LINEARLY IMPLICIT METHODS FOR NONLINEAR EVOLUTION EQUATIONS

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Abstract. We construct and analyze combinations of rational implicit and explicit multistep methods for nonlinear evolution equations and extend thus recent results concerning the discretization of nonlinear parabolic equations. The resulting schemes are linearly implicit and include as particular cases implicit–explicit multistep schemes as well as the combination of implicit Runge–Kutta schemes and extrapolation. We establish optimal order error estimates. The abstract results are applied to a third–order evolution equation arising in the modelling of flow in a fluidized bed. We discretize this equation in space by a Petrov–Galerkin method. The resulting fully discrete schemes require solving some linear systems to advance in time with coefficient matrices the same for all time levels.

1. Introduction

In [1], a wide class of linearly implicit methods were applied to the following initial value problem

$$\begin{aligned} &v'(t) + \mathcal{A}v(t) = \mathcal{B}(t, v(t)), \quad 0 < t < T,\\ &v(0) = v^0, \end{aligned} \tag{1.1}$$

with $\mathcal{A}$ a positive definite, selfadjoint, linear operator on a Hilbert space $(H, (\cdot,\cdot))$ with domain $D(\mathcal{A})$ dense in H, and $\mathcal{B}(t,\cdot) : D(\mathcal{A}) \to H$, $t \in [0,T]$, a (possibly) nonlinear operator and initial datum $v^0 \in H$.

This paper is concerned with the construction and analysis of linearly implicit schemes for more general equations. For $T > 0$ and $u^0 \in H$, we consider the initial value problem of seeking $u : [0,T] \to D(A)$ satisfying

$$\begin{aligned} &Lu'(t) + Au(t) = B(t, u(t)), \quad 0 < t < T,\\ &u(0) = u^0, \end{aligned} \tag{1.2}$$

with L and A positive definite, selfadjoint, linear operators on H with domain $D(A)$ dense in H, $D(A) \subset D(L)$, and $B(t,\cdot) : D(A) \to H$, $t \in [0,T]$, a (possibly) nonlinear operator.

To discretize (1.2) by the general class of schemes analyzed in [1], we first rewrite it in the form (1.1). To this end we let

$$\begin{aligned} &\Lambda := L^{1/2},\ \mathcal{A} := \Lambda^{-1}A\Lambda^{-1},\ \mathcal{B}(t,\cdot) := \Lambda^{-1}B(t,\Lambda^{-1}\cdot),\\ &v(t) := \Lambda u(t) \text{ and } v^0 := \Lambda u^0. \end{aligned} \tag{1.3}$$

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It is then easily seen that (1.2) can be written in the form (1.1).

Following [1], we express the numerical schemes in terms of bounded rational functions $\rho_i, \sigma_i : [0,\infty] \to \mathbb{R}$, $i = 0,\dots,q$, with $\rho_q = 1$ and $\sigma_q = 0$; we assume that the functions σ_i vanish at infinity, $\sigma_i(\infty) = 0$.

Let $N \in \mathbb{N}$, $k := \frac{T}{N}$ be the time step, and $t^n := nk, n = 0,\dots,N$. We recursively define a sequence of approximations $V^m \in \mathcal{V}, \mathcal{V} := D(\mathcal{A}^{1/2})$, to $v^m := v(t^m)$ by

$$\sum_{i=0}^{q} \rho_i(k\mathcal{A})V^{n+i} = k\sum_{i=0}^{q-1} \sigma_i(k\mathcal{A})\mathcal{B}(t^{n+i}, V^{n+i}), \tag{1.4}$$

assuming that starting approximations $V^0,\dots,V^{q-1}$ are given. The approximations $U^m \in V$, $V := D(A^{1/2})$, to $u^m := u(t^m)$, i.e., to the value of the solution of (1.2) at the time level t^m, are then defined by $U^m := \Lambda^{-1}V^m$, that is as solutions of the equations $\Lambda U^m = V^m$; alternatively, bypassing V^m, we may directly define the approximations U^m by

$$\sum_{i=0}^{q} \rho_i(k\mathcal{A})\Lambda U^{n+i} = k\sum_{i=0}^{q-1} \sigma_i(k\mathcal{A})\mathcal{B}(t^{n+i}, \Lambda U^{n+i}),$$

i.e., by

$$\sum_{i=0}^{q} \rho_i(k\mathcal{A})\Lambda U^{n+i} = k\sum_{i=0}^{q-1} \sigma_i(k\mathcal{A})\Lambda^{-1}B(t^{n+i}, U^{n+i}). \tag{1.5}$$

Let $|\cdot|$ denote the norm of H, and introduce in $\mathcal{H}, \mathcal{H} := D(\Lambda)$, and V the norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$, respectively, by $\|w\| := |\Lambda w|$ and $|\!|\!|w|\!|\!| := |A^{1/2}w|$; we assume that $|\!|\!|\cdot|\!|\!|$ dominates $\|\cdot\|$ in V, and $\|\cdot\|$ dominates $|\cdot|$.

We identify H with its dual, and denote by V' the dual of V, and by $|\!|\!|\cdot|\!|\!|_\star, |\!|\!|w|\!|\!|_\star := |A^{-1/2}w|$, the dual norm on V'. For stability purposes, we assume that $B(t,\cdot)$ can be extended to an operator from V into V' —this is actually the condition needed in the sequel; the hypothesis $B(t,\cdot) : D(A) \to H$, $t \in [0,T]$, has only been made for simplicity— and an estimate of the form

$$|\!|\!|B(t,w) - B(t,\tilde{w})|\!|\!|_\star \le \lambda|\!|\!|w-\tilde{w}|\!|\!| + \mu\|w-\tilde{w}\| \qquad \forall w,\tilde{w} \in T_u \tag{1.6}$$

holds in a tube $T_u, T_u := \{w \in V : \min_t |\!|\!|u(t) - w|\!|\!| \le 1\}$, around the solution u, uniformly in t, with the *stability constant* λ and a constant μ.

