FINITE DIFFERENCE DISCRETIZATION WITH VARIABLE MESH OF THE SCHRÖDINGER EQUATION IN A VARIABLE DOMAIN

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Abstract. We consider a partial differential equation of Schrödinger type, known as the 'parabolic' approximation to the Helmholtz equation in the theory of sound propagation in an underwater, range- and depth-dependent environment with a variable bottom. We solve an associated initial- and boundary-value problem by a finite difference scheme of Crank-Nicolson type on a variable mesh. We prove that the method is stable in ℓ_2, establish optimal, second-order error estimates and show results of relevant numerical experiments.

1. Introduction

The partial differential equation of Schrödinger type

$$u_r = \mathrm{i}\alpha u_{zz} + \mathrm{i}\beta(z, r)u \tag{1.0}$$

derived as a 'parabolic' approximation to the Helmholtz equation with cylindrical symmetry, is widely used as a model in numerical computations of long-range, low-frequency sound propagation in underwater acoustics, [7], [5]. Here $u = u(z, r)$ is a complex-valued function of two real variables, the *depth* z and the *range* r, α is a real constant and β a real-valued function of z and r, reflecting the fact that the speed of sound in the sea is supposed to be both depth- and range-dependent. In this paper we shall consider a finite-difference scheme for approximating the solution of (1.0) posed in a variable, range-dependent domain like the one shown in Figure 1.

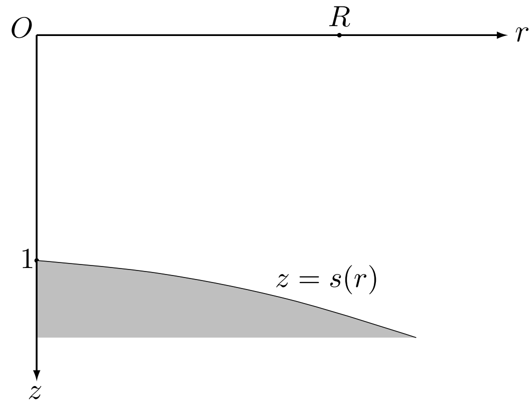

Figure 1. The range-dependent domain of integration

Specifically, given $R > 0$, let $s = s(r),\ r \in [0, R]$, be the (rigid) bottom of the sea, assumed to be a known, real-valued, continuous and piecewise C^1 function, strictly positive on $[0, R]$, such that $s(0) = 1$. Supplementing (1.0) with appropriate auxiliary conditions at the boundary of the domain shown in Figure 1, we consider the initial- and boundary-value problem of finding $u = u(z, r),\ 0 \le z \le s(r),\ 0 \le r \le R$, satisfying

$$\begin{cases} u_r = \mathrm{i}\alpha u_{zz} + \mathrm{i}\beta(z,r)u, & 0 \le r \le R\,, \quad 0 \le z \le s(r), \\ u(0,r) = u(s(r),r) = 0, & 0 \le r \le R, \\ u(z,0) = u_0(z), & 0 \le z \le s(0) = 1. \end{cases} \tag{1.1}$$

We shall suppose that (1.1) possesses a unique solution which is smooth enough for the purposes of its numerical approximation. The boundary condition $u(0, r) = 0$ corresponds to a pressure-release condition on the surface $z = 0$, while setting $u = 0$ at the bottom $z = s(r)$ is not so realistic for applications in underwater acoustics. An appropriate local boundary condition at the bottom would be a mixed-type condition with complex coefficients, cf. e.g. equation (29) in [6]. Recently however, some questions have been raised, cf. [1], regarding the well-posedness of the problem under such bottom conditions. We will treat $u(s(r), r) = 0$ here as a first step in the analysis of finite difference methods for initial- and boundary-value problems for (1.0) noting that (1.1) is certainly well-posed.

