^{x',y'}$ are subsets of $V_{\lhd i}$ of maximum weight such that their union with $\{i, x, y\}$ and $\{x', y'\}$, respectively, induce an S-forest of G. Let z' be the vertex of $\{i, x, y\} \setminus \{x', y'\}$. Observe that no vertex of x', y', z' belongs to S by the definitions of x, y and the hypothesis for i. Let Y be a subset of $V_{\lhd i}$ such that $Y \cup \{x', y'\}$ induces an S-forest of G. We show that $Y \cup \{x', y', z'\}$ induces an S-forest of G. Assume for contradiction that a subset of $Y \cup \{x', y', z'\}$ induces an S-triangle $\langle v_1, v_2, z' \rangle$ of G. Since $z' \notin S$, without loss of generality, assume that $v_1 \in S$. This particularly means that $v_1 \in Y$, because $x', y' \notin S$ as well. Notice that $v_1 z' \in E$ implies that $\ell(z') < r(v_1)$. By the fact that $v_1 \in V_{\lhd i}$, we have $v_1 <_r \{x', y', z'\}$. Also, recall that $x' <_\ell y' <_\ell z'$. Put together, the previous inequalities imply that $v_1 x', v_1 y' \in E$. Thus, $\langle v_1, x', y' \rangle$ is an S-triangle of G, leading to a contradiction that $Y \cup \{x', y'\}$ induces an S-forest. Moreover, given a subset Y of $V_{\lhd i}$ such that $Y \cup \{x', y', z'\}$ induces an S-forest of G, it is clear that $Y \cup \{x', y'\}$ induces an S-forest as well. Therefore, $C_i^{x,y} = C_{\lhd i}^{x',y'} \cup \{i\}$ as desired. $\square$

Now we are equipped with the necessary tools to obtain the main result of this section, namely a polynomial-time algorithm for SFVS on interval graphs.

Theorem 3.6. Subset Feedback Vertex Set *can be solved in $O(mn)$ time on interval graphs.*

Proof. We briefly describe such an algorithm based on Lemmas 3.2, 3.4, and 3.5. In a preprocessing step, we compute $\lhd i$ and $\lll i$ for all intervals $i \in V \setminus \{0\}$. We visit all intervals from 0 to n in ascending order with respect to $<_r$. For every interval i that we visit, we first compute A_i according to Lemma 3.2 and, then, compute B_i^x and $C_i^{x,y}$ for every x, y such that $i <_r \{x, y\}$, $x <_\ell y$, and $xy \in E$ according to Lemmas 3.4 and 3.5, respectively. We output $V \setminus A_n$, as already explained. The correctness of the algorithm follows from the aforementioned Lemmas.

Regarding its running time, recall that $n \leq m$. Computing $\lhd i$ and $\lll i$ can be done in $O(m)$ time, because the intervals are sorted with respect to $<_r$. The computation of a single A-set, B-set or C-set takes constant time. Moreover, for each i, the number of B_i^x and $C_i^{x,y}$ are at most $n + m$, based on the appropriate x and y. Therefore, the overall running time of the algorithm is $O(mn)$. $\square$

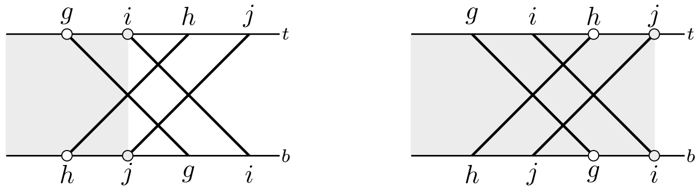


Figure 5: Illustrating the two partial orders $\leq_\ell$ and $\leq_r$ among crossing pairs.

4 Computing SFVS on permutation graphs

Let $\pi = \pi(1), \ldots, \pi(n)$ be a permutation over $\{1, \ldots, n\}$. The position of an integer i in π is denoted by $\pi^{-1}(i)$. Given a permutation π, the *inversion graph* of π, denoted by $G(\pi)$, has vertex set $\{1, \ldots, n\}$ and two vertices i, j are adjacent if $(i - j)(\pi(i) - \pi(j)) < 0$. A graph is a *permutation graph* if it is isomorphic to the inversion graph of a permutation [7, 22]. Permutation graphs are the intersection graphs of segments between two horizontal parallel lines, that is, there is a one-to-one mapping from the segments onto the vertices of a graph such that there is an edge between two vertices of the graph if and only if their corresponding segments intersect. We refer to the two horizontal lines as *top* and *bottom* lines. This representation is called a *permutation diagram* and a graph is a permutation graph if and only if it has a permutation diagram. Whether a given graph is a permutation graph can be decided in linear time and if so, its permutation diagram can be constructed in linear time [30]. Note that every induced subgraph of a permutation graph is a permutation graph. It is also important to note that any induced cycle of a permutation graph is either an triangle or a square [4, 5, 6, 26, 33].

We assume that we are given a permutation graph $G = (V, E)$ such that $G = G(\pi)$ along with $S \subseteq V$ and a weight function $w : V \to \mathbb{R}^+$ as input. We add an isolated vertex 0 in G and augment π to π^+ so that $\pi^+(0) = 0$. Further, we assign zero value for 0's weight and assume that $0 \notin S$. To simplify notations, hereafter we assume that G is a permutation graph that contains the isolated vertex 0 and corresponds to π^+.

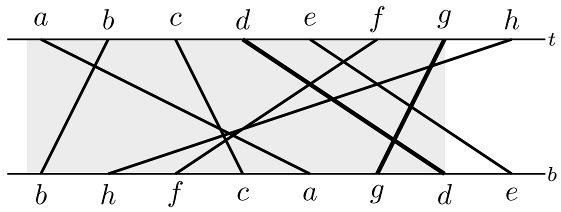
The orderings of the segments' endpoints on the top and bottom lines of the permutation diagram induce two total orders on the vertices of G which we denote by $\leq_t$ and $\leq_b$, respectively. That is, $i \leq_t j \Leftrightarrow i \leq j$ and $i \leq_b j \Leftrightarrow \pi^{-1}(i) \leq \pi^{-1}(j)$ for all $i, j \in V$.

Two vertices $i, j \in V$ are called *crossing pair*, denoted by ij, if $i \leq_t j$ and $j \leq_b i$. We denote by $\mathcal{X}$ the set of all crossing pairs in G. Let $\mathcal{I} = \{ii \in \mathcal{X} : i \in V\}$. It is clear that for any edge $ij \in E$ either $ij \in \mathcal{X} \setminus \mathcal{I}$ or $ji \in \mathcal{X} \setminus \mathcal{I}$. Given two crossing pairs $gh, ij \in \mathcal{X}$, we define two partial orders $\leq_\ell$ and $\leq_r$:

$$gh \leq_\ell ij \quad \Leftrightarrow \quad g \leq_t i \text{ and } h \leq_b j \qquad \text{and} \qquad gh \leq_r ij \quad \Leftrightarrow \quad g \leq_b i \text{ and } h \leq_t j.$$

Intuitively, each crossing pair consists of its leftmost endpoints and its rightmost endpoints. Thus $\leq_\ell$ corresponds to the ordering with respect to the leftmost endpoints, whereas $\leq_r$ corresponds to the ordering with respect to the rightmost endpoints (Figure 5 illustrates the two partial orders). We stress that we write $<_\ell$ and $<_r$ to denote that the inequalities concerning the top and bottom endpoints are both strict. Given a vertex set $X \subseteq V$, we denote by $\mathcal{X}[X]$ the set of all crossing pairs of $G[X]$ under the same permutation diagram. It is not difficult to see that the *minimum* crossing pair of $\mathcal{X}[X]$ with respect to $\leq_\ell$ and the *maximum* crossing pair of $\mathcal{X}[X]$ with respect to $\leq_r$ are both well defined; we write $\ell\text{-}\min$ and $r\text{-}\max$ to denote them respectively.

We next define the predecessors of a crossing pair with respect to $\leq_r$, which correspond to the subproblems that our dynamic programming algorithm wants to solve. Let $ij \in \mathcal{X}$. We define the set of vertices that induces the subproblem that we consider at the crossing



$$\begin{array}{|r|}\hline \overline{\lhd} dg = df \\ \unlhd dg = gg \\ \lhd dg = af \\ \lhd\!\!\lhd dg = ac \\ dg \cdot hh = bb \\ \hline \end{array} \qquad \begin{array}{|l|}\hline V_{dg} = \{a,b,c,d,f,g\} \\ V_{\overline{\lhd} dg} = \{a,b,c,d,f\} \\ V_{\unlhd dg} = \{a,b,c,f,g\} \\ V_{\lhd dg} = \{a,b,c,f\} \\ V_{\lhd\!\!\lhd dg} = \{a,b,c\} \\ \hline \end{array}$$

Figure 6: A permutation graph given by its permutation diagram and the set V_{dg} of the crossing pair dg together with the corresponding predecessors of dg. Observe that the line segments that are properly contained within the gray area form the set V_{dg}. Moreover, regarding the ordering with respect to $\leq_r$ among the given predecessors of dg, notice that $bb \leq_r ac \leq_r af \leq_r \{gg, df\}$.

pair ij to be $V_{ij} := \{h \in V : hh \leq_r ij\}$. This implies that for any $x \in V$, we have $x \notin V_{ij}$ if and only if $i <_b x$ or $j <_t x$. The predecessors of the crossing pair ij are defined as follows:

(i) $\overline{\lhd} ij := r\text{-}\max \mathcal{X}[V_{ij} \setminus \{j\}]$,

(ii) $\unlhd ij := r\text{-}\max \mathcal{X}[V_{ij} \setminus \{i\}]$,

(iii) $\lhd ij := r\text{-}\max \mathcal{X}[V_{ij} \setminus \{i, j\}]$ and

(iv) $\lhd\!\!\lhd ij := r\text{-}\max \mathcal{X}[V_{ij} \setminus (\{i, j\} \cup N(\{i, j\}))]$.

Those are precisely the greatest predecessors of ij with respect to $<_r$ having no member being (i) j, (ii) i, (iii) either i or j and (iv) either i or j or any of their neighbors. We also define *the product* of two crossing pairs gh and ij,

$$gh \cdot ij := r\text{-}\max \mathcal{X}[V_{gh} \cap V_{ij}],$$

which is the greatest common predecessor of gh and ij with respect to $<_r$. An example of a permutation graph that illustrates the defined predecessors is given in Figure 6. Having defined the above predecessors of ij, it is not difficult to show how V_{ij} can be partitioned into smaller sets of vertices.