We will assume in the sequel that (1.2) possesses a solution which is sufficiently regular for our results to hold. Local uniqueness of smooth solutions follows easily in view of (1.6).

Stability assumptions. For $x \in [0,\infty]$, we introduce the polynomials $\rho(x,\cdot)$ and $\sigma(x,\cdot)$ by

$$\rho(x,\zeta) := \sum_{i=0}^{q} \rho_i(x)\zeta^i, \quad \sigma(x,\zeta) := \sum_{i=0}^{q-1} \sigma_i(x)\zeta^i.$$

We order the roots $\zeta_j(x), j = 1,\dots,q$, of $\rho(x,\cdot)$ in such a way that the functions ζ_j are continuous in $[0,\infty]$ and the roots $\xi_j := \zeta_j(0), j = 1,\dots,s$, satisfy $|\xi_j| = 1$;

these unimodular roots are the *principal roots* of $\rho(0,\cdot)$ and the complex numbers $\lambda_j := \frac{\partial_1\rho(0,\xi_j)}{\xi_j\,\partial_2\rho(0,\xi_j)}$ (with ∂_1 denoting differentiation with respect to the first variable) are the *growth factors* of ξ_j. We assume that the method described by the rational functions $\rho_0,\dots,\rho_q$ is *strongly* $A(0)-$*stable*, cf. [1], i.e.,

$$(i)\qquad \begin{array}{l}\text{for all } 0 < x \le \infty \text{ and for all } j = 1,\dots,q,\\ \text{there holds } |\zeta_j(x)| < 1,\end{array}$$

and

$$(ii)\qquad \begin{array}{l}\text{the principal roots of } \rho(0,\cdot) \text{ are simple and their growth}\\ \text{factors have positive real parts, } \operatorname{Re}\lambda_j > 0, j = 1,\dots,s.\end{array}$$

Depending on the particular scheme we will use for discretizing (1.1) in time, it will be essential for our analysis that λ be appropriately small. More precisely, with

$$K_{(\rho,\sigma)} := \sup_{x>0}\max_{\zeta\in S_1}\Big|\frac{x\sigma(x,\zeta)}{\rho(x,\zeta)}\Big|, \tag{1.7}$$

which, under our assumptions, is finite, we will assume for stability purposes that

$$\lambda < \frac{1}{K_{(\rho,\sigma)}}; \tag{1.8}$$

here S_1 denotes the unit circle in the complex plane, $S_1 := \{z \in \mathbb{C} : |z| = 1\}$. The tube T_u is defined in terms of the norm of V for concreteness. The analysis may be modified to yield convergence under conditions analogous to (1.6) for w and $\tilde{w}$ belonging to tubes defined in terms of other norms, not necessarily the same for both arguments.

Consistency assumptions. We first state the consistency hypotheses for the discretization in time. Let $p \ge 1$, and functions $\varphi_\ell : [0,\infty) \to \mathbb{R}, \ell = 0,\dots,p$, be defined by

$$\varphi_\ell(x) := \sum_{i=0}^{q}\big[i^\ell\rho_i(x) - (\ell i^{\ell-1} + xi^\ell)\sigma_i(x)\big],\ \ \ell = 0,\dots,p-1,$$

and

$$\varphi_p(x) := \sum_{i=0}^{q}\big[i^p\rho_i(x) - pi^{p-1}\sigma_i(x)\big].$$

We assume that the order of the scheme is p, i.e.,

$$(C_p)\qquad \varphi_\ell(x) = O(x^{p+1-\ell}) \ \text{ as } x \to 0+, \quad \ell = 0,\dots,p,$$

and its polynomial order $\tilde{p} \le p$, i.e.,

$$(\tilde{C}_{\tilde{p}})\qquad \varphi_\ell = 0, \quad \ell = 0,\dots,\tilde{p}-1,$$

see [1].

Next, we state the consistency assumptions for the discretization in space. For the space discretization we use a family V_h, $0 < h < 1$, of finite dimensional subspaces of V. In the sequel the following discrete operators will play an essential role: Define $P_o : V' \to V_h$, $L_h, A_h : V \to V_h$ and $B_h(t,\cdot) : V \to V_h$ by

$$\begin{aligned} (P_o w, \chi) &= (w, \chi) && \forall \chi \in V_h \\ (L_h\varphi, \chi) &= (L\varphi, \chi) && \forall \chi \in V_h \\ (A_h\varphi, \chi) &= (A\varphi, \chi) && \forall \chi \in V_h \\ (B_h(t,\varphi), \chi) &= (B(t,\varphi), \chi) && \forall \chi \in V_h. \end{aligned}$$

Thus, we are led to a semidiscrete problem approximating (1.2): we seek a function u_h, $u_h(t) \in V_h$, defined by

$$\begin{aligned} &L_h u_h'(t) + A_h u_h(t) = B_h(t, u_h(t)), \quad 0 < t < T, \\ &u_h(0) = u_h^0; \end{aligned} \tag{1.11}$$

here $u_h^0 \in V_h$ is a given approximation to u^0. The semidiscrete approximation v_h, $v_h(t) \in V_h$, to v is then defined by

$$\begin{aligned} &v_h'(t) + \mathcal{A}_h v_h(t) = \mathcal{B}_h(t, v_h(t)), \quad 0 < t < T, \\ &v_h(0) = v_h^0 \end{aligned} \tag{1.12}$$

with $v_h^0 := \Lambda_h u_h^0$,

$$\Lambda_h := L_h^{1/2},\ \mathcal{A}_h := \Lambda_h^{-1} A_h \Lambda_h^{-1},\ \mathcal{B}_h(t,\cdot) := \Lambda_h^{-1} B_h(t, \Lambda_h^{-1}\cdot), \tag{1.9}$$

and Λ_h^{-1} is considered an operator from V_h onto itself.