We solve (1.1) numerically on a grid which has a uniform step k in range. For N a positive integer, let $k = R/N$ and define $r^n = nk$, $r^{n+1/2} = r^n + \frac{k}{2}, n = 0, 1, 2, \dots$. At each range level r^n we shall partition the depth interval $[0, s(r^n)]$ into $J + 1$ equal subintervals (J will be a fixed positive integer) of length $h_n = s(r^n)/(J + 1)$ and let $z_j^n = jh_n$, $z_j^{n+1/2} = (z_j^n + z_j^{n+1})/2, 0 \le j \le J + 1$. In addition let C_0^{J+2} denote the complex $J + 2$-vectors $g = (g_0, \dots, g_{J+1})^T$ with $g_0 = g_{J+1} = 0$. For $0 \le n \le N$ our scheme will yield approximations $U^n = (U_0^n, \dots, U_{J+1}^n)^T \in \mathbb{C}_0^{J+2}$ to the values $u^n = (u_0^n, \dots, u_{J+1}^n)^T$ of the solution u of (1.1), where $u_j^n = u(z_j^n, r^n)$. The approximations will be defined for $n = 0$ by $U^0 = u^0$, where $u_j^0 = u_0(z_j^0)$, and, for $0 \le n \le N - 1$, by the scheme

$$\begin{aligned} L_h^n(U_j^n) := {} & \big(h_{n+1}\, U_j^{n+1} - h_n\, U_j^n\big) \\ & - \frac{1}{4}(h_{n+1} - h_n) \Big[(j+1)\left(U_{j+1}^{n+1} + U_{j+1}^n\right) - (j-1)\left(U_{j-1}^{n+1} + U_{j-1}^n\right)\Big] \\ & - \frac{\mathrm{i}\alpha k}{2}\Big(\frac{1}{h_{n+1}}\delta_h^2\, U_j^{n+1} + \frac{1}{h_n}\delta_h^2 U_j^n\Big) \\ & - \frac{\mathrm{i}k}{2}\beta_j^{n+1/2}\Big(h_{n+1}U_j^{n+1} + h_n U_j^n\Big) = 0\,, \qquad 1 \le j \le J, \end{aligned} \tag{1.2}$$

where we denote $\delta_h^2\, U_j^n = U_{j+1}^n - 2U_j^n + U_{j-1}^n$, $\beta_j^{n+1/2} = \beta\left(z_j^{n+1/2}, r^{n+1/2}\right)$. This is an implicit, single-step method that requires solving a $J \times J$ tridiagonal linear system of equations at each range step. The scheme was derived in [3] and analyzed by Jamet in [4] in the case of the heat equation in a variable domain. (It may actually be derived by lumping a space-time finite element scheme, cf. [3].) It can be easily seen that when $h_n = h_{n+1}$ —flat bottom— (1.2) reduces to the conservative Crank-Nicolson scheme, [2], for the Schrödinger equation.

First we make some observations and assumptions regarding the variable mesh. Since $h_n = s(r^n)/(J+1)$ and J is constant, we see that the ratio h_n/h_m is uniformly bounded above and below by positive constants. Specifically,

$$\sigma^{-1} \le h_n/h_m \le \sigma, \qquad 0 \le n, m \le N, \tag{1.3}$$

where $\sigma = \max_{0\le r\le R} s(r)/\min_{0\le r\le R} s(r)$. In particular, each h_n is in this sense comparable to $h := \max_n h_n$ and $\underline{h} := \min_n h_n$. Since now $h_{n+1} - h_n = \big(s(r^{n+1}) - s(r^n)\big)/(J+1)$, observing that $s(r)$ is Lipschitz continuous on $[0, R]$ and denoting by L its Lipschitz constant, we obtain that $|h_{n+1} - h_n| \le L\,k/(J+1)$. As a consequence, we have

$$|h_{n+1} - h_n| \le C_0\, k\, \underline{h}, \qquad 0 \le n \le N-1, \tag{1.4}$$

where $C_0 = L/\underline{s}$, $\underline{s} := \min_{0\le r\le R} s(r)$. In the sequel it will be convenient to assume a weak mesh condition, namely that there exists a positive constant a such that

$$k \le ah. \tag{1.5}$$

An immediate consequence of (1.3) and (1.4) is that for $0 \le n \le N$,

$$(J+1)|h_{n+1} - h_n| \le C_1 k \underline{h}, \tag{1.6}$$

where $C_1 = \sigma a L$.