Observation 4.1. *Let $ij \in \mathcal{X}$ and let $x \in V \setminus V_{ij}$. Then,*

(1) $V_{ij} = V_{\overline{\lhd} ij} \cup \{j\} = V_{\unlhd ij} \cup \{i\} = V_{\lhd ij} \cup \{i, j\}$,

(2) $V_{\lhd ij} = V_{\lhd\!\!\lhd jj} \cup (N(j) \cap V_{\lhd ij}) = V_{\lhd\!\!\lhd ii} \cup (N(i) \cap V_{\lhd ij})$,

(3) $V_{\lhd\!\!\lhd ii} = V_{\lhd\!\!\lhd ij} \cup (N(j) \cap V_{\lhd\!\!\lhd ii})$,

(4) $V_{\lhd\!\!\lhd jj} = V_{\lhd\!\!\lhd ij} \cup (N(j) \cap V_{\lhd\!\!\lhd ii})$*, and*

(5) $V_{\lhd ij} = V_{\lhd ij \lhd\!\!\lhd xx} \cup (N(x) \cap V_{\lhd ij})$.

Proof. Let ij_1 be the predecessor $\overline{\lhd} ij$. By the r-max choice of j_1, there is no vertex j' such that $j_1 <_t j' <_t j$. Thus $V_{ij_1} \cup \{j\}$ is the set V_{ij}. The rest of the equalities in the first statement follow in a similar way.

Let $i_1 j_1$ be the predecessor $\lhd\!\!\lhd jj$. Then both i_1 and j_1 are non-adjacent to j and have the maximum values such that $i_1 <_b j$ and $j_1 <_t j$, respectively. This particularly means that $i_1 <_t j_1 <_t j$ and $j_1 <_b i_1 <_b j$. Thus any vertex $i' \in V_{ij} \setminus \{i, j\}$ with $j_1 <_t i' <_t j$ or $i_1 <_b i' <_b j$ must be adjacent to j which implies that $V_{\lhd ij} \setminus V_{\lhd\!\!\lhd jj}$ contains exactly the neighbors of j in $V_{\lhd ij}$. These arguments imply the second, third, and fourth statements.

For the last statement, notice that $V_{\lhd ij}$ can be partitioned into the neighbors and the non-neighbors of x. By definition, $V_{\lhd ij \lll xx}$ contains the non-neighbors of x so that every vertex of $V_{\lhd ij} \setminus V_{\lhd ij \lll xx}$ is adjacent to x. $\square$

Our dynamic programming algorithm relies on sets similar to the ones we used for the case of interval graphs. That is, we need to describe appropriate sets of the considered subproblems that are extended to candidate solutions of certain extended subproblems. Although, for interval graphs we showed that adding two vertices into such sets is sufficient, the equivalent for permutation graphs is that we consider at most two newly crossing pairs which correspond to at most four newly vertices. This is implied by the following lemma.

Lemma 4.2. *Let $gh \in \mathcal{X}$ and let $ab \in \mathcal{X} \setminus \mathcal{I}$ such that $gh <_r ab$ and $a, b \notin S$. Moreover, let $cd \in \mathcal{X}$ such that $\begin{cases} gh <_r cd, & \text{if } cd \notin \mathcal{I} \\ g <_b c \text{ or } h <_t d, & \text{if } cd \in \mathcal{I} \end{cases}$, $ab <_\ell cd$, and $c, d \notin S$, and let $ef \in \mathcal{X}$ such that $\begin{cases} gh <_r ef, & \text{if } ef \notin \mathcal{I} \\ g <_b e \text{ or } h <_t f, & \text{if } ef \in \mathcal{I} \end{cases}$, $cd <_\ell ef$, and $e, f \notin S$. Then, for all $Y \subseteq V_{gh}$, the following are equivalent:*

(i) $G[Y \cup \{a, b, c, d, e, f\}] \in \mathcal{F}_S$ *(ii)* $G[Y \cup \{a, b, c, d, e\}] \in \mathcal{F}_S$
(iii) $G[Y \cup \{a, b, c, d, f\}] \in \mathcal{F}_S$ *(iv)* $G[Y \cup \{a, b, c, d\}] \in \mathcal{F}_S$

Proof. In the context of this proof, we will consider statement (i) only if $ef \notin \mathcal{I}$, statement (ii) only if $ef \notin \mathcal{I}$ or $ef \in \mathcal{I}$ such that $g <_b e$, and statement (iii) only if $ef \notin \mathcal{I}$ or $ef \in \mathcal{I}$ such that $h <_b f$. Notice that this is sufficient, because (i), (ii), and (iii) are equivalent whenever $ef \in \mathcal{I}$. We next show all directions among the four statements.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These facts are trivial because an induced subgraph of an S-forest of G is also an S-forest of G.

(iv) $\Rightarrow$ (ii). Assume for contradiction that a subset of $Y \cup \{a, b, c, d, e\}$ containing e induces an S-cycle of G. Since an induced cycle of a permutation graph can only be a triangle or a square, we assume that that S-cycle is an S-triangle or an S-square.

- Let $\langle v_1, v_2, e\rangle$ be an S-triangle. Since $e \notin S$, without loss of generality, assume that $v_1 \in S$. Then $v_1 \in Y \subseteq V_{gh}$ because a, b, c, d are also not in S, which along with $gh <_r ab$, $\begin{cases} gh <_r cd, & \text{if } cd \notin \mathcal{I} \\ g <_b c, & \text{if } cd \in \mathcal{I} \end{cases}$ and $\begin{cases} gh <_r ef, & \text{if } ef \notin \mathcal{I} \\ g <_b e, & \text{if } ef \in \mathcal{I} \end{cases}$ imply $v_1 \leq_t h <_t b$ and $v_1 \leq_b g <_b \{a, c, e\}$. Since $v_1 e \in E$ and we already know that $v_1 <_b e$, we get $e <_t v_1$. By $ab <_\ell cd <_\ell ef$, we have that $a <_t c <_t e$ and $b <_b d \leq_b c$. Putting it all together gives $a <_t c <_t v_1 <_t b$ and $\{v_1, b\} <_b \{a, c\}$, which in particular show that $v_1 a, v_1 c, bc \in E$.
  - If $v_1 b \in E$, then $\langle v_1, a, b\rangle$ is an S-triangle of G.
  - If $ac \in E$, then $\langle v_1, a, c\rangle$ is an S-triangle of G.
  - If $v_1 b, ac \notin E$, then $\langle v_1, a, b, c\rangle$ is an S-square of G.
- Let $\langle v_1, v_2, v_3, e\rangle$ be an S-square. By the previous arguments, if $v_1 \in S$ or $v_3 \in S$, then we obtain an S-cycle in G induced by vertices of $Y \cup \{a, b, c, d\}$. So, let us assume that $v_2 \in S$. Since $a, b, c, d \notin S$, we have $v_2 \in Y \subseteq V_{gh}$, which along with $gh <_r ab$ gives $v_2 \leq_t h <_t b$ and $v_2 \leq_b g <_b a$. By $ab <_\ell cd <_\ell ef$, we have $a <_t e$ and $b <_b f \leq_b e$. Moreover, as $\langle v_1, v_2, v_3, e\rangle$ is a square, we conclude that either $\{v_1, v_3\} <_t \{v_2, e\}$ and $\{v_2, e\} <_b \{v_1, v_3\}$, or $\{v_2, e\} <_t \{v_1, v_3\}$ and $\{v_1, v_3\} <_b \{v_2, e\}$. Assume the former. Then $\{v_1, v_3\} <_t b$ and $b <_b \{v_1, v_3\}$, so that $v_1 b, v_3 b \in E$.
  - If $v_2 b \in E$, then $\langle v_1, v_2, b\rangle$ is an S-triangle of G.

- If $v_2b \notin E$, then $\langle v_1, v_2, v_3, b\rangle$ is an S-square of G.

Assume the latter. Then $a <_t \{v_1, v_3\}$ and $\{v_1, v_3\} <_b a$, so that $v_1a, v_3a \in E$.

- If $v_2a \in E$, then $\langle v_1, v_2, a\rangle$ is an S-triangle of G.
- If $v_2a \notin E$, then $\langle v_1, v_2, v_3, a\rangle$ is an S-square of G.

Thus, in all cases we obtain an S-cycle of G induced by a subset of $Y \cup \{a, b, c, d\}$, resulting in a contradiction to (iv). Therefore, no subset of $Y \cup \{a, b, c, d, e\}$ induces an S-cycle of G.

(iv) $\Rightarrow$ (iii). Symmetrical arguments to the previous case show this direction.

(ii) $\Rightarrow$ (i). Assume for contradiction that a subset of $Y \cup \{a, b, c, d, e, f\}$ containing f induces an S-cycle of G. The S-cycle is either an S-triangle or an S-square.

- Let $\langle v_1, v_2, f\rangle$ be an S-triangle. Since $f \notin S$, without loss of generality, assume that $v_1 \in S$. Then $v_1 \in Y \subseteq V_{gh}$ because a, b, c, d, e are also not in S. By $v_1 \in V_{gh}$ and $gh <_r \{ab, ef\}$, we get $v_1 \leq_t h <_t \{b, d, f\}$ and $v_1 \leq_b g <_b a$. Since $v_1f \in E$ and we already know that $v_1 <_t f$, we get that $f <_b v_1$. By $ab <_\ell cd <_\ell ef$, we have that $a <_t c <_t d$ and $b <_b d <_b f$. Putting it all together gives $\{v_1, a\} <_t \{b, d\}$ and $b <_b d <_b v_1 <_b a$, which in particular show that $v_1b, v_1d, ad \in E$.
  - If $v_1a \in E$, then $\langle v_1, a, b\rangle$ is an S-triangle of G.
  - If $bd \in E$, then $\langle v_1, b, d\rangle$ is an S-triangle of G.
  - If $v_1a, bd \notin E$, then $\langle v_1, b, a, d\rangle$ is an S-square of G.
- Let $\langle v_1, v_2, v_3, f\rangle$ be an S-square. By the previous arguments, if $v_1 \in S$ or $v_3 \in S$, then we obtain an S-cycle in G induced by vertices of $Y \cup \{a, b, c, d, e\}$. Assume that $v_2 \in S$. Since $a, b, c, d, e \notin S$, we have $v_2 \in Y \subseteq V_{gh}$, which along with $gh <_r ab$ gives $v_2 \leq_t h <_t b$ and $v_2 \leq_b g <_b a$. By $ab <_\ell cd <_\ell ef$, we have that $a <_t e <_t f$ and $b <_b f$. Moreover, as $\langle v_1, v_2, v_3, f\rangle$ is a square, we conclude that either $\{v_1, v_3\} <_t \{v_2, f\}$ and $\{v_2, f\} <_b \{v_1, v_3\}$, or $\{v_2, f\} <_t \{v_1, v_3\}$ and $\{v_1, v_3\} <_b \{v_2, f\}$. Assume the former. Then $\{v_1, v_3\} <_t b$ and $b <_b \{v_1, v_3\}$, so that $v_1b, v_3b \in E$.
  - If $v_2b \in E$, then $\langle v_1, v_2, b\rangle$ is an S-triangle of G.
  - If $v_2b \notin E$, then $\langle v_1, v_2, v_3, b\rangle$ is an S-square of G.