In analogy to (1.4), we recursively define a sequence of fully discrete approximations $V_h^m \in V_h$ to v^m by

$$\sum_{i=0}^{q} \rho_i(k\mathcal{A}_h) V_h^{n+i} = k \sum_{i=0}^{q-1} \sigma_i(k\mathcal{A}_h) \mathcal{B}_h(t^{n+i}, V_h^{n+i}), \tag{1.10}$$

assuming that starting approximations $V_h^0, \dots, V_h^{q-1} \in V_h$ are given. The fully discrete approximations $U_h^m \in V_h$ to u^m are then defined by $U_h^m := \Lambda_h^{-1} V_h^m$, that is as solutions of the equations $\Lambda_h U_h^m = V_h^m$; alternatively, cf. (1.5), the approximations U_h^m may be directly defined by

$$\sum_{i=0}^{q} \rho_i(k\mathcal{A}_h) \Lambda_h U_h^{n+i} = k \sum_{i=0}^{q-1} \sigma_i(k\mathcal{A}_h) \Lambda_h^{-1} B_h(t^{n+i}, U_h^{n+i}). \tag{1.11}$$

Let $B(t,\cdot) : V \to V'$ be differentiable, and assume that the linear operator $M(t)$, $M(t) := A - B'(t, u(t)) + \kappa I$, is uniformly positive definite, for an appropriate constant κ. We introduce the 'elliptic' projection operator $R_h(t) : V \to V_h, t \in [0,T]$, by

$$P_o M(t) R_h(t) w = P_o M(t) w. \tag{1.12}$$

We assume that $R_h(t)$ has the following approximation properties

$$|u(t) - R_h(t)u(t)| + h^{d/2} \| u(t) - R_h(t)u(t) \| \le C\, h^r, \tag{1.13}$$

and

$$\|\!|L(u - R_h u)'(t)\|\!|_\star \le C\, h^r, \tag{1.14}$$

with two integers r and d, $2 \le d \le r$. We further assume that

$$\|\!|\frac{d^j}{dt^j}[R_h(t)u(t)]\|\!| \le C, \quad j = 1, \dots, p+1. \tag{1.15}$$

For consistency purposes, we assume for the nonlinear part the estimate

$$\begin{aligned} \|\!|B(t, u(t)) - B(t, R_h(t)u(t)) - B'(t, u(t))(u(t) - R_h(t)u(t))\|\!|_\star \\ \le C h^r. \end{aligned} \tag{1.16}$$

Let us note here that condition (1.14) essentially means that, if A is a differential operator of order d, then L is a differential operator of order at most $d/2$. Condition (1.16) on the other hand may be satisfied even if B contains derivatives of order d.

If both operators A and B are dominated by L, then the differential equation in (1.2) is nonstiff and problem (1.2) may be integrated in a stable way by explicit schemes, cf. [5]; therefore, in this case there is no need to resort to linearly implicit schemes.

Let the order and the polynomial order of the method be p and $p-1$, respectively, i.e., $\tilde p = p - 1$. For initial approximations $U^0, \dots, U^{q-1} \in V$ to $u(t^0), \dots, u(t^{q-1})$ such that

$$\sum_{j=0}^{q-1} \big(\|u(t^j) - U^j\| + k^{1/2}\|\!|u(t^j) - U^j\|\!|\big) \le ck^p, \tag{1.17}$$

we shall prove, for sufficiently small k, the error estimate

$$\max_{0\le n\le N} \|u(t^n) - U^n\| \le Ck^p. \tag{1.18}$$

Concerning the fully discrete approximations, letting $W(t) := R_h(t)u(t)$, for starting approximations $U_h^0, \dots, U_h^{q-1} \in V_h$ such that

$$\sum_{j=0}^{q-1} \big(\|W(t^j) - U_h^j\| + k^{1/2}\|\!|W(t^j) - U_h^j\|\!|\big) \le c(k^p + h^r), \tag{1.19}$$

we shall prove, for $k^{-1}h^{2r}$ and k sufficiently small, the error estimate

$$\max_{0\le n\le N} \|W(t^n) - U^n\| \le C(k^p + h^r), \tag{1.20}$$

which combined with (1.13) yields

$$\max_{0\le n\le N} |u(t^n) - U^n| \le C(k^p + h^r). \tag{1.21}$$

As already mentioned the implicit–explicit multistep schemes are particular cases of the schemes considered in this paper. Indeed, if we let (α, β) be a strongly $A(0)-$stable $q-$step scheme and (α, γ) be an explicit $q-$step scheme, characterized by three polynomials α, β and γ,

$$\alpha(\zeta) = \sum_{i=0}^{q} \alpha_i \zeta^i\,, \quad \beta(\zeta) = \sum_{i=0}^{q} \beta_i \zeta^i\,, \quad \gamma(\zeta) = \sum_{i=0}^{q-1} \gamma_i \zeta^i,$$

then the corresponding implicit–explicit (α,β,γ) scheme for (1.11) is

$$\sum_{i=0}^{q}(\alpha_i L_h + k\beta_i A_h)U_h^{n+i} = k\sum_{i=0}^{q-1}\gamma_i B_h(t^{n+i},U_h^{n+i}). \tag{1.26}$$

Letting now

$$\rho_i(x) := \frac{\alpha_i+\beta_i x}{\alpha_q+\beta_q x}, i=0,\dots,q, \quad \text{and} \quad \sigma_i(x) := \frac{\gamma_i}{\alpha_q+\beta_q x},$$

$i=0,\dots,q-1$, it is easily seen that the scheme (1.11) reduces to (1.26). For specific examples of implicit–explicit multistep schemes we refer to [2] and [3].

