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★ **Άσκηση 6.3**

$n\in\mathbb{N}$, $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$, $\lambda_1+\lambda_2+\dots+\lambda_n=1$

$\varphi:[a,b]\to\mathbb{R}$, $x_1,\dots,x_n\in[a,b]$

ΝΔΟ: $\exists\, \xi\in[a,b] : \lambda_1\varphi_1(x_1)+\dots+\lambda_n\varphi(x_n)=\varphi(\xi)$

Απόδειξη

$$\lambda_1\varphi(x_1)+\dots+\lambda_n\varphi(x_n)\leq\lambda_1\max_{a\leq x\leq b}\varphi(x)+\dots+\lambda_n\max_{a\leq x\leq b}\varphi(x)=$$

$$=\underbrace{(\lambda_1+\dots+\lambda_n)}_{=1}\max_{a\leq x\leq b}\varphi(x)=$$

$$=\max_{a\leq x\leq b}\varphi(x)$$

Όμοια: $\min\limits_{a\leq x\leq b}\varphi(x)\leq\dots\leq\lambda_1\varphi(x_1)+\dots+\lambda_n\varphi(x_n)$

Άρα:

$$\min_{a\leq x\leq b}\varphi(x)\leq\lambda_1\varphi(x_1)+\dots+\lambda_n\varphi(x_n)\leq\max_{a\leq x\leq b}\varphi(x)$$

Σύμφωνα με το θεώρημα της ενδιάμεσης τιμής υπάρχει $\xi\in[a,b]$ τέτοιο ώστε:

$$\lambda_1\varphi(x_1)+\dots+\lambda_n\varphi(x_n)=\varphi(\xi)$$

* Άσκηση 6.4

$n \in \mathbb{N}$, $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$

$\varphi : [a, b] \to \mathbb{R}$, $x_1, \dots, x_n \in [a, b]$

$$(\lambda_1 + \dots + \lambda_n) \min_x \varphi(x) \leq \lambda_1 \varphi(x_1) + \dots + \lambda_n \varphi(x_n) \leq (\lambda_1 + \dots + \lambda_n) \max_x \varphi(x)$$

$$\Rightarrow \min_x \varphi(x) \leq \underbrace{\frac{\lambda_1 \varphi(x_1) + \dots + \lambda_n \varphi(x_n)}{\lambda_1 + \dots + \lambda_n}}_{= \varphi(\xi)} \leq \max_x \varphi(x)$$

$$\Rightarrow \lambda_1 \varphi(x_1) + \dots + \lambda_n \varphi(x_n) = (\lambda_1 + \dots + \lambda_n) \varphi(\xi)$$

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Άσκηση 6.8

Q_n^T, Q_m^S στο $[-1, 1]$

$f(x) = \frac{x^6}{30} - x^2$

<u>ΝΔΟ</u>: $Q_n^T(f) \leq \int_{-1}^{1} f(x)\,dx \leq Q_m^S(f)$

$\exists\, \xi \in (-1,1): \int_{-1}^{1} f(x)\,dx - Q_n^T(f) = -\underbrace{\frac{1}{6}\left(\frac{2}{n-1}\right)^2}_{>0}\underbrace{f''(\xi)}_{<0} > 0$

$f'(x) = \frac{x^5}{5} - 2x$

$f''(x) = x^4 - 2 \leq -1.$

$\Rightarrow \int_{-1}^{1} f(x)\,dx \geq Q_n^T(f)$

$\exists\, \xi \in (-1,1): \int_{-1}^{1} f(x)\,dx - Q_m^S(f) = -\underbrace{\frac{1}{90}\left(\frac{2}{m-1}\right)^4}_{>0}\underbrace{f^{(4)}(\xi)}_{\geq 0} \leq 0$

$f'''(x) = 4x^3$

$f^{(4)}(x) = 12x \geq 0$

$\Rightarrow \int_{-1}^{1} f(x)\,dx \leq Q_m^S(f)$

Ασκηση 6.9

$[a,b]$ $\quad Q$

$$R(f) = \int_a^b f(x)dx - Q(f)$$

ΝΔΟ: Υπάρχει το πολύ ένας $\kappa \in \mathbb{N}$ τέτοιος ώστε:

(*) $\quad \exists C_\kappa \in \mathbb{R} \; \forall f \in C^\kappa[a,b] \; \exists \xi \in [a,b]:$

$$R(f) = C_\kappa f^{(\kappa)}(\xi)$$

$C_\kappa = 0$; Αδύνατον, γιατί τότε θα ίσχυε $R(p)=0$ για κάθε πολυώνυμο p.