  Assume the latter. Then $a <_t \{v_1, v_3\}$ and $\{v_1, v_3\} <_b a$, so that $v_1a, v_3a \in E$.
  - If $v_2a \in E$, then $\langle v_1, v_2, a\rangle$ is an S-triangle of G.
  - If $v_2a \notin E$, then $\langle v_1, v_2, v_3, a\rangle$ is an S-square of G.

Thus, in all cases we obtain an S-cycle of G induced by a subset of $Y \cup \{a, b, c, d, e\}$, resulting in a contradiction to (ii). Therefore, no subset of $Y \cup \{a, b, c, d, e, f\}$ induces an S-cycle of G.

(iii) $\Rightarrow$ (i). Symmetrical arguments to the previous case show this direction.

Notice that all other directions ((ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii), and (iv) $\Rightarrow$ (i)), follow from the previous cases. Therefore, all four statements are equivalent, as desired. □

We now introduce the A-sets, B-sets and C-sets for this section.

***A*-sets** Let $ij \in \mathcal{X}$. Then,

$$A_{ij} = \max_w \{Y \subseteq V_{ij} : G[Y] \in \mathcal{F}_S\}.$$

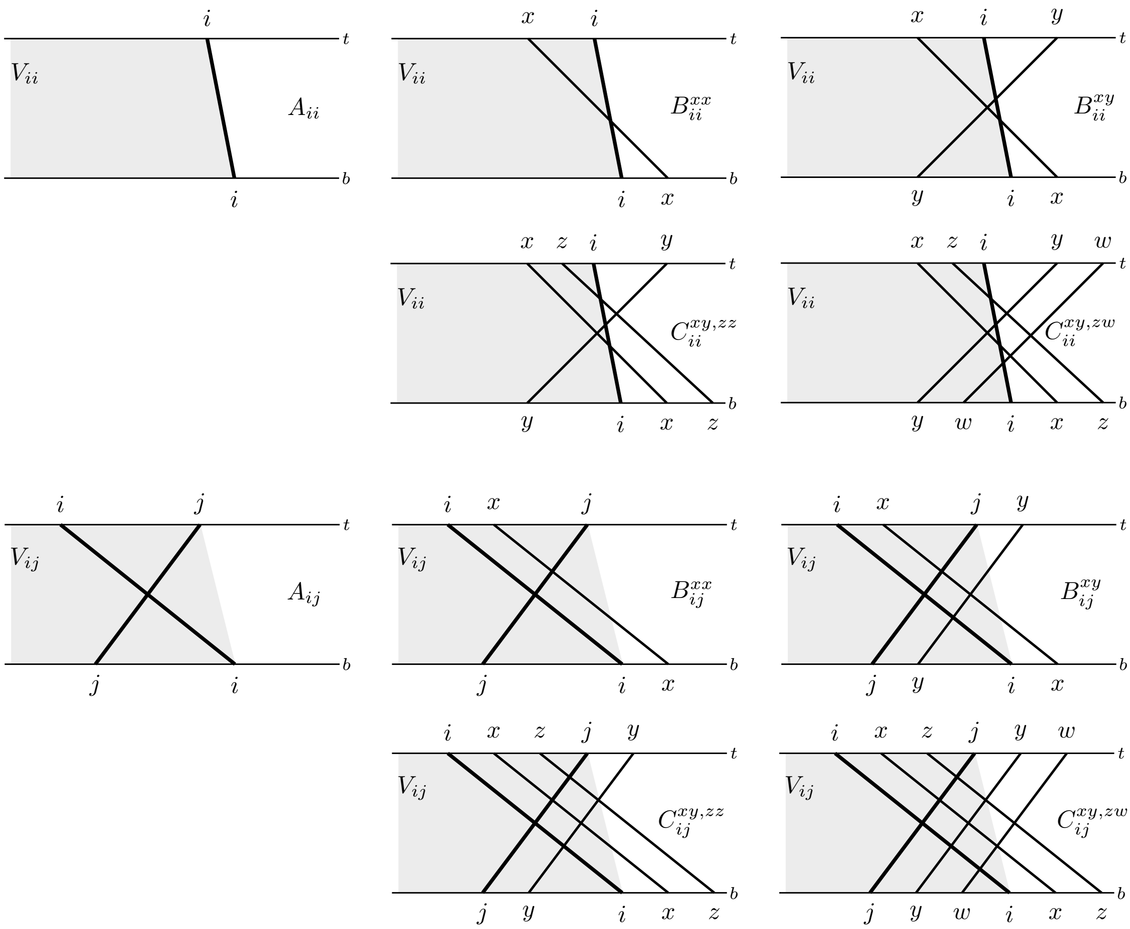


Figure 7: Illustrating the three sets A_{ij}, B_{ij}^{xy}, and $C_{ij}^{xy,zw}$ defined for permutation graphs. The upper and lower parts show the sets for crossing pairs $ii \in \mathcal{I}$ and $ij \in \mathcal{X} \setminus \mathcal{I}$, respectively. Each figure depicts its respective set whenever $\{uv \in E : u, v \in \{i, j, x, y, z, w\}\}$ constitutes the set $\{uv : u \in \{i, x, z\}$ and $v \in \{j, y, w\}\}$.

B**-sets** Let $ij \in \mathcal{X}$ and let $x \in V \setminus V_{ij}$. Then,

$$B_{ij}^{xx} := \max_w \{Y \subseteq V_{ij} : G[Y \cup \{x\}] \in \mathcal{F}_S\}.$$

Moreover, let $xy \in \mathcal{X} \setminus \mathcal{I}$ such that $ij <_r xy$ and $x, y \notin S$. Then,

$$B_{ij}^{xy} := \max_w \{Y \subseteq V_{ij} : G[Y \cup \{x, y\}] \in \mathcal{F}_S\}.$$

C**-sets** Let $ij \in \mathcal{X}$, $xy \in \mathcal{X} \setminus \mathcal{I}$, and let $z \in V \setminus (V_{ij} \cup S)$ such that $ij <_r xy$, $j <_t z$ or $i <_b z$, $xy <_\ell zz$, and $x, y \notin S$. Then,

$$C_{ij}^{xy,zz} := \max_w \{Y \subseteq V_{ij} : G[Y \cup \{x, y, z\}] \in \mathcal{F}_S\}.$$

Moreover, let $zw \in \mathcal{X} \setminus \mathcal{I}$ such that $ij <_r zw$, $xy <_\ell zw$ and $z, w \notin S$. Then,

$$C_{ij}^{xy,zw} := \max_w \{Y \subseteq V_{ij} : G[Y \cup \{x, y, z, w\}] \in \mathcal{F}_S\}.$$

The corresponding sets are shown in Figure 7. Observe that, since $V_{00} = \{0\}$ and $w(0) = 0$, we have $A_{00} = \emptyset$ and, since $V_{\pi(n)n} = V$, we have $A_{\pi(n)n} = \max_w \{X \subseteq V : G[X] \in \mathcal{F}_S\}$. The following lemmas state how to recursively compute all A-sets, B-sets, and C-sets, other than A_{00}. We first consider the crossing pairs ii for the sets A_{ii}, B_{ii}^{xx}, B_{ii}^{xy}, $C_{ii}^{xy,zz}$, and $C_{ii}^{xy,zw}$.

Lemma 4.3. *Let $i \in V \setminus \{0\}$. Then $A_{ii} = A_{\triangleleft ii} \cup \{i\}$.*

Proof. Notice that i has no neighbor in $G[V_{ii}]$ and, thus, no subset of V_{ii} containing i induces an S-cycle of G. Therefore, if Y is a subset of $V_{\triangleleft ii}$ such that $G[Y] \in \mathcal{F}_S$, then $G[Y \cup \{i\}] \in \mathcal{F}_S$ as well. By Observation 4.1 (1), we get $A_{ii} = A_{\triangleleft ii} \cup \{i\}$. □

Lemma 4.4. *Let $i \in V$ and let $x \in V \setminus V_{ii}$.*

1. *If $ix \notin E$, then $B_{ii}^{xx} = A_{ii}$.*

2. *If $ix \in E$, then $B_{ii}^{xx} = B_{\triangleleft ii}^{xx} \cup \{i\}$.*

Proof. First assume that $ix \notin E$. Since $x \notin V_{ii}$, we know that $i <_t x$ or $i <_b x$. As $ix \notin E$, we have $i <_t x$ and $i <_b x$, so x has no neighbor in $G[V_{ii} \cup \{x\}]$. Thus, no subset of $V_{ii} \cup \{x\}$ containing x induces an S-cycle of G. Hence, $B_{ii}^{xx} = A_{ii}$ follows.