An outline of the paper is as follows: In Section 2 we show that the assumptions of [1] for problem (1.1) are satisfied under our hypotheses for (1.2) and this allows us to derive optimal order estimates. In Section 3 we apply our abstract results to a periodic initial value problem for a third–order evolution equation arising in the modelling of flow in a fluidized bed; in space the equation is discretized by a Petrov–Galerkin method.

2. Error estimates

It is easily seen that the operator $\mathcal{A}$ given in (1.3) is selfadjoint and positive definite. Further,

$$|\mathcal{A}^{1/2}w|^2 = (\mathcal{A}w,w) = (A^{1/2}\Lambda^{-1}w, A^{1/2}\Lambda^{-1}w),$$

and, thus,

$$|\mathcal{A}^{1/2}w| = |A^{1/2}\Lambda^{-1}w|. \tag{2.1}$$

In particular, between V and $\mathcal{V}$, the domains of $A^{1/2}$ and $\mathcal{A}^{1/2}$, respectively, we have the relation $\mathcal{V}=\Lambda(V)$. Similarly, we have

$$|\mathcal{A}^{-1/2}w| = |A^{-1/2}\Lambda w|. \tag{2.2}$$

Obviously, $\mathcal{V}'=\Lambda^{-1}(V')$. Let $w\in\mathcal{V}$. Then $\Lambda^{-1}w\in V$ and thus $B(t,\Lambda^{-1}w)\in V'$. Therefore, $\Lambda^{-1}B(t,\Lambda^{-1}w)\in\Lambda^{-1}(V')=\mathcal{V}'$, and we conclude that the operator $\mathcal{B}(t,\cdot)$, $t\in[0,T]$, given in (1.3), maps $\mathcal{V}$ into $\mathcal{V}'$. Next, we show a local Lipschitz condition for $\mathcal{B}$. First, we rewrite (1.6) in the form

$$|A^{-1/2}\big(B(t,w)-B(t,\tilde w)\big)| \le \lambda|A^{1/2}(w-\tilde w)| + \mu|\Lambda(w-\tilde w)| \tag{1.6$'$}$$

for all $w,\tilde w\in T_u$. Let $\mathcal{T}_u := \{w\in\mathcal{V} : \Lambda^{-1}w\in T_u\}$. Now, using (2.2) and (1.6′), for $w,\tilde w\in\mathcal{T}_u$, we have

$$\begin{aligned}|\mathcal{A}^{-1/2}\big(\mathcal{B}(t,w)-\mathcal{B}(t,\tilde w)\big)| &= |A^{-1/2}\Lambda\big(\mathcal{B}(t,w)-\mathcal{B}(t,\tilde w)\big)|\\ &= |A^{-1/2}\big(B(t,\Lambda^{-1}w)-B(t,\Lambda^{-1}\tilde w)\big)|\\ &\le \lambda|A^{1/2}\Lambda^{-1}(w-\tilde w)| + \mu|\Lambda\Lambda^{-1}(w-\tilde w)|,\end{aligned}$$

i.e., in view of (2.1),

$$|\mathcal{A}^{-1/2}\big(\mathcal{B}(t,w)-\mathcal{B}(t,\tilde w)\big)| \le \lambda|\mathcal{A}^{1/2}(w-\tilde w)| + \mu|w-\tilde w|, \tag{2.3}$$

for all $w,\tilde w\in\mathcal{T}_u$.

Let the consistency error $E^n, n = 0, \dots, N-q$, of the scheme (1.4) for the solution v of (1.1) be given by

$$\begin{aligned} &k(I + k\mathcal{A})^{-1}E^n = \\ &\sum_{i=0}^{q} \rho_i(k\mathcal{A})v(t^{n+i}) - k\sum_{i=0}^{q-1} \sigma_i(k\mathcal{A})\mathcal{B}(t^{n+i}, v(t^{n+i})). \end{aligned} \tag{2.4}$$

It is easily seen that $E^n, n = 0, \dots, N-q$, is also the consistency error of the scheme (1.5) for the solution u of (1.2),

$$\begin{aligned} &k(I + k\mathcal{A})^{-1}E^n = \\ &\sum_{i=0}^{q} \rho_i(k\mathcal{A})\Lambda u(t^{n+i}) - k\sum_{i=0}^{q-1} \sigma_i(k\mathcal{A})\Lambda^{-1}B(t^{n+i}, u(t^{n+i})). \end{aligned} \tag{2.5}$$

Theorem 4.2 of [1] yields the following result:

Theorem 2.1. *Let the order and the polynomial order of the scheme be p and $p-1$, respectively. Assume that* (1.6) *—and hence also* (2.3)*— is satisfied with a constant λ satisfying* (1.8)*. Let starting approximations $V^0, V^1, \dots, V^{q-1} \in \mathcal{V}$ to $v(t^0), \dots, v(t^{q-1})$ be given such that*

$$\sum_{j=0}^{q-1} \big(|v(t^j) - V^j| + k^{1/2}|\mathcal{A}^{1/2}\big(v(t^j) - V^j\big)|\big) \le Ck^p \tag{2.6}$$

and $V^n \in \mathcal{V}$, $n = q, \dots, N$, be recursively defined by (1.4)*. Let $\vartheta^n = v(t^n) - V^n, n = 0, \dots, N$. Then, there exist constants C and c, independent of k and n, such that, for k sufficiently small,*

$$\begin{aligned} &|\vartheta^n|^2 + k\sum_{\ell=0}^{n} \|\vartheta^\ell\|^2 \le \\ &Ce^{c\mu^2 t^n}\Big\{\sum_{j=0}^{q-1}\big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + k\sum_{\ell=0}^{n-q}\|\Lambda E^\ell\|_\star^2\Big\}, \end{aligned} \tag{2.7}$$