Using (1.6) we prove in section 2 that the scheme (1.2) is *stable* in the ℓ_2 sense. Specifically, we show in Proposition 2.1 that, for k sufficiently small, there exists a positive constant c, independent of h_n and k, such that $\|U^n\|_h \le c\|U^0\|_h$, $1 \le n \le N$, where U^n is any solution of (1.2) and $\|U\|_h := \Big[(J+1)^{-1}\sum_{j=1}^J |U_j|^2\Big]^{1/2}$ for $U \in \mathbb{C}_0^{J+2}$. As a consequence, for each n, the matrix of the tridiagonal linear system represented by (1.2) is invertible and the solution U^n exists uniquely. In section 3 we study the *consistency* of the scheme and derive an optimal-order bound for the local truncation error. We then go on to prove in Theorem 3.1 that the optimal-order error estimate

$$\max_n \|U^n - u^n\|_h \le c(k^2 + h^2)$$

holds, where here, and in the sequel, we denote by c generic positive constants independent of h_n and k. We close the paper with a section of numerical experiments that verify the order of convergence for various bottom topographies $s(r)$.

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2. Stability

First, let us observe that it may be easily established that (1.1) is *conservative in* L^2, in the sense that the integral $\int_0^{s(r)} |u(z,r)|^2\,dz$ remains constant on $[0,R]$. Indeed (denoting by $\bar z$ the complex conjugate of $z\in\mathbb{C}$), multiply (1.0) by $\bar u$ and integrate with respect to z on $[0,s(r)]$ to obtain for $r\ge 0$

$$\int_0^{s(r)} u_r\,\bar u\,dz = \mathrm{i}\alpha\int_0^{s(r)} u_{zz}\,\bar u\,dz + \mathrm{i}\int_0^{s(r)} \beta\,|u|^2\,dz.$$

Integrating by parts using the boundary conditions in (1.2) we then see that $\int_0^{s(r)} u_{zz}\,\bar u\,dz = -\int_0^{s(r)} |u_z|^2\,dz$. Hence, taking real parts in the equation above yields

$$\mathrm{Re}\int_0^{s(r)} u_r\,\bar u\,dz = 0. \tag{2.1}$$

On the other hand, by Leibniz's rule

$$\begin{aligned}\frac{d}{dr}\int_0^{s(r)} |u|^2\,dz &= \int_0^{s(r)} (u_r\,\bar u + u\,\bar u_r)\,dz + s'(r-)\,|u(s(r),r)|^2\\ &= 2\,\mathrm{Re}\int_0^{s(r)} u_r\bar u\,dz,\end{aligned}$$

where use has been made of the boundary condition $u=0$ at the bottom. Hence, for $r\ge 0$, (2.1) yields

$$\int_0^{s(r)} |u(z,r)|^2\,dz = \int_0^{s(0)} |u(z,0)|^2\,dz = \int_0^{1} |u_0(z)|^2\,dz. \tag{2.2}$$

To establish the ℓ_2-stability of the finite difference scheme (1.2) we shall roughly follow in the discrete mode the steps that led to (2.2). To this end, fix n and let $\Gamma_j := h_n U_j^{n+1} + h_{n+1} U_j^n$. Then, summation by parts easily yields

$$\sum_{j=1}^{J}\Big(\delta_h^2\Gamma_j\Big)\bar\Gamma_j = \sum_{j=1}^{J}\Big(\Gamma_{j-1}\bar\Gamma_j + \bar\Gamma_{j-1}\Gamma_j - 2|\Gamma_j|^2\Big)\in\mathbb{R}. \tag{2.3}$$

In addition, note that

$$\begin{aligned}&\sum_{j=1}^{J}\Big(h_{n+1}U_j^{n+1} - h_n U_j^n\Big)\bar\Gamma_j\\ &\quad= \sum_{j=1}^{J}\Big[h_{n+1}h_n\Big(|U_j^{n+1}|^2 - |U_j^n|^2\Big) + (h_{n+1}^2 - h_n^2)\,U_j^n\,\bar U_j^{n+1}\Big]\\ &\qquad + h_{n+1}^2\sum_{j=1}^{J}\Big(U_j^{n+1}\,\bar U_j^n - U_j^n\,\bar U_j^{n+1}\Big),\end{aligned} \tag{2.4}$$

and that the last sum of the right-hand side is purely imaginary. Denote now $A_j := U_j^{n+1} + U_j^n$ and obtain, after long but straightforward computations, the identity

$$\begin{aligned}\operatorname{Re}\sum_{j=1}^{J}\big[(j+1)A_{j+1}-(j-1)A_{j-1}\big]\bar{\Gamma}_j &= \operatorname{Re}\sum_{j=1}^{J}A_{j+1}\bar{\Gamma}_j\\ &-\frac{1}{2}(h_{n+1}-h_n)\sum_{j=1}^{J}j\big[(U_{j+1}^n\bar{U}_j^{n+1}+\bar{U}_{j+1}^nU_j^{n+1})-(U_{j+1}^{n+1}\bar{U}_j^n+\bar{U}_{j+1}^{n+1}U_j^n)\big].\end{aligned}\tag{2.5}$$

