ΑΣΚΗΣΕΙΣ 3ου ΚΕΦΑΛΑΙΟΥ

03/12/2015

Άσκηση 3.12

$p \geq 1 \Leftrightarrow b_1 + \dots + b_q = 1.$

$$\delta^n = y(t^n) + h\sum_{i=1}^{q} b_i f(t^{n,i}, \int^{n,i}) - y(t^{n+1})$$

$$= y(t^n) + h\sum_{i=1}^{q} b_i f\left(t^n + c_i h,\ y(t^n) + h\sum_{j=1}^{q} a_{ij} f(t^{n,j}, \int^{n,j})\right) - y(t^{n+1})$$

(Κάνω Taylor)

$$= y(t^n) + h\sum_{i=1}^{q} b_i \left[\underbrace{f(t^n, y(t^n))}_{= y'(t^n)} + O(h)\right] - y(t^{n+1})$$

$$= y(t^n) + h y'(t^n)\sum_{i=1}^{q} b_i + O(h^2) - y(t^{n+1})$$

$$= y(t^n) + h y'(t^n)\sum_{i=1}^{q} b_i + O(h^2) - [y(t^n) + h y'(t^n) + O(h^2)]$$

$$= h y'(t^n)\left(\sum_{i=1}^{q} b_i - 1\right) + O(h^2)$$

• Για $y'(t^n) \neq 0$: $\delta^n = O(h^2) \Leftrightarrow \sum_{i=1}^{q} b_i - 1 = 0$

Άσκηση 3.13

$$\begin{cases} y'(t) = 1, & 0 \leq t \leq 1 \\ y(0) = 0 \end{cases}$$

Λύση: $y(t) = t$

$N \in \mathbb{N}$, $h = \frac{1}{N}$, $t^n = nh$, $n = 0, \dots, N$

RK N.ΔΟ.: $y^N \to 1$, $N \to \infty \Rightarrow$ μέθοδος συνεπής!

$y^0 = 0$

$$y^{n+1} = y^n + h\sum_{i=1}^{q} b_i \quad , n=0,\dots,N-1$$

$\Rightarrow$ (τετριμμένη επαγωγή)

$$y^n = nh\sum_{i=1}^{q} b_i$$

$$\Rightarrow y^N = N \cdot h \sum_{i=1}^{q} b_i \Rightarrow y^N = \sum_{i=1}^{q} b_i$$

"1"

Προφανώς, $y^N \to 1 \Leftrightarrow \sum_{i=1}^{q} b_i = 1$.

10/12/2015

Άσκηση 3.2

1/3	1/3
	1

Ν.δ.ο. $p=1$.

Απόδειξη

$$y^{n,1} = y(t^n) + \frac{h}{3} f(t^n + \frac{h}{3}, y^{n,1})$$

$$\delta^n = y(t^n) + hf(t^n + \frac{h}{3}, y^{n,1}) - y(t^{n+1})$$

Έχουμε, $y^{n,1} = y(t^n) + O(h)$, οπότε

$$\delta^n = y(t^n) + hf(t^n + \frac{h}{3}, y(t^n) + O(h)) - y(t^{n+1}) \Rightarrow$$

$$\delta^n = y(t^n) + h[f(t^n, y(t^n) + O(h)] - y(t^{n+1}) \Rightarrow$$

$$\delta^n = y(t^n) + hy'(t^n) + O(h^2) - y(t^{n+1})$$

$$= y(t^n) + hy'(t^n) + O(h^2) - [y(t^n) + hy'(t^n) + O(h^2)]$$

$$= O(h^2) \Rightarrow \boxed{p \geq 1}$$

$$\begin{cases} y'(t) = 2t, & 0 \le t \le 1 \\ y(0) = 0 \end{cases}$$ Λύση: $y(t) = t^2$

τότε $\delta^n = y(t^n) + h2(t^n + \frac{h}{3}) - y(t^{n+1}) = (t^n)^2 + 2t^n h + \frac{2}{3}h^2 - (t^n + h)^2$

$= (t^n)^2 + 2t^n h + \frac{2}{3}h^2 - (t^n)^2 - 2t^n h - h^2$

$= -\frac{1}{3}h^2$

$\Rightarrow |\delta^n| \ge \frac{1}{3}h^2 \Rightarrow \boxed{p \le 1}$

άρα $\boxed{p = 1}$

Άσκηση 3.3

0	0	0
a_{21}	0	τ_2
b_1	b_2	

Να βρείτε τα a_{21}, τ_2, b_1, b_2, έτσι ώστε η αντίστοιχη μέθοδος που προκύπτει να έχει $p = 2$

▷ $y^{n,1} = y(t^n)$

▷ $y^{n,2} = y(t^n) + h a_{21} f(t^n, y^{n,1})$

▷ $\delta^n = y(t^n) + b_1 h f(t^{n,1}, y^{n,1}) + b_2 h f(t^{n,2}, y^{n,2}) - y(t^{n+1})$

$= y(t^n) + b_1 h \underbrace{f(t^n, y(t^n))}_{y'(t^n)} + b_2 h f(t^n + \tau_2 h, \underbrace{y(t^n) + h a_{21} f(t^n, \underbrace{y^{n,1}}_{y(t^n)})}_{y'(t^n)}) - y(t^{n+1})$

$= y(t^n) + b_1 h y'(t^n) + b_2 h f(t^n + \tau_2 h, y(t^n) + h a_{21} y'(t^n)) - y(t^{n+1}) =$

$= y(t^n) + b_1 h y'(t^n) + b_2 h [f(t^n, y(t^n)) + \tau_2 h f_t(t^n, y(t^n)) + a_{21} h y'(t^n) \cdot f_y(t^n, y(t^n)) + O(h^2)] - y(t^n) - h y'(t^n) - \frac{h^2}{2} y''(t^n) - O(h^3) =$