Next assume that $ix \in E$. Then the neighbourhood of i in $G[V_{ii} \cup \{x\}]$ is $\{x\}$. This means that no subset of $V_{ii} \cup \{x\}$ containing i induces an S-cycle of G. By Observation 4.1 (1), it follows that $B_{ii}^{xx} = B_{\triangleleft ii}^{xx} \cup \{i\}$. □

Lemma 4.5. *Let $i \in V$ and let $xy \in \mathcal{X} \setminus \mathcal{I}$ such that $ii <_r xy$ and $x, y \notin S$.*

1. *If $iy \notin E$, then $B_{ii}^{xy} = B_{ii}^{xx}$.*

2. *If $ix \notin E$, then $B_{ii}^{xy} = B_{ii}^{yy}$.*

3. *If $ix, iy \in E$, then* $B_{ii}^{xy} = \begin{cases} B_{\triangleleft ii}^{xy} &, \text{ if } i \in S \\ B_{\triangleleft ii}^{xy} \cup \{i\}, & \text{if } i \notin S. \end{cases}$

Proof. By $xy \in \mathcal{X} \setminus \mathcal{I}$ and $ii <_r xy$, we have $\{i, x\} <_t y$ and $\{i, y\} <_b x$. First assume that i is non-adjacent to at least one of x and y. Let $iy \notin E$. Then $\{i, x\} <_t y$ and $i <_b y <_b x$, so that the neighborhood of y in $G[V_{ii} \cup \{x, y\}]$ is $\{x\}$. Thus, no subset of $V_{ii} \cup \{x, y\}$ containing y induces an S-cycle of G, which implies that $B_{ii}^{xy} = B_{ii}^{xx}$. If $ix \notin E$, completely symmetrical arguments apply in showing the second statement.

Next assume that $ix, iy \in E$. Then $x <_t i <_t y$ and $y <_b i <_b x$, so that the neighborhood of i in $G[X \cup \{x, y\}]$ is $\{x, y\}$. We distinguish two cases according to whether i is in S. Suppose that $i \in S$. Then $\langle i, x, y\rangle$ is an S-triangle of G, so that $i \notin B_{ii}^{xy}$. By this fact and Observation 4.1 (1), we get that $B_{ii}^{xy} = B_{\triangleleft ii}^{xy}$.

Next suppose that $i \notin S$. We will show that no subset of $V_{ii} \cup \{x, y\}$ that contains i induces an S-cycle of G. Recall that an induced cycle of a permutation graph is either a triangle or a square. Assume that $\langle v_1, v_2, i\rangle$ is an S-triangle of G induced by a subset of $V_{ii} \cup \{x, y\}$. Since the neighborhood of i in $G[V_{ii} \cup \{x, y\}]$ is $\{x, y\}$, we obtain that $\{v_1, v_2\} = \{x, y\}$, which implies a contradiction because $i, x, y \notin S$. Next assume that $\langle v_1, v_2, v_3, i\rangle$ is an S-square of G induced by a subset of $V_{ii} \cup \{x, y\}$. As before, we obtain that $\{v_1, v_3\} = \{x, y\}$. This, however, implies a contradiction, since $xy \in E$ and $v_1 v_3 \notin E$ by the S-square. Therefore, no subset of $V_{ii} \cup \{x, y\}$ containing i induces an S-cycle of G, so that $i \in B_{ii}^{xy}$. By this fact and Observation 4.1 (1), we get that $B_{ii}^{xy} = B_{\triangleleft ii}^{xy} \cup \{i\}$ and this completes the proof. □

Lemma 4.6. *Let $i \in V$, $xy \in \mathcal{X} \setminus \mathcal{I}$, and let $z \in V \setminus (V_{ii} \cup S)$ such that $ii <_r xy$, $i <_t z$ or $i <_b z$, $xy <_\ell zz$, and $x, y \notin S$.*

1. *If $iz \notin E$, then $C_{ii}^{xy,zz} = B_{ii}^{xy}$.*

2. *If $iz \in E$, then* $C_{ii}^{xy,zz} = \begin{cases} C_{\triangleleft ii}^{xy,zz}, & \text{if } i \in S \\ C_{\triangleleft ii}^{xy,zz} \cup \{i\}, & \text{if } i \notin S. \end{cases}$

Proof. First assume that $iz \notin E$. This fact along with $z \notin V_{ii}$ gives $i <_t z$ and $i <_b z$, which implies that z is non-adjacent to any vertex of V_{ii}, so the neighborhood of z in $G[V_{ii} \cup \{x, y, z\}]$ is a subset of $\{x, y\}$. Since $x, y, z \notin S$ and $xy \in E$, no subset of $V_{ii} \cup \{x, y, z\}$ containing z induces an S-cycle of G. Thus $C_{ii}^{xy,zz} = B_{ii}^{xy}$.

Next assume that $iz \in E$. Then, either $i <_t z$ and $z <_b i$ hold, or $z <_t i$ and $i <_b z$ hold. Since $xy \in \mathcal{X} \setminus \mathcal{I}$ such that $ii <_r xy$ and $xy <_l zz$, we have $\{i, x\} <_t y$ and $\{i, y\} <_b x$ as well as $x <_t z$ and $y <_b z$. Putting together, we get either $\{i, x\} <_t \{y, z\}$ and $y <_b z <_b i <_b x$, or $x <_t z <_t i <_t y$ and $\{i, y\} <_b \{x, z\}$. In particular, the former implies that $iy, xz \in E$ and the latter implies that $ix, yz \in E$. We distinguish two cases depending on whether i is in S.

- Let $i \in S$. We will show that $i \notin C_{ii}^{xy,zz}$. For the sake of contradiction, let $i \in C_{ii}^{xy,zz}$. If both x and y are adjacent to i, then $\langle i, x, y \rangle$ is an S-triangle of G. If x is non-adjacent to i, then $iy, xz \in E$ must hold. If we additionally have $yz \in E$, then $\langle i, y, z \rangle$ is an S-triangle of G; otherwise, $\langle i, y, x, z \rangle$ is an S-square of G. The case for y being non-adjacent to i is completely symmetric. Therefore, we obtain an S-cycle of G induced by a subset of $\{i, x, y, z\}$, which is a contradiction, so that $i \notin C_{ii}^{xy,zz}$. By this fact and Observation 4.1 (1), we get $C_{ii}^{xy,zz} = C_{\lhd ii}^{xy,zz}$.

- Let $i \notin S$. Let $Y \subseteq V_{\lhd ii}$ such that $G[Y \cup \{x, y, z\}] \in \mathcal{F}_S$. We will show that $G[Y \cup \{i, x, y, z\}] \in \mathcal{F}_S$. Assume for contradiction that there is an S-triangle $\langle v_1, v_2, i \rangle$ or an S-square $\langle v_1, v_2, v_3, i \rangle$ of G induced by a subset of $Y \cup \{i, x, y, z\}$. Then, $\{v_1, v_2\} \subset \{x, y, z\}$ or $\{v_1, v_3\} \subset \{x, y, z\}$, respectively, since i is non-adjacent to any vertex of Y. In the former case, we have a contradiction because $i, x, y, z \notin S$. In the latter case, notice that v_2 must be in S and, consequently, in Y. If $\{v_1, v_3\} = \{x, y\}$, then we reach a contradiction to the S-square $\langle v_1, v_2, v_3, i \rangle$ because $xy \in E$. Thus, either $\{v_1, v_3\} = \{y, z\}$, or $\{v_1, v_3\} = \{x, z\}$. Without loss of generality, assume that $\{v_1, v_3\} = \{y, z\}$. Then, $yz \notin E$ holds due to the square $\langle v_1, v_2, v_3, i \rangle$ and, consequently, $iy, xz \in E$ also hold. This, however, implies that either $\langle y, v_2, x \rangle$ is an S-triangle or $\langle y, v_2, z, x \rangle$ is an S-square of G depending on whether x is adjacent to v_2, which is a contradiction to $v_2 \in Y$. Therefore, we obtain a contradiction in all cases, which means that no S-cycle of G is induced by a subset of $Y \cup \{i, x, y, z\}$ containing i. By Observation 4.1 (1), we get $C_{ii}^{xy,zz} = C_{\lhd ii}^{xy,zz} \cup \{i\}$.

In each case, we have shown the described equations and this completes the proof. $\square$

Lemma 4.7. *Let $i \in V$ and let $xy, zw \in \mathcal{X} \setminus \mathcal{I}$ such that $ii <_r \{xy, zw\}$, $xy <_\ell zw$, and $x, y, z, w \notin S$.*

1. *If $iw \notin E$, then $C_{ii}^{xy,zw} = C_{ii}^{xy,zz}$.*
2. *If $iz \notin E$, then $C_{ii}^{xy,zw} = C_{ii}^{xy,ww}$.*
3. *If $iz, iw \in E$, then* $C_{ii}^{xy,zw} = \begin{cases} C_{\lhd ii}^{xy,zw}, & \text{if } i \in S \\ C_{\lhd ii}^{xy,zw} \cup \{i\}, & \text{if } i \notin S. \end{cases}$

Proof. First assume that $iw \notin E$. Since $ii <_r zw$, we get $i <_t w$ and $i <_b w$, which implies that w has no neighbor in $G[V_{ii}]$. Thus, the neighborhood of w in $G[V_{ii} \cup \{x, y, z, w\}]$ is a subset of $\{x, y, z\}$. Let $Y \subseteq V_{ii}$ such that $G[Y \cup \{x, y, z\}] \in \mathcal{F}_S$. We will show that $G[Y \cup \{x, y, z, w\}] \in \mathcal{F}_S$. Assume for contradiction that a subset of $Y \cup \{x, y, z, w\}$ containing w induces an S-cycle of G. Such an S-cycle is either an S-triangle or an S-square. If $\langle v_1, v_2, w \rangle$ is an S-triangle, then $\{v_1, v_2\} \subset \{x, y, z\}$, which is a contradiction because $x, y, z, w \notin S$. Suppose that $\langle v_1, v_2, v_3, w \rangle$ is an S-square. Then $\{v_1, v_3\} \subset \{x, y, z\}$ and, since $x, y, z, w \notin S$, we get that v_2 is in S and, thus, in $Y \subseteq V_{ii}$.

- Assuming that $\{v_1, v_3\} = \{x, y\}$ results in a contradiction to the square $\langle v_1, v_2, v_3, w\rangle$ because $xy \in E$.

- Assume that $\{v_1, v_3\} = \{x, z\}$. If $xz \in E$, then $\langle v_1, v_2, v_3, w\rangle$ is not a square, which is a contradiction, so let $xz \notin E$. This fact, along with $xy <_\ell zw$, gives $x <_t z$ and $y <_b x <_b z$. By the square and $ii <_r xy, zw$, we get $\{x, z\} <_t v_2 \leq_t i <_t y$ and $v_2 \leq_b i <_b \{x, z\}$. The previous inequalities imply that $yz \in E$. Then, either $\langle x, v_2, y\rangle$ is an S-triangle or $\langle x, v_2, z, y\rangle$ is an S-square of G depending on whether y is adjacent to v_2, which is a contradiction to $v_2 \in Y$.