$n = q-1, \dots, N$, cf. (2.2)*, and*

$$\max_{0\le n\le N} |v(t^n) - V^n| \le Ck^p. \tag{2.8}$$

For the approximations $U^n = \Lambda^{-1}V^n$, given by (1.5)*, to $u(t^n)$ the condition* (2.6)*, the local stability estimate* (2.7)*, with $\zeta^n = u(t^n) - U^n$, and the error estimate* (2.8) *read, respectively,*

$$\sum_{j=0}^{q-1} \big(\|u(t^j) - U^j\| + k^{1/2}|\!|\!|u(t^j) - U^j|\!|\!|\big) \le Ck^p \tag{2.9}$$

$$\begin{aligned} &\|\zeta^n\|^2 + k\sum_{\ell=0}^{n} |||\zeta^\ell|||^2 \leq \\ &\qquad C\mathrm{e}^{c\mu^2 t^n}\Big\{\sum_{j=0}^{q-1}\big(\|\zeta^j\|^2 + k|||\zeta^j|||^2\big) + k\sum_{\ell=0}^{n-q}\|\varLambda E^\ell\|_\star^2\Big\}, \end{aligned} \tag{2.10}$$

$n = q-1,\dots,N,$ *and*

$$\max_{0\le n\le N}\|u(t^n) - U^n\| \le Ck^p. \qquad \square \tag{2.11}$$

Let $E_h(t) \in V_h$ denote the consistency error of the semidiscrete equation (1.11) for W, the elliptic projection of the solution u of (1.2),

$$E_h(t) := L_h W'(t) + A_h W(t) - B_h(t, W(t)), \quad 0 \le t \le T. \tag{2.12}$$

From the definition of W we easily conclude

$$(A_h W(t), \chi) = (Au(t) - \big[B'(t,u(t)) - \kappa I\big](u(t) - W(t)), \chi) \tag{2.13}$$

for all $\chi \in V_h$. Therefore, using (1.2),

$$\begin{aligned} E_h(t) &= L_h W'(t) - P_o L u'(t) + \kappa\big[P_o u(t) - W(t)\big] \\ &\quad + P_o\big[B(t,u(t)) - B(t,W(t)) - B'(t,u(t))(u(t) - W(t))\big], \end{aligned}$$

and, in view of (1.14), (1.13) and (1.16), we easily obtain the following optimal order estimate for the consistency error E_h,

$$\max_{0\le t\le T}\|E_h(t)\|_\star \le Ch^r. \tag{2.14}$$

The main result in this paper is given in the following theorem:

Theorem 2.2. *Let the order and the polynomial order of the scheme be* p *and* $p-1$*, respectively. Assume we are given initial approximations* $U_h^0, U_h^1, \dots, U_h^{q-1} \in V_h$ *to* $u(t^0),\dots,u(t^{q-1})$ *such that*

$$\sum_{j=0}^{q-1}\big(\|W(t^j) - U_h^j\| + k^{1/2}|||W(t^j) - U_h^j|||\big) \le C(k^p + h^r). \tag{2.15}$$

Let $U_h^n \in V_h$, $n = q,\dots,N$, *be recursively defined by* (1.11). *Then, there exists a constant* C*, independent of* k *and* h*, such that, for* k *and* $h^{2r}k^{-1}$ *sufficiently small,*

$$\max_{0\le n\le N}|u(t^n) - U^n| \le C(k^p + h^r). \tag{2.16}$$

Proof. Let $\rho^n := u(t^n) - W(t^n), n = 0,\dots,N$. In view of (1.13), we have

$$\max_{0\le n\le N}|\rho^n| \le Ch^r. \tag{2.17}$$

Obviously, $\tilde B(t,v) := B(t,v) + E_h(t)$, cf. (2.12), satisfies (1.6) with the same constants λ and μ. Now let $\tilde W^j := W(t^j), j = 0,\dots,q-1$, and define $\tilde W^n, n = q,\dots,N$, by applying the time discretization scheme to the equation (2.12), i.e. by

$$\sum_{i=0}^{q}\rho_i(k\mathcal{A}_h)\varLambda_h\tilde W^{n+i} = k\sum_{i=0}^{q-1}\sigma_i(k\mathcal{A}_h)\varLambda_h^{-1}\tilde B_h(t^{n+i}, \tilde W^{n+i}), \tag{2.18}$$

with $\tilde{B}_h(t,v) = B_h(t,v) + E_h(t)$. Then, according to (2.11), and in view of (1.15),

$$\max_{0\le n\le N} \|W^n - \tilde{W}^n\| \le Ck^p. \tag{2.19}$$

In view of (2.17) and (2.19), it remains to estimate $\zeta^n := \tilde{W}^n - U_h^n$. Subtracting (1.11) from (2.18), we obtain

$$\begin{aligned}\sum_{i=0}^{q} \rho_i(kA_h)\Lambda_h\zeta^{n+i} &= k\sum_{i=0}^{q-1} \sigma_i(kA_h)\Lambda_h^{-1}E_h(t^{n+i})\\ &+ k\sum_{i=0}^{q-1} \sigma_i(kA_h)\Lambda_h^{-1}\big[B_h(t^{n+i},\tilde{W}^{n+i}) - B_h(t^{n+i},U_h^{n+i})\big]\end{aligned} \tag{2.20}$$

Using now the boundedness of σ_i and (2.10), we get

$$\begin{aligned}&\|\zeta^n\|^2 + k\sum_{\ell=0}^{n} |||\zeta^\ell|||^2 \le\\ &\qquad Ce^{c\mu^2 t^n}\Big\{\sum_{j=0}^{q-1}\big(\|\zeta^j\|^2 + k|||\zeta^j|||^2\big) + k\sum_{\ell=0}^{n-q}\|E_h(t^\ell)\|_\star^2\Big\}.\end{aligned} \tag{2.21}$$