Finally, note that

$$\begin{aligned}\sum_{j=1}^{J}\beta_j^{n+1/2}(h_{n+1}U_j^{n+1}+h_nU_j^n)\bar{\Gamma}_j &= -(h_{n+1}^2-h_n^2)\sum_{j=1}^{J}\beta_j^{n+1/2}U_j^n\bar{U}_j^{n+1}\\ &+\sum_{j=1}^{J}\beta_j^{n+1/2}\Big[h_{n+1}h_n\left(|U_j^{n+1}|^2+|U_j^n|^2\right)+h_{n+1}^2\big(U_j^{n+1}\bar{U}_j^n+\bar{U}_j^{n+1}U_j^n\big)\Big],\end{aligned}\tag{2.6}$$

with the last sum of the right-hand side being of course real.

Multiply now both sides of (1.2) by $\bar{\Gamma}_j$, sum from $j=1$ to J using the identities (2.3)–(2.6) and obtain, taking real parts in the end:

$$\begin{aligned}h_nh_{n+1}\sum_{j=1}^{J}\left(|U_j^{n+1}|^2-|U_j^n|^2\right) &= -(h_{n+1}^2-h_n^2)\operatorname{Re}\sum_{j=1}^{J}U_j^n\bar{U}_j^{n+1}\\ &+\frac{1}{4}(h_{n+1}-h_n)\operatorname{Re}\sum_{j=1}^{J}A_{j+1}\bar{\Gamma}_j\\ &-\frac{1}{8}(h_{n+1}-h_n)^2\sum_{j=1}^{J}j\big[(U_{j+1}^n\bar{U}_j^{n+1}+\bar{U}_{j+1}^nU_j^{n+1})-(U_{j+1}^{n+1}\bar{U}_j^n+\bar{U}_{j+1}^{n+1}U_j^n)\big]\\ &+\frac{1}{2}(h_{n+1}^2-h_n^2)\operatorname{Im}\sum_{j=1}^{J}\beta_j^{n+1/2}U_j^n\bar{U}_j^{n+1}.\end{aligned}\tag{2.7}$$

Applying the Cauchy–Schwarz and the arithmetic–geometric mean inequalities in the right-hand side of (2.7), and using the mesh relations (1.3)–(1.6) we obtain for k sufficiently small:

$$\sum_{j=1}^{J}|U_j^{n+1}|^2 \le \frac{1+ck}{1-ck}\sum_{j=1}^{J}|U_j^n|^2.\tag{2.8}$$

Finally, use of (2.8) and the discrete Gronwall inequality gives the result of

Proposition 2.1 (Stability)**.** *Let* (1.5) *hold. Then if* U^n *is any solution of* (1.2)*, it satisfies*

$$\max_n \|U^n\|_h \le c\|U^0\|_h.\tag{2.9}$$

□

3. Consistency and convergence

We first estimate the local truncation error of the difference scheme (1.2):

Proposition 3.1 (Consistency)**.** *Let* (1.5) *hold, assume that* u*, the solution of* (1.1)*, is sufficiently smooth and recall the notation* L_h^n *for the difference operator in* (1.2)*. Then*

$$|L_h^n(u_j^n)| \le ckh(k^2+h^2), \qquad 1\le j\le J, \quad 0\le n\le N-1. \tag{3.1}$$