$= (b_1 + b_2 - 1) h y'(t^n) + h^2 (b_2 \tau_2 f_t(t^n, y(t^n)) + b_2 a_{21} y'(t^n) f_y(t^n, y(t^n)) - \frac{1}{2} y''(t^n)) + O(h^3)$

$$\Rightarrow \delta^n = (b_1+b_2-1)hy'(t^n) + h^2\left(b_2\tau_2 f_t(t^n,y(t^n)) + b_2 a_{21} y'(t^n) f_y(t^n,y(t^n))\right.$$
$$\left. - \frac{1}{2} f_t(t^n,y(t^n)) - \frac{1}{2} y'(t^n) f_y(t^n,y(t^n))\right) + O(h^3)$$

$$\Rightarrow \delta^n = (b_1+b_2-1)hy'(t^n) + h^2\left(b_2\tau_2 - \frac{1}{2}\right) f_t(t^n,y(t^n)) + h^2\left(b_2 a_{21} - \frac{1}{2}\right) y'(t^n) f_y(t^n,y(t^n))$$
$$+ O(h^3)$$

$$\Rightarrow \delta^n = O(h^3) \Rightarrow \begin{cases} b_1+b_2-1=0 \\ b_2\tau_2 - 1/2 = 0 \\ b_2 a_{21} - 1/2 = 0 \end{cases}$$

Άρα, $p \geq 2 \Rightarrow \left.\begin{matrix} b_1+b_2=1 \\ b_2\tau_2 = 1/2 \\ b_2 a_{21} = 1/2 \end{matrix}\right\} \overset{b_2\neq 0}{\Longleftrightarrow} \begin{matrix} a_{21} = \tau_2 = \frac{1}{2b_2} \\ b_1 = 1 - b_2 \end{matrix}$

Για οποιαδήποτε επιλογή $b_2 \neq 0$ και $a_{21} = \tau_2 = \frac{1}{2b_2}$ και $b_1 = 1-b_2$, η μέθοδος έχει τάξη τουλάχιστον 2

<u>Παράδειγμα</u>

$$\begin{cases} y' = y, \quad 0 \leq t \leq 1 \\ y(0) = 1 \end{cases} \qquad \text{Λύση: } y(t) = e^t$$

$$\delta^n = y(t^n) + b_1 h \int^{n,1} + b_2 h \int^{n,2} - y(t^{n+1})$$
$$= y(t^n) + b_1 h y(t^n) + b_2 h\left[y(t^n) + a_{21} h y(t^n)\right] - y(t^{n+1})$$
$$= \left[(1+b_1+b_2)h + b_2 a_{21} h^2\right] y(t^n) - y(t^{n+1}) =$$
$$= \left[1 + h + \frac{h^2}{2}\right] e^{t^n} - e^{t^n} e^h = \left(1 + h + \frac{h^2}{2} - e^h\right) e^{t^n}$$

<u>1ος τρόπος:</u> $e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + O(h^n)$

τότε $\delta^n = -\frac{h^3}{6} e^{t^n} + O(h^n)$, άρα δ^n δεν μπορεί να είναι $O(h)$

2ος τρόπος • $e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} e^{\xi}$, με $\xi \in (0,h)$

άρα, $\delta^n = -\frac{h^3}{6} e^{\xi} \cdot e^{t^n} \Rightarrow |\delta^n| = \frac{h^3}{6} \underbrace{e^{t^n+\xi}}_{>1} \geq \frac{h^3}{6}$

$\Rightarrow p \leq 2$, άρα $\boxed{p=2}$

Άσκηση 3.14

Ν.δ.ο. (a) $\max_{n,i} |y(t^{n,i}) - Y^{n,i}| \leq ch$

(b) $\max_{n} |y(t^{n,i}) - Y^{n,i}| \leq ch^2 \Leftrightarrow \sum_{j=1}^{n} a_{ij} = \tau_i$

(a) $Y^{n,i} = y(t^n) + h \sum_{j=1}^{q} a_{ij} f(t^{n,j}, Y^{n,j})$

$\Rightarrow Y^{n,i} = y(t^n) + O(h) = y(t^{n,i}) + O(h) + O(h) \Rightarrow |y(t^{n,i}) - Y^{n,i}| \leq ch$

(b) $Y^{n,i} = y(t^n) + h \sum_{j=1}^{q} a_{ij} f(t^{n,j}, Y^{n,j}) = y(t^n) + h \sum_{j=1}^{q} a_{ij} f(t^n + \tau_j h, y(t^n)) + O(h)$

$= y(t^n) + h \sum_{j=1}^{q} a_{ij} [f(t^n, y(t^n)) + O(h)] = y(t^n) + h y'(t^n) \sum_{j=1}^{q} (a_{ij}) + O(h^2)$

* $y(t^{n,i}) - \tau_i h y'(t^n) + O(h^2)$

$Y^{n,i} - y(t^{n,i}) = h \left[\sum_{j=1}^{q} (a_{ij}) - \tau_i \right] y'(t^n) + O(h^2)$

$\Rightarrow Y^{n,i} - y(t^{n,i}) = O(h^2) \Leftrightarrow \sum_{j=1}^{q} (a_{ij}) - \tau_i = 0$

Άσκηση 3.15

[illegible]	[illegible]
$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{6}$	

Είναι συνεπής? (Ελέγχουμε αν $\sum_{i=1}^{q} b_i = 1$)

$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{3}{6} + \frac{2}{6} + \frac{1}{6} = 1 \Rightarrow p \geq 1 \Rightarrow$ Είναι συνεπής!