From this estimate, in view of (2.14) and our condition on the starting approximations, we easily conclude

$$\max_{0\le n\le N} \|\tilde{W}^n - U_h^n\| \le C(k^p + h^r). \tag{2.22}$$

Let us note that it is in the derivation of (2.21) and (2.22) where we need the mesh-condition "$h^{2r}k^{-1}$ sufficiently small"; this is due to the fact that in the course of the proof we use the estimates, cf. (4.9) and the proof of Theorem 5.1 in [1],

$$\max_{0\le j\le n-1} |||\zeta^j||| \le C_\star(k^{p-1/2} + h^r k^{-1/2}) \le 1/2,$$

and for the last estimate to be satisfied we need to assume k and $h^{2r}k^{-1}$ to be sufficiently small; this ensures $U^j \in T_u, j = 0,\dots,n-1$. From (2.17), (2.19) and (2.22) the desired estimate (2.16) follows and the proof is complete. □

Remark 2.1. Assuming that (1.6) holds with $\lambda = 0$, i.e.,

$$|||B(t,w) - B(t,\tilde{w})|||_\star \le \mu\|w - \tilde{w}\| \qquad \forall w,\tilde{w} \in T_u, \tag{2.23}$$

and that $\rho_0,\dots,\rho_{q-1}$ vanish at infinity, one can derive the error estimate (2.16) under the milder condition

$$\sum_{j=0}^{q-1} \|W(t^j) - U_h^j\| \le c(k^p + h^r), \quad U_h^0,\dots,U_h^{q-1} \in T_u, \tag{2.24}$$

on the starting approximations. This is due to the fact that in this case (2.10) takes the form, see Remark 7.2 in [1],

$$\|\zeta^n\|^2 + k\sum_{\ell=q}^{n} |||\zeta^\ell|||^2 \le Ce^{c\mu^2 t^n}\Big\{\sum_{j=0}^{q-1}\|\zeta^j\|^2 + k\sum_{\ell=0}^{n-q}\|\Lambda E^\ell\|_\star^2\Big\}. \tag{2.25}$$

Remark 2.2. Let $\tau \in \mathbb{R}$ be such that $A + \tau I$ is positive semidefinite. It is then easily seen that the error estimate (2.16) holds also for the scheme

$$\begin{aligned} &\sum_{i=0}^{q} \rho_i(k\tilde{\mathcal{A}}) \Lambda U^{n+i} = \\ &\qquad k \sum_{i=0}^{q-1} \sigma_i(k\tilde{\mathcal{A}}) \Lambda^{-1} \big[B(t^{n+i}, U^{n+i}) + \tau U^{n+i}\big] \end{aligned} \tag{2.26}$$

with $\tilde{\mathcal{A}} := \Lambda^{-1}(A + \tau I)\Lambda^{-1}$.

Remark 2.3. The mild meshcondition "$k^{-1}h^{2r}$ small" is used only to show that $|||\zeta^n||| \le 1/2$ which combined with (1.13) and (2.19) implies $U_h^n \in T_u$. If the estimate (1.6) holds in tubes around u defined in terms of weaker norms, not necessarily the same for both arguments w and $\tilde{w}$, one may get by with an even milder meshcondition. Assume, for instance, that (1.6) holds for $w, \tilde{w} \in T_u^\star := \{\omega \in V : \min_t \|u(t) - \omega\|^\star \le 1\}$ —or for $w \in T_u$, cf. (1.13), and $\tilde{w} \in T_u^\star$— and the norm $\|\cdot\|^\star$ satisfies an inequality of the form

$$\|w\|^\star \le \|w\| + \|w\|^{1-a} |||w|||^a, \quad w \in V,$$

for some constant $a, 0 \le a < 1$. Then, a condition of the form "k and $k^{-a}h^{2r}$ sufficiently small" suffices for (2.16) to hold.

Similarly, when the relation (1.6) is satisfied in tubes around u defined in terms of stronger norms, not necessarily the same for both arguments, the error estimate (2.16) may still be valid but under *stronger* meshconditions, cf. [1].

3. Application to the third–order fluidization equation

In this section we consider the following periodic initial value problem for a third–order evolution equation: For $T > 0$, we seek a real-valued function u defined on $\mathbb{R} \times [0, T]$, 1−periodic in the space variable and satisfying

$$u_t + u_{xxx} + \kappa(u^2)_x + \nu(u^2)_{xx} + \varepsilon u_{xx} - \delta u_{tx} = 0, \quad 0 < t < T, \tag{3.1}$$

and

$$u(\cdot, 0) = u^0 \quad \text{in} \quad \mathbb{R}, \tag{3.2}$$

with u^0 a given, smooth 1−periodic function. Here, κ, ν, ε and δ are real constants, and ε, δ are positive.

Equation (3.1) arises in the modelling of flow in a fluidized bed, see [6] and [9]. The unknown u represents the value of a small perturbation of the concentration of particles. For numerical methods for problem (3.1)–(3.2) we refer to [8] and the references therein.

The standard Galerkin finite element method for (3.1)–(3.2) is unstable. A stable semidiscrete Petrov–Galerkin method is proposed and analyzed in [8]. In this section we shall combine the Petrov–Galerkin method of [8] with the time–stepping schemes considered in the previous sections to obtain efficient fully discrete schemes. We establish optimal-order error estimates in the L^2−norm for the fully discrete approximations.

For $s \in \mathbb{N}_0$, let H^s_{per} denote the periodic Sobolev space of order s, consisting of the $1-$periodic elements of $H^s_{\text{loc}}(\mathbb{R})$, and let $\|\cdot\|_{H^s}$ be the norm over a period in H^s_{per}. The inner product in $H := L^2_{\text{per}} = H^0_{\text{per}}$ is denoted by $(\cdot,\cdot)$, and the induced norm by $|\cdot|$.