- Assume that $\{v_1, v_3\} = \{y, z\}$. If $yz \in E$, then $\langle v_1, v_2, v_3, w\rangle$ is not a square, which is a contradiction. If $yz \notin E$, then, by this fact and $ij <_r \{xy, zw\}$, we get $v_2 \leq_t j <_t y <_t z$ and $v_2 \leq_b i <_b w <_b z$, which imply that $v_2 z \notin E$, resulting in a contradiction to the square $\langle v_1, v_2, v_3, w\rangle$.

Thus, no subset of $Y \cup \{x, y, z, w\}$ induces an S-cycle of G. Therefore, $C_{ii}^{xy,zw} = C_{ii}^{xy,zz}$. The case for $iz \notin E$ is completely symmetric in showing the second statement.

Now assume that $iz, iw \in E$. Together with the inequalities implied by $iz, iw \in E$, $ii <_r \{xy, zw\}$, and $xy <_\ell zw$, give $x <_t z <_t i <_t \{y, w\}$ and $y <_b w <_b i <_b \{x, z\}$. Thus, the neighborhood of i in $G[V_{ii} \cup \{x, y, z, w\}]$ is $\{x, y, z, w\}$. Assume that $i \in S$. Then $\langle i, x, y\rangle$ is an S-triangle of G, which implies that $i \notin C_{ii}^{xy,zw}$. By this fact and Observation 4.1 (1), we get $C_{ii}^{xy,zw} = C_{\lhd ii}^{xy,zw}$. Now let us assume that $i \notin S$. Let $Y \subseteq V_{\lhd ii}$ such that $G[Y \cup \{x, y, z, w\}] \in \mathcal{F}_S$. We will show that $G[Y \cup \{i, x, y, z, w\}] \in \mathcal{F}_S$. Assume for contradiction that a subset of $Y \cup \{i, x, y, z, w\}$ containing i induces an S-cycle of G.

- Let $\langle v_1, v_2, i\rangle$ be an S-triangle of G induced by a subset of $Y \cup \{i, x, y, z, w\}$. Then $\{v_1, v_2\} \subset \{x, y, z, w\}$, which yields a contradiction because $i, x, y, z, w \notin S$.

- Let $\langle v_1, v_2, v_3, i\rangle$ be an S-square of G induced by a subset of $Y \cup \{i, x, y, z, w\}$. Then $\{v_1, v_3\} \subset \{x, y, z, w\}$. Since $i, x, y, z, w \notin S$, we get $v_2 \in S$ and, thus, $v_2 \in Y$. By the square, v_1 and v_3 are non-adjacent, so we have either $\{v_1, v_3\} = \{x, z\}$ or $\{v_1, v_3\} = \{y, w\}$. If $\{v_1, v_3\} = \{x, z\}$, then either $\langle x, v_2, y\rangle$ is an S-triangle or $\langle x, v_2, z, y\rangle$ is an S-square of G depending on whether y is adjacent to v_2, which is a contradiction to $v_2 \in Y$. The case for $\{v_1, v_3\} = \{y, w\}$ is completely symmetric.

Thus, no subset of $Y \cup \{i, x, y, z, w\}$ induces an S-cycle of G. Therefore, by Observation 4.1 (1), we obtain $C_{ii}^{xy,zw} = C_{\lhd ii}^{xy,zw} \cup \{i\}$. □

Based on Lemmas 4.3–4.7, for each crossing pair of the form ii we can describe its subsolution by using appropriate formulations of the A-, B-, or C-sets. In the forthcoming lemmas we give the recursive formulations for the sets A_{ij}, B_{ij}^{xx}, B_{ij}^{xy}, $C_{ij}^{xy,zz}$, and $C_{ij}^{xy,zw}$ whenever $ij \in \mathcal{X} \setminus \mathcal{I}$ which particularly means that i and j are distinct vertices in G.

Lemma 4.8. *Let $ij \in \mathcal{X} \setminus \mathcal{I}$. Then,*

$$A_{ij} = \begin{cases} \max_w \left\{ A_{\bar{\lhd} ij}, A_{\unlhd ij}, B_{\lll jj}^{ii} \cup \{i, j\}, B_{\lll ii}^{jj} \cup \{i, j\} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ A_{\bar{\lhd} ij}, A_{\unlhd ij}, B_{\lhd ij}^{ij} \cup \{i, j\} \right\}, & \text{if } i, j \notin S. \end{cases}$$

Proof. Let $j \notin A_{ij}$. Then by Observation 4.1 (1) it follows that $A_{ij} = A_{\bar{\lhd} ij}$. Similarly, if $i \notin A_{ij}$ then $A_{ij} = A_{\unlhd ij}$. For the rest of the proof, we assume that $i, j \in A_{ij}$. We distinguish two cases according to whether i or j belong to S.

- Assume that $i, j \notin S$. By the definition of $B_{\lhd ij}^{ij}$, we get $A_{ij} = B_{\lhd ij}^{ij} \cup \{i, j\}$, which completes the second description in the formula.

- Assume that $i \in S$ or $j \in S$. We show that all the vertices of $A_{ij} \setminus \{i,j\}$ are non-adjacent to i or non-adjacent to j. Let $h \in A_{ij} \setminus \{i,j\}$. If $hi, hj \in E$, then $\langle h,i,j\rangle$ is an S-triangle in G, resulting in a contradiction to $h,i,j \in A_{ij}$. Thus for every $h \in A_{ij} \setminus \{i,j\}$ we know that $hi \notin E$ or $hj \notin E$. Let $g,h \in A_{ij} \setminus \{i,j\}$ such that $gj, hi \in E$ and $gi, hj \notin E$. Observe that $\{g,h\} <_b i$ and $\{g,h\} <_t j$, as $g,h \in V_{\lhd ij}$. It follows that $j <_b g$ and $i <_t h$, so that $gj, hi \in E$. If $i <_t g$ or $j <_b h$, then $gi \in E$ or $hj \in E$, which contradict our assumption, so $g <_t i$ and $h <_b j$. Thus, $g <_t i <_t h$ and $h <_b j <_b g$ hold, which in turn imply that $gh \in E$. Hence, $\langle g,h,i,j\rangle$ is an S-square in G, leading to a contradiction to $g,h,i,j \in A_{ij}$. Thus, if a vertex of $A_{ij} \setminus \{i,j\}$ is adjacent to i (resp. to j), then all its vertices are non-adjacent to j (resp. to i). By Observation 4.1 (2), it follows that either $A_{ij} \setminus \{i,j\} \subseteq V_{\lll jj}$ or $A_{ij} \setminus \{i,j\} \subseteq V_{\lll ii}$.

  Suppose that the former holds. Observe that the neighborhood of j in $G[V_{\lll jj} \cup \{i,j\}]$ is $\{i\}$. Thus, no subset of $V_{\lll jj} \cup \{i,j\}$ containing j induces an S-cycle of G. This means that $A_{ij} = B^{ii}_{\lll jj} \cup \{i,j\}$, as described in the first case of the given formula. Symmetrically, if $A_{ij} \setminus \{i,j\} \subseteq V_{\lll ii}$ then we have $A_{ij} = B^{jj}_{\lll ii} \cup \{i,j\}$.

Therefore, the corresponding formulas given in the statement follow. □

With the next two lemmas we describe recursively the sets B^{xx}_{ij} and B^{xy}_{ij}.

Lemma 4.9. *Let $ij \in \mathcal{X} \setminus \mathcal{I}$ and let $x \in V \setminus V_{ij}$. Moreover let $x'y' = \ell\text{-}\min \mathcal{X}[\{i,j,x\}]$ and let z' be the remaining vertex of $\{i,j,x\}$.*

1. *If $ix, jx \notin E$ then $B^{xx}_{ij} = A_{ij}$.*
2. *If $ix \in E$ and $jx \notin E$ then*

$$B^{xx}_{ij} = \begin{cases} \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}, B^{ii}_{\lll jj} \cup \{i,j\}, B^{jj}_{\lll ix} \cup \{i,j\}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}, B^{ij}_{\lhd ij \lll xx} \cup \{i,j\}\right\}, & \text{if } i,j \notin S, x \in S \\ \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}, C^{x'y',z'z'}_{\lhd ij} \cup \{i,j\}\right\}, & \text{if } i,j,x \notin S. \end{cases}$$

3. *If $ix \notin E$ and $jx \in E$ then*

$$B^{xx}_{ij} = \begin{cases} \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}, B^{ii}_{\lll xj} \cup \{i,j\}, B^{jj}_{\lll ii} \cup \{i,j\}\right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}, B^{ij}_{\lhd ij \lll xx} \cup \{i,j\}\right\}, & \text{if } i,j \notin S, x \in S \\ \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}, C^{x'y',z'z'}_{\lhd ij} \cup \{i,j\}\right\}, & \text{if } i,j,x \notin S. \end{cases}$$

4. *If $ix, jx \in E$ then*

$$B^{xx}_{ij} = \begin{cases} \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}\right\}, & \text{if } i \in S \text{ or } j \in S \text{ or } x \in S \\ \max_w \left\{B^{xx}_{\lhd ij}, B^{xx}_{\unlhd ij}, C^{x'y',z'z'}_{\lhd ij} \cup \{i,j\}\right\}, & \text{if } i,j,x \notin S. \end{cases}$$

Proof. Let us assume first that $ix, jx \notin E$. Since $i <_t j$, $j <_b i$, and $x \in V \setminus V_{ij}$, we know that $i <_t j <_t x$ and $j <_b i <_b x$. Thus, x has no neighbor in $G[V_{ij} \cup \{x\}]$. Hence, no subset of $V_{ij} \cup \{x\}$ containing x induces an S-cycle of G and it follows that $B^{xx}_{ij} = A_{ij}$ as described in the first statement.

Assume next that $ix \in E$ or $jx \in E$. Let $j \notin B^{xx}_{ij}$. By Observation 4.1 (1), we get $B^{xx}_{ij} = B^{xx}_{\lhd ij}$. Similarly, if $i \notin B^{xx}_{ij}$ then $B^{xx}_{ij} = B^{xx}_{\unlhd ij}$. So, suppose next that $i,j \in B^{xx}_{ij}$. We distinguish the following cases, according to whether ix and jx belong to E.