$C_\kappa \neq 0$

Έστω ότι ισχύει η (*).
Τότε, αν $f \in \mathbb{P}_{\kappa-1}$ θα έχουμε $R(f)=0$
Για $f(x) := x^\kappa$ θα έχουμε $R(f) = C_\kappa \cdot \kappa! \neq 0$

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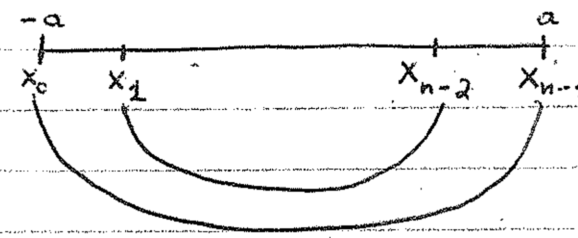
Άσκηση 6.10

$[-a, a]$ Q_n ο τύπος των Newton-Cotes με n κόμβους

$$Q_n(f) = w_0 f(x_0) + w_1 f(x_1) + \dots + w_{n-1} f(x_{n-1})$$



Έστω x_i και x_j τέτοια ώστε $x_i = -x_j$

<u>ΝΔΟ</u>: $w_i = w_j$

Συμμετρικός τύπος!

Απόδειξη

$$(*) \quad \forall p \in \mathbb{P}_{n-1} : \int_{-a}^{a} p(x)\,dx = Q_n(p)$$

Για $L_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n-1} \dfrac{x - x_k}{x_i - x_k}$ έχουμε

$L_i \in \mathbb{P}_{n-1}$ και σύμφωνα με την $(*)$

$$\boxed{w_i = \int_{-a}^{a} L_i(x)\,dx}$$

Εντελώς παρόμοια:

$$w_j = \int_{-a}^{a} L_j(x)\,dx$$

με $L_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{n-1} \dfrac{x - x_k}{x_j - x_k}$

Άρα:

$$w_i = \int_{-a}^{a} L_i(x)\,dx = \int_{-a}^{a} \prod_{\substack{k=0 \\ k\neq i}}^{n-1} \frac{x - x_k}{x_i - x_k}\,dx =$$

$$= \int_{-a}^{a} \prod_{\substack{k=0 \\ k\neq j}}^{n-1} \frac{x + x_k}{x_i + x_k}\,dx \underset{t=-x}{=} -\int_{a}^{-a} \prod_{\substack{k=0 \\ k\neq j}}^{n-1} \frac{-t + x_k}{x_i + x_k}\,dt =$$

$$= \int_{-a}^{a} \prod_{\substack{k=0 \\ k\neq j}}^{n-1} \frac{-t + x_k}{x_i + x_k}\,dt = \int_{-a}^{a} \prod_{\substack{k=0 \\ k\neq j}}^{n-1} \frac{t - x_k}{x_j - x_k}\,dt =$$

$$= \int_{-a}^{a} L_j(t)\,dt = w_j$$

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Άσκηση 6.11

$[-a, a]$ Q_n ο τύπος των Newton-Cotes

φ περιττή

ΝΔΟ: $\int_{-a}^{a} \varphi(x)dx = Q_n(\varphi)$

- $\int_{-a}^{a} \varphi(x)dx = 0$

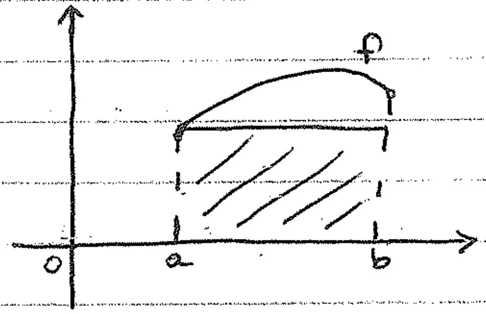
- $Q_n(\varphi) = w_1\varphi(x_1) + w_2\varphi(x_2) + \dots + w_n\varphi(x_n) =$