For the space discretization, we let $0 = x_0 < x_1 < \cdots < x_J = 1$ be a partition of $[0,1]$, and $h := \max_j(x_{j+1} - x_j)$. Setting $x_{jJ+s} := j + x_s,\ j \in \mathbb{Z},\ \ s = 0,\dots,J-1$, this partition is periodically extended to a partition of $\mathbb{R}$. For integer $r \geq 4$, let V_h denote a space of at least once continuously differentiable, $1-$periodic splines of degree $r-1$, in which approximations to the solution $u(\cdot,t)$ of (3.1)–(3.2) will be sought for $0 \leq t \leq T$. The following approximation property of the family $\{V_h\}_{0<h<1}$ is well known, cf., e.g., [11],

$$\inf_{\chi \in V_h} \sum_{j=0}^{2} h^j \|v - \chi\|_{H^j} \leq ch^s \|v\|_{H^s},\ \ v \in H^s_{\text{per}},\ \ 2 \leq s \leq r. \tag{3.3}$$

The standard Galerkin finite element method for (3.1)–(3.2) is unstable, see [8]. The Petrov–Galerkin finite element method, based on the weak formulation of (3.1) obtained by taking the inner product of (3.1) with $\chi - \delta\chi', \chi \in V_h$, and integrating by parts, is as follows: Seek $u_h(t) \in V_h$, $t \in [0,T]$, satisfying

$$\begin{gathered}(u_{ht},\chi) + \delta^2(u_{hxt},\chi') + \delta(u_{hxx},\chi'') - (1+\delta\varepsilon)(u_{hxx},\chi') + \\ (\nu + \kappa\delta)(u_h^2,\chi'') - \kappa(u_h^2,\chi') + \nu\delta((u_h^2)_x,\chi'') - \varepsilon(u_{hx},\chi') = 0,\end{gathered} \tag{3.4}$$

for all $\chi \in V_h$ and for all $t \in [0,T]$, and

$$u_h(\cdot,0) = u_h^0 \tag{3.5}$$

with $u_h^0 \in V_h$ an approximation to the initial value u^0,

$$|u^0 - u_h^0| \leq Ch^r. \tag{3.6}$$

Let $H := L^2_{\text{per}}$, and the operators $A : H^4_{\text{per}} \to H$ and $L : V \to H$ be defined by $Av := \delta v_{xxxx} + \tau v$, with sufficiently large τ, and $Lv := v - \delta^2 v_{xx}$. Then $V := D(A^{1/2}) = H^2_{\text{per}}$, $\mathcal{H} := D(L^{1/2}) = H^1_{\text{per}}$, and the norms in V and $\mathcal{H}$ are given by $\|\!|v|\!\| = (\delta|v_{xx}|^2 + \tau|v|^2)^{1/2}$ and $\|v\| = (|v|^2 + \delta^2|v_x|^2)^{1/2}$, respectively. Let $B : V \to V'$ be given by $B(v) := -(1+\delta\varepsilon)v_{xxx} - \delta\nu(v^2)_{xxx} - (\nu + \delta\kappa)(v^2)_{xx} - \kappa(v^2)_x - \varepsilon v_{xx} + \tau v$. Differentiating equation (3.1) with respect to x, multiplying the result by δ and adding to the original equation, we can rewrite (3.1) in the form

$$Lu_t + Au = B(u), \quad 0 < t < T. \tag{3.7}$$

It is easily seen that the standard Galerkin method for (3.7), with approximating space V_h, coincides with (3.4).

Following the corresponding abstract setting in the Introduction, we define the operators $P_o : V' \to V_h$, $L_h, A_h : V \to V_h$ and $B_h(t,\cdot) : V \to V_h$ by

$$\begin{aligned}
(P_o w, \chi) &= (w, \chi) && \forall \chi \in V_h\\
(L_h \varphi, \chi) &= (\varphi, \chi) + \delta^2(\varphi_x, \chi_x) && \forall \chi \in V_h\\
(A_h \varphi, \chi) &= \delta(\varphi_{xx}, \chi_{xx}) + \tau(\varphi, \chi) && \forall \chi \in V_h\\
(B_h(t,\varphi), \chi) &= (B(t,\varphi), \chi) && \forall \chi \in V_h.
\end{aligned}$$

Let $N \in \mathbb{N}$, $k := \frac{T}{N}$ be the time step, and $t^n := nk, n = 0, \dots, N$. With the notation of the Introduction, we define a sequence of approximations U_h^n, $U_h^n \in V_h$, to $u(t^n) := u(\cdot, t^n)$, $n = q, \dots, N$, by

$$\sum_{i=0}^{q} \rho_i(k\mathcal{A}_h)\Lambda_h U_h^{n+i} = k \sum_{i=0}^{q-1} \sigma_i(k\mathcal{A}_h)\Lambda_h^{-1} B_h(t^{n+i}, U_h^{n+i}). \tag{3.8}$$

Assume that the time–stepping scheme satisfies the stability assumptions of the Introduction and that its order and polynomial order are p and $p-1$, respectively. Then, for starting approximations $U_h^0, \dots, U_h^{p-1}$ satisfying (3.15) (or ((3.16)) below, and for sufficiently small k and h, the analysis of the previous sections can be used to establish the optimal order error estimate

$$\max_{0 \le n \le N} |u(t^n) - U_h^n| \le c(k^p + h^r). \tag{3.9}$$

Indeed, to prove (3.9), we have only to verify the hypotheses on L, A and B of the previous sections.

First, by periodicity, for $w, \tilde{w}, \omega \in V$,

$$\begin{aligned}
(B(w) - B(\tilde{w}), \omega) = &-(1 + \delta\varepsilon)((w - \tilde{w})_x, \omega_{xx})\\
&- \delta\nu((w^2 - \tilde{w}^2)_x, \omega_{xx}) - (\nu + \delta\kappa)(w^2 - \tilde{w}^2, \omega_{xx})\\
&+ \kappa(w^2 - \tilde{w}^2, \omega_x) - \varepsilon(w - \tilde{w}, \omega_{xx}) + \tau(w - \tilde{w}, \omega)
\end{aligned}$$

and, using the fact that the H^1-norm dominates in one dimension the $L^\infty-$norm, we easily see that

$$|\!|\!| B(w) - B(\tilde{w}) |\!|\!|_\star \le \mu \|w - \tilde{w}\|, \qquad \forall w, \tilde{w} \in T_u^\star, \tag{3.10}$$

with $T_u^\star := \{v \in V : \min_t \|u(\cdot,t) - v\| \le 1\}$.