- Assume that $ix \in E$ and $jx \notin E$. Since $x \notin V_{ij}$, $j <_t x$ or $i <_b x$. If $i <_b x$, then $x <_t i$ as $ix \in E$, so we have $x <_t i <_t j$ and $j <_b i <_b x$. However, this means that $jx \in E$ which contradicts our assumption. Thus $j <_t x$ holds. Since $ix \in E$ and $jx \notin E$, we have $i <_t j <_t x$ and $j <_b x <_b i$. We further reduce to subcases depending on whether i, j, x belong to S.
  - Let $i \in S$ or $j \in S$. We show that all the vertices of $B_{ij}^{xx} \setminus \{i, j\}$ are non-adjacent to i or non-adjacent to j. Let $h \in B_{ij}^{xx} \setminus \{i, j\}$ such that $hi, hj \in E$. Then $\langle h, i, j\rangle$ is an S-triangle of G, a contradiction. So, $hi \notin E$ or $hj \notin E$ for every $h \in B_{ij}^{xx} \setminus \{i, j\}$. Let $g, h \in B_{ij}^{xx} \setminus \{i, j\}$ such that $gj, hi \in E$. Since $\{g, h\} <_b i$ and $\{g, h\} <_t j$ by the fact that $g, h \in V_{\lhd ij}$, it follows that $g <_t i <_t h$ and $h <_b j <_b g$. Thus $gh \in E$. This, however, results in an S-square $\langle g, h, i, j\rangle$ of G, leading to a contradiction to $g, h, i, j \in B_{ij}^{xx} \setminus \{i, j\}$. By Observation 4.1 (2), it follows that either $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\lll jj}$ or $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\lll ii}$.
    In the former case, notice that both j and x in $G[V_{\lll jj} \cup \{i, j, x\}]$ are adjacent only to i. Thus no subset of $V_{\lll jj} \cup \{i, j, x\}$ that contains j or x induces an S-cycle of G so that $B_{ij}^{xx} = B_{\lll jj}^{ii} \cup \{i, j\}$, as described.
    In the latter case we have $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\lll ii}$. Let $h \in B_{ij}^{xx} \setminus \{i, j\}$. Observe that $h <_t i$ and $h <_b i$. We show that $hx \notin E$. Assume for contradiction that $hx \in E$. This means that either $h <_t x$ and $x <_b h$, or $x <_t h$ and $h <_b x$. Observe that $h <_t j$ and $h <_b i$. Since $i <_t j <_t x$ and $j <_b x <_b i$, we get $h <_t i <_t j <_t x$ and $j <_b x <_b h <_b i$. Thus $hj \in E$. This, however, shows that $\langle h, j, i, x\rangle$ is an S-square of G, a contradiction, so $hx \notin E$ for every $h \in B_{ij}^{xx} \setminus \{i, j\}$. By Observation 4.1 (3), it follows that $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\lll ix}$. Notice that i and x are adjacent only to j in $G[V_{\lll ix} \cup \{i, j, x\}]$. Hence, no subset of $V_{\lll ix} \cup \{i, j, x\}$ that contains i or x induces an S-cycle in G, so that $B_{ij}^{xx} = B_{\lll ix}^{jj} \cup \{i, j\}$, as described.
  - Let $i, j \notin S$ and $x \in S$. Let $h \in B_{ij}^{xx} \setminus \{i, j\}$. Observe that $h <_t j$ and $h <_b i$. We show that $hx \notin E$. Assume for contradiction that $hx \in E$. Then either $h <_t x$ and $x <_b h$ hold, or $x <_t h$ and $h <_b x$ hold. Since $i <_t j <_t x$ and $j <_b x <_b i$, we have $\{h, i\} <_t j <_t x$ and $j <_b x <_b h <_b i$, implying that $hj \in E$. If $hi \in E$, then $\langle h, i, x\rangle$ is an S-triangle, whereas if $hi \notin E$, then $\langle h, j, i, x\rangle$ is an S-square. Thus, we reach a contradiction, so $hx \notin E$ for every $h \in B_{ij}^{xx} \setminus \{i, j\}$. Notice that the neighborhood of x in $G[V_{\lhd ij \lll xx} \cup \{i, j, x\}]$ is $\{i\}$, so no subset of $V_{\lhd ij \lll xx} \cup \{i, j, x\}$ containing x induces an S-cycle of G. By Observation 4.1 (5), we get $B_{ij}^{xx} = B_{\lhd ij \lll xx}^{ij} \cup \{i, j\}$, as described.
  - Let $i, j, x \notin S$. By the fact $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\lhd ij}$, we have $B_{ij}^{xx} = C_{\lhd ij}^{x'y', z'z'} \cup \{i, j\}$.
- Assume that $ix \notin E$ and $jx \in E$. This case is symmetric to the one above, which proves the stated formulas.
- Assume that both $ix, jx \in E$. Then no vertex of $\{i, x, y\}$ belong to S, as $\langle i, j, x\rangle$ is a triangle in G and $i, j \in B_{ij}^{xx}$. Since $B_{ij}^{xx} \setminus \{i, j\} \subseteq V_{\lhd ij}$, we get $B_{ij}^{xx} = C_{\lhd ij}^{x'y', z'z'} \cup \{i, j\}$.

Therefore, we obtain the described formulas in all cases. $\square$

Let $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ such that $ij <_r xy$. For the crossing pair $uv = \ell\text{-}\min \mathcal{X}[\{i, j, x, y\}]$, observe that $u \in \{i, x\}$ and $v \in \{j, y\}$. This means that $uv \in \mathcal{X} \setminus \mathcal{I}$. Such an observation reassures that the crossing pair $x'y'$ defined in the following three lemmas belongs to $\mathcal{X} \setminus \mathcal{I}$.

Lemma 4.10. *Let $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ such that $ij <_r xy$ and $x, y \notin S$. Moreover, if $iy, jx \in E$ then let $x'y' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y\}]$ and let $z'w' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y\} \setminus \{x', y'\}]$.*

1. *If $iy \notin E$, then $B_{ij}^{xy} = B_{ij}^{xx}$.*

2. *If $jx \notin E$, then $B_{ij}^{xy} = B_{ij}^{yy}$.*

3. *If $iy, jx \in E$, then*

$$B_{ij}^{xy} = \begin{cases} \max_w \left\{ B_{\overline{\lhd} ij}^{xy}, B_{\underline{\lhd} ij}^{xy} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ B_{\overline{\lhd} ij}^{xy}, B_{\underline{\lhd} ij}^{xy}, C_{\lhd ij}^{x'y',z'w'} \cup \{i,j\} \right\}, & \text{if } i,j \notin S. \end{cases}$$

Proof. Assume that $iy \notin E$. Then $j <_b i <_b y$, because $i <_t j <_t y$, so that the neighborhood of y in $G[V_{ij} \cup \{x,y\}]$ is $\{x\}$. Thus no subset of $V_{ij} \cup \{x,y\}$ that contains y induces an S-cycle in G. Therefore $B_{ij}^{xy} = B_{ij}^{xx}$ as described. If $jx \notin E$ then i is non-adjacent to x and, similar to the previous case, we obtain $B_{ij}^{xy} = B_{ij}^{yy}$.

Assume that $iy, jx \in E$. We distinguish cases depending on whether i or j belong to the solution. Assume first that at least one of i or j does not belong to B_{ij}^{xy}. If $j \notin B_{ij}^{xy}$ then we have $B_{ij}^{xy} = B_{\overline{\lhd} ij}^{xy}$ by Observation 4.1 (1); if $i \notin B_{ij}^{xy}$ then we get $B_{ij}^{xy} = B_{\underline{\lhd} ij}^{xy}$.

Next assume that $i, j \in B_{ij}^{xy}$. Notice that by Observation 4.1 (1), we have $B_{ij}^{xy} \setminus \{i,j\} \subseteq V_{\lhd ij}$. Let us show that $i, j \notin S$. If $ix \in E$ or $jy \in E$ then $\langle i,x,y \rangle$ or $\langle j,x,y \rangle$ is a triangle in G, since $iy, jx \in E$. Otherwise, $ix, jy \notin E$, so that $\langle i,j,x,y \rangle$ is a square in G. Thus, if $i \in S$ or $j \in S$ then a subset of $\{i,j,x,y\}$ induces an S-cycle in G, which is a contradiction to $i, j \in B_{ij}^{xy}$. Hence $i, j \notin S$. Since $B_{ij}^{xy} \setminus \{i,j\} \subseteq V_{\lhd ij}$, it follows $B_{ij}^{xy} = C_{\lhd ij}^{x'y',z'w'} \cup \{i,j\}$ as required. $\square$

Lemma 4.11. *Let $ij, xy \in \mathcal{X} \setminus \mathcal{I}$ and let $z \in V \setminus (V_{ij} \cup S)$ such that $ij <_r xy$, $xy <_\ell zz$, $j <_t z$ or $i <_b z$, and $x, y \notin S$. Moreover, if $iz \in E$ or $jz \in E$ then let $x'y' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y,z\}]$ and let $z'w' = \ell\text{-}\min \mathcal{X}[\{i,j,x,y,z\} \setminus \{x',y'\}]$.*

1. *If $iz, jz \notin E$ then $C_{ij}^{xy,zz} = B_{ij}^{xy}$.*

2. *If $iz \in E$ or $jz \in E$ then*

$$C_{ij}^{xy,zz} = \begin{cases} \max_w \left\{ C_{\overline{\lhd} ij}^{xy,zz}, C_{\underline{\lhd} ij}^{xy,zz} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ C_{\overline{\lhd} ij}^{xy,zz}, C_{\underline{\lhd} ij}^{xy,zz}, C_{\lhd ij}^{x'y',z'w'} \cup \{i,j\} \right\}, & \text{if } i,j \notin S. \end{cases}$$

Proof. First assume that $iz, jz \notin E$. Then $i <_t j <_t z$ and $j <_b i <_b z$. This means that the neighborhood of z in $G[V_{ij} \cup \{x,y,z\}]$ is a subset of $\{x,y\}$. We will show that no subset of $V_{ij} \cup \{x,y,z\}$ that contains z induces an S-cycle of G.