$\boxed{\varphi(0)=0}$ $= \underbrace{w_1[\varphi(x_1) + \varphi(x_n)]}_{=0} + w_2\underbrace{[\varphi(x_2) + \varphi(x_{n-1})]}_{=0} + \dots =$

$= 0$

★ Άσκηση 6.13

$Q(f) = (b-a) f(a)$



Αριστερός τύπος του ορθογωνίου

$R(f) = \int_a^b f(x)dx - Q(f)$

a) $\forall p \in \mathbb{P}_0 : R(f) = 0$

$p(x) = \gamma$

$R(f) = \int_a^b \gamma dx - (b-a)\gamma =$

$= \gamma(b-a) - (b-a)\gamma =$

$= 0$

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β) $\forall f \in C^1[a,b] \ \exists \xi \in (a,b)$:

$$R(f) = \frac{(b-a)^2}{2} f'(\xi)$$

$$R(f) = \int_a^b f(x)\,dx - (b-a) f(a) =$$

$$= \int_a^b f(x)\,dx - \int_a^b f(a)\,dx =$$

$$= \int_a^b [f(x) - f(a)]\,dx$$

$$= \int_a^b f'(\xi(x)) \underbrace{(x-a)}_{\geq 0}\,dx$$

$\boxed{\text{Εξήγηση!}}$ $= f'(\xi) \int_a^b (x-a)\,dx =$

$$= \frac{(b-a)^2}{2} f'(\xi)$$

γ) $n \in \mathbb{N}$, $h = \frac{b-a}{n}$, $x_i = a + ih$, $i = 0, \ldots, n$

<u>ΝΔΟ</u>: Για $f \in C^1[a,b]$ υπάρχει $\xi \in (a,b)$ τέτοιο ώστε:

$$\int_a^b f(x)dx - h\sum_{i=0}^{n-1} f(x_i) = \frac{b-a}{2} h f'(\xi)$$

$$\int_a^b f(x)dx - h\sum_{i=0}^{n-1} f(x_i) =$$

$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx - h\sum_{i=0}^{n-1} f(x_i) =$$

$$= \sum_{i=0}^{n-1} \left[\int_{x_i}^{x_{i+1}} f(x)dx - h f(x_i) \right] =$$

$$= \sum_{i=0}^{n-1} \left[\frac{(x_{i+1}-x_i)^2}{2} \cdot f'(\xi_i) \right] =$$

$$= \frac{h^2}{2} \sum_{i=0}^{n-1} f'(\xi_i) =$$

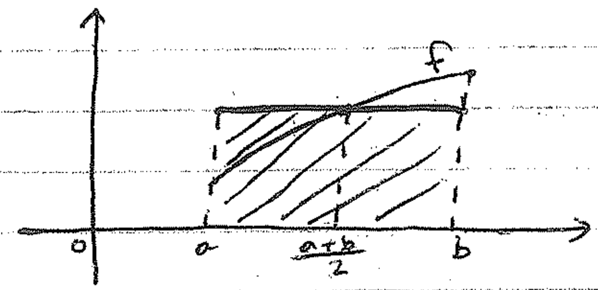
$\boxed{\text{Εξήγηση!}}$ $= \frac{h^2}{2} n \frac{1}{n} \sum_{i=0}^{n-1} f'(\xi_i) =$

$$= \frac{b-a}{2} h f'(\xi)$$

με $\xi_i \in (x_i, x_{i+1})$

Άσκηση 6.14

$$Q(f) = (b-a)\, f\left(\frac{a+b}{2}\right)$$



Τύπος του μέσου

$$R(f) = \int_a^b f(x)\,dx - Q(f)$$

a) $\forall p \in \mathbb{P}_1 : R(f) = 0$
(Είναι ο τύπος του Gauss με $n=1$ και $w(x)=1$)