Proof. Long but straightforward Taylor expansions yield (in view of the mesh relations), for $1\le j\le J,\ 0\le n\le N-1$, the estimates

$$\begin{aligned} h_{n+1}u_j^{n+1} - h_n u_j^n - \frac14(h_{n+1}-h_n)\big[(j+1)(u_{j+1}^{n+1}+u_{j+1}^n)-(j-1)(u_{j-1}^{n+1}+u_{j-1}^n)\big]\\ = \frac12 k(h_{n+1}+h_n)u_r(P_j^{n+1/2}) + O(kh^3+k^3h),\end{aligned} \tag{3.2}$$

$$-\mathrm{i}\alpha\frac{k}{2}\Big[\frac{1}{h_{n+1}}\delta_h^2\, u_j^{n+1}+\frac{1}{h_n}\delta_h^2 u_j^n\Big] = -\mathrm{i}\alpha\frac{k}{2}\big(h_{n+1}+h_n\big)u_{zz}(P_j^{n+1/2})+O(kh^3+k^3h), \tag{3.3}$$

$$-\frac{\mathrm{i}k}{2}\beta_j^{n+1/2}(h_{n+1}u_j^{n+1}+h_n u_j^n) = -\frac{\mathrm{i}k}{2}(h_{n+1}+h_n)\beta_j^{n+1/2}u(P_j^{n+1/2}) + O(k^3h), \tag{3.4}$$

where the point $P_j^{n+1/2} := \big(z_j^{n+1/2}, r^{n+1/2}\big)$ can always be made to lie in the domain $0\le z\le s(r),\ 0\le r\le R$, by assuming that k or h is sufficiently small. (3.1) then follows from (3.2)–(3.4) and (1.0). □

Finally, putting together (3.1) and the energy method of the stability proof we may prove the following optimal-order ℓ_2 error estimate for our problem:

Theorem 3.1. *Assume that* (1.5) *holds and* u*, the solution of* (1.1)*, is sufficiently smooth. If* U^n *is the solution of the finite difference scheme* (1.2)*, we have*

$$\max_{0\le n\le N}\|U^n-u^n\|_h \le c(k^2+h^2). \tag{3.5}$$

Proof. Let $e^n = U^n - u^n$ and define $\rho_j^n = L_h^n(e_j^n), 1\le j\le J, 0\le n\le N-1$. Use of the linearity of L_h^n, (1.2) and (3.1) yields

$$\max_{n,j}|\rho_j^n| \le ckh\,(k^2+h^2)\,. \tag{3.6}$$

Now fix n and let $\gamma_j := h_n e_j^{n+1} + h_{n+1}e_j^n,\ a_j := e_j^{n+1}+e_j^n$. Then, we may rewrite the equation $L_h^n(e_j^n) = \rho_j^n$ for $1\le j\le J,\ 0\le n\le N-1$, as

$$\begin{aligned} h_{n+1}e_j^{n+1} - h_n e_j^n =& \frac14(h_{n+1}-h_n)\big[(j+1)a_{j+1}-(j-1)a_{j-1}\big]\\ &- \frac{\mathrm{i}\alpha k}{2h_n h_{n+1}}\delta_h^2\gamma_j + \frac{\mathrm{i}k}{2}\beta_j^{n+1/2}\,(h_{n+1}e_j^{n+1}+h_n e_j^n) + \rho_j^n.\end{aligned} \tag{3.7}$$

In analogy with similar computations made in the course of the energy proof of the stability of our scheme in section 2 we obtain now

$$\sum_{j=1}^{J}(\delta_h^2\gamma_j)\,\bar\gamma_j = \sum_{j=1}^{J}\big(\gamma_{j-1}\,\bar\gamma_j + \bar\gamma_{j-1}\gamma_j - 2|\gamma_j|^2\big) \in \mathbb{R}, \tag{3.8}$$

$$\begin{aligned}\sum_{j=1}^{J}(h_{n+1}e_j^{n+1} - h_n e_j^n)\bar\gamma_j &= h_n h_{n+1}(J+1)\big(\|e^{n+1}\|_h^2 - \|e^n\|_h^2\big)\\ &\quad + (h_{n+1}^2 - h_n^2)\sum_{j=1}^{J} e_j^n \bar e_j^{n+1} + h_{n+1}^2\sum_{j=1}^{J}(e_j^{n+1}\bar e_j^n - e_j^n\,\bar e_j^{n+1}),\end{aligned} \tag{3.9}$$