- Let $\langle v_1, v_2, z \rangle$ be an S-triangle induced by a subset of $V_{ij} \cup \{x,y,z\}$. Then $\{v_1, v_2\} = \{x,y\}$, which leads to a contradiction because $x, y, z \notin S$.

- Let $\langle v_1, v_2, v_3, z \rangle$ be an S-square induced by a subset of $V_{ij} \cup \{x,y,z\}$. Then $\{v_1, v_3\} = \{x,y\}$, which leads to a contradiction because $xy \in E$.

Thus, no subset of $V_{ij} \cup \{x,y,z\}$ containing z induces an S-cycle of G. Therefore, $C_{ij}^{xy,zz} = B_{ij}^{xy}$ holds.

Assume that $iz \in E$ or $jz \in E$. We show the described formula given in the second statement. We distinguish cases depending on whether i or j belong to $C_{ij}^{xy,zz}$. If $j \notin C_{ij}^{xy,zz}$ or $i \notin C_{ij}^{xy,zz}$ then by Observation 4.1 (1) we get $C_{ij}^{xy,zz} = C_{\overline{\lhd} ij}^{xy,zz}$ or $C_{ij}^{xy,zz} = C_{\underline{\lhd} ij}^{xy,zz}$, respectively. The remaining case is $i, j \in C_{ij}^{xy,zz}$. By Observation 4.1 (3), notice that $C_{ij}^{xy,zz} \setminus \{i,j\} \subseteq V_{\lhd ij}$.

Assume that $i \in S$ or $j \in S$. We will show that there is always an S-cycle of G induced by a subset of $\{i,j,x,y\}$ containing i and j, which is a contradiction to $i, j \in C_{ij}^{xy,zz}$.

- If $ix \in E$, then $\langle i, j, x \rangle$ is an S-triangle.
- If $jy \in E$, then $\langle i, j, y \rangle$ is an S-triangle.
- If $ix, jy \notin E$, then $\langle i, j, y, x \rangle$ is an induced S-square.

Thus, $i, j \notin S$. Let a' be the vertex of $\{i, j, x, y, z\} \setminus \{x', y', z', w'\}$. Observe that $a' \notin S$, since $S \cap \{i, j, x, y, z\} = \emptyset$. Applying Lemma 4.2 with $gh = \lhd ij$, $ab = x'y'$, $cd = z'w'$, and $ef = a'a'$, shows that for all $Y \subseteq V_{\lhd ij}$, $G[Y \cup \{x', y', z', w'\}] \in \mathcal{F}_S$ if and only if $G[Y \cup \{x', y', z', w', a'\}] \in \mathcal{F}_S$. This particularly implies that $C_{ij}^{xy,zz} = C_{\lhd ij}^{x'y',z'w'} \cup \{i, j\}$, as described in the second statement. □

The next lemma shows how to recursively compute $C_{ij}^{xy,zw}$, where $ij, xy, zw \in \mathcal{X} \setminus \mathcal{I}$. Note that in each case we describe $C_{ij}^{xy,zw}$ as a predefined smaller set of a subsolution that is either of the same form or has already been described in one of the previous lemmas.

Lemma 4.12. *Let $ij, xy, zw \in \mathcal{X} \setminus \mathcal{I}$ such that $ij <_r \{xy, zw\}$, $xy <_\ell zw$, and $x, y, z, w \notin S$. Moreover, if $iw, jz \in E$ then let $x'y' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y, z, w\}]$ and $z'w' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y, z, w\} \setminus \{x', y'\}]$.*

1. *If $iw \notin E$ then $C_{ij}^{xy,zw} = C_{ij}^{xy,zz}$.*
2. *If $jz \notin E$ then $C_{ij}^{xy,zw} = C_{ij}^{xy,ww}$.*
3. *If $iw, jz \in E$ then*

$$C_{ij}^{xy,zw} = \begin{cases} \max_w \left\{ C_{\bar{\lhd} ij}^{xy,zw}, C_{\underline{\lhd} ij}^{xy,zw} \right\}, & \text{if } i \in S \text{ or } j \in S \\ \max_w \left\{ C_{\bar{\lhd} ij}^{xy,zw}, C_{\underline{\lhd} ij}^{xy,zw}, C_{\lhd ij}^{x'y',z'w'} \cup \{i, j\} \right\}, & \text{if } i, j \notin S. \end{cases}$$

Proof. First assume that $iw \notin E$. Since $i <_t j <_t w$, we have $i <_b w$. Thus, the neighborhood of w in $G[V_{ij} \cup \{x, y, z, w\}]$ is a subset of $\{x, y, z\}$. Let $Y \subseteq V_{ij}$ such that $G[Y \cup \{x, y, z\}] \in \mathcal{F}_S$. We will show that $G[Y \cup \{x, y, z, w\}] \in \mathcal{F}_S$. Assume for contradiction that a subset of $Y \cup \{x, y, z, w\}$ containing w, induces an S-cycle. Since an S-cycle in G is either an S-triangle or an S-square, we distinguish the following two cases:

- Let $\langle v_1, v_2, w \rangle$ be an S-triangle such that $v_1, v_2 \in Y \cup \{x, y, z\}$. Then $\{v_1, v_2\} \subset \{x, y, z\}$, which leads to a contradiction because $x, y, z, w \notin S$.
- Let $\langle v_1, v_2, v_3, w \rangle$ be an S-square such that $v_1, v_2, v_3 \in Y \cup \{x, y, z\}$. Then $\{v_1, v_3\} \subset \{x, y, z\}$. Since $x, y, z, w \notin S$, we have $v_2 \in Y$ so that $v_2 \in S$.
  - Assume that $\{v_1, v_3\} = \{x, y\}$. Then we reach a contradiction because $xy, yz \in E$ and the S-square is induced.
  - Assume that $\{v_1, v_3\} = \{x, z\}$. If $xz \in E$ then we have a contradiction to the S-square, so $xz \notin E$. This fact along with $xy <_\ell zw$ gives $x <_t z$ and $y <_b x <_b z$. By the S-square and $ij <_r \{xy, zw\}$, we get $\{x, z\} <_t v_2 \leq_t j <_t y$ and $v_2 \leq_b i <_b \{x, z\}$. The previous inequalities imply that $yz \in E$. Then either $\langle x, v_2, y \rangle$ is an S-triangle, or $\langle x, v_2, z, y \rangle$ is an S-square of G depending on whether y is adjacent to v_2, which is a contradiction to $v_2 \in Y$.
  - Assume that $\{v_1, v_3\} = \{y, z\}$. If $yz \in E$ then we have a contradiction to the S-square. If $yz \notin E$ then, by $ij <_r \{xy, zw\}$, we get $v_2 \leq_t j <_t y <_t z$ and $v_2 \leq_b i <_b w <_b z$, which imply that $v_2 z \notin E$, again a contradiction to the S-square.

Thus, no subset of $Y \cup \{x, y, z, w\}$ induces an S-cycle of G. Therefore, $C_{ij}^{xy,zw} = C_{ij}^{xy,zz}$ holds.

If $jz \notin E$ then similar arguments to the previous case for $iw \notin E$ show that $C_{ij}^{xy,zw} = C_{ij}^{xy,ww}$.

Our remaining case is $iw, jz \in E$. If $j \notin C_{ij}^{xy,zw}$ then by Observation 4.1 (1) we get $C_{ij}^{xy,zw} = C_{\overline{\lhd} ij}^{xy,zw}$. Similarly, if $i \notin C_{ij}^{xy,zw}$ then $C_{ij}^{xy,zw} = C_{\unlhd ij}^{xy,zw}$. So, let us assume that $i, j \in C_{ij}^{xy,zw}$. Notice that, by Observation 4.1 (1), we know that $C_{ij}^{xy,zw} \setminus \{i, j\} \subseteq V_{\lhd ij}$. We distinguish two cases depending on whether i, j belong to S. If $i \in S$ or $j \in S$ then the following S-cycles show that we reach a contradiction to $i, j \in C_{ij}^{xy,zw}$:

- If $iz \in E$ then $\langle i, j, z\rangle$ is an S-triangle of G.
- If $jw \in E$ then $\langle i, j, w\rangle$ is an S-triangle of G.
- If $iz, jw \notin E$ then $\langle i, j, z, w\rangle$ is an S-square of G.

So, let $i, j \notin S$ and let $a'b' = \ell\text{-}\min \mathcal{X}[\{i, j, x, y, z, w\} \setminus \{x', y', z', w'\}]$. Observe that $a', b' \notin S$, since $S \cap \{i, j, x, y, z, w\} = \emptyset$. Applying Lemma 4.2 with $gh = \lhd ij$, $ab = x'y'$, $cd = z'w'$, and $ef = a'b'$, shows that for all $Y \subseteq V_{\lhd ij}$, $G[Y \cup \{x', y', z', w'\}] \in \mathcal{F}_S$ if and only if $G[Y \cup \{x', y', z', w', a', b'\}] \in \mathcal{F}_S$. By this fact, it follows that $C_{ij}^{xy,zw} = C_{\lhd ij}^{x'y',z'w'} \cup \{i, j\}$, as described in the third statement. $\square$

It is important to notice that all described formulations are given recursively based on Lemmas 4.3–4.12. Now, we are in position to state our claimed polynomial-time algorithm for the SFVS problem on permutation graphs.