Further,

$$\begin{aligned}
B'(w)\tilde{w} = &-(1 + \delta\varepsilon)\tilde{w}_{xxx} - 2\delta\nu(w\tilde{w})_{xxx} - 2(\nu + \delta\kappa)(w\tilde{w})_{xx}\\
&- 2\kappa(w\tilde{w})_x - \varepsilon\tilde{w}_{xx} + \tau\tilde{w},
\end{aligned}$$

and we easily see that $M(t) := A - B'(u(t)) + \tau I$ is uniformly positive definite in H^2_{per}, for appropriately large τ.

We define the elliptic projection operator $R_h(t) : V \to V_h$, $t \in [0,T]$, by

$$P_o M(t) R_h(t) v = P_o M(t) v.$$

It is easily seen, cf. Theorem 3.1 in [8], that

$$|u(\cdot,t)-R_h(t)u(\cdot,t)|+h^2\|u(\cdot,t)-R_h(t)u(\cdot,t)\|\le Ch^r, \tag{3.11}$$

$$\|\!|L\frac{\partial}{\partial t}[u(\cdot,t)-R_h(t)u(\cdot,t)]\|\!|_\star\le Ch^r, \tag{3.12}$$

and

$$\|\!|\frac{\partial^j}{\partial t^j}[R_h(t)u(\cdot,t)]\|\!|\le C,\quad j=1,\dots,p+1; \tag{3.13}$$

thus, (1.13), (1.14) and (1.15) are satisfied with $d=4$.

Further, with $W(t):=R_h(t)u(\cdot,t)$, for $w\in H^2_{\text{per}}$,

$$\begin{aligned}(B(u(\cdot,&t))-B(W(t))-B'(u(\cdot,t))(u(\cdot,t)-W(t)),w)\\ &=2\delta\nu((u(\cdot,t)-W(t))(u_x(\cdot,t)-W_x(t)),w'')\\ &+(\nu+\delta\kappa)((u(\cdot,t)-W(t))^2,w'')-\kappa((u(\cdot,t)-W(t))^2,w')\end{aligned}$$

and thus, in view of (3.11),

$$\begin{aligned}\|\!|B(u(\cdot,t))-B(W(t))-B'(u(\cdot,t))(u(\cdot,t)-W(t))\|\!|_\star\\ \le Ch^r,\end{aligned} \tag{3.14}$$

i.e., (1.16) is satisfied.

Assume now that we are given starting approximations $U_h^0,\dots,U_h^{q-1}\in V_h\cap T_u^\star$ to $u^0,\dots,u^{q-1}$ such that

$$\sum_{j=0}^{q-1}\big(\|W(t^j)-U_h^j\|+k^{1/2}\|\!|W(t^j)-U_h^j\|\!|\big)\le c(k^p+h^r) \tag{3.15}$$

or, for the schemes mentioned in Remark 2.1,

$$\sum_{j=0}^{q-1}\|W(t^j)-U_h^j\|\le c(k^p+h^r). \tag{3.16}$$

Then, for the approximations $U_h^q,\dots,U_h^N$ defined by (3.8) we have the estimate (3.9), in view of the results of Section 2, for k and h sufficiently small. Let us emphasize that no meshcondition is needed, since $T_u^\star$ in (3.10) is defined in terms of the norm $\|\cdot\|$, cf. Remark 2.3, and also that the estimate (3.9) holds for all schemes of order and polynomial order p and $p-1$, respectively, considered in this paper, since $\lambda=0$ in (3.10).

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References

1. G. Akrivis and M. Crouzeix *Linearly implicit methods for nonlinear parabolic equations*, Math. Comp. (to appear).
2. G. Akrivis, M. Crouzeix and Ch. Makridakis, *Implicit-explicit multistep finite element methods for nonlinear parabolic problems*, Math. Comp. **67** (1998) 457–477.
3. G. Akrivis, M. Crouzeix and Ch. Makridakis, *Implicit-explicit multistep methods for quasilinear parabolic equations*, Numer. Math. **82** (1999) 521–541.
4. G. A. Baker, V. A. Dougalis and O. Karakashian, *On multistep–Galerkin discretizations of semilinear hyperbolic and parabolic equations*, Nonl. Anal., Theory, Methods & Applic. **4** (1980) 579–597.
5. P. Chatzipantelidis, *Explicit multistep for nonstiff partial differential equations*, Appl. Numer. Math. **27** (1998) 13–31.
6. I. Christie and G. Ganser, *A numerical study of nonlinear waves arising in a one-dimensional model of a fluidized bed*, J. Comput. Phys. **81** (1989) 300–318.
7. M. Crouzeix, *Une méthode multipas implicite-explicite pour l'approximation des équations d'évolution paraboliques*, Numer. Math. **35** (1980) 257–276.
8. G. Fairweather, J. M. Sanz–Serna and I. Christie, *A stabilized Galerkin method for a third-order evolutionary problem*, Math. Comp. **55** (1990) 497–507.
9. G. H. Ganser and D. A. Drew, *Nonlinear periodic waves in a two-phase flow model*, SIAM J. Appl. Math. **47** (1987) 726–736.
10. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II*: *Stiff and Differential-Algebraic Problems*. Springer–Verlag, Berlin Heidelberg, Springer Series in Computational Mathematics v. 14, 1991.
11. L. L. Schumaker, *Spline Functions*: *Basic Theory*. Wiley, New York, 1980.
12. V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*. Springer–Verlag, Berlin, Springer Series in Computational Mathematics v. 25, 1997.

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