(with the last term being of course purely imaginary),

$$\begin{aligned}\mathrm{Re}\sum_{j=1}^{J}\big[(j+1)a_{j+1} - (j-1)a_{j-1}\big]\bar\gamma_j &= \mathrm{Re}\sum_{j=1}^{J} a_{j+1}\,\bar\gamma_j\\ &- \frac{1}{2}(h_{n+1}-h_n)\sum_{j=1}^{J} j\Big[\big(e_{j+1}^n\bar e_j^{n+1} + \bar e_{j+1}^n e_j^{n+1}\big) - \big(e_{j+1}^{n+1}\bar e_j^n + \bar e_{j+1}^{n+1}e_j^n\big)\Big],\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\sum_{j=1}^{J}\beta_j^{n+1/2}(h_{n+1}e_j^{n+1} + h_n e_j^n)\bar\gamma_j &= -(h_{n+1}^2 - h_n^2)\sum_{j=1}^{J}\beta_j^{n+1/2}e_j^n\bar e_j^{n+1}\\ &+ \sum_{j=1}^{J}\beta_j^{n+1/2}\Big[h_{n+1}h_n\big(|e_j^{n+1}|^2 + |e_j^n|^2\big) + h_{n+1}^2\big(e_j^{n+1}\bar e_j^n + \bar e_j^{n+1}e_j^n\big)\Big]\end{aligned} \tag{3.11}$$

(the last term of the right-hand side being of course real).

Hence, multiplying both sides of (3.7) by $\bar\gamma_j$, summing from $j = 1$ to J and taking real parts we conclude, using (3.8)–(3.11), much as in (2.7):

$$\begin{aligned}h_{n+1}h_n(J+1)\big(\|e^{n+1}\|_h^2 - \|e^n\|_h^2\big) &= -(h_{n+1}^2 - h_n^2)\,\mathrm{Re}\sum_{j=1}^{J} e_j^n\,\bar e_j^{n+1}\\ &+ \frac{1}{4}(h_{n+1}-h_n)\,\mathrm{Re}\sum_{j=1}^{J} a_{j+1}\bar\gamma_j\\ &- \frac{1}{8}(h_{n+1}-h_n)^2\sum_{j=1}^{J} j\Big[\big(e_{j+1}^n\bar e_j^{n+1} + \bar e_{j+1}^n e_j^{n+1}\big) - \big(e_{j+1}^{n+1}\bar e_j^n + \bar e_{j+1}^{n+1}e_j^n\big)\Big]\\ &+ \frac{k}{2}(h_{n+1}^2 - h_n^2)\,\mathrm{Im}\sum_{j=1}^{J}\beta_j^{n+1/2}e_j^n\bar e_j^{n+1} + \mathrm{Re}\sum_{j=1}^{J}\rho_j^n\bar\gamma_j.\end{aligned}$$

Hence, using our mesh estimates (1.3)–(1.6) we see that the above implies

$$\begin{aligned}\|e^{n+1}\|_h^2 - \|e^n\|_h^2 \leq & ckh\Big|\,\mathrm{Re}\sum_{j=1}^{J} e_j^n\,\bar{e}_j^{n+1}\Big| + ck\,\Big|\,\mathrm{Re}\sum_{j=1}^{J} a_{j+1}\,\bar{\gamma}_j\Big| \\ & + ckh\Big|\sum_{j=1}^{J}\Big[\big(e_{j+1}^n\,\bar{e}_j^{n+1} + \bar{e}_{j+1}^n\,e_j^{n+1}\big) - \big(e_{j+1}^{n+1}\,\bar{e}_j^n + \bar{e}_{j+1}^{n+1}\,e_j^n\big)\Big]\Big| \\ & + ck^2h\Big|\,\mathrm{Im}\sum_{j=1}^{J}\beta_j^{n+1/2}\,e_j^n\bar{e}_j^{n+1}\Big| + c(J+1)\Big|\,\mathrm{Re}\sum_{j=1}^{J}\rho_j^n\bar{\gamma}_j\Big|.\end{aligned}$$

Using now the Cauchy–Schwarz and the arithmetic–geometric mean inequalities in the right-hand side of the above, as well as the bound (3.6), we obtain

$$\|e^{n+1}\|_h^2 \leq \frac{1+ck}{1-ck}\|e^n\|_h^2 + ck(k^2+h^2)^2,$$

for k sufficiently small; (3.5) follows in view of Gronwall's discrete inequality. □