Theorem 4.13. Subset Feedback Vertex Set *can be solved in $O(m^3)$ time on permutation graphs.*

Proof. Let us describe such an algorithm. Recall that we consider connected graphs with $n \leq m$ for the analysis of its running time. Given a permutation π, that is, the orderings $<_t$ and $<_b$ on the vertices of the input graph, we first compute all crossing pairs $ij \in \mathcal{X}$. Observe that $|\mathcal{X}| = |\mathcal{I}| + |\mathcal{X} \setminus \mathcal{I}| = n + m$. For each crossing pair ij, we compute $\overline{\lhd} ij$, $\unlhd ij$, $\lhd ij$, $\lll ij$, and $\lhd ij \lll xx$ for all $x \in V \setminus V_{ij}$. Note that such a simple application requires $O(n^2)$ time for every crossing pair ij, giving a total running time of $O(n^2 m)$. Next, we scan all crossing pairs of $\mathcal{X}$ according to their ascending order with respect to $<_r$. For every crossing pair ij, we compute A_{ij} according to Lemmas 4.3 and 4.8. That is, for every crossing pair xy of $V \setminus V_{ij}$ in descending order with respect to $<_\ell$ we compute B_{ij}^{xy} according to Lemmas 4.4, 4.5, 4.9, and 4.10. By the recursive formulations of B_{ij}^{xy}, for every crossing pair zw of $V \setminus V_{xy}$ in descending order with respect to $<_\ell$ we compute $C_{ij}^{xy,zw}$ according to Lemmas 4.6, 4.7, 4.11, and 4.12. In total, such computations require $O(m^3)$ time. At the end, the set $A_{\pi(n)n}$ is the maximum weighted S-forest for G, so that $V \setminus A_{\pi(n)n}$ is exactly the minimum subset feedback vertex set of (G, S). $\square$

5 Concluding remarks

We have given the first polynomial-time algorithms for SFVS on subclasses of AT-free graphs. Despite our positive results, the complexity of SFVS on AT-free graphs still remains unresolved. Due to Theorem 2.1, we believe that such an approach towards AT-free graphs should deal first with the complexity of the unweighted version of SFVS. Moreover, it is interesting to settle the complexity of SFVS on other related graph classes such as strongly chordal graphs or subclasses of AT-free graphs like trapezoid graphs or complements of triangle-free graphs. Regarding graphs of bounded structural parameter and due to the nature of the dynamic programming used for SFVS on interval and permutation graphs, it is interesting to consider graphs of bounded maximum induced matching width introduced in [3].

Another interesting open question is concerned with problems related to *terminal-sets* such as the Multiway Cut problem in which we want to disconnect a given set of terminals by removing vertices of minimum total weight. As already mentioned in the Introduction, the Multiway Cut problem reduces to the SFVS problem by adding a vertex s with $S = \{s\}$ that is adjacent to all terminals and whose weight is larger than the sum of the weights of all vertices in the original graph [17]. Notice that such a reduction does not directly work on interval or permutation graphs, since the augmented graph might not belong to the same graph class. Nonetheless, and even though polynomial-time algorithms exist for the Multiway Cut problem on interval graphs [24] and permutation graphs [31], it is still interesting to explore whether we can apply our algorithms for the SFVS problem to the Multiway Cut problem.

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A Appendix: enumeration of maximal S-forests of a co-bipartite graph

Here we analytically enumerate the $22n^4$ maximal S-forests claimed in the proof of Theorem 2.1 regarding a co-bipartite graph $G = (V, E)$. We remind that (A, B) be a partition of V such that such that $G[A]$ and $G[B]$ are cliques. Also, recall that V is further partitioned as (A_S, A_R, B_S, B_R), where $A_S = A \cap S$, $A_R = A \setminus S$, $B_S = B \cap S$ and $B_R = B \setminus S$. For a vertex v of G and a set $U \in \{A_S, B_S, A_R, B_R\}$ we denote by $N_U(v)$ the neighbors of v in the set U, that is, $N_U(v) := N(v) \cap U$. Moreover, the *symmetric difference* of two sets L and R is the set $(L \setminus R) \cup (R \setminus L)$ and is denoted by $L \triangle R$. Recall that (X, Y, Z, W) is the partition of the vertex set of a maximal S-forest of G such that $X \subseteq A_S$, $Y \subseteq A_R$, $Z \subseteq B_S$ and $W \subseteq B_R$ (see Figure 2). It is clear that $|X| \leq 2$ and $|Z| \leq 2$. Thus, it is sufficient to consider the following cases with respect to X and Z:

- Let $X = \emptyset$ and $Z = \emptyset$. Then the maximal S-forest contains no vertex of S, so we can safely include all vertices of $V \setminus S$. Thus the following set is a maximal S-forest of G:
  1. $(\emptyset, A_R, \emptyset, B_R)$.
- Let $X = \{a_S\}$ and $Z = \emptyset$. Observe that $|Y| \leq 1$, since $G[X \cup Y]$ is a clique. If $Y = \emptyset$ then including at least two neighbors of a_S that are contained in B_R leads to an S-cycle. Thus, we can safely include all non-neighbors of a_S and exactly one neighbor of a_S contained in B_R in the maximal S-forest. If $Y = \{a_R\}$ then including a neighbor of a_S and a neighbor of a_R (may well be the same) that are contained in B_R leads to an S-cycle. If we do not include a neighbor of a_S then we can safely include all other vertices of B_R. However, if we include a neighbor of a_S that is non-adjacent to a_R then we can safely include all other vertices that are non-adjacent to both. Thus, the following sets induce the corresponding maximal S-forests of G:
  2. $(\{a_S\}, \emptyset, \emptyset, B_R)$, where $N_{B_R}(a_S) = \emptyset$;
  3. $(\{a_S\}, \emptyset, \emptyset, \{b_R\} \cup (B_R \setminus N(a_S)))$, where $b_R \in N_{B_R}(a_S)$;
  4. $(\{a_S\}, \{a_R\}, \emptyset, B_R \setminus N(a_S))$;
  5. $(\{a_S\}, \{a_R\}, \emptyset, \{b_R\} \cup (B_R \setminus N(\{a_S, a_R\})))$, where $b_R \in N_{B_R}(a_S) \setminus N_{B_R}(a_R)$.
- Let $X = \emptyset$ and $Z = \{b_S\}$. Completely symmetric arguments with the previous case imply that the following sets induce the corresponding maximal S-forests of G:
  6. $(\emptyset, A_R, \{b_S\}, \emptyset)$, where $N_{A_R}(b_S) = \emptyset$;
  7. $(\emptyset, \{a_R\} \cup (A_R \setminus N(b_S)), \{b_S\}, \emptyset)$, where $a_R \in N_{A_R}(b_S)$;
  8. $(\emptyset, A_R \setminus N(b_S), \{b_S\}, \{b_R\})$;
  9. $(\emptyset, \{a_R\} \cup (A_R \setminus N(\{b_S, b_R\})), \{b_S\}, \{b_R\})$, where $a_R \in N_{A_R}(b_S) \setminus N_{A_R}(b_R)$.
- Let $X = \{a_S\}$ and $Z = \{b_S\}$. Then both $|Y| \leq 1$ and $|W| \leq 1$. Thus, the following sets induce the maximal S-forest of G:
  10. $(\{a_S\}, \emptyset, \{b_S\}, \emptyset)$, where $\{a_S, b_S\} \in E$ and $V \setminus S \subseteq N(a_S) \cap N(b_S)$;
  11. $(\{a_S\}, \{a_R\}, \{b_S\}, \emptyset)$, where $G[\{a_S, a_R, b_S\}]$ is acyclic and $B_R \subseteq N(a_S) \cup N(a_R)$;
  12. $(\{a_S\}, \emptyset, \{b_S\}, \{b_R\})$, where $G[\{a_S, b_S, b_R\}]$ is acyclic and $A_R \subseteq N(b_S) \cup N(b_R)$;
  13. $(\{a_S\}, \{a_R\}, \{b_S\}, \{b_R\})$, where $G[\{a_S, a_R, b_S, b_R\}]$ is acyclic.
- Let $X = \{a_S, a'_S\}$ and $Z = \emptyset$. Then $|Y| = 0$, since $G[X \cup Y]$ is a clique. Adding a vertex of B_R that is adjacent to both a_S and a'_S leads to an S-cycle. If we add a vertex of B_R that is adjacent to either a_S or a'_S then adding another such vertex leads to an S-cycle. Thus, we can safely include all other vertices that are non-adjacent to either a_S or a'_S.

14. $(\{a_S, a'_S\}, \emptyset, \emptyset, B_R \setminus (N(a_S) \cap N(a'_S)))$, where $N_{B_R}(a_S) \vartriangle N_{B_R}(a'_S) = \emptyset$;
15. $(\{a_S, a'_S\}, \emptyset, \emptyset, \{b_R\} \cup (B_R \setminus N(\{a_S, a'_S\})))$, where $b_R \in N_{B_R}(a_S) \vartriangle N_{B_R}(a'_S)$.

- Let $X = \emptyset$ and $Z = \{b_S, b'_S\}$. Completely symmetric arguments with the previous case imply that the following sets induce the corresponding maximal S-forest of G:

16. $(\emptyset, A_R \setminus (N(b_S) \cap N(b'_S)), \{b_S, b'_S\}, \emptyset)$, where $N_{A_R}(b_S) \vartriangle N_{A_R}(b'_S) = \emptyset$;
17. $(\emptyset, \{a_R\} \cup (A_R \setminus (N(b_S) \cup N(b'_S))), \{b_S, b'_S\}, \emptyset)$, where $a_R \in N_{A_R}(b_S) \vartriangle N_{A_R}(b'_S)$.

- Let $X = \{a_S, a'_S\}$ and $Z = \{b_S\}$. Then $|Y| = 0$ and $|W| \leq 1$. Thus, the following sets induce the maximal S-forest of G:

18. $(\{a_S, a'_S\}, \emptyset, \{b_S\}, \emptyset)$, where $G[\{a_S, a'_S, b_S\}]$ is acyclic;
19. $(\{a_S, a'_S\}, \emptyset, \{b_S\}, \{b_R\})$, where $G[\{a_S, a'_S, b_S, b_R\}]$ is acyclic.

- Let $X = \{a_S\}$ and $Z = \{b_S, b'_S\}$. Then similarly to the previous case we obtain the following:

20. $(\{a_S\}, \emptyset, \{b_S, b'_S\}, \emptyset)$, where $G[\{a_S, b_S, b'_S\}]$ is acyclic.
21. $(\{a_S\}, \{a_R\}, \{b_S, b'_S\}, \emptyset)$, where $G[\{a_S, b_S, b'_S, a\}]$ is acyclic.

- Let $X = \{a_S, a'_S\}$ and $Z = \{b_S, b'_S\}$. Then $|Y| = 0$ and $|W| = 0$, so that the following set induces such a maximal S-forest:

22. $(\{a_S, a'_S\}, \emptyset, \{b_S, b'_S\}, \emptyset)$, where $G[\{a_S, a'_S, b_S, b'_S\}]$ is acyclic.

Because $|X|, |Y|, |Z|, |W| \leq n$, each described maximal S-forest gives at most n^4 maximal S-forests. Therefore, in total there are at most $22n^4$ maximal S-forests that correspond to each particular case.