$$p(x) = \gamma x + \delta$$

$(p(x) = 1,\ p(x) = x)$

$$R(p) = \int_a^b (\gamma x + \delta)\,dx - (b-a)\left[\gamma \frac{a+b}{2} + \delta\right] =$$

$$= \gamma \frac{b^2 - a^2}{2} + (b-a)\delta - \gamma \frac{(a+b)(a-b)}{2} - \delta(b-a) =$$

$$= 0$$

β) $\forall f \in C^2[a,b] \quad \exists \xi \in (a,b):$

$$R(f) = \frac{(b-a)^3}{24} f''(\xi)$$

$p \in \mathbb{P}_1$

$p\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$

$p'\left(\frac{a+b}{2}\right) = f'\left(\frac{a+b}{2}\right)$

$\forall x \in [a,b] \quad \exists \xi \in (a,b):$

$$f(x) - p(x) = \frac{f''(\xi(x))}{2}\left(x - \frac{a+b}{2}\right)^2$$

$R(f) = \int_a^b f(x)\,dx - Q(f) =$ $\boxed{(\text{χωρίς το } p)}$

$$= \int_a^b f(x)\,dx - \int_a^b f\left(\frac{a+b}{2}\right)dx =$$

$$= \int_a^b \left[f(x) - f\left(\frac{a+b}{2}\right)\right]dx =$$

$$\underset{\text{Taylor}}{=} \int_a^b \left[f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{f''(\xi(x))}{2}\left(x - \frac{a+b}{2}\right)^2\right]dx =$$

$$= \frac{1}{2}\int_a^b f''(\xi(x))\underbrace{\left(x - \frac{a+b}{2}\right)^2}_{\geq 0}dx =$$

$$= \frac{1}{2} f''(\xi) \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$$

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$$R(f) = \int_a^b f(x)dx - Q(f) = \boxed{(\mu\varepsilon\ \tau o\ p)}$$

$$= \int_a^b f(x)dx - \int_a^b p(x)dx =$$

$$= \int_a^b [f(x) - p(x)]dx =$$

$$= \frac{1}{2}\int_a^b f''(\xi(x)) \underbrace{\left(x - \frac{a+b}{2}\right)^2}_{\geq 0} dx =$$

$$= \frac{1}{2} f''(\xi) \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$$

γ) $n \in \mathbb{N}$, $h = \frac{b-a}{n}$, $x_i = a + ih$, $i = 0, \ldots, n$

<u>ΝΔΟ</u>: Για $f \in C^2[a,b]$ $\exists \, \xi \in (a,b)$:

$$\int_a^b f(x)dx - h\sum_{i=0}^{n-1} f(x_i + \tfrac{h}{2}) = \frac{b-a}{24} h^2 f''(\xi)$$

$$\int_a^b f(x)dx - h\sum_{i=0}^{n-1} f(x_i + \tfrac{h}{2}) =$$

$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx - h\sum_{i=0}^{n-1} f(x_i + \tfrac{h}{2}) =$$

$$= \sum_{i=0}^{n-1} \left[\int_{x_i}^{x_{i+1}} f(x)dx - h f(x_i + \tfrac{h}{2}) \right] =$$

$$= \sum_{i=0}^{n-1} \left[\frac{(x_{i+1} - x_i)^3}{24} f''(\xi_i) \right] =$$

$$= \frac{h^3}{24} n \frac{1}{n} \sum_{i=0}^{n-1} f''(\xi_i) =$$

$$= \frac{h^3}{24} n f''(\xi) =$$

$$= \frac{b-a}{24} h^2 f''(\xi)$$

Άσκηση 6.15

$$Q(f) = (b-a)f(a) + \frac{(b-a)^2}{2} f'(a)$$

$$R(f) = \int_a^b f(x)dx - Q(f)$$

$$Q(f) = \int_a^b f(a)dx + \int_a^b f'(a)(x-a)dx =$$

$$= \int_a^b \underbrace{[f(a) + f'(a)(x-a)]}_{\substack{\text{πολυώνυμο Taylor} \\ \text{βαθμού } \leq 1 \text{ της } f \\ \text{ως προς το σημείο } a}} dx$$

a) $\forall p \in \mathbb{P}_1 : R(p) = 0$

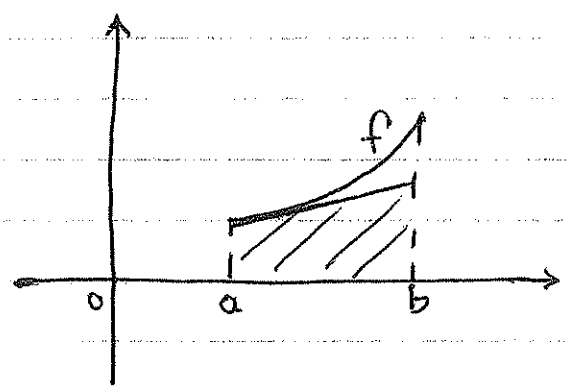
1ος τρόπος: $p(x) = \gamma x + \delta$

πράξεις....

2ος τρόπος: $p(x) = p(a) + p'(a)(x-a)$

$$\Rightarrow \int_a^b p(x)dx = \int_a^b [p(a) + p'(a)(x-a)]dx =$$

$$= Q(p)$$



β) **ΝΔΟ**: $\forall f \in C^2[a,b] \ \exists \xi \in (a,b)$:

$$R(f) = \frac{(b-a)^3}{6} f''(\xi)$$

i) $p \in \mathbb{P}_1$

$p(a) = f(a)$
$p'(a) = f'(a)$

$\forall x \in [a,b] \ \exists \xi(x) \in (a,b)$:

$$f(x) - p(x) = \frac{f''(\xi(x))}{2} (x-a)^2$$

$$R(f) = \int_a^b f(x)\,dx - Q(f) =$$

$$= \int_a^b f(x)\,dx - \int_a^b p(x)\,dx =$$

$$= \int_a^b [f(x) - p(x)]\,dx =$$

$$= \int_a^b \frac{f''(\xi(x))}{2} \underbrace{(x-a)^2}_{\geq 0} dx =$$

$$= \frac{f''(\xi)}{2} \int_a^b (x-a)^2 dx =$$

$$= \frac{f''(\xi)}{2} \cdot \frac{(b-a)^3}{3} =$$

$$= \frac{(b-a)^3}{6} f''(\xi)$$

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ii) $R(f) = \int_a^b f(x)\,dx - \int_a^b [f(a) + f'(a)(x-a)]\,dx =$

$= \int_a^b \{ f(x) - [f(a) + (x-a) f'(a)] \} dx =$

$= \int_a^b \frac{f''(\xi(x))}{2} (x-a)^2 dx =$

Taylor

$= \frac{(b-a)^3}{6} f''(\xi)$

γ) $n \in \mathbb{N}$, $h = \frac{b-a}{n}$, $x_i = a + ih$, $i = 0, \ldots, n$

<u>ΝΔΟ</u>: Για $f \in C^2[a,b]$ $\exists \xi \in (a,b)$ τέτοιο ώστε:

$$\int_a^b f(x)dx - \sum_{i=0}^{n-1}\left[hf(x_i) + \frac{h^2}{2}f'(x_i)\right] = \frac{b-a}{6}h^2 f''(\xi)$$

$$\int_a^b f(x)dx - \sum_{i=0}^{n-1}\left[hf(x_i) + \frac{h^2}{2}f'(x_i)\right] =$$

$$\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx - \sum_{i=0}^{n-1}\left[hf(x_i) + \frac{h^2}{2}f'(x_i)\right] =$$

$$= \sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} f(x)dx - \left[hf(x_i) + \frac{h^2}{2}f'(x_i)\right]\right) =$$

$$= \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^3}{6} f''(\xi_i) =$$

$$= \frac{h^3}{6}\sum_{i=0}^{n-1} f''(\xi_i) =$$

$$= \frac{h^3}{6}\, n \,\frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i) =$$

$$= \frac{h^3}{6}\, n\, f''(\xi) =$$

$$= \frac{b-a}{6}h^2 f''(\xi)$$