4. Numerical experiments

In this section we present the results of some simple numerical experiments that were run (using double precision in Fortran on a VAX 8600 at the University of Crete) with various bottom boundaries $s(r)$ to verify the orders of convergence proved in section 3. We considered the non-homogeneous equation

$$u_r = \mathrm{i}u_{zz} + \mathrm{i}\beta(z,r)u + f(z,r) \tag{4.1}$$

for $0 \leq r \leq 1,\ 0 \leq z \leq s(r)$, with boundary and initial conditions as in (1.1). (It is straightforward to extend the error estimate (3.5) to the case of a nonhomogeneous equation such as (4.1), provided we add to the difference scheme (1.2) the forcing term $-kf(z_j^{n+1/2}, r^{n+1/2})$.) We experimented with six choices of the function $s(r), 0 \leq r \leq 1$, namely:

$$s(r) = 1, \tag{4.2}$$

$$s(r) = 1 + r, \tag{4.3}$$

$$s(r) = 1 + r^2, \tag{4.4}$$

$$s(r) = 1 + 0.3\sin 2\pi r, \tag{4.5}$$

$$s(r) = 1 - \tfrac{1}{2}r \qquad \text{(converging duct)}, \tag{4.6}$$

$$s(r) = \begin{cases} 1 & \text{if } \ 0 \leq r \leq \frac{1}{2} \\ \frac{1}{2} + r & \text{if } \ \frac{1}{2} \leq r \leq 1 \end{cases} \qquad \text{(piecewise linear)}. \tag{4.7}$$

In all cases the exact solution of (4.1) was taken to be

$$u(z,r) = (z - s(r))\sin \pi z \sin r + \mathrm{i}z(z - s(r))\cos r,$$

while the coefficient $\beta(z,r)$ was equal to $\sin(rz)$. The nonhomogeneous term f and the initial condition u_0 were computed by (4.1) and the exact solution. We computed with

$k = h_0 = (J+1)^{-1}$, where $J+1 = 10, 20, \ldots, 100$, with the difference scheme (solving the tridiagonal linear system at each range step by the LINPACK subroutine ZGTSL) and recorded the error $E^N := \|U^N - u^N\|_h$ at the final range level $r^N = 1$. In table 1 we show the errors E^N for all six $s(r)$'s and the resulting convergence rates between successive runs. The predicted second-order rate of convergence emerges clearly from these experiments. (Note that since the number of range steps was always even, a node was always placed at the point $r = 1/2$, where the derivative s' is discontinuous in example (4.7).)

	(4.2)		(4.3)		(4.4)	
$J+1$	E^N	rate	E^N	rate	E^N	rate
10	.32856-2		.29844-1		.30134-1	
20	.88455-3	1.893	.74279-2	2.006	.74421-2	2.018
30	.39726-3	1.974	.33186-2	1.987	.32997-2	2.006
40	.22573-3	1.965	.18650-2	2.003	.18547-2	2.003
50	.14419-3	2.009	.11906-2	2.011	.11862-2	2.003
60	.99898-4	2.013	.82589-3	2.006	.82359-3	2.001
70	.73296-4	2.009	.60642-3	2.004	.60507-3	2.000
80	.56077-4	2.005	.46416-3	2.002	.46323-3	2.000
90	.44291-4	2.003	.36667-3	2.001	.36598-3	2.000
100	.35869-4	2.002	.29696-3	2.001	.29642-3	2.001

	(4.5)		(4.6)		(4.7)	
$J+1$	E^N	rate	E^N	rate	E^N	rate
10	.90194-2		.22914-2		.96149-2	
20	.18574-2	2.280	.56627-3	2.017	.28714-2	1.744
30	.86354-3	1.889	.25335-3	1.984	.12746-2	2.003
40	.48501-3	2.005	.14540-3	1.930	.70387-3	2.064
50	.31024-3	2.002	.93999-4	1.955	.44357-3	2.069
60	.21582-3	1.991	.65782-4	1.958	.30556-3	2.044
70	.15893-3	1.985	.48407-4	1.990	.22338-3	2.032
80	.12211-3	1.973	.37028-4	2.006	.17051-3	2.022
90	.96769-4	1.975	.29341-4	1.975	.13448-3	2.015
100	.78441-4	1.993	.23782-4	1.994	.10882-3	2.010

Table 1. Errors E^N and convergence rates for the examples (4.2)–(4.